UNIVERSITA' DEGLI STUDI DI FERRARA

DIPARTIMENTO D'INGEGNERIA

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Tesi di Laurea

"Some example of one-dimensional variational problem"

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Prefazione

In questa tesi abbiamo considerato problemi classici di calcolo di variazioni consistenti nella ricerca dei minimi funzionali del tipo:

$$J(Y) = \int_a^b F\left(x, Y(x), Y'(x)\right) dx$$

In tali tipi di problemi rientrano ad esempio il problema della geodetica, brachistocrona, e problemi isoperimetrici.
Per studiare tali problemi abbiamo richiamato alcuni concetti di Analisi I,II ed abbiamo accennato come tali concetti si estendono nel calcolo delle variazioni.

Preface

In this thesis we considered some classical problems of calculus of variations consistent in research of the minimum of functionals of the type:

$$J(Y) = \int_a^b F\left(x, Y(x), Y'(x)\right) dx$$

In these types of problems we are going to see problem of geodesic, Brachistochrone, and Isoperimetric problems.
To study these problems we have reviewed some concepts from Calculus I,II and we mentioned how these concepts extend in the calculus of variations.

Table of contents

Chapter 1

1. Introduction

We are going to introduce some example at the beginning in order to present some application of problems in Calculus of Variations.

Example 1.1 (Shortest Path Problem). Let A and B be two fixed points in a space. Then we want to find the shortest curve between these two points. We can construct the problem diagrammatically as below.

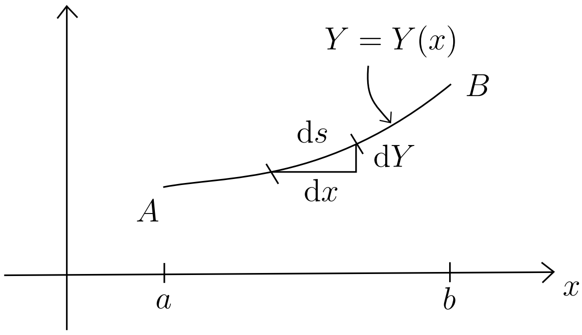

Figure 1. A simple curve.

From basic geometry (i.e. Pythagoras' Theorem) we know that

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}x^2 + \mathrm{d}Y^2 \\ &= \big\{1 + (Y')^2\big\}\mathrm{d}x^2. \end{aligned} \tag{1.1}$$

The second line of this is achieved by noting $Y' = \frac{\mathrm{d}Y}{\mathrm{d}x}$. Now to find the path between the points A and B we integrate $\mathrm{d}s$ between A and B, i.e. $\int_A^B \mathrm{d}s$. We however replace $\mathrm{d}s$ using equation (1.1) above and hence get the expression of the length of our curve

$$J(Y) = \int_a^b \sqrt{1 + (Y')^2}\ \mathrm{d}x.$$

To find the shortest path, i.e. to minimise J, we need to find the extremal function.

Example 1.2 (Minimal Surface of Revolution - Euler). This problem is very similar to the above but instead of trying to find the shortest distance, we are trying to find the smallest surface to cover an area. We again display the problem diagrammatically.

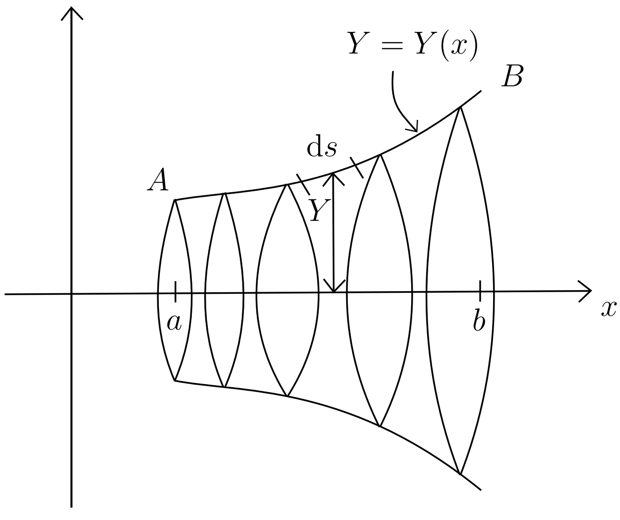

FIGURE 2. A volume of revolution, of the curve $Y(x)$, around the line $y = 0$.

To find the surface of our shape we need to integrate $2\pi Y \mathrm{d}s$ between the points A and B, i.e. $\int_A^B 2\pi Y \mathrm{d}s$. Substituting $\mathrm{d}s$ as above with equation (1.1) we obtain our expression of the size of the surface area

$$J(Y) = \int_a^b 2\pi Y \sqrt{1 + (Y')^2}\,\mathrm{d}x.$$

To find the minimal surface we need to find the extremal function.

EXAMPLE 1.3 (Brachistochrone). This problem is derived from Physics. If I release a bead from O and let it slip down a frictionless curve to point B, accelerated only by gravity, what shape of curve will allow the bead to complete the journey in the shortest possible time. We can construct this diagrammatically below.

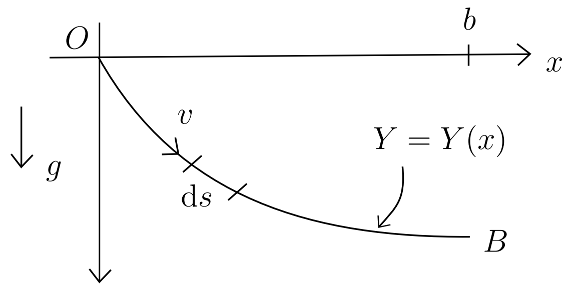

FIGURE 3. A bead slipping down a frictionless curve from point O to B.

In the above problem, we want to minimise the variable of time. So, we construct our integral accordingly and consider the total time taken T as a function of the curve Y.

$$T(Y) = \int_0^b \, \mathrm{d}t$$

now using $v = \frac{\mathrm{d}s}{\mathrm{d}t}$ and rearranging we achieve

$$= \int_0^b \frac{\mathrm{d}s}{v}.$$

Finally using the formula $v^2 = 2gY$ we obtain

$$= \int_0^b \sqrt{\frac{1 + (Y')^2}{2gY}} \, \mathrm{d}x.$$

Thus to find the smallest possible time taken we need to find the extremal function.

EXAMPLE 1.4 (Isoperimetric problems). These are problems with constraints. A simple example of this is trying to find the shape that maximises the area enclosed by a rectangle of fixed perimeter p.

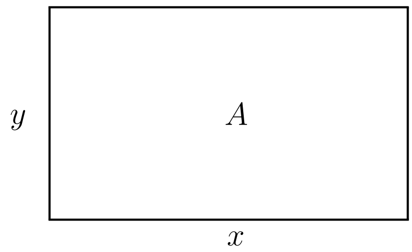

FIGURE 4. A rectangle with sides of length x and y.

We can see clearly that the constraint equations are

$$A = xy \tag{1.2}$$
$$p = 2x + 2y. \tag{1.3}$$

By rearranging equation (1.3) in terms of y and substituting into (1.2) we obtain that

$$A = \frac{x}{2}p - 2x$$
$$\Rightarrow \frac{\mathrm{d}A}{\mathrm{d}x} = \frac{p}{2} - 2x$$

we require that $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$ and thus

$$\frac{p}{2} - 2x = 0 \Rightarrow x = \frac{p}{4}$$

and finally substituting back into equation (1.3) gives us

$$y = \frac{1}{2}\left(p - \frac{1}{2}p\right) = \frac{p}{4}.$$

Thus a square is the shape that maximises the area.

