UNIVERSITÀ DEGLI STUDI DI FERRARA

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STUDY OF STABILITY FOR AN ELASTIC ROD

Tesi di Laurea

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Sunto

In questa tesi abbiamo studiato il problema della individuazione e della classificazione dei punti stazionari di una sbarra elastica soggetta ad un carico concentrato in una delle due estremità. Tale studio ha portato alla ricerca delle soluzioni di una equazione molto classica, l'equazione del pendolo fisico.

Per arrivare alla formulazione rigorosa del problema, abbiamo dimostrato una generalizzazione del Teorema della Funzione Implicita, strumento fondamentale per lo sviluppo dei metodi di Calcolo delle Variazioni qui utilizzati.

Abbiamo approcciato il problema dello studio della stabilità delle configurazioni di equilibrio.

Abstract

In this dissertation we were study the problem of determination and classification of stationary points for an elastic rod subjected to an external force acting on one of the two extremities. This study took us to the research of solutions of a very classical equation, that is the equation of the physical pendulum.

To reach the correct formulation of the problem, we were demonstrate a generalization of the Implicit Function Theorem, fundamental tool for the development of methods of Calculus of Variations needed here.

Finally we have approached the stability's problem of the equilibrium configurations.

Contents

Chapter 1

Introduction

In this dissertation we have considered the problem of finding and classifying the stationary configuration of an elastic rod subjected to an external force. This study has many application in the theory of constructions; for instance, it can be the study of the deformation of a bar or of a column at which an external weight is applied.

We have found all the possible configuration; they come out by imposing the equilibrium condition, that is the first variation of the energy functional equals to 0. Such variation, or better the Euler-Lagrange equation associated to the energy, give rise to the characteristic equation of a pendulum; for this reason we have inserted in the Appendix A a detailed description of the pendulum.

In the next section, we start by translating in a mathematical language the problem of the elastic rod; then we shall study, in the following chapters, the definition of stationary point and show that they are determined by the solution of the pendulum equation.

In the last part of the thesis we give some idea on how to study the stability of the stationary configurations. This part is not exhaustive; the main reason of this is that a complete characterization is not even present in the literature. We give some possible approach that can prove the classification.

1.1 Definition of the problem

The problem is then the study of the stability of an elastic rod of length ℓ subjected to an external force $\vec{F}$. We may assume that the rod has the initial point positioned at the origin of the axis and that the end point is constrained to lie on the x-axis. By this, assuming the rod to be arc length parametrized by $\gamma : [0, \ell] \to \mathbb{R}^2$ with

$$\gamma(0) = (0,0), \qquad \gamma(\ell) = (\gamma_1(\ell), 0).$$

The fact that we are assuming arc length parametrization means that

$$\|\gamma'(s)\| = 1;$$

otherwise stated, this means that we may define the function $\vartheta : [0, \ell] \to \mathbb{R}$ such that

$$\gamma'(s) = (\cos\vartheta(s), \sin\vartheta(s)).$$

The elastic energy of the rod, assuming the rod to be homogeneous with constant elastic density given by B, is given by

$$\mathscr{E}_{\text{el}} = \frac{B}{2}\int_0^\ell k_\gamma(s)^2 ds,$$

where k_γ is the curvature of the rod; by the fact that

$$\gamma'(s) = \vartheta'(s)(-\sin\vartheta(s), \cos\vartheta),$$

whence

$$\mathscr{E}_{\text{el}} = \frac{B}{2}\int_0^\ell \vartheta'(s)^2 ds.$$

Assuming now that the external force is horizontal $\vec{F} = (-F, 0)$, the total work done by the force is given in part by the work done give rise to the elastic energy plus the work done moving the end point of the curve of a length Δx, that is

$$\begin{aligned}\frac{B}{2}\int_0^\ell \vartheta'(s)^2 ds - F\Delta x =& \frac{B}{2}\int_0^\ell \vartheta'(s)^2 ds - F\left(\ell - \int_0^\ell \gamma'(s)\cdot e_1 ds\right)\\ =& \frac{B}{2}\int_0^\ell \vartheta'(s)^2 ds - F\left(\ell - \int_0^\ell \cos\vartheta(s) ds\right).\end{aligned}$$

Introducing the parameter $q = \frac{F}{B}$, the total work done by the external force, divided by B, is a function of ϑ as follows

$$\mathscr{E}(\vartheta) = \frac{1}{2}\int_0^\ell \vartheta'(s)^2 ds - q\left(\ell - \int_0^\ell \cos\vartheta(s) ds\right).$$

We notice that such a functional is bounded below by

$$\mathscr{E}(\vartheta) \geq -2q\ell$$

and that such value is assumed when $\gamma(\ell) = (-\ell, 0)$; this is then the only global miminum for $\mathscr{E}$. We shall see that it is not the only one and also that there are infinitely many stationary points. The right space where to define the functional $\mathscr{E}$ is the space $L^2([0,\ell])$; the exact domain $\mathscr{E}$ is given by the functions ϑ such that $\mathscr{E}(\vartheta) < +\infty$, that is $W^{1,2}([0,\ell])$.

The assumption that the end point of the curve belongs to the x-axis is the requirement that $\vartheta \in E$, with

$$E = \left\{ \vartheta \in L^2([0,\ell]) : \int_0^\ell \sin\vartheta(s) ds \right\}.$$

We shall then start by stundying the properties of the functional defining E, that is the functional $\mathscr{G} : L^2([0,\ell]) \to \mathbb{R}$

$$\mathscr{G}(\vartheta) = \int_0^\ell \sin\vartheta(s) ds.$$

After this, we shall study the problem of constrained stationary points of $\mathscr{E}$, that is the possibility of defining the Lagrange multiplier

$$\mathscr{E}(\vartheta) + \lambda \mathscr{G}(\vartheta).$$

In the last chapter we shall also introduce the problem of the study of the stability of the stationary points; this is not an exhaustive analysis, since it is still an open problem. We suggest a possible approach, alternative to the eigenvalues analysis.

Chapter 2

Implicit Function Theorem

In this chapter we shall prove the Implicit Function Theorem; the space where the functional $\mathscr{G}$ is defined is the space $L^2([0,\ell])$, that is the space of measurable functions $u : [0,\ell] \to \mathbb{R}$ such that

$$\int_0^\ell u(s)^2 ds < +\infty.$$

Such space is an Hilbert space with the inner product

$$\langle u, v\rangle = \int_0^\ell u(s)v(s)ds$$

and norm given by $\|u\|_2 = \langle u, u\rangle^{1/2}$. The space $L^2([0,\ell])$ contains for instance all the continuous functions on $[0,\ell]$ and the piecewise constant functions on $[0,\ell]$. The domain where $\mathscr{E}$ is finite and differentiable is the space $W^{1,2}([]0,\ell)$, that is the measurable functions $u \in L^2([0,\ell])$ such that there exists a weak derivative $u' \in L^2([0,\ell])$. Such space contains all the continuous functions that are piecewise regular.

