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Introduzione

I modelli variazionali per la segmentazione, introdotti alla fine degli anni '80 da D. Mumford, J. Shah, A. Blake e A. Zisserman, hanno permesso di ambientare alcuni problemi di cruciale importanza come la segmentazione di segnali e immagini, il riconoscimento automatico o lo studio delle fratture dei materiali, in ambito puramente matematico. Lo studio di questi modelli ha permesso di sviluppare tecniche variazionali per la soluzione dei problemi cosiddetti *a discontinuità libera*.
Per la formulazione variazionale della segmentazione di immagini proposta da D. Mumford e J. Shah si parte da una funzione $g : \Omega \to [0, 1]$ rappresentante l'immagine in tonalità di grigio da segmentare e si vuole che il risultato del processo di segmentazione fornisca una funzione u che sia la versione *regolarizzata* di g ed un insieme $K \subset \mathbb{R}^N$ rappresentante i bordi degli oggetti distinguibili nell'immagine g.
L'idea è quella di trovare tali oggetti in seguito ad un processo di minimizzazione del funzionale

$$\mathcal{MS}(u, K) = \int_{\Omega\setminus K} |\nabla u|^2 \, dx + \mu \int_{\Omega} |u - g|^2 \, dx + \alpha \mathcal{H}^{N-1}(K \cap \Omega)$$

fra tutti i possibili $K \subset \overline{\Omega}$ chiusi e $u \in C^1(\Omega \setminus K)$. I parametri $\alpha, \mu > 0$ sono inseriti al fine di regolare le caratteristiche del minimizzante ottenuto.
Grazie al processo di minimizzazione il termine $\int_\Omega |u - g|^2$ costringe u ad essere vicina al dato g da segmentare, mentre il termine contenente $|\nabla u|^2$ fa sì che essa sia il più regolare possible sull'insieme $\Omega \setminus K$, ignorando le discontinuità meno rilevanti e costringendo la formazione di discontinuità solo sull'insieme K. Il termine $\mathcal{H}^{N-1}(K \cap \Omega)$ penalizza la formazione di insiemi di discontinuità troppo grandi. L'insieme K rappresenta il bordo degli oggetti presenti in g.
Questi problemi di minimo sono caratterizzati dal fatto che la competizione per la minimizzazione avviene contemporaneamente su energie di volume ed energie di superficie, in particolare tali energie sono definite su supporti che costituiscono a loro volta un'incognita del problema. La terminologia *a discontinuità libera* si riferisce al fatto che l'insieme di discontinuità di u è sconosciuto a priori.
Poichè tale formulazione si rivela troppo forte per poter applicare i risultati classici del calcolo delle variazioni, al fine di dimostrare l'esistenza dei minimi è necessario un rilassamento del funzionale che porta il problema nello spazio delle funzioni speciali a variazione limitata $SBV(\Omega)$, giungendo al funzionale

$$F(u) := \int_{\Omega} |\nabla u|^2 \, dx + \mu \int_{\Omega} |u - g|^2 \, dx + \alpha \mathcal{H}^{N-1}(S_u \cap \Omega).$$

con $u \in SBV(\Omega)$ ed S_u l'insieme dei salti di u. A causa del termine $\mathcal{H}^{N-1}(S_u \cap \Omega)$ il funzionale non è differenziabile quindi il problema del calcolo effettivo di minimi non può essere risolto direttamente con i classici metodi basati sulla discesa del gradiente.
Viene proposta nel 1990 da L. Ambrosio e V. M. Tortorelli una approssimazione in Γ-convergenza del funzionale con funzionali ellittici definiti su opportuni spazi di Sobolev, per i quali è possible applicare i metodi numerici classici per la minimizzazione. Le proprietà della Γ-convergenza garantiscono inoltre che tali minimi convergono ad un minimo del funzionale F.
Modelli del primo ordine come quello di Mumford e Shah hanno alcuni difetti. Anzitutto, alcune caratteristiche delle immagini come le pieghe (discontinuità del gradiente) non vengono percepite, inoltre il termine gradiente induce il fenomeno della cosiddetta *sovrasegmentazione dei gradienti ripidi*, ovvero tratti di funzione con gradiente molto ripido vengono approssimati con funzioni a scalino, con conseguente perdita di definizione. Un'altra caratteristica del modello è quella di condurre ad insiemi di discontinutà formati da unioni di archi C^1 con giunzioni al più tri-ramificate, ed in tal caso tali giunzioni formano 3 angoli di $2/3\pi$ di ampiezza ciascuno.
Per ovviare a tali difetti viene proposto da A. Blake e A. Zisserman un modello variazionale del secondo ordine. La formulazione debole nello spazio delle funzioni speciali a variazione limitata generalizzate $GSBV^2(\Omega)$