Example 1.5 (Chord and Arc Problem). Here we are seeking the curve of a given length that encloses the largest area above the x-axis. So, we seek the curve $Y = y$ (y is reserved for the solution and Y is used for the general case). We describe this diagrammatically below.

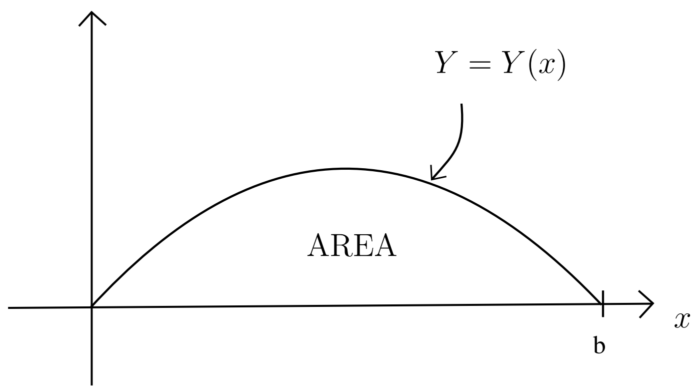

Figure 5. A curve $Y(x)$ above the x-axis.

We have the area of the curve $J(Y)$ to be

$$J(Y) = \int_0^b Y \, \mathrm{d}x \qquad Y \geq 0$$

where $J(Y)$ is maximised subject to the length of the curve

$$K(Y) = \int_0^b \sqrt{1 + (Y')^2} \, \mathrm{d}x = c,$$

where c is a given constant.

To solve the descriped problems we first prove that the problem is well posed; that is the minimum exists and then find the satisfied neccessary conditions that allows us to characterize the solutions.

Existence of minimum

Suppose that the lagrangian $F(x, Y, p)$ satisfies the following conditions:

i) $F(x, Y, p)$ and $F_p(x, Y, p)$ are continuous in (x, Y, p);

ii) $F(x, Y, p)$ is convex in p;

iii) $F(x, Y, p)$ has a superlinear growth.
then there exists a minimizer of

$$J(Y) := \int_I F(x, Y, Y')dx$$

in the class $C(\alpha, \beta) := \{Y \in H^{1,1}((a, b), \mathbb{R}) : Y(a) = \alpha, Y(b) = \beta\}$

Where α, β are fixed real numbers.

Here $H^{1,1}\{(a, b), \mathbb{R}\}$ is Sobolev space of functions

$Y = (a, b) \to \mathbb{R}$ such that $Y(x)^2, Y'(x)^2$ are integrable in (a, b).
i.e

$$\int_a^b Y(x)^2 dx < +\infty$$

$$\int_a^b Y'(x)^2 dx < +\infty.$$

It is possible to prove that if $Y(x, Y, p)$ end $p \to F_p(x, Y, p)$ are C^1 functions and then the minima of J are C^2 functions and then Euler-Lagrange equations that we are going to write are well defined.

2. The Simplest / Fundamental Problem

Examples 1.1, 1.2 and 1.3 are all special cases of the simplest/fundamental problem.

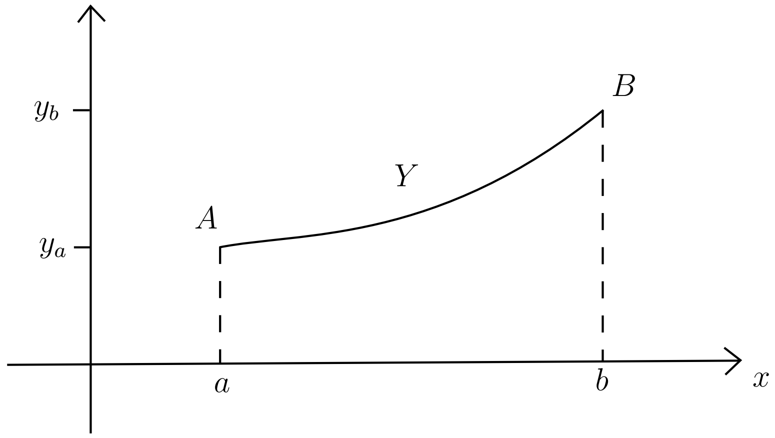

Figure 6. The Simplest/Fundamental Problem.

Suppose $A(a, y_a)$ and $B(b, y_b)$ are two fixed points and consider a set of curves

$$Y = Y(x) \tag{2.1}$$

joining A and B. Then we seek a member $Y = y(x)$ of this set which minimises the integral

$$J(Y) = \int_a^b F(x, Y, Y')\ \mathrm{d}x \tag{2.2}$$

where $Y(a) = y_a$, $Y(b) = y_b$. We note that examples 1.1 to 1.3 correspond to a specification of the above general case with the integrand

$$F = \sqrt{1 + (Y')^2},$$
$$F = 2\pi Y \sqrt{1 + (Y')^2},$$
$$F = \sqrt{\frac{1 + (Y')^2}{2gY}}.$$

An extra example is

$$J(Y) = \int_a^b \left\{ \frac{1}{2} p(x)(Y')^2 + \frac{1}{2} q(x) Y^2 + f(x) Y \right\}\ \mathrm{d}x.$$

Now, the curves Y in (2.2) may be continuous, differentiable or neither and this affects the problem for $J(Y)$. We shall suppose that the functions $Y = Y(x)$ are continuous and have continuous derivatives a suitable number of times. Thus the functions (2.1) belong to a set Ω of admissible functions. We define Ω precisely as

$$\Omega = \left\{ Y \,\middle|\, Y \text{ continuous and } \frac{\mathrm{d}Y}{\mathrm{d}x} \text{ continuous, } k = 1 \right\}. \tag{2.3}$$

So the problem is to minimise $J(Y)$ in (2.2) over the functions Y in Ω where

$$Y(a) = y_a \quad \text{and} \quad Y(b) = y_b.$$

This basic problem can be extended to much more complicated problems.

EXAMPLE 2.1. We can extend the problem by considering more derivatives of Y. So the integrand becomes

$$F = F(x, Y, Y', Y''),$$

i.e. F depends on Y'' as well as x, Y, Y' and here $\Omega = C^2([a, b])$

EXAMPLE 2.2. We can consider more than one function of x. Then the integrand becomes

$$F = F(x, Y_1, Y_2, Y_1', Y_2'),$$

so F depends on two (or more) functions Y_k of x.

EXAMPLE 2.3. Finally we can consider functions of more than one independent variable. So, the integrand would become

$$F = F(x, y, \Phi, \Phi_x, \Phi_y),$$

where subscripts denote partial derivatives. So, F depends on functions $\Phi(x, y)$ of two independent variables x, y. This would mean that to calculate $J(Y)$ we would have to integrate more than once, for example

$$J(Y) = \iint F \,\mathrm{d}x\mathrm{d}y.$$

Note. The integral $J(Y)$ is a *numerical-valued* function of Y, which is an example of a *functional.*

Definition: Let $\mathbb{R}$ be the real numbers and Ω a set of functions. Then the function $J : \Omega \to \mathbb{R}$ is called a *functional.*

Then we can say that the calculus of variations is concerned with *maxima* and *minima* (*extremum*) of functionals.

Chapter 2

3. Maxima and Minima

3.1. The First Necessary Condition

(i) We use ideas from elementary calculus of functions $f(u)$.