2.1 Regularity of the constraint

In this section we prove the differentiability of $\mathscr{G}$; we remind that a functional $\mathscr{G} : L^2([0,\ell]) \to \mathbb{R}^2$ said to be Frechet differentiable at $\vartheta \in L^([0,\ell])$ if there exists a linear functional $\mathscr{G} : L^2([0,\ell]) \to \mathbb{R}^2$ such that:

$$\lim_{v\in L^2([0,\ell]), \|v\|_2 \to 0} \frac{\mathscr{G}(\vartheta + v) - \mathscr{G}(v) - \mathscr{G}'(\vartheta)[v]}{\|v\|_2} = 0. \tag{2.1}$$

We can state the following Proposition.

Proposition 2.1. *The function $\mathscr{G} : L^2([0,\ell]) \to \mathbb{R}^2$ is differentiable and the differential is represented by $\nabla\mathscr{G}(\vartheta) = \cos\vartheta$ in the sense that*

$$\mathscr{G}'(\vartheta)[v] = d_\vartheta\mathscr{G}(v) = \int_0^\ell v(s)\cos\vartheta(s)ds$$

for any $v \in L^2([0,\ell])$.

Proof. It is enough to prove the identity

$$\mathscr{G}'(\vartheta)[v] = \int_0^\ell v(s)\cos\vartheta(s)ds,$$

for any $v \in L^2([0,\ell])$. Let us fix $v \in L^2([0,\ell])$ with small L^2-norm and define $\varepsilon^2 = \|v\|_2$. We define the set

$$I_\varepsilon = \{s \in [0,\ell] : |v(s)| < \varepsilon\}.$$

We point out that on I_ε^c, $|v(s)| \geq \varepsilon$ and then by Hölder inequality

$$|I_\varepsilon^c| = \int_{I_\varepsilon^c} ds \leq \frac{1}{\varepsilon}\int_{I_\varepsilon^c} |v(s)|ds \leq \frac{1}{\varepsilon}|I_\varepsilon|^{1/2}\|v\|_2,$$

that implies $|I_\varepsilon^c| \leq \varepsilon^2$. In addition, on I_ε we can use the Taylor expansion

$$\sin(\vartheta(s)+v(s)) = \sin\vartheta(s) + \cos\vartheta(s)v(s) - \frac{1}{2}\sin\vartheta(s)v(s)^2 + o(\varepsilon^2).$$

We then obtain

$$\begin{aligned}
&\mathscr{G}(\vartheta+v) - \mathscr{G}(\vartheta) - \int_0^\ell v(s)\cos\vartheta(s)ds = \\
&\quad= \int_0^\ell \Big(\sin(\vartheta(s)+v(s)) - \sin\vartheta(s) - v(s)\cos\vartheta(s)\Big)ds \\
&\quad= \int_{I_\varepsilon} \Big(\sin(\vartheta(s)+v(s)) - \sin\vartheta(s) - v(s)\cos\vartheta(s)\Big)ds+ \\
&\qquad+ \int_{I_\varepsilon^c} \Big(\sin(\vartheta(s)+v(s)) - \sin\vartheta(s) - v(s)\cos\vartheta(s)\Big)ds \\
&\quad= \int_{I_\varepsilon^c} \Big(\sin(\vartheta(s)+v(s)) - \sin\vartheta(s) - v(s)\cos\vartheta(s)\Big)ds+ \\
&\qquad- \frac{1}{2}\int_{I_\varepsilon} v(s)^2\sin\vartheta(s)ds + o(\varepsilon^2);
\end{aligned}$$

from this and the fact that $|\sin(\vartheta(s) - v(s)) - \sin\vartheta(s)| \leq |v(s)|$, we obtain that

$$\begin{aligned}
&\left|\mathscr{G}(\vartheta + v) - \mathscr{G}(\vartheta) - \int_0^\ell v(s)\cos\vartheta(s)ds\right| \leq \\
&\quad \leq \int_{I_\varepsilon^c} |\sin(\vartheta(s) + v(s)) - \sin\vartheta(s)|ds + \int_{I_\varepsilon} |v(s)||\cos\vartheta(s)|ds + \\
&\quad\quad + \frac{1}{2}\int_{I_\varepsilon} v(s)^2|\sin\vartheta(s)|ds + o(\varepsilon^2) \\
&\quad \leq 2\int_{I_\varepsilon^c} |v(s)|ds + \frac{1}{2}\|v\|_2^2 + o(\varepsilon^2) \\
&\quad \leq 2\|v\|_2|I_\varepsilon^c|^{1/2} + \frac{1}{2}\|v\|_2^2 + o(\varepsilon^2) \\
&\quad \leq \frac{5}{2}\|v\|_2^2 + o(\varepsilon^2)
\end{aligned}$$

and from this, using the definition in (2.1), the proof follows. □

As a corollary of the previous Proposition we have also the fact that $\mathscr{G}$ is of class C^1, that the differential is continuous. This is an easy consequence of the formula

$$|\cos\vartheta_1 - \cos\vartheta_2| \leq |\vartheta_1 - \vartheta_2|$$

since by this we get

$$\begin{aligned}
\|\nabla\mathscr{G}(\vartheta_1) - \nabla\mathscr{G}(\vartheta_2)\|_2 &= \left(\int_0^\ell |\cos\vartheta_1(s) - \cos\vartheta_2(s)|^2 ds\right)^{1/2} \\
&\leq \left(\int_0^\ell |\vartheta_1(s) - \vartheta_2(s)|^2 ds\right)^{1/2} = \|\vartheta_1 - \vartheta_2\|_2.
\end{aligned}$$

It is possible to prove that $\mathscr{G}$ is differentiable infinitly many times and its differentials are continuous functions. We don't need to compute the differentials different from the first one and its continuity is fundamental in the proof of the implicit Function Theorem.

2.2 The Implicit Function Theorem

We want to prove in this section the Implicit Function Theorem for any point

$$\vartheta \in E = \{\vartheta \in L^2([0,\ell]) : \mathscr{G}(\vartheta) = 0\}.$$

To apply the Theorem we have to assume $\cos\vartheta(s) \not\equiv 0$; but this is not too restrictive, since in this case we would have that

$$\vartheta(s) \equiv \frac{\pi}{2} + k\pi, \qquad k \in \mathbb{Z}$$

that doesn't satisfy $\mathscr{G}(\vartheta) = 0$ and so $\vartheta \notin E$.

We also set $c = \|\cos\vartheta\|_2$ so that the element

$$e(s) = \frac{1}{c}\cos\vartheta(s)$$

is e versor in $L^2([0,\ell])$, i.e. $\|e\|_2 = 1$; in this way we obtain the orthogonal decomposition

$$L^2([0,\ell]) = \langle e \rangle \oplus H,$$

with $H = e^{\perp}$.