$$F(u) = \int_\Omega |\nabla^2 u|^2 + \mu(u-g)^2 \, dx + \alpha \mathcal{H}^{N-1}(S_u) + \beta \mathcal{H}^{N-1}(S_{\nabla u} \setminus S_u)$$

permette di provare l'esistenza di minimi, inoltre in dimesione 2 è possible dimostrare che tali minimi rappresentano (attraverso una semplice identificazione) minimi per la formulazione forte.
Ancora una volta è necessaria una approssimazione in Γ-convergenza con funzionali regolari per poter calcolare i minimi del funzionale. Tale approssimazione viene fatta nel 2001 da L. Ambrosio, L. Faina e R. March, i quali, adattando opportunamente le tecniche sviluppate per dimostrare la Γ-convergenza nel caso del MS, propongono come approssimanti i funzionali ellittici

$$\begin{aligned} F_\epsilon(u,s,z) = & \int_\Omega z^2 |\nabla^2 u|^2 \, dx + \xi_\epsilon \int_\Omega (s^2 + o_\epsilon)|\nabla u|^2 \, dx \\ & + (\alpha - \beta) \int_\Omega \epsilon |\nabla s|^2 + \frac{1}{4\epsilon}(s-1)^2 \, dx + \beta \int_\Omega \epsilon |\nabla z|^2 + \frac{1}{4\epsilon}(z-1)^2 \, dx \\ & + \mu \int_\Omega (u-g)^2 \, dx \end{aligned}$$

definiti su opportuni spazi di Sobolev. α, β, μ sono paramtri positivi, ξ_ϵ, o_ϵ sono infitesimi per ϵ.

Questa tesi si propone principalmente due scopi. Inizialmente si vuole illustrare in maniera il più esaustiva e ricca di dettagli possibile la dimostrazione per la Γ-convergenza nel caso del funzionale di Mumford e Shah (vedi Capitolo 2). Successivamente, dopo aver illustrato i risultati della Γ-convergenza nel caso del funzionale di Blake e Zisserman (vedi Capitolo 3), si vuole proporre un algoritmo per la computazione numerica dei minimi di quest'ultimo attraverso una opportuna discretizzazione dei funzionali F_ϵ (vedi Capitolo 4).

Introduction

Variational models for segmentation, proposed by D. Mumford, J. Shah, A. Blake and A. Zisserman, allowed for a mathematical formulation of several significant problems such as signal segmentation, automatic recognition and material fractures analysis. Research on these models led to the development of variational methods for the so-called *free discontinuity problems*.
In the variational formulation for image segmentation proposed by Mumford and Shah, function $g : \Omega \to [0, 1]$ is the grayscale representation of the given image and the results of the segmentation process are a function u and a set $K \subset \mathbb{R}^N$. The former is a *regularized* version of g and the latter represents the edges of the distinguishable objects in g.
The basic idea is to identify these objects by minimizing the functional

$$\mathcal{MS}(u, K) = \int_{\Omega \setminus K} |\nabla u|^2 \, dx + \mu \int_{\Omega} |u - g|^2 \, dx + \alpha \mathcal{H}^{N-1}(K \cap \Omega)$$