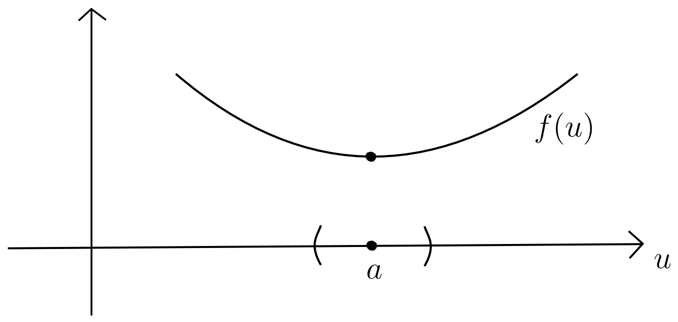

Figure 7. Plot of a function $f(u)$ with a minimum at $u = a$.

If $f(u) \geqslant f(a)$ for all u near a on both sides of $u = a$ this means that there is a minimum at $u = a$. The consequences of this are often seen in an expansion. Let us assume that there is a minimum at $f(a)$ and a *Taylor expansion* exists about $u = a$ such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2} f''(a) + \mathrm{O}_3 \qquad (h \neq 0). \tag{3.1}$$

Note that we define $\mathrm{d}f(a,h) := hf'(a)$ to be the first differential. As there exists a minimum at $u = a$ we must have

$$f(a+h) \geqslant f(a) \quad \text{for} \quad h \in (-\delta, \delta) \tag{3.2}$$

by the above comment. Now, if $f'(a) \neq 0$, say it is positive and h is sufficiently small, then

$$\operatorname{sign}\{\triangle f = f(a+h) - f(a)\} = \operatorname{sign}\{\mathrm{d}f = hf'(a)\} \quad (\neq 0). \tag{3.3}$$

In equation (3.3) the L.H.S. $\geqslant 0$ because f has a minimum at a and hence equation (3.2) holds and also the R.H.S. > 0 if $h > 0$. However this is a contradiction, hence $\mathrm{d}f = 0$ which $\Rightarrow f'(a) = 0$.

(ii) For functions $f(u,v)$ of *two* variables; similar ideas hold. Thus if (a,b) is a minimum then $f(u,v) \geqslant f(a,b)$ for all u near a and v near b. Then for some intervals $(-\delta_1, \delta_1)$ and $(-\delta_2, \delta_2)$ we have that

$$\begin{cases} a - \delta_1 \leqslant u \leqslant a + \delta_1 \\ b - \delta_2 \leqslant v \leqslant b + \delta_2 \end{cases} \tag{3.4}$$

gives a minimum / maximum at $(a, b) \leqslant f(a, b)$. The corresponding Taylor expansion is

$$f(a+h, b+k) = f(a, b) + hf_u(a, b) + kf_v(a, b) + \mathrm{O}_2. \tag{3.5}$$

We note that in this case the first derivative is $\mathrm{d}f(a, b, h, k) := hf_u(a, b) + kf_v(a, b)$. For a minimum (or a maximum) at (a, b) it follows, as in the previous case that a necessary condition is

$$\mathrm{d}f = 0 \Rightarrow \frac{\partial f}{\partial u} = \frac{\partial f}{\partial v} = 0 \text{ at } (a, b). \tag{3.6}$$

(iii) Now considering functions of multiple (say n) variables, i.e. $f = f(u_1, u_2, \ldots, u_n) = f(\mathbf{u})$ we have the Taylor expansion to be

$$f(\mathbf{a} + \mathbf{h}) = f(\mathbf{a}) + \mathbf{h} \cdot \nabla f(\mathbf{a}) + \mathrm{O}_2. \tag{3.7}$$

Thus the statement from the previous case (3.6) becomes

$$\mathrm{d}f = 0 \Rightarrow \nabla f(\mathbf{a}) = 0. \tag{3.8}$$

3.2. **Calculus of Variations**

Now we consider the integral

$$J(Y) = \int_a^b F(x, Y, Y')\ \mathrm{d}x. \tag{3.9}$$

Suppose $J(Y)$ has a minimum for the curve $Y = y$. Then

$$J(Y) \geqslant J(y) \tag{3.10}$$

for $Y \in \Omega = \{Y \mid Y \in C_2, Y(a) = y_a, Y(b) = y_b\}$. To obtain information from (3.10) we *expand* $J(Y)$ about the curve $Y = y$ by taking the so-called *varied curves*

$$Y = y + \varepsilon\xi \tag{3.11}$$

like $u = a + h$ in (3.2). We can represent the consequences of this expansion diagrammatically.

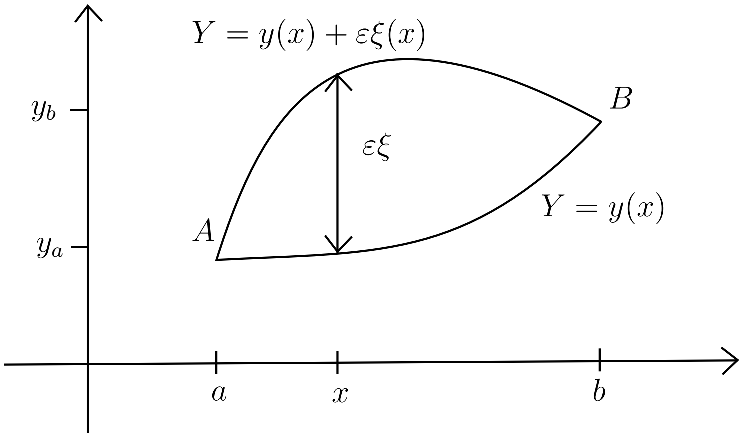

FIGURE 8. Plot of $y(x)$ and the expansion $y(x) + \varepsilon\xi(x)$.

Since all curves Y, including y, go through A and B, it follows that

$$\xi(a) = 0 \quad \text{and} \quad \xi(b) = 0. \tag{3.12}$$

Now substituting equation (3.11) into (3.10) gives us

$$J(Y) = J(y + \varepsilon\xi) \geqslant J(y) \tag{3.13}$$

for all $y + \varepsilon\xi \in \Omega$ and substituting (3.11) into (3.9) gives us

$$J(y + \varepsilon\xi) = \int_a^b F(x, y + \varepsilon\xi, y' + \varepsilon\xi')\ \mathrm{d}x. \tag{3.14}$$

Now to deal with this expansion we take a *fixed* x in (a, b) and treat y and y' as independent variables. Recall Y and Y' are independent and the Taylor expansion of two variables (equation (3.5)) from above. Then we have

$$f(u + h, v + k) = f(u, v) + h\frac{\partial f}{\partial u} + k\frac{\partial f}{\partial v} + \mathrm{O}_2. \qquad O_2 = O\left(||(h,k)||\right) \tag{3.15}$$

Now we take $u = y$, $h = \varepsilon\xi$, $v = y'$, $k = \varepsilon\xi'$, $f = F$. Then (3.14) implies

$$\begin{aligned} J(Y) = J(y + \varepsilon\xi) &= \int_a^b \left\{ F(x, y, y') + \varepsilon\xi\frac{\partial F}{\partial y} + \varepsilon\xi'\frac{\partial F}{\partial y'} + \mathrm{O}(\varepsilon^2) \right\}\ \mathrm{d}x \\ &= J\left(y\right) + \varepsilon\,\delta J + \mathrm{O}_2. \end{aligned}$$

We note that δJ is calculus of variations notation for $\mathrm{d}J$ where we have

$$\delta J = \int_a^b \left\{ \xi \frac{\partial F}{\partial y} + \xi' \frac{\partial F}{\partial y'} \right\} \, \mathrm{d}x \tag{3.16}$$
$$= \text{ linear terms in } \varepsilon = \text{ first } \textit{variation} \text{ of } J$$

and we also have that

$$\frac{\partial F}{\partial y} = \left\{ \frac{\partial F(x, Y, Y')}{\partial Y} \right\}_{Y=y, Y'=y'} . \tag{3.17}$$

Now δJ is analogous to the linear terms in (3.1), (3.5), (3.7). Then if $J(Y)$ has a minimum at $Y = y$, then

$$\boxed{\delta J = 0} \tag{3.18}$$

the first necessary condition for a minimum.