Theorem 2.2 (Implicit function)**.** *Let $\vartheta \in E$ be a fixed point; then, there exists $\varepsilon > 0$ and a function $\varphi : H \cap B_\varepsilon(0) \to \mathbb{R}$ with $\varphi(0) = 0$ such that $E \cap B_\varepsilon(\vartheta)$ is given by the graph of φ, that is*

$$E \cap B_\varepsilon(\vartheta) = \{\eta \in L^2([0,\ell]) : \eta(s) = \theta(s) + v(s) + \varphi(v)e(s), v \in H \cap B_\varepsilon(0)\}.$$

The function φ is twice differentiable with

$$d_0\varphi(v) = \int_0^\ell v(s)\cos\vartheta(s)ds = 0$$

and

$$d_0^2\varphi(v) = c\int_0^\ell v(s)^2 \sin\vartheta(s)ds;$$

as a consequence, $T_\vartheta E = H$.

Proof. The proof is rather classical, but we repeat here the argument since we are working in a function space. The map

$$g_0(t) = \int_0^\ell \sin(\vartheta(s) + te(s))ds = \mathscr{G}(\vartheta + te)$$

is such that

$$\begin{cases} g_0(0) = 0 \\ g_0'(t) = \displaystyle\int_0^\ell e(s)\cos(\vartheta(s) + te(s))ds, \end{cases}$$

that is

$$g_0'(0) = \frac{1}{c}\int_0^\ell \cos^2\vartheta(s)ds = c > 0.$$

This implies that g_0 is positive for $t > 0$ and negative for $t < 0$, if $|t|$ is small enough. Let us fix $\delta_1 > 0$ such that $g_0(t) > 0$ for $t \in (0, \delta_1]$ and $g_0(t) < 0$ for $t \in [-\delta_1, 0)$. The continuity of $\mathscr{G}$ implies that there exists $\delta_2 > 0$ such that if $\|v\|_2 < \delta_2$ then

$$\mathscr{G}(\vartheta + v + \delta_1 e) > 0, \qquad \mathscr{G}(\vartheta + v - \delta_1 e) < 0.$$

Let us now consider the map $g_v : [-\delta_1, \delta_1] \to \mathbb{R}$ given by

$$g_v(t) = \mathscr{G}(\vartheta + v + te);$$

this map is differentiable with

$$g_v'(t) = \int_0^\ell e(s) \cos(\vartheta(s) + v(s) + te(s)) ds.$$

Since $\mathscr{G}$ is of class C^1, we deduce that there exists $\delta_3 > 0$ such that if $\|v\|_2 < \delta_3$, then $g_v'(t) > 0$. Then the map

$$t \mapsto g_v(t)$$

admits an unique t_v such that $g_v(t_v) = 0$; such value t_v defines the implicit function, i.e. the implicit function is defined by $\varphi(v) = t_v$. It is easy to show that φ is twice differentiable (the most difficult part is the continuity, which we omit here, whether the differentiability is easier). The formulae defining $d_0\varphi(v)$ and $d_0^2\varphi(v)$ are obtained by considering the maps

$$\Phi_t(s) = \vartheta(s) + tv(s) + \varphi(tv)e(s), \qquad v \in H$$

with t such that $tv \in H \cap B_\varepsilon(0)$. Indeed, from one side we have that

$$\frac{d}{dt}\Phi_t(s)_{|t=0} = v(s) + d_0\varphi(v)e(s)$$

and

$$\frac{d^2}{dt^2}\Phi_t(s)_{|t=0} = d_0^2\varphi(v)e(s),$$

and on the other side

$$\begin{aligned} 0 =& \frac{d}{dt}\mathscr{G}(\Phi_t)_{|t=0} = \frac{d}{dt}\int_0^\ell \sin(\vartheta(s) + tv(s) + \varphi(tv)e(s)) ds_{|t=0} \\ =& \underbrace{\int_0^\ell v(s)\cos\vartheta(s) ds}_{=0} + d_0\varphi(v) \underbrace{\int_0^\ell e(s)\cos\vartheta(s) ds}_{=c} \end{aligned}$$

and

$$\begin{aligned}
0 =& \frac{d^2}{dt^2}\mathscr{G}(\Phi_t)_{|t=0} \\
=& -\int_0^\ell \sin\vartheta(s)\left(\frac{d}{dt}\Phi_t(s)_{|t=0}\right)^2 ds + \int_0^\ell \cos\vartheta(s)ds\frac{d^2}{dt^2}\Phi_t(s)_{|t=0}ds \\
=& -\int_0^\ell v(s)\sin\vartheta(s)ds + d_0^2\varphi(v)\int_0^\ell e(s)\cos\vartheta(s)ds.
\end{aligned}$$

□

Chapter 3

Stationary points and stability

In this chapter we find and study the stability of stationary points of the functional $\mathscr{E}$. In order to do this, we shall consider the function Φ_t defined in the previous chapter so that the derivatives are:

$$\frac{d}{dt}\mathscr{E}(\Phi_t)_{|t=0}, \qquad \frac{d^2}{dt^2}\mathscr{E}(\Phi_t)_{|t=0}.$$

To compute these derivatives, we have to assume that $\mathscr{E}(\Phi_t) < 0$ and this is true if $\vartheta \in W^{1,2}([0,\ell])$ and $v \in H \cap W^{1,2}([0,\ell])$.

Definition 3.1. *The function $\vartheta \in W^{1,2}([0,\ell])$ is said to be a stationary point for $\mathscr{E}$ if*

$$\frac{d}{dt}\mathscr{E}(\Phi_t)_{|t=0} = 0, \qquad \forall v \in H \cap W^{1,2}([0,\ell]).$$

Moreover, ϑ is said to be stable if

$$\frac{d^2}{dt^2}\mathscr{E}(\Phi_t)_{|t=0} > 0, \qquad \forall v \in H \cap W^{1,2}([0,\ell]),$$

unstable otherwise, that is if there exists $v \in H \cap W^{1,2}([0,\ell])$ such that

$$\frac{d^2}{dt^2}\mathscr{E}(\Phi_t)_{|t=0} < 0.$$

To compute explicitly the first and second variation of $\mathscr{E}$, we recall some facts; if we denote by $\Phi_t'(s)$ and $\Phi_t''(s)$ the first and second derivative of $\Phi_t(s)$ respectively with respect to s, we have that

$$\Phi_t'(s) = \vartheta'(s) + tv'(s) + \varphi(tv)e'(s), \quad \Phi_t''(s) = \vartheta''(s) + tv''(s) + \varphi(tv)e''(s);$$

in addition

$$\frac{d}{dt}\Phi_t(s)_{|t=0} = v(s) + d_0\varphi(v)e(s) = 0,$$

$$\frac{d}{dt}\Phi_t'(s)_{|t=0} = v(s) + d_0\varphi(v)e'(s) = 0,$$

$$\frac{d^2}{dt^2}\Phi_t(s)_{|t=0} = d_0^2\varphi(v)e(s) = \cos\vartheta(s)\int_0^\ell \sin\vartheta(\tau)d\tau,$$

and

$$\frac{d^2}{dt^2}\Phi_t'(s)_{|t=0} = d_0^2\varphi(v)e'(s) = -\sin\vartheta(s)\int_0^\ell \sin\vartheta(\tau)d\tau.$$