among any closed set $K \subset \overline{\Omega}$ and any function $u \in C^1(\Omega \setminus K)$. Parameters $\alpha, \mu > 0$ are introduced in order to properly adjust the characteristics of the minimizers.
In the minimization process, the term $\int_\Omega |u - g|^2$ forces u to be close to the datum g. On the other hand, the term containing $|\nabla u|^2$ preserves the smoothness of the solution u on set $\Omega \setminus K$, by neglecting small discontinuities so that only discontinuities on K are allowed and their size is controlled by the term $\mathcal{H}^{N-1}(K \cap \Omega)$. The set K represents the edges of objects in g.
In these optimization problems, both bulk and surface energies are minimized simultaneously and their supports are also undefined, being unknowns of the problem theirself. Consequently, the discontinuity set of u is not known *a priori*, thus leading to the expression *free discontinuities problems*.
However, classical results of Calculus of Variations cannot be applied in this framework, since the formulation turns out to be too strong. A relaxation of the functional is necessary in order to prove the existence of minima and the problem is then moved to the space of bounded variation functions $SBV(\Omega)$, leading to the functional

$$F(u) := \int_{\Omega} |\nabla u|^2 \, dx + \mu \int_{\Omega} |u - g|^2 \, dx + \alpha \mathcal{H}^{N-1}(S_u \cap \Omega).$$

where $u \in SBV(\Omega)$ and S_u is the jump set of u. Because of the term $\mathcal{H}^{N-1}(S_u \cap \Omega)$, the functional is not differentiable, hence classical gradient descent methods cannot be directly applied for computing the minima.

In 1990, L. Ambrosio and V. M. Tortorelli proposed a Γ-convergence approximation of the functional, realized with elliptical functionals defined on proper Sobolev spaces that can be minimized by applying classical numerical methods. Moreover, properties of the Γ-convergence ensure that the minima obtained converge to a minimum of the functional F itself.
First-order models, such as Mumford and Shah model, have some drawbacks. Firstly, some features of the image, such as creases (gradient discontinuities), are not sensed. Secondly, the gradient term leads to the so-called *steep gradient oversegmentation*, that is regions with very steep gradient are approximated by step functions, thus decreasing the definition of the image. In addition, the model leads to discontinuity sets composed of unions of C^1 arcs with at most 3-points junctions (in this case arcs are $2/3\pi$ wide).
In order to overcome these limitations, A. Blake and A. Zisserman proposed a second-order variational model. The weak formulation in the space of generalized special bounded variation functions $GSBV(\Omega)$

$$F(u) = \int_\Omega |\nabla^2 u|^2 + \mu(u-g)^2\, dx + \alpha\mathcal{H}^{N-1}(S_u) + \beta\mathcal{H}^{N-1}(S_{\nabla u} \setminus S_u)$$

allows the proof of minima existence. In the 2-dimensional case, a corrispondence with minima for the strong formulation can also be proved by operating a simple identification.
Again, a Γ-convergence approximation is necessary in order to compute the minima. L. Ambrosio, L. Faina e R. March realized such approximation in 2000, again using elliptical functionals and properly adapting the techniques developed for the MS case:

$$\begin{aligned} F_\epsilon(u,s,z) = &\int_\Omega z^2|\nabla^2 u|^2\, dx + \xi_\epsilon \int_\Omega (s^2 + o_\epsilon)|\nabla u|^2\, dx \\ &+ (\alpha - \beta)\int_\Omega \epsilon|\nabla s|^2 + \frac{1}{4\epsilon}(s-1)^2\, dx + \beta\int_\Omega \epsilon|\nabla z|^2 + \frac{1}{4\epsilon}(z-1)^2\, dx \\ &+ \mu\int_\Omega (u-g)^2\, dx \end{aligned}$$

defined on proper Sobolev spaces. α, β, μ are positive parameters, ξ_ϵ, o_ϵ are infinitesimals as ϵ tends to 0.

In this thesis we propose to achieve two main goals. First, we want to present an exhaustive and detailed version of the Γ-convergence proof for the Mumford and Shah case (see Chapter 2). Then, after having stated the necessary Γ-convergence results for the Blake-Zisserman functional (see Chapter 3), we propose an algorithm for the numerical minimization of this functional by performing a proper discretization of the functionals F_ϵ (see Chapter 4).

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Chapter 1

Elements

This Chapter is entirely devoted to the presentation of the most important topics to know when dealing with minimum problems in spaces of functions.
First a brief introduction to the Direct Methods of the Calculus of Variation is presented. Then the notion of Γ-convergence is given and some of its properties related to the minimum problems are investigated.
The Geometric Measure Theory Section contains some fundamental results that are very useful to develope both the theory of SBV functions, which are described in the last Section of this Chapter, and the Γ-convergence demonstrations, showed in the following two chapters.