Proof. Suppose $\delta J \neq 0$ then $J(Y) - J(y) = \delta J + \mathrm{O}_2$. For small enough $\varepsilon\xi$ then

$$\operatorname{sign}\{J(Y) - J(y)\} = \operatorname{sign}\{\delta J\}.$$

We have that $\delta J > 0$ or $\delta J < 0$ for some varied curves corresponding to $\varepsilon\xi$. However there is a minimum of J at $Y = y \Rightarrow$ L.H.S. $\geqslant 0$. This is a contradiction. □

For our J, we have by (3.17)

$$\delta J = \int_a^b \left(\xi \frac{\partial F}{\partial y} + \xi' \frac{\partial F}{\partial y'} \right) \, \mathrm{d}x. \tag{3.19}$$

If we integrate 2nd term by parts we obtain

$$\delta J = \int_a^b \xi \left\{ \frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial F}{\partial y'} \right\} \, \mathrm{d}x + \underbrace{\left[\xi \frac{\partial F}{\partial y'} \right]_a^b}_{=0} \tag{3.20}$$

and so we have

$$\boxed{\delta J = \int_a^b \xi \left\{ \frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial F}{\partial y'} \right\} \, \mathrm{d}x.} \tag{3.21}$$

Note. For notational purposes we write

$$\langle f, g \rangle = \int_a^b f(x) g(x) \, \mathrm{d}x \tag{3.22}$$

which is an inner (or scalar) product. Also, for our J, write

$$J'(y) = \frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} \tag{3.23}$$

as the derivative of J. Then we can express δJ as

$$\delta J = \big\langle \xi, J'(y) \ \big\rangle. \tag{3.24}$$

This gives us the Taylor expansion

$$J(y + \varepsilon\xi) = J(y) + \varepsilon\underbrace{\big\langle \ \xi, J'(y) \ \big\rangle}_{\delta J} + \mathrm{O}_2. \tag{3.25}$$

We compare this with the previous cases of Taylor expansion (3.1), (3.5) and (3.7). Now collecting our results together we get

Theorem 3.1. *A necessary condition for $J(Y)$ to have an extremum (maximum or minimum) at $Y = y$ is*

$$\boxed{\delta J = \big\langle \ \xi, J'(y) \ \big\rangle = 0} \tag{3.26}$$

for all admissible ξ. i.e.

$$\text{“}J(Y) \text{ has an extremum at } Y = y\text{”} \Rightarrow \text{“}\delta J(y, \xi) = 0\text{”}.$$

Definition: y is a *critical curve* (or an *extremal*), i.e. y is a solution of $\delta J = 0$. $J(y)$ is a *stationary value* of J and (3.26) is a *stationary condition.*

To establish an extremum, we need to examine the sign of

$$\triangle J = J(y + \ \xi) - J(y) = \text{total variation of } J.$$

We consider this later in part two of this thesis .

4. Stationary Condition ($\delta J = 0$)

Our next step is to see what can be deduced from the condition

$$\delta J = 0. \tag{4.1}$$

In detail this is

$$\big\langle \ \xi, J'(y) \ \big\rangle = \int_a^b \ \xi J'(y) \ \mathrm{d}x = 0. \tag{4.2}$$

For our case $J\ (Y) = \int_a^b F(x, Y, Y') \ \mathrm{d}x$, we have

$$J'(y) = \frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'}. \tag{4.3}$$

To deal with equations (4.2) and (4.3) we use the Euler-Legrange Lemma. Using this Lemma we can state

Theorem 4.1. *A necessary condition for*

$$J(Y) = \int_a^b F(x, Y, Y')\ \mathrm{d}x, \tag{4.4}$$

with $Y(a) = y_a$ *and* $Y(b) = y_b$*, to have an extremum at* $Y = y$ *is that* y *is a solution of*

$$J'(y) = \frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} = 0 \tag{4.5}$$

with $a < x < b$ *and* $F = F(x, y, y')$*. This is known as the Euler-Lagrange equation.*

The above theorem is the Euler-Lagrange variational principle and it is a *stationary* principle.

Example 4.1 (Shortest Path Problem). We now revisit Example 1.1 with the mechanisms that we have just set up. So we had that $F(x, Y, Y') = \sqrt{1 + (Y')^2}$ previously but instead we write $F(x, y, y') = \sqrt{1 + (y')^2}$. Now

$$\frac{\partial F}{\partial y} = 0, \qquad \frac{\partial F}{\partial y'} = \frac{2y'}{2\sqrt{1 + (y')^2}} = \frac{y'}{\sqrt{1 + (y')^2}}.$$

The Euler-Lagrange equation for this example is

$$\begin{aligned} 0 - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y'}{\sqrt{1 + (y')^2}}\right) &= 0 \qquad a < x < b \\ \Rightarrow \frac{y'}{\sqrt{1 + (y')^2}} &= \text{const.} \\ \Rightarrow y' &= \alpha \\ \Rightarrow y &= \alpha x + \beta \end{aligned}$$

where α, β are arbitrary constants. So, $y = \alpha x + \beta$ defines the critical curves. We require more information to establish a minimum.

Example 4.2 (Minimum Surface Problem). Now revisiting Example 1.2, we had $F(x, Y, Y') = 2\pi Y\sqrt{1 + (Y')^2}$ but again we write $F(x, y, y') = 2\pi y\sqrt{1 + (y')^2}$. We drop the 2π to give us

$$\frac{\partial F}{\partial y} = \sqrt{1+(y')^2}, \qquad \frac{\partial F}{\partial y'} = \frac{yy'}{\sqrt{1+(y')^2}}.$$

So, we are left with the Euler-Lagrange equation

$$\sqrt{1+(y')^2} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{yy'}{\sqrt{1+(y')^2}}\right) = 0.$$

Now solving the differential equation leaves us with

$$\left(1+(y')^2\right)^{-\frac{3}{2}}\left\{1+(y')^2 - yy''\right\} = 0,$$

for finite y' in (a,b). We therefore have

$$\boxed{1+(y')^2 = yy''.} \tag{4.6}$$

Start by rewriting y'' in terms of y and y'. Then substituting gives us

$$\begin{aligned} y'' &= \frac{\mathrm{d}y'}{\mathrm{d}x} \\ &= \frac{\mathrm{d}y'}{\mathrm{d}y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x} \\ &= y'\frac{\mathrm{d}y'}{\mathrm{d}y} = \frac{1}{2}\frac{\mathrm{d}(y')^2}{\mathrm{d}y}. \end{aligned} \tag{4.7}$$

So, substituting (4.7) into (4.6), the Euler-Lagrange equation implies that

$$\begin{aligned} 1+(y')^2 &= \frac{1}{2}y\frac{\mathrm{d}(y')^2}{\mathrm{d}y} \\ \Rightarrow \frac{\mathrm{d}y}{y} &= \frac{1}{2}\frac{\mathrm{d}(y')^2}{1+(y')^2} = \frac{1}{2}\frac{\mathrm{d}z}{1+z} \\ \Rightarrow \ln y &= \frac{1}{2}\ln\left(1+(y')^2\right) + C \\ \Rightarrow y &= C\sqrt{1+(y')^2}, \end{aligned}$$

which is a first integral.