From these equations we obtain that

$$\begin{aligned}\frac{d}{dt}\mathscr{E}(\Phi_t) =& \frac{d}{dt}\left(\frac{1}{2}\int_0^\ell \Phi_t'(s)^2 ds - q\ell + q\int_0^\ell \cos\Phi_t(s)ds\right)\\ =& \int_0^\ell \Phi_t'(s)\frac{d}{dt}\Phi_t'(s)ds - q\int_0^\ell \sin\Phi_t(s)\frac{d}{dt}\Phi_t(s)ds\end{aligned}$$

and

$$\begin{aligned}\frac{d^2}{dt^2}\mathscr{E}(\Phi_t) =& \int_0^\ell \left(\frac{d}{dt}\Phi_t'(s)\right)^2 ds + \int_0^\ell \Phi_t'(s)\frac{d^2}{dt^2}\Phi_t'(s)ds +\\ & - q\int_0^\ell \cos\Phi_t(s)\left(\frac{d}{dt}\Phi_t(s)\right)^2 ds - q\int_0^\ell \sin\Phi_t(s)\frac{d^2}{dt^2}\Phi_t(s)ds.\end{aligned}$$

So the first variation of $\mathscr{E}$ is given by

$$\frac{d}{dt}\mathscr{E}(\Phi_t)_{|t=0} = \int_0^\ell \vartheta'(s)v'(s)ds - q\int_0^\ell v(s)\sin\vartheta(s)ds$$

and the second variation is given by

$$\begin{aligned}\frac{d^2}{dt^2}\mathscr{E}(\Phi_t)_{|t=0} =& \int_0^\ell v'(s)^2 ds + c\int_0^\ell \vartheta'(s)e'(s)ds\int_0^\ell \sin\vartheta(s)v(s)^2 ds +\\ & - q\int_0^\ell cos\vartheta(s)v(s)^2 ds +\\ & - q\int_0^\ell \sin\vartheta(s)\cos\vartheta(s)ds\int_0^\ell \sin\vartheta(s)v(s)^2 ds.\end{aligned}$$

3.1 Stationary points

In this section we shall find all the functions $\vartheta \in W^{1,2}([0\ell])$ satisfying the condition

$$\frac{d}{dt}\mathscr{E}(\Phi_t) = 0, \qquad \forall v \in H.$$

A little remark on the regularity; the functions ϑ that are the solution of the previous equation are weak solutions of a second order differential equation with good coefficients, so ϑ is analytic, since we can use classical results on regularity for differential equations. So, in the sequel we shall write without any other discussion, the derivatives of any order of ϑ and use the fact that they are continuous on the closed interval $[0, \ell]$.

We start with the following Lemma.

Lemma 3.2. *Let ϑ a stationary point for $\mathscr{E}$; then $\vartheta'(0) = \vartheta'(\ell) = 0$.*

Proof. Let us assume that $\vartheta'(0) \neq 0$; then the function $\cos\vartheta$ has a well defined sign around the point $s_0 = 0$. In addition, there exists a second point $s_1 \in (0, \ell)$ for which $\cos\vartheta(s_1) \neq 0$. We then define (see Figure3.1) the function

$$v_h(s) = (1 - hs)\chi_{[0,\frac{1}{h}]}(s) + \sigma\left(1 - k_h \frac{s - s_1}{|s - s_1|}\right)\chi_{[s_1 - \frac{1}{k_h}, s_1 + \frac{1}{k_h}]}(s);$$

here σ depends on the sign of $\cos\vartheta(s_1)$ and on the sign of $\cos\vartheta(s)$ on $[0, \frac{1}{h}]$

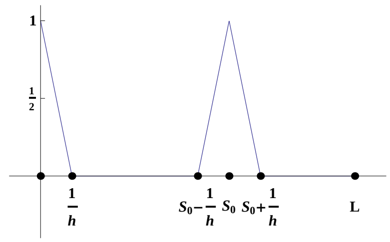

Figure 3.1: The graph of the function v_h.

and k_h is defined in such a way that $\frac{1}{h} < s_1 - \frac{1}{k_h}$, $s_1 + \frac{1}{k_h} < \ell$ and

$$\int_0^\ell v_h(s)\cos\vartheta(s)ds = 0.$$

The existence of such k_h follows by the application of the Implicit Function Theorem applied to the function

$$f(x, y) = \int_0^\ell \left(\left(1 - \frac{s}{x}\right)\chi_{[0,x]}(s) + \sigma\left(1 - \frac{s - s_1}{y|s - s_1|}\right)\chi_{[s_1 - y, s_1 + y]}\right)\cos\vartheta(s)ds.$$

It is easy indeed to show that

$$\lim_{(x,y)\to 0} f(x, y) = 0$$

and that f is of class C^1 with $\nabla f(0, 0) \neq 0$.

Inserting such v_h in the first variation, we obtain that

$$
\begin{aligned}
0 = &- h\int_0^{\frac{1}{h}} \vartheta'(s)ds - \sigma k_h \int_{s_1-\frac{1}{k_h}}^{s_1+\frac{1}{k_h}} \frac{s-s_1}{|s-s_1|}\vartheta'(s)ds+ \\
= &- h\int_0^{\frac{1}{h}} \vartheta'(s)ds + \sigma k_h \int_{s_1-\frac{1}{k_h}}^{s_1} \vartheta'(s)ds+ \\
&- \sigma k_h \int_{s_1}^{s_1+\frac{1}{k_h}} \vartheta'(s)ds - q\int_0^\ell v_h(s)\sin\vartheta(s)ds.
\end{aligned}
$$

Since

$$\lim_{h\to+\infty} h\int_0^{\frac{1}{h}} \vartheta'(s)ds = \vartheta'(0),$$

$$\lim_{h\to+\infty} k_h \int_{s_1-\frac{1}{k_h}}^{s_1} \vartheta'(s)ds = \lim_{h\to+\infty} k_h \int_{s_1} s_1 + \frac{1}{k_h}^{s_1} \vartheta'(s)ds = \vartheta'(s_1)$$

and

$$\lim_{h\to+\infty} \int_0^\ell v_h(s)\sin\vartheta(s)ds = 0$$

we obtain that $\vartheta'(0) = 0$, a contradiction. □

We are now in the position to prove the following Theorem.

Theorem 3.3. *A function $\vartheta \in W^{1,2}([0,\ell])$ is a stationary point of $\mathscr{E}$ if and only if is a solution of the problem*

$$\begin{cases} \vartheta''(s) + q\sin\vartheta(s) = \lambda\cos\vartheta(s) \\ \vartheta'(0) = \vartheta'(\ell) = 0, \end{cases}$$

for some suitable $\lambda \in \mathbb{R}$.