1.1 Direct Methods of the Calculus of Variations

Let X be a separable metric space endowed with a convergence defined by sequences, and let $F : X \to [0, +\infty]$ be a funtional $F \not\equiv +\infty$, the aim of calculus of variations is to find solutions for the problem

$$\min\{F(u) : u \in X\}. \tag{1.1}$$

Because of its non-negativity, F is bounded from below and there exists $\inf\{F(u) : u \in X\}$, so we have the existence of a sequence $u_j \subset X$ such that

$$\lim_j F(u_j) = \inf\{F(u) : u \in X\}.$$

The key idea is to find a solution of the problem (1.1) following the limit (in some sense) of the sequence u_j; we want to characterize some conditions on F and X that allow to control the limit behaviour of the sequence, and this has to be a general approach not depending on the choice of the minimizing sequence.
One way could be to require that in X every minimizing sequence (in general every bounded sequence) lies in a sequentially compact set $K \subset X$ so that we can extract a subsequence $u_{j_k} \subset u_j$ converging to some $\overline{u} \in K$.
This compactness requirement can be directly made on the functional F by asking it to be sequentially coercive, that is, for all sequences u_h such that $\sup_h F(u_h) < +\infty$ there exists a converging subsequence.

Anyway, once we have a subsequence u_{j_k} converging to some $\overline{u} \in X$, it becomes the best candidate to be a minimizer of F on X. The only thing that remains to check is the limit behaviour of F when u_{j_k} tends to $\overline{u}$; more precisely we require that

$$F(\overline{u}) = \lim_j F(u_j) = \inf\{F(u) : u \in X\}. \tag{1.2}$$

Trivially the inequality $\inf\{F(u) : u \in X\} \leq F(\overline{u})$ holds, the other one has to be formulated as a property of F for the solvability of the problem. A functional F is said to be lower semicontinuous at $u \in X$ if for any u_h converging at u

$$F(u) \leq \liminf_h F(u_h).$$

Given a functional $F : X \to [0, +\infty]$ not necessarly lower semicontinuous there is always a way to obtain a lower semicontinuous one from it.

Definition 1.1.1 *Let $F : X \to [0, +\infty]$ be a functional $F \not\equiv +\infty$ and $Y \subset X$ a subset. The lower semicontinuous envelope of F at $u \in X$ on Y is defined by*

$$\overline{F}(u) := \inf\{\liminf_j F(u_j) : u_j \to u \quad u_j \in Y\}.$$

It is clearly lower semicontinuous on Y, it is also called the relaxation of the functional F on the set Y.

Requiring F to be lover semicontinuous on X, together with other compactness properties, leads to the existence of solutions for the problem (1.1) by following the behaviour of minimizing sequences.

1.2 Γ-convergence in minimum problems

Γ-convergence is a convergence defined on families of functionals and it has some fine properties strictly related to minimum problems. Its definition can be derived by following an approach similar to the one seen in the direct methods.
The aim is to find a solution for the problem (1.1), even when F lacks of some regularity properties, by approximating it with a family depending on a parameter of more regular (or more easily computable) problems of the type

$$\alpha_j := \inf\{F_j(u) : u \in X\}. \tag{1.3}$$

Let us consider an asimptotically minimizing sequence for the family F_j, that is, a sequence $(u_j)_j$ such that

$$\lim_j [F_j(u_j) - \alpha_j] = 0,$$

we want to characterize some properties ensuring the existence of a minimum following this sequence of minimizers.
In order to find a converging subsequence we require compactness properties on the space X by asking that any minimizing sequence admits converging subsequences, or on the family F_j by asking it to be equi-coercive.
The limit $\overline{u}$ reached up to subsequences becomes the best candidate for the solution of the problem, the definition of Γ-convergence of the family F_j to the functional F is precisely built to ensure $\overline{u}$ to be a solution of (1.1).

We are going to define two properties that control the limit behaviour of the values of $F_j(u_j)$ approaching this limit. The first one is a lower bound and it is formulated similarly to the semicontinuity:

Definition 1.2.1 (liminf inequality) *The family F_j satisfies the liminf inequality if for every $v \in X$ the inequality*

$$F(v) \leq \liminf_j F_j(v_j)$$

holds for every sequence $v_j \subset X$ such that $v_j \to v$.