Now integrating again gives us

$y'' = \frac{1+(y'^2)}{y}$ the autonomous differential equation

suppose that $y' = v(y)$

then we have $y'' = v'(y).y' = v'(y)v(y)$

so we get $vv' = \frac{1+v^2}{y}$

$\Longrightarrow$ $\mathrm{v}'\frac{v}{1+v^2} = \frac{1}{y}$

Integrate both sides we get $\frac{1}{2}\log(1+\mathrm{v}^2) = log|y|+c_1 \;\Leftrightarrow\; log(1+v^2) = logy^2+c_1$

and $1+v^2 = c_1y^2$, $\mathrm{v}=\sqrt{c_1y^2-1}$, so $\mathrm{y}' = \sqrt{c_1y^2-1}$

divide by $\sqrt{c_1y^2-1}$

we get $\frac{y'}{\sqrt{c_1y^2-1}} = 1$

thus $\mathrm{c}_1y^2 = cosh^2\alpha$, $y = \frac{1}{\sqrt{c_1}}cosh\alpha$, $dy = \frac{1}{\sqrt{c_1}}senh\alpha$

integrating

$$\frac{1}{\sqrt{c_1}}\int\frac{senh\alpha}{senh\alpha}d\alpha = x+c_2$$

then

$$\frac{1}{\sqrt{c_1}}\alpha = x+c_2$$

we have $\alpha = \sqrt{c_1}x+c_2$, $\alpha = arccosh(\sqrt{c_1}y)$

$$\Longrightarrow \qquad \mathrm{y}=\cosh(\sqrt{c_1}x+c_2)$$

where C_1 and C are arbitrary constants. We note that this is a *catenary* curve.

Chapter 3

5. Special Cases of the Euler-Lagrange Equation for the Simplest Problem

For $F(x, y, y')$ the Euler-Lagrange equation is a 2nd order differential equation in general. There are special cases worth noting.

(1) $\frac{\partial F}{\partial y} = 0 \Rightarrow F = F(x, y')$ and hence y is missing. The Euler-Lagrange equation is

$$
\begin{aligned}
\underbrace{\frac{\partial F}{\partial y}}_{=0} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} &= 0 \\
\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} &= 0 \\
\Rightarrow \frac{\partial F}{\partial y'} &= c
\end{aligned}
$$

on orbits (extremals or critical curves), first integral. If this can be for y' thus

$$y' = f(x, c)$$

then $y(x) = \int f(x, c)\ \mathrm{d}x + c_1$.

Example 5.1. $F = x^2 + x^2(y')^2$. Hence $\frac{\partial F}{\partial y} = 0$. So Euler-Lagrange equation has a first integral $\frac{\partial F}{\partial y'} = c \Rightarrow 2x^2 y' = c \Rightarrow y' = \frac{c}{2x^2} \Rightarrow y' = -\frac{c}{2x} + c_1$, which is the solution.

(2) $\frac{\partial F}{\partial x} = 0$ and so $F = F(y, y')$ and hence x is missing. For any differentiable $F(x, y, y')$ we have by the chain rule

$$
\begin{aligned}
\frac{\mathrm{d}F}{\mathrm{d}x} &= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}y' + \frac{\partial F}{\partial y'}y'' \\
&= \frac{\partial F}{\partial x} + \left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'}\right)y' + \frac{\partial F}{\partial y'}y'' \\
&= \frac{\partial F}{\partial x} + \frac{\mathrm{d}}{\mathrm{d}x}\left(y'\frac{\partial F}{\partial y'}\right)
\end{aligned}
$$

on orbits. Now, we have

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\left(F - y'\frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial x}$$

on orbits. When $\frac{\partial F}{\partial x} = 0$, this gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(F - y'\frac{\partial F}{\partial y'}\right) = 0 \text{ on orbits}$$

$$\Rightarrow \boxed{F - y'F_{y'} = c}$$

on orbits. First integral.

Note. $G := F - y'F_y = G(x, y, y')$ is the Jacobi function.

EXAMPLE 5.2. $F = 2\pi y\sqrt{1 + (y')^2}$. An example of case 2. So we have

$$F_{y'} = \frac{2\pi y y'}{\sqrt{1 + (y')^2}}.$$

then the first integral is

$$\begin{aligned}
F &= y'F_{y'} \\
&= 2\pi y\left\{\sqrt{1 + (y')^2} - \frac{(y')^2}{\sqrt{1 + (y\prime)2}}\right\} \\
&= \frac{2\pi y}{\sqrt{1 + (y')^2}} \\
&= \text{const.} = 2\pi c
\end{aligned}$$

$\Rightarrow y = c\sqrt{1 + (y')^2}$ - First Integral.

This arose in Example 4.2 as a result of integrating the Euler Lagrange equation once.

(3) $\frac{\partial F}{\partial y'} = 0$ and hence $F = F(x, y)$, i.e. y' is missing. The Euler-Lagrange equation is

$$0 = \frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} = \frac{\partial F}{\partial y} = 0$$

but this isn't a differential equation in y.

EXAMPLE 5.3. $F(x, y, y') = -y\ln y + xy$ and then the Euler-Lagrange equation gives

$$\begin{aligned}
\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} &= 0 \Rightarrow -\ln y - 1 + x = 0 \\
\ln y = -1 + x &\Rightarrow y = e^{-1+x}
\end{aligned}$$

6. Change of Variables

In the shortest path problem above, we note that $J(Y)$ does not depend on the coordinate system chosen. Suppose we require the extremal for

$$J(r) = \int_{\theta_0}^{\theta_1} \sqrt{r^2 + (r')^2}\, \mathrm{d}\theta \tag{6.1}$$

where $r = r(\theta)$ and $r' = \frac{\mathrm{d}r}{\mathrm{d}\theta}$. The Euler-Lagrange equation is

$$\frac{\partial F}{\partial r} - \frac{\mathrm{d}}{\mathrm{d}\theta}\frac{\partial F}{\partial r'} = 0 \Rightarrow \frac{r}{\sqrt{r^2 + (r')^2}} - \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{r'}{\sqrt{r^2 + (r')^2}}\right) = 0. \tag{6.2}$$

To simplify this we can

(a) change variables in (6.2), or
(b) change variables in (6.1).

For example, in (6.1) take

$$\begin{aligned} x &= r\cos\theta & \qquad \mathrm{d}x &= \mathrm{d}r\cos\theta - r\sin\theta\,\mathrm{d}\theta \\ y &= r\sin\theta & \qquad \mathrm{d}y &= \mathrm{d}r\sin\theta + r\cos\theta\,\mathrm{d}\theta. \end{aligned}$$

So, we obtain

$$\begin{aligned} \mathrm{d}x^2 + \mathrm{d}y^2 &= \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 \\ \big(1 + (y')^2\big)\mathrm{d}x^2 &= \left\{\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 + r^2\right\}\, \mathrm{d}\theta^2 \\ \sqrt{1 + (y')^2}\mathrm{d}x &= \sqrt{r^2 + (r')^2}\mathrm{d}\theta. \end{aligned}$$

This is just the smallest path problem, hidden in polar coordinates. We know the solution to this is

$$J(r) \to \bar{J}(y) = \int \sqrt{1 + (y')^2}\, \mathrm{d}x.$$

Now the Euler-Lagrange equation is $y'' = 0 \Rightarrow y = \alpha x + \beta$. Now,

$$r\sin\theta = \alpha r\cos\theta + \beta$$

$$\boxed{r = \frac{\beta}{\sin\theta - \alpha\cos\theta}}$$

7. Several Independent functions of One Variable

Consider a general solution of the simplest problem.