Proof. The boundary condition $\vartheta'(0) = \vartheta'(\ell)$ come from Lemma 3.2; the stationary condition implies that for any $v \in H \cap W^{1,2([0,\ell])}$

$$\int_0^\ell \vartheta'(s)v'(s)ds - q\int_o^\ell v(s)\sin\vartheta(s)ds = 0.$$

Integrating by parts and using the fact that $\vartheta'(0) = \vartheta'(\ell) = 0$, we obtain that

$$\int_0^\ell v(s)\Big(\vartheta''(s) + q\sin\vartheta(s)\Big)ds = 0, \qquad \forall v \in H.$$

This means that $\vartheta''(s) + q\sin\vartheta(s)$ is orthogonal to H, that is there exists $\lambda \in \mathbb{R}$ such that

$$\vartheta''(s) + q\sin\vartheta(s) = \lambda\cos\vartheta(s) \tag{3.1}$$

and this concludes the proof. □

Integrating (3.1) we find that

$$\underbrace{\int_0^\ell \vartheta''(s)ds}_{=0} + q\underbrace{\int_0^\ell \sin\vartheta(s)ds}_{=0} = \lambda\int_0^\ell \cos\vartheta(s)ds = \lambda\gamma_1(\ell).$$

This means that if $\gamma_1(\ell) \neq 0$, then $\lambda = 0$; for this reason, we shall consider in the present work only the case $\lambda = 0$ on stationary points. Then we want to describe the solutions of the problem

$$\begin{cases} \vartheta''(s) + q\sin\vartheta(s) = 0 \\ \vartheta'(0) = \vartheta'(\ell) = 0; \end{cases}$$

this is the classical equation of the pendulum and a lot of material can be found in literature. We collect some of these results in the Appendix A.

Multiplying the previous equation by $\vartheta'(s)$, we find that

$$\frac{d}{ds}\left(\frac{1}{2}\vartheta'(s)^2 - q\cos\vartheta(s)\right) = 0,$$

that is, along the rod the quantity

$$E = \frac{1}{2}\vartheta'(s)^2 - q\cos\vartheta(s)$$

is constant and in particular is equal to its value at $s = 0$, that is

$$\frac{1}{2}\vartheta'(s)^2 - q\cos\vartheta(s) = -q\cos\vartheta_0$$

with initial angle $\vartheta_0 = \vartheta(0) \in [-\pi, \pi]$ (see Figure 3.2). In particular, we can

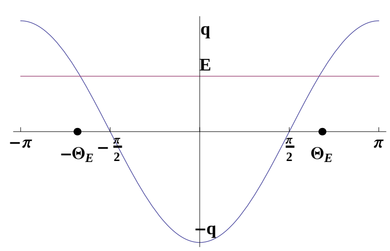

Figure 3.2: The graph of the energy.

write

$$|\vartheta'(s)| = \sqrt{2q\cos\vartheta(s) - 2q\cos\vartheta_0};$$

without loss of generality, we can assume $\vartheta_0 > 0$ (the problem is completely symmetric with respect to the x–axis); so there exists a first value $s_1 > 0$ for which ϑ assume its minumum value $-\vartheta_0$. Since ϑ is monotone decreasing on $[0, s_1]$, we can invert the map $\alpha = \vartheta(s)$ to obtain $s = \vartheta^{-1}(\alpha)$. By the formula for the inverse function

$$\frac{d}{d\alpha}\vartheta^{-1}(s) = \frac{1}{\vartheta'(\vartheta^{-1}(\alpha))} = \frac{1}{\vartheta'(s)},$$

we obtain that

$$\begin{aligned} T :=s_1 = & \int_0^{s_1} ds = \int_{\vartheta_0}^{-\vartheta_0} \frac{1}{\vartheta'(\vartheta^{-1}(\alpha))} d\alpha \\ = & -\int_{\vartheta_0}^{-\vartheta_0} \frac{1}{\sqrt{2q\cos\alpha - 2q\cos\vartheta_0}} d\alpha \\ = & \sqrt{\frac{2}{q}} \int_0^{\vartheta_0} \frac{1}{\sqrt{\cos\alpha - \cos\vartheta_0}} d\alpha. = \sqrt{\frac{2}{q}} f(\vartheta_0), \end{aligned}$$

where we have defined

$$f(t) = \int_0^t \frac{1}{\sqrt{\cos\alpha - \cos t}} d\alpha, \qquad t \in (0, \pi).$$

It is an easy computation to show that

$$f'(t) > 0, \qquad t \in (0, \pi);$$

in addition, using a Taylor expansion for $t \to 0$ and the fact that

$$\frac{1}{\cos\vartheta - \cos t}$$

is asymptotic to

$$\frac{\sqrt{2}}{\sqrt{t^2 - \vartheta^2}},$$

we obtain the following limit:

$$\lim_{t\to 0} f(t) = \frac{\pi}{\sqrt{2}}.$$

Finally, using the fact that for $t \to \pi$ the integral is asymptotic to

$$\int_0^{\pi} \frac{1}{\sqrt{\cos\vartheta + 1}} d\vartheta = +\infty,$$

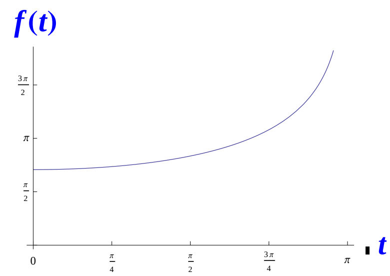

Figure 3.3: The graph of f.

we also have that:

$$\lim_{t\to\pi} f(t) = +\infty.$$

The properties of f are summarized in Figure 3.3. The fact that $\vartheta'(\ell) = 0$ implies in particular that there exists $n \in \mathbb{N}$ such that $\ell = nT$ (a finite number of oscillations). Notice then that from $f(t) \geq \frac{2}{\sqrt{\pi}}$ we obtain the condition

$$\ell = nT \geq T = \sqrt{\frac{2}{q}} f(\vartheta_0) \geq \frac{\pi}{\sqrt{q}},$$

which means

$$q \geq \frac{\pi^2}{\ell^2},$$

so that

$$F \geq F_0 := \frac{B\pi^2}{\ell^2}. \tag{3.2}$$

Then, if the external force is below the quantity F_0, the rod remains in its resting position.