The second one is an upper bound.

Definition 1.2.2 (limsup inequality) *The family F_j satisfies the limsup inequality if for every $v \in X$ there is a sequence $v_j \subset X$ such that $v_j \to v$ and*

$$\limsup_j F_j(v_j) \leq F(v).$$

Such a sequence is called a recovery sequence.

We observe that, together with the liminf inequality

$$F(u) \leq \liminf_j F_j(v_j) \leq \limsup_j F_j(v_j) \leq F(u), \tag{1.4}$$

the recovery sequence actually realize a limit.

Definition 1.2.3 *If the liminf and limsup inequalities are verified we say that the family F_j Γ-converges to F and we write*

$$\Gamma\text{-}\lim_j F_j = F;$$

the functional F is called the Γ-limit of F_j.

1.2.1 Properties of Γ-convergence

To better understand the behaviour of Γ-convergence we first observe an important fact. Not always the Γ-limit of a constant sequence exists, and if it exists it has to be lower semiconinuous.

Remark 1.2.4 *If the constant sequence $F_j := F$ admits some Γ-limit F_∞, then it is exactly the relaxation of F, that is $\overline{F} = F_\infty$. In particular, this limit equals F if and only if F is lower semicontinuous.*

Proof: We consider $\overline{F}$ the relaxation of F on the whole X. Let $u \in X$, beause of the liminf inequality we have $F_\infty(u) \leq \liminf_j F(u_j)$ for every $u_j \to u$, so by the inf properties is $F_\infty(u) \leq \overline{F}(u)$.
Now let v_j a recovery sequence, then

$$\overline{F}(u) \geq F_\infty(u) \geq \limsup_j F(v_j) \geq \liminf_j F(v_j) \geq \overline{F}(u)$$

hence $F_\infty(u) = \overline{F}(u)$. ∎

The verification of the Γ-convergence leads to some crucial facts concerning the framework of minimum problems. Let F_j be a family of functionals Γ-convergent to F. If $u_j \subset X$ is an asymptotically minimizing sequence, by the limsup inequality we have

$$\limsup_j \inf\{F_j(v) : v \in X\} \leq F(u)$$

for every $u \in X$, hence

$$\begin{aligned}\limsup_j \inf\{F_j(u) : u \in X\} &\leq \inf\{F(u) : u \in X\} \\ &\leq F(\overline{u}) \leq \liminf_j F_j(u_j) \leq \liminf_j \inf\{F_j(u) : u \in X\}.\end{aligned}$$

This leads not only to the existence of solutions of the problem

$$\min\{F(u) : u \in X\}$$

but also to the existence of some solution $\overline{u} \in X$ such that

$$F(\overline{u}) = \lim_j \inf\{F_j(u) : u \in X\}.$$

So, this solution can be viewed as a limit of the minimum problems which approximtate it. Moreover in general, every minimizing sequences (up to a subsequence) converge to a minimizer of F on X.

An equivalent definition of Γ-limit is possible as an equivalence of well-defined Γ-lim inf and Γ-lim sup .

Definition 1.2.5 *Let F_j be a sequence of functionals, we define the following quantities*

$$\Gamma\text{-}\liminf_j \ F_j(u) := \inf\{\liminf_j F_j(u_j) : u_j \to u\}$$

$$\Gamma\text{-}\limsup_j \ F_j(u) := \inf\{\limsup_j F_j(u_j) : u_j \to u\}$$

for every $u \in X$.

Corollary 1.2.6 *The* Γ-lim inf *functional is lower semicontinuous.*

Remark 1.2.7 *A sequence of functionals F_j is Γ-converging (to some limit F_∞) if and only if $\Gamma\text{-}\liminf_j F_j = \Gamma\text{-}\limsup_j F_j (= F_\infty)$.*

Proof: (⇒) Fix $u \in X$. By the liminf inequality we have $F_\infty(u) \leq \liminf_j F_j(u_j)$ for every $u_j \to u$, it follows that

$$F_\infty(u) \leq \ \Gamma\text{-}\liminf_j \ F_j(u) \leq \ \Gamma\text{-}\limsup_j \ F_j(u).$$