$$J(Y_1, Y_2, \dots, Y_n) = \int_a^b F(x, Y_1, \dots, Y_n, Y_1', \dots, Y_n')\,\mathrm{d}x. \tag{7.1}$$

Here each curve Y_k goes through given end points

$$Y_k(a) = y_{ka},\ Y_k(b) = y_{kb} \qquad \text{for } k = 1, \dots, n. \tag{7.2}$$

Find the critical curves y_k, $k = 1, \dots, n$. Take

$$Y_k = y_k(x) + \varepsilon \xi_k(x) \tag{7.3}$$

where $\xi_k(a) = 0 = \xi_k(b)$. By taking a Taylor expansion around the point $(x, y_1, \dots, y_n, y_1', \dots, y_n')$ we find that the first variation of J is

$$\delta J = \sum_{k=1}^{n} \big\langle\, \xi_k, J'(y_1, \dots, y_n)\,\big\rangle. \tag{7.4}$$

with

$$J_k'(y_1, \dots, y_n) = \frac{\partial F}{\partial y_k} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y_k'} \tag{7.5}$$

for $k = 1, \dots, n$. The stationary condition $\delta J = 0 \Rightarrow$

$$\sum_{k=1}^{n} \langle\, \xi_k, {J'}_k \rangle = 0 \tag{7.6}$$

$$\Rightarrow \langle\, \xi_k, {J'}_k \rangle = 0 \tag{7.7}$$

for all $k = 1, \dots, n$. Thus, by the Euler-Lagrange Lemma, this implies

$$J_k' = 0 \tag{7.8}$$

for $k = 1, \dots, n$. i.e.

$$\frac{\partial F}{\partial y_k} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y_k'} = 0 \tag{7.9}$$

for $k = 1, \dots, n$. (7.9) is a system of n Euler-Lagrange equations. These are solved subject to (7.3).

EXAMPLE 7.1. $F = y_1 y_2^2 + y_1^2 y_2 + y_1' y_2'$ and so we obtain equations

$$\left\{\begin{aligned} J_1' = 0 \Rightarrow y_2^2 + 2y_1 y_2 - \frac{\mathrm{d}}{\mathrm{d}x}(y_2') = 0 \\ J_2' = 0 \Rightarrow 2y_1 y_2 + y_1^2 - \frac{\mathrm{d}}{\mathrm{d}x}(y_1') = 0 \end{aligned}\right\}$$

Consider

$$\frac{\mathrm{d}F}{\mathrm{d}x} = \frac{\partial F}{\partial x} + \sum_{k=1}^{n} \left(\frac{\partial F}{\partial y_k} y_k' + \frac{\partial F}{\partial y_k'} y_k'' \right)$$

the general chain rule.

$$= \frac{\partial F}{\partial x} + \sum_{k=1}^{n} \left(y_k' \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial F}{\partial y_k'} + \frac{\partial F}{\partial y_k'} y_k'' \right)$$

on orbits.

$$= \frac{\partial F}{\partial x} + \sum_{k=1}^{n} \frac{\mathrm{d}}{\mathrm{d}x} \left(y_k' \frac{\partial F}{\partial y_k'} \right)$$

and this is again on orbits. Now we obtain

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x} \left\{ F - \sum_{k=1}^{n} y_k' \frac{\partial F}{\partial y_k'} \right\} = \frac{\partial F}{\partial x}$$

on orbits.

We define $G = -\left\{ F - \sum y_k' \dfrac{\partial F}{\partial y'} \right\}$. Then

$$-\frac{\mathrm{d}G}{\mathrm{d}x} = \frac{\partial F}{\partial x}.$$

If $\frac{\partial F}{\partial x} = 0$, i.e. F does not depend *implicitly* on x, we have

$$-\frac{\mathrm{d}G}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} \left\{ F - \sum y_k' \frac{\partial F}{\partial y'} \right\} = 0$$

on orbits. i.e.

$$\boxed{F - \sum_{k=1}^{n} y'_k \frac{\partial F}{\partial y'_k} = c}$$

on orbits, where c is a constant. This is a first integral. i.e. G is constant on orbits. This is the Jacobi integral.

8. Double Integrals

Here we look at functions of 2 variables, i.e. surfaces,

$$\Phi = \Phi(x, y). \tag{8.1}$$

We then take the integral

$$J(\Phi) = \iint_R F(x, y, \Phi, \Phi_x, \Phi_y)\ \mathrm{d}x\mathrm{d}y \tag{8.2}$$

with $\Phi - \varphi_B =$ given on the boundary ∂R of R. Suppose $J(\Phi)$ has a minimum for $\Phi = \varphi$. Hence R is some closed regioon in the xy-plane and $\Phi_x = \frac{\partial \Phi}{\partial x}$, $\Phi_y = \frac{\partial \Phi}{\partial y}$. Assume F has continuous first and second derivatives with respect to x, y, Φ, Φ_x, Φ_y. Consider

$$J(\varphi + \varepsilon\xi) = \iint_R F(x, y, \varphi + \varepsilon\xi, \varphi_x + \varepsilon\xi_x, \varphi_y + \varepsilon\xi_y)\ \mathrm{d}x\mathrm{d}y$$

and expand in Taylor series

$$\begin{aligned} &= \iint_R \left\{ F(x, y, \varphi, \varphi_x, \varphi_y) + \varepsilon\xi\frac{\partial F}{\partial \varphi} + \varepsilon\xi_x\frac{\partial F}{\partial \varphi_x} + \varepsilon\xi_y\frac{\partial F}{\partial \varphi_y} + \mathrm{O}_2 \right\}\ \mathrm{d}x\mathrm{d}y \\ &= J(\varphi) + \delta J + \mathrm{O}_2 \end{aligned} \tag{8.3}$$

where we have

$$\delta J = \varepsilon \iint_R \left\{ \xi\frac{\partial F}{\partial \varphi} + \xi_x\frac{\partial F}{\partial \varphi_x} + \xi_y\frac{\partial F}{\partial \varphi_y} \right\}\ \mathrm{d}x\mathrm{d}y \tag{8.4}$$

is the first variation. For an extremum at φ, it is necessary that

$$\delta J = 0. \tag{8.5}$$

To use this we rewrite (8.4)

$$\delta J = \varepsilon \iint_R \left\{ \xi\frac{\partial F}{\partial \varphi} + \frac{\partial}{\partial x}\left(\xi\frac{\partial F}{\partial \varphi_x}\right) + \frac{\partial}{\partial y}\left(\xi\frac{\partial F}{\partial \varphi_y}\right) - \xi\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial \varphi_x}\right) - \xi\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial \varphi_y}\right) \right\}\ \mathrm{d}x\mathrm{d}y. \tag{8.6}$$

Now by Green's Theorem we have

$$\begin{aligned} &\iint_R \frac{\partial P}{\partial x}\ \mathrm{d}x\mathrm{d}y = \int_{\partial R} P\ \mathrm{d}y && P = \xi\frac{\partial F}{\partial \varphi_x} \\ &\iint_R \frac{\partial Q}{\partial y}\ \mathrm{d}x\mathrm{d}y = -\int_{\partial R} Q\ \mathrm{d}x && Q = \xi\frac{\partial F}{\partial \varphi_y} \end{aligned} \tag{8.7}$$

So,

$$\delta J = \varepsilon \iint_R \xi \left\{ \frac{\partial F}{\partial \varphi} - \frac{\partial}{\partial x}\frac{\partial F}{\partial \varphi_x} - \frac{\partial}{\partial y}\frac{\partial F}{\partial \varphi_y} \right\} \,\mathrm{d}x\mathrm{d}y + \varepsilon \int_{\partial R} \left(\xi \frac{\partial F}{\partial \varphi_x}\mathrm{d}y - \xi \frac{\partial F}{\partial \varphi_y}\mathrm{d}x \right). \tag{8.8}$$