In addition, if the force F satisfies $F_0 \leq F \leq 2F_0$, then there is an unique stationary position determined by the unique initial angle ϑ_0 such that

$$T = T(\vartheta_0) = \ell,$$

that is the unique $\vartheta_0 \in [0, \pi)$ such that

$$f(\vartheta_0) = \ell\sqrt{\frac{F}{2B}}.$$

If $2F_0 \leq F \leq 3F_0$ there are two possible stationary position; the first one determined by the unique angle $\vartheta_0 \in [0, \pi)$ such that

$$f(\vartheta_0) = \ell\sqrt{\frac{F}{2B}},$$

and a second one determined by $2T = \ell$, that is the one determined by $\vartheta_0 \in [0, \pi)$ such that

$$f(\vartheta_0) = \frac{\ell}{2}\sqrt{\frac{F}{2B}}.$$

Generalizing, if the external force satisfies $nF_0 \leq F \leq (n+1)F_0$, then there are n stationary positions, determined by the n angles $\vartheta_k \in [0, \pi)$ for which

$$f(\vartheta_k) = \frac{\ell}{k}\sqrt{\frac{F}{2B}}, \qquad k = 1, \ldots, n.$$

3.2 Stability

The second variation of $\mathscr{E}$ in direction v is given by the quadratic form

$$Q(v) = \frac{d^2}{dt^2}\mathscr{E}(\Phi_t)_{|t=0} = \int_0^\ell v'(s)^2 ds - q\int_0^\ell v(s)^2 \cos\vartheta(s) ds$$

so the study of stability is just the study of Q. There are several approaches to investigate if Q is positive definite or not. The first is to reduce the problem to a Sturm-Liouville problem as follows. If we assume $v(0) = v(\ell) = 0$, then with an integration by parts we can write

$$Q(v) = -\int_0^\ell v(s)\Big(v''(s) + qv(s)\cos\vartheta(s)\Big)ds.$$

We can then define the operator $L : D(L) \subset L^2([0,\ell]) \to L^2([0,\ell])$ where $D(L)$ is given by the functions $v \in W_0^{1,2}([0,\ell])$ for which there exists $\psi \in L^2([0,\ell])$ s.t.

$$\int_0^\ell \psi(s)\eta(s)ds = -\int_0^\ell \eta'(s)v'(s)ds + q\int_0^\ell v(s)\eta(s)\cos\vartheta(s)ds, l$$

for all $\eta \in W_0^{1,2}([0,\ell])$ and

$$Lv := \psi.$$

Since $[0,\ell]$ is bounded, it can be proved that L is a compact operator, so its spectrum is discrete with the only accumulation point is $-\infty$; this means that there exists $(\lambda_j)_{j\in\mathbb{N}}$ decreasing with $\lambda_j \to -\infty$ as $j \to +\infty$ and an orthonormal basis $(v_j)_{j\in\mathbb{N}}$ of $L^2([0,\ell])$ belonging to $W_0^{1,2}([0,\ell])$ such that

$$Lv_j = \lambda_j v_j.$$

So, on this orthonormal basis we have that

$$Q(v_j) = -\lambda_j \int_o^\ell v_j(s)^2 ds = -\lambda_j;$$

as a consequence, the operator cannot be positive definite. The question then becomes whether or not Q is completely unstable, that is if Q is negative definite, or there exists v such that

$$Q(v) < 0.$$

The problem can be rephrased by asking whether or not there are negative eigenvalues λ of L, that is if $\lambda_1 > 0$. Notice that by the Poincaré inequality

$$\int_0^\ell v(s)^2 ds \leq \frac{\ell^2}{\pi^2} \int_0^\ell v'(s)^2 ds$$

we get that

$$Q(v) \geq \int_0^\ell \left(\frac{\pi^2}{\ell^2} - q \cos \vartheta(s) \right) v(s)^2 ds;$$

then, if $q \leq \frac{\pi^2}{\ell^2}$, that is if

$$F \leq \frac{\pi^2 B}{\ell^2},$$

Q turns out to be definite positive, whence stability. Notice that the previous value is just the F_0 that we have found in (3.2); so it seems that the only stable configurations are A second approach could be the following; assuming that $T = \ell$, we have that $\vartheta : [0, \ell] \to [-\vartheta_0, \vartheta_0]$ is monotone decreasing and then it define a change of variable. One can then consider the orthonormal frame

$$1, \sin \frac{k\pi s}{\vartheta_0}, \cos \frac{k\pi s}{\vartheta_0};$$

pulling them back via ϑ, we have an orthonormal frame in $L^2([0, \ell])$; with this basis, computations are a little easier, so one can try to classify Q in this way.

Appendix A

The pendulum

Since the fundamental equation determining the stationary configurations is the pendulum equation, we summarize here the main properties of the solutions of such equation. These notes are kindly offered by prof. Paolo Codecà, that we acknowledge here.

A.1 The period of the pendulum

The ideal pendulum is a point P of mass m, constrained to move on a circumference vertically under the action of its weight $\overrightarrow{mg}$. To derive the differential equation that governs the motion of the pendulum equation we start key dynamics, that is

$$m \cdot \overrightarrow{a} = \overrightarrow{F} + \overrightarrow{\Phi} \tag{A.1}$$

where $\overrightarrow{F}$ is the active force acting on the point, $\overrightarrow{\Phi}$ is the reaction force and $\overrightarrow{a}$ is the acceleration. We place the origin of the arches in the lowest point B of the circle, we expect a positive arcs in a counterclockwise direction, we call O the center of the circle, its radius l, and φ the central angle B $\widehat{O}$ P s corresponding to the arc (see drawing A.1).

Projecting the (A.1) on the unit vector tangent to the circle we get (assuming the bond smooth, that is, Φ orthogonal to the constraint)

$$s''(t) = -mg \cdot \sin\left(\varphi(t)\right) \tag{A.2}$$

where $s(t)$ indicates the length of the arc instantly t and $\varphi(t) = \frac{1}{l}s(t)$. From (A.2) follows

$$\varphi''(t) = -\frac{g}{l}\sin(\varphi(t)) \tag{A.3}$$

which is the differential equation of time of the pendulum.

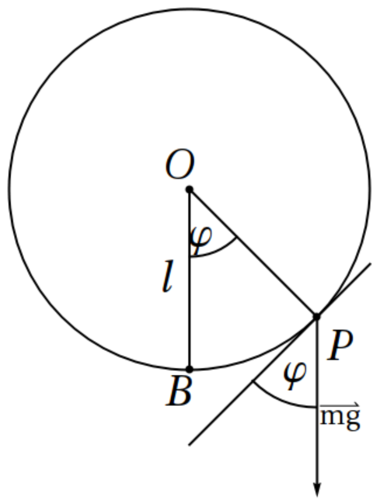

Figure A.1:

One can show that (A.3) does not integrate in an elementary way: it can be linearized, assuming "small" fluctuations, that is replacing $\sin(\varphi(t))$ the subject $\varphi(t)$. In this way the (A.3) becomes

$$\varphi''(t) + \frac{g}{l} \cdot \varphi(t) = 0 \tag{A.4}$$

which is homogeneous linear equation of the second order. The (A.5) integrates easily, in fact, its characteristic equation is

$$\lambda^2 + \frac{g}{l} = 0 \tag{A.5}$$

which has two complex conjugate roots, $\lambda = \pm i\sqrt{\frac{g}{l}}$. So the general solution of (A.4) is given by

$$\varphi(t) = c_1 \cos(\sqrt{\frac{g}{l}}\, t) + c_2 \sin(\sqrt{\frac{g}{l}}\, t) \ \ c_1\, ,\, c_2 \in \mathbb{R} \tag{A.6}$$

and then the movement is harmonious.

By way of example we find explicitly the law of time of " small" oscillations with the initial conditions $\varphi(0) = \varphi_0$ (with $-\frac{\pi}{2} < \varphi_0 < \frac{\pi}{2}$ and $\varphi'(0) = 0$ (The pendulum is left in its original position φ_0 with speed equal to zero). It comes to solving the Cauchy problem.

$$\begin{cases} \varphi''(t) + \dfrac{g}{l} \cdot \varphi(t) = 0 \\ \varphi(0) = \varphi_0 \\ \varphi'(0) = 0 \end{cases} \tag{A.7}$$

Given that $\varphi(t)$ is given by (A.6) must be

$$\varphi(0) = c_1 = \varphi_0 \tag{A.8}$$

and since
$\varphi'(t) = c_1(-\sin(\sqrt{\frac{g}{l}}\, t))(\sqrt{\frac{g}{l}}) + c_2 \cos(\sqrt{\frac{g}{l}}\, t)(\sqrt{\frac{g}{l}})$

$$\varphi'(0) = c_2\sqrt{\frac{g}{l}} = 0 \iff c_2 = 0 \tag{A.9}$$

Then the solution of the Cauchy problem (A.7) is given from

$$\varphi(t) = \varphi_0 \cos(\sqrt{\frac{g}{l}} t) \tag{A.10}$$

from (A.10) follows that the function $\varphi(t)$ is periodic with period T given by

$$T = \frac{2\pi}{\sqrt{g/l}} = 2\pi\sqrt{\frac{l}{g}} \tag{A.11}$$

and then the small oscillations of the pendulum are isochronous (that is, they require the same time regardless initial amplitude φ_0) and the period is directly proportional to the square root of the length of the pendulum and inversely proportional to the square root of the force of gravity (law of Galileo). Obviously, these considerations are valid for small oscillations, in fact the $\varphi(t)$ given from (A.6) is the general solution of the equation linearized (A.4) and not the true equation of the pendulum, that is, the (A.3). We also observe that, since the length of a pendulum and its period can be measured, the formula (A.11) can be used to calculate, with good approximation, the acceleration of gravity.

A.2 The period; general case

To address this problem we start from the equation of motion (A.3): multiplying both members by $\varphi'(t)$ and bringing everything to the first member, you get.

$$\varphi''(t)\varphi'(t) + \frac{g}{l}\sin(\varphi(t))\varphi'(t) = 0 \tag{A.12}$$

But the (A.12) can be rewritten in the form

$$\left(\frac{1}{2}(\varphi'(t))^2 - \frac{g}{l}\cos(\varphi(t))\right)' = 0 \tag{A.13}$$

(to justify (A.13) just remember the theorem derivation of composite function). The (A.13), in turn, implies that

$$\frac{1}{2}(\varphi'(t))^2 - \frac{g}{l}\cos(\varphi(t)) = c \tag{A.14}$$

let c be a constant. It must therefore have, by placing $t = 0$,

$$\frac{1}{2}(\varphi'(t))^2 - \frac{g}{l}\cos(\varphi(t)) = \frac{1}{2}(\varphi'(0))^2 - \frac{g}{l}\cos(\varphi(0)) \tag{A.15}$$

the physical meaning of (A.15) can be clarified as follows.

Multiplying the (A.15) by ml^2 we get

$$\frac{ml^2}{2}(\varphi'(t))^2 - mgl\cos(\varphi(t)) = \frac{ml^2}{2}(\varphi'(0))^2 - mgl\cos(\varphi(0)) \tag{A.16}$$

and adding member to member mgl to (A.16) it has

$$\begin{aligned}&\frac{ml^2}{2}(\varphi'(t))^2 + mgl\,(1-\cos(\varphi(t)))\\ &\qquad = \frac{ml^2}{2}(\varphi'(0))^2 + mgl\,(1-\cos(\varphi(0))) = E\end{aligned} \tag{A.17}$$

We report now the potential energy of the pendulum at the lowest point of the circle (see figure A.2, where $h(t)$ represents the distance from the point $P(t)$ the horizontal plane through B)

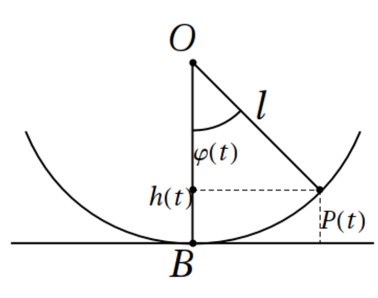

Figure A.2:

Since $s(t) = l \cdot \varphi(t)$ and then $v(t) = s'(t) = l \cdot \varphi'(t)$, and also $h(t) = l\,(1 - \cos(\varphi(t))\,)$, the (A.17) can be rewritten in the form

$$\frac{m}{2}(v'(t))^2 + mgh(t) = \frac{m}{2}(v'(0))^2 + mgh(0) = E \tag{A.18}$$

where $\frac{1}{2}mv^2(t)$ is the kinetic energy of the pendulum at instant t, $mgh(t)$ is the potential energy with respect to said plane, and with E we have indicated the total energy. So the (A.18), which is a consequence of the (A.15), is the theorem of conservation of energy.

Let us return to the formula (A.17) : dividing both sides by ml^2 we get

$$\frac{1}{2}(\varphi'(t))^2 + \frac{g}{l}\,(1 - \cos(\varphi(t))\,) = \frac{E}{ml^2} \tag{A.19}$$

where E is the second member of the (A.17), that is the total energy. Since $1-\cos(\varphi) = 2\cdot\sin^2(\frac{\varphi}{2})$ (fact that the following equality $\cos(\varphi) = \cos(\frac{\varphi}{2}+\frac{\varphi}{2}) = \cos^2(\frac{\varphi}{2}) - \sin^2(\frac{\varphi}{2}) = 1 - 2\sin^2(\frac{\varphi}{2})$), substituting in (A.19) we get

$$\frac{1}{2}(\varphi'(t))^2 + \frac{g}{l}2\sin^2(\frac{\varphi(t)}{2}) = \frac{E}{ml^2} \tag{A.20}$$

which is equivalent to

$$\left(\frac{1}{2}\varphi'(t)\right)^2 = \frac{g}{l}\left(\frac{E}{2mgl} - \sin^2(\frac{\varphi(t)}{2})\right) \tag{A.21}$$

Now suppose that we abandon the pendulum to its initial position $\varphi(0) = -\varphi_0$ (with $0 < \varphi_0 < \frac{\pi}{2}$) with speed equal to zero , that is $\varphi'(0) = 0$, see figure A.3.