If we choose all functions Φ including φ to statisfy

$$\begin{aligned} \varphi &= \varphi_B && \text{on } \partial R \\ \text{then } \varphi + \varepsilon\xi &= \varphi_B && \text{on } \partial R \\ \Rightarrow \xi &= 0 && \text{on } \partial R. \end{aligned}$$

Hence (8.8) simplifies to

$$\begin{aligned} \delta J = &\iint_R \xi(x,y) \left\{ \frac{\partial F}{\partial \varphi} - \frac{\partial}{\partial x}\frac{\partial F}{\partial \varphi_x} - \frac{\partial}{\partial y}\frac{\partial F}{\partial \varphi_y} \right\} \,\mathrm{d}x\mathrm{d}y \\ \equiv &\left\langle \, \xi, J'(\varphi) \, \right\rangle. \end{aligned}$$

By a simple extension of the Euler-Lagrange lemma, since ξ is arbitrary in R, we have $\delta J = 0 \to \varphi(x,y)$ is a solution of

$$\boxed{J'(\varphi) = \frac{\partial F}{\partial \varphi} - \frac{\partial}{\partial x}\frac{\partial F}{\partial \varphi_x} - \frac{\partial}{\partial y}\frac{\partial F}{\partial \varphi_y} = 0.} \tag{8.9}$$

This is the Euler-Lagrange Equation - a partial differential equation. We seek the solution φ which takes the given values φ_B on ∂R.

Example 8.1. $F = F(x,y,\varphi,\varphi_x,\varphi_y) = \frac{1}{2}\varphi_x^2 + \frac{1}{2}\varphi_y^2 + f\varphi$, where $f = f(x,y)$ is given. The Euler-Lagrange equation is

$$\frac{\partial F}{\partial \varphi} = f \qquad \frac{\partial F}{\partial \varphi_x} = \varphi_x \qquad \frac{\partial F}{\partial \varphi_y} = \varphi_y.$$

So,

$$\frac{\partial F}{\partial \varphi} - \frac{\partial}{\partial x}\frac{\partial F}{\partial \varphi_x} - \frac{\partial}{\partial y}\frac{\partial F}{\partial \varphi_y} = 0 \Rightarrow f - \frac{\partial}{\partial x}\varphi_x - \frac{\partial}{\partial y}\varphi_y = 0.$$

So, i.e. we are left with

$$\boxed{\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = f,}$$

which is Poisson's Equation.

EXAMPLE 8.2. $F = F(x, t, \varphi, \varphi_x, \varphi_t) = \frac{1}{2}\varphi_x^2 - \frac{1}{2c^2}\varphi_t^2$. The Euler-Lagrange equation is

$$\begin{aligned} \frac{\partial F}{\partial x} - \frac{\partial}{\partial x}\frac{\partial F}{\partial \varphi_x} - \frac{\partial}{\partial t}\frac{\partial F}{\partial \varphi_t} &= 0 \\ \Rightarrow 0 - \frac{\partial}{\partial x}\varphi_x - \frac{\partial}{\partial t}\left(\frac{1}{c^2}\varphi_t\right) &= 0 \\ \Rightarrow \boxed{\frac{\partial^2 \varphi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2}} \end{aligned}$$

This is the classical wave equation.

Chapter 4

9. Canonical Euler-Equations (Euler-Hamilton)

In this chapter we sketch a different approach to some problems in calculus of variations, the Hamiltonian approach.

9.1. The Hamiltonian

We have the Euler-Lagrange equations

$$\frac{\partial F}{\partial y_k} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y_{k'}} = 0 \tag{9.1}$$

which give the critical curves $y_1, \dots, y_n$ of

$$J(Y_1, \dots, Y_n) = \int F(x, Y_1, \dots, Y_n, Y_1', \dots, Y_n')\ \mathrm{d}x \tag{9.2}$$

Equations (9.1) form a system of n 2nd order differential equations. We shall now rewrite this as a system of $2n$ first order differential equations. First we introduce a new variable

$$p_i = \frac{\partial F}{\partial y_i} \qquad i = 1, \dots, n. \tag{9.3}$$

p_i is said to be the variable *conjugate* to y_i. We suppose that equations (9.3) can be solved to give y' as a function ψ_i of x, y_j, p_j $(j = 1, \dots, n)$. Then it is possible to define a new function H by the equation

$$H(x, y_1, \dots, y_n, p_1, \dots, p_n) = \sum_{i=1}^{n} p_i y_i' - F(x, y_1, \dots, y_n, y_1', \dots, y_n') \tag{9.4}$$

where $y_i' = \psi_i(x, y_j, p_j)$. The function H is called the *Hamiltonian* corresponding to (9.2). Now look at the differential of H which by (9.4) is

$$
\begin{aligned}
dH &= \sum_{i=1}^{n}(\cancel{p_i \mathrm{d}y_i'} + y_i'\mathrm{d}p_i) - \frac{\partial F}{\partial x}\mathrm{d}x - \sum_{i=1}^{n}\left(\frac{\partial F}{\partial y_i}\mathrm{d}y_i + \cancel{\frac{\partial F}{\partial y_i'}\mathrm{d}y_i'}\right) \\
&= -\frac{\partial F}{\partial x}\mathrm{d}x + \sum_{i=1}^{n}\left(y_i'\mathrm{d}p_i - \frac{\partial F}{\partial y_i}\right) \qquad (9.5)
\end{aligned}
$$

using (9.3). For $H = (x, y_1, \ldots, y_n, p_1, \ldots, p_n)$ then we have

$$
\mathrm{d}H = \frac{\partial H}{\partial x}\mathrm{d}x + \sum_{i=1}^{n}\left(\frac{\partial H}{\partial y_i}\mathrm{d}y_i + \frac{\partial H}{\partial p_i}\mathrm{d}p_i\right).
$$

Comparison with (9.5) gives

$$
\begin{aligned}
y_i' &= \frac{\partial H}{\partial p_i} & -\frac{\partial F}{\partial y_i} &= \frac{\partial H}{\partial y_i} \\
\Rightarrow \frac{\mathrm{d}y_i}{\mathrm{d}x} &= \frac{\partial H}{\partial p_i} & -\frac{\mathrm{d}p_i}{\mathrm{d}x} &= \frac{\partial H}{\partial y_i} \qquad (9.6)
\end{aligned}
$$

for $i = 1, \ldots, n$. Equations (9.6) are the *canonical Euler-Lagrange equations* associated with the integral (9.2).