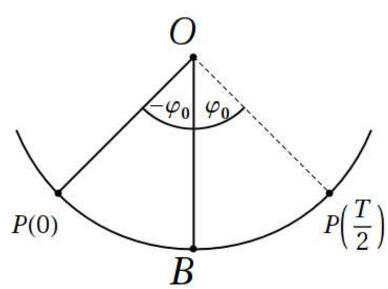

Figure A.3:

Then we have (see the second member of (A.17)),

$$\frac{E}{2mgl} = \frac{1 - \cos(\varphi(0))}{2} = \sin^2(\frac{\varphi(0)}{2}) = \sin^2(\frac{\varphi_0}{2}) \tag{A.22}$$

substituting of (A.22) in (A.21) we get

$$\left(\frac{1}{2}\varphi'(t)\right)^2 = \frac{g}{l}\left(\sin^2(\frac{\varphi_0}{2}) - \sin^2(\frac{\varphi(t)}{2})\right) \tag{A.23}$$

If T is the period , from (A.23) follows

$$\frac{\frac{1}{2}\varphi'(t)}{\sqrt{\frac{g}{l}}\sqrt{\sin^2(\frac{\varphi_0}{2}) - \sin^2(\frac{\varphi(t)}{2})}} \tag{A.24}$$

(in fact for $0 < t < \frac{T}{2}$ it has $\varphi'(t) > 0$ since $\varphi(t)$ is growing increasely and passes from the value $-\varphi_0$ to φ_0). By integrating the (A.24) between 0 and $\frac{T}{2}$ we obtained

$$\sqrt{\frac{l}{g}}\int_0^{\frac{T}{2}} \frac{\varphi'(t)}{\sqrt{\sin^2(\frac{\varphi_0}{2}) - \sin^2(\frac{\varphi(t)}{2})}}\,dt = 2\int_0^{\frac{T}{2}} 1\,dt = T \tag{A.25}$$

We observe now that the integral on the left of (A.25) is equal to

$$\sqrt{\frac{l}{g}}\int_{-\varphi_0}^{\varphi_0} \frac{1}{\sqrt{\sin^2(\frac{\varphi_0}{2}) - \sin^2(\frac{\varphi}{2})}}\,d\varphi = T \tag{A.26}$$

where the time variable is missing t. To justify this statement just make the change of variable $\varphi = \varphi(t)$ in (A.26) and remember that $\varphi(0) = -\varphi_0$ and that $\varphi(\frac{T}{2}) = \varphi_0$.

In this way the integral (A.26) and turns in the integral on the left of (A.25) (Theorem of integration by substitution). The integral (A.26) is, in fact, a generalized integral (The denominator of the integrand is zero for $\varphi = \pm\varphi_0$), but it is an integral convergent (as one might try independently (A.25) or (A.24)).
Now let $\sin(\frac{\varphi_0}{2}) = k$ and $\sin(\frac{\varphi}{2}) = k\sin(\psi)$, from which follows

$$\varphi = 2\arcsin(k\sin(\psi))$$

with this change of variable the integral (A.26) becomes

$$\begin{aligned}&\sqrt{\tfrac{l}{g}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tfrac{1}{\sqrt{k^2-k^2\sin^2(\psi)}}\cdot\tfrac{2}{\sqrt{1-k^2\sin^2(\psi)}}\cdot k\cos(\psi)\,d\psi\\ &= 2\sqrt{\tfrac{l}{g}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tfrac{1}{\sqrt{1-k^2\sin^2(\psi)}}\,d\psi = T\end{aligned} \tag{A.27}$$

in fact for $\varphi = \varphi_0$ it has $\sin(\psi) = -1$ and then $\psi = -\frac{\pi}{2}$, while for $\varphi = \varphi_0$ it has $= \sin(\psi) = 1$ and then $\psi = \frac{\pi}{2}$. Also $\sqrt{k^2 - k^2\sin^2(\psi)} = \sqrt{k^2\cos^2(\psi)} = k\cos(\psi)$ since $k > 0$ and that $\cos(\psi) \geq 0$ as $-\frac{\pi}{2} \leq \psi \leq \frac{\pi}{2}$.
The integral in the second member of (A.27) is an "elliptic integral" and the primitive is not expressible with elementary functions. To calculate the integral (A.27) remember the series expansion (binomial series $\alpha = -\frac{1}{2}$) following

$$\frac{1}{\sqrt{1-x^2}} = 1+\frac{1}{2}x^2+\frac{1\cdot 3}{2\cdot 4}x^4+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6+\cdots\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}x^{2n} \quad \text{(A.28)}$$

valid for $|x| < 1$.
Placing $x = k\cdot\sin\psi$ we get

$$\begin{aligned}\frac{1}{\sqrt{1-k^2\sin^2\psi}} =& 1+\frac{1}{2}k^2\sin^2\psi+\frac{1\cdot 3}{2\cdot 4}k^4\sin^4\psi+\cdots+\\ &+\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}k^{2n}\sin^{2n}\psi+\cdots \end{aligned} \quad \text{(A.29)}$$

Substituting the expansion of (A.29) in the integral of the second member (A.27) and integrating term by term (as it is reasonable because the convergence (A.29) is uniform for the values $-\frac{\pi}{2} \leq \psi \leq \frac{\pi}{2}$)

$$\begin{aligned}T = 2\sqrt{\tfrac{l}{g}}\Big(&\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 d\psi + \tfrac{1}{2}k^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2(\psi)d\psi + \tfrac{1\cdot 3}{2\cdot 4}k^4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^4(\psi)d\psi + \\ &\cdots + \tfrac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}k^{2n}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^{2n}(\psi)d\psi+\cdots\Big)\end{aligned} \quad \text{(A.30)}$$

But there is, as we shall see later that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^{2n}(x)dx = (\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n})\pi \quad \forall n \geq 1 \quad \text{(A.31)}$$

and replacing the (A.31) in (A.30) we get

$$\begin{aligned}T = 2\pi\sqrt{\frac{l}{g}}\Big(1+\Big(\frac{1}{2}\Big)^2k^2+\Big(\frac{1\cdot 3}{2\cdot 4}\Big)^2k^4+\cdots\\ +\Big(\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\Big)^2k^{2n}+\cdots\Big)\end{aligned} \quad \text{(A.32)}$$

where $k^2 = \sin^2(\frac{\varphi_0}{2})$.

From the formula (A.32) follows that the period T depends on the amplitude of oscillation of the pendulum $\varphi(0) = -\varphi_0$, which represents the maximum angular deviation from the equilibrium position, also the coefficients $\left(\frac{1\cdot3\cdots(2n-1)}{2\cdot4\cdots2n}\right) < 1$.

Usually if the amplitudes are " small" we cannot ignore the power of k at the right of (A.32), finding the formula (A.11) of Galileo, that is the isochronism of small oscillations.