Example 9.1. Take

$$
J(Y) = \int_a^b \left(\alpha(Y')^2 + \beta Y^2\right) \mathrm{d}x \qquad (9.7)
$$

where α, β are given functions of x. For this $F(x, y, y') = \alpha(y')^2 + \beta y^2$ if so

$$
\phi = \frac{\partial F}{\partial y'} = 2\alpha y' \Rightarrow y' = \frac{1}{2\alpha}\phi.
$$

The Hamiltonian H is, by (9.4),

$$
\begin{aligned}
H &= py' - F \\
&= py' - \alpha(y')^2 - \beta y^2 \qquad \text{with } y' = \frac{1}{2\alpha}\phi \\
&= p\frac{1}{2\alpha}p - \alpha\frac{1}{4\alpha^2}p^2 - \beta y^2 \\
&= \frac{1}{4\alpha}p^2 - \beta y^2
\end{aligned}
$$

in correct variables x, y, p. Canonical equations are then

$$
\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\partial H}{\partial p} = \frac{1}{2\alpha}p \qquad -\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{\partial H}{\partial y} = -2\beta y. \qquad (9.8)
$$

The ordinary Euler-Lagrange equation for $J(Y)$ is

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} = 0 \Rightarrow 2\beta y - \frac{\mathrm{d}}{\mathrm{d}x}(2\alpha y') = 0$$

which is equivalent to (9.8)

9.2. **The Euler-Hamilton (Canonical) variational principle.**

Let

$$I(P,Y) = \int_a^b \left\{ P\frac{\mathrm{d}Y}{\mathrm{d}x} - H(x,Y,P) \right\} \,\mathrm{d}x$$

be defined for any admissible independent functions P and Y with $Y(a) = y_a$, $Y(b) = y_b$, i.e.

$$Y = y_B = \begin{cases} y_a & \text{at } x = a \\ y_b & \text{at } x = b. \end{cases}$$

Suppose $I(P,Y)$ is stationary at $Y = y$, $P = p$. Take varied curves

$$Y = y + \varepsilon\xi \qquad P = p + \varepsilon\eta.$$

Then we obtain

$$\begin{aligned} I(p+\varepsilon\eta, y+\varepsilon\xi) &= \int_a^b \left\{ (p+\varepsilon\eta)\frac{\mathrm{d}}{\mathrm{d}x}(y+\varepsilon\xi) - H(x, y+\varepsilon\xi, p+\varepsilon\eta) \right\} \,\mathrm{d}x \\ &= \int_a^b \left\{ p\frac{\mathrm{d}y}{\mathrm{d}x} + \varepsilon\eta\frac{\mathrm{d}y}{\mathrm{d}x} + \varepsilon p\frac{\mathrm{d}\xi}{\mathrm{d}x} + \varepsilon^2\eta\frac{\mathrm{d}\xi}{\mathrm{d}x} - H(x,y,p) - \varepsilon\xi\frac{\partial H}{\partial y} - \varepsilon\eta\frac{\partial H}{\partial p} - \mathrm{O}_2 \right\} \,\mathrm{d}x \\ &= I(p,y) + \delta I + \mathrm{O}_2. \end{aligned}$$

where we have

$$\begin{aligned} \delta I &= \varepsilon \int_a^b \left\{ \eta\frac{\mathrm{d}y}{\mathrm{d}x} + p\frac{\mathrm{d}\xi}{\mathrm{d}x} - \xi\frac{\partial H}{\partial y} - \eta\frac{\partial H}{\partial p} \right\} \,\mathrm{d}x \\ &= \varepsilon \int_a^b \left\{ \eta\left(\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{\partial H}{\partial p}\right) - \xi\left(\frac{\mathrm{d}p}{\mathrm{d}x} + \frac{\partial H}{\partial y}\right) \right\} \,\mathrm{d}x + \underbrace{\left[\varepsilon p\xi\right]_a^b}_{=0} \end{aligned}$$

If all curves Y, including y, go through (a, y_a) and (b, y_b) then $\xi = 0$ at $x = a$ and $x = b$. Then $\delta I = 0 \Rightarrow (p, y)$ are solutions of

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\partial H}{\partial p} \qquad -\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{\partial H}{\partial y} \qquad (a < x < b)$$

with

$$y = y_B = \begin{cases} y_a & \text{at } x = a \\ y_b & \text{at } x = b \end{cases}$$

Note. If $Y = y_B$ no boundary conditions are required on p.

9.3. **An extension.**

Modify $I(P, Y)$ to be

$$I_{\text{mod}}(P, Y) = \int_a^b \left\{ P\frac{\mathrm{d}Y}{\mathrm{d}x} - H(x, P, Y) \right\} \mathrm{d}x - \Big[P(Y - y_B) \Big]_a^b$$

Here P and Y are any admissible functions. I_{mod} is stationary at $P = p$, $Y = y$. $Y = y + \varepsilon\xi$, $P = p + \varepsilon y$ and then make $\delta I_{\text{mod}} = 0$. You should find that (y, p) solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\partial H}{\partial p} \qquad -\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{\partial H}{\partial y}$$

with $y = y_B$ on $[a, b]$.

10. First Integrals of the Canonical Equations

A first integral of a system of differential equations is a function, which has constant value along each solution of the differential equations. We now look for first integrals of the canonical system

$$\frac{\mathrm{d}y_i}{\mathrm{d}x} = \frac{\partial H}{\partial p_i} \qquad -\frac{\mathrm{d}p_i}{\mathrm{d}x} = \frac{\partial H}{\partial y_i} \qquad (i = 1, \dots, n) \tag{10.1}$$

and hence of the system

$$\frac{\mathrm{d}y_i}{\mathrm{d}x} = y_i' \qquad \frac{\partial F}{\partial y_i} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y_i'} = 0 \qquad (i = 1, \dots, n) \tag{10.2}$$

which is equivalent to (10.1). Take the case where

$$\frac{\partial F}{\partial x} = 0.$$

i.e. $F = F(y_1, \dots, y_n, y_1', \dots, y_n')$. Then

$$H = \sum_{i=1}^{n} p_i y_i' - F$$

is such that $\frac{\partial H}{\partial x} = 0$ and hence

$$\frac{\mathrm{d}H}{\mathrm{d}x} = \cancelto{0}{\frac{\partial H}{\partial x}} + \sum_{i=1}^{n} \left(\frac{\partial H}{\partial y_i} \frac{\mathrm{d}y_i}{\mathrm{d}x} + \frac{\partial H}{\partial p_i} \frac{\mathrm{d}p_i}{\mathrm{d}x} \right).$$

On critical curves (orbits) this gives

$$\begin{aligned} &= \sum_{i=1}^{n} \left(\frac{\partial H}{\partial y_i} \frac{\partial H}{\partial p_i} + \frac{\partial H}{\partial p_i} (-1) \frac{\partial H}{\partial y_i} \right) \\ &= 0 \end{aligned}$$

which implies H is constant on orbits. Consider now an arbitrary differentiable function $W = W()x, y_1, \ldots, y_n, p_1, \ldots, p_n)$. Then

$$\frac{\mathrm{d}W}{\mathrm{d}x} = \frac{\partial W}{\partial x} + \sum_{i=1}^{n} \left(\frac{\partial W}{\partial y_i} \frac{\mathrm{d}y_i}{\mathrm{d}x} + \frac{\partial W}{\partial p_i} \frac{\mathrm{d}p_i}{\mathrm{d}x} \right) = \frac{\partial W}{\partial x} + \sum_{i=1}^{n} \left(\frac{\partial W}{\partial y_i} \frac{\partial H}{\partial p_i} - \frac{\partial W}{\partial p_i} \frac{\partial H}{\partial y_i} \right)$$

on orbits.

Definition: We define the *Poisson bracket* of X and Y to be

$$[X, Y] = \sum_{i=1}^{n} \left(\frac{\partial X}{\partial y_i} \frac{\partial Y}{\partial p_i} - \frac{\partial X}{\partial p_i} \frac{\partial Y}{\partial y_i} \right)$$

with $X = X(y_i, p_i)$ and $Y = Y(y_i, p_i)$.

Then we have

$$\frac{\mathrm{d}W}{\mathrm{d}x} = \frac{\partial W}{\partial x} + [W, H]$$

on orbits. In the case when $\frac{\partial W}{\partial x} = 0$, we have

$$\frac{\mathrm{d}W}{\mathrm{d}x} = [W, H].$$

So,

$$\frac{\mathrm{d}W}{\mathrm{d}x} = 0 \Leftrightarrow [W, H] = 0.$$