On the other hand, if v_j is a recovery sequence, by the definition of Γ-lim sup and the limsup inequality, we get

$$\Gamma\text{-}\limsup_j \ F_j(u) \leq \limsup_j F_j(v_j) \leq F_\infty(u).$$

(⇐) Fix $u \in X$ and let $\Gamma\text{-}\liminf F_j = \Gamma\text{-}\limsup F_j = F_\infty$. Trivially for every $u_j \to u$ we have

$$F_\infty(u) = \inf\{\liminf_j F(v_j) : v_j \to u\} \le \liminf_j F_j(u_j),$$

that is, the liminf inequality is verified. Moreover, by the inf properties, for every $h \ge 1$ we know there exists a sequence $(v_j^h)_j$ converging to u as $j \to +\infty$ such that

$$\limsup_j F_j(u_j^h) \le F_\infty(u) + \frac{1}{h} \tag{1.5}$$

then the diagonal sequence $v_j := u_j^j$ is a recovery sequence; also the limsup inequlity is verified. ■

Corollary 1.2.8 *If the* Γ*-limit exists, then it is unique.*

Often the limsup inequality in the Γ-convergence is replaced by the condition (1.5) because it can result more convenient in some calculations.

The last property of Γ-convergence that we present is its stability under continuous perturbations (with respect to the convergence considered on the space X).

Remark 1.2.9 *Let* $F_j : X \to [0, +\infty]$ *be* Γ*-convergent to* F_∞ *and let* $G : X \to [0, +\infty]$ *be a continuous functional, then*

$$\Gamma\text{-}\lim_j \,[F_j + G] = F_\infty + G$$

that is, Γ*-convergence is stable under additive continuous perturbations.*

Proof: Fix $u \in X$ and a sequence $u_j \to u$, then by the liminf inequality and the continuity of G we have

$$[F_\infty + G](u) \le \liminf_j F_j(u_j) + \lim_j G(u_j) = \liminf_j [F_j + G](u_j).$$

As in (1.4) any recovery sequence does realize a limit, hence

$$\limsup_j [F_j + G](v_j) = \lim_j F_j(v_j) + \lim_j G(v_j) = F_\infty(u) + G(u) = [F_\infty + G](u)$$

so we are done. ■

1.3 Geometric measure theory

Starting with an outer measure on Ω we see how to obtain a measure by restricting it to the σ-algebra of the open set of Ω. Then a very useful measure on $\mathbb{R}^N$, the Hausdorff measure, is defined, and some Theorems of the Calculus (Coarea formula) are stated for the Lipschitz functions.

1.3.1 Basics of the measure theory

Let μ^* be an outer measure on a set Ω. Given a subset $A \subset \Omega$ such that for every $T \subset \Omega$

$$\mu^*(A) = \mu^*(A \cap T) + \mu^*(A \setminus T)$$

holds, we say that A is μ^*-measureable and we write $A \in \mathfrak{M}(\mu^*)$. The family of the μ^*-measureable sets is a σ-algebra and the map

$$\mu := \mu|_{\mathfrak{M}(\mu^*)} : \mathfrak{M}(\mu^*) \longrightarrow [0, +\infty]$$

is a measure.
Given a topological space (Ω, τ) we consider the σ-algebra generated by the open sets of τ, we denote it as $\mathcal{B}(\Omega) = \sigma(\tau)$. It is called the Borel σ-algebra on Ω, and every $A \subset \mathcal{B}(\Omega)$ is called a Borel set.
An outer measure μ^* is called a Borel measure if $\mathcal{B}(\Omega) \subset \mathfrak{M}(\mu^*)$. Generally the inclusion is strict, given a $x_0 \in \Omega$ then the measure δ_{x_0}, is such that $\mathfrak{M}(\delta_{x_0}) = \mathcal{P}(\Omega)$, but for instance in the usual topology is $\mathcal{P}(\Omega) \not\equiv \mathcal{B}(\Omega)$.
Given a set $M \subset \Omega$ and an outer measure μ^* we denote by $\mu^* \llcorner M$ the restriction to M of μ^*, that is, the map obtained by setting

$$\mu^* \llcorner M(A) = \mu^*(A \cap M) \qquad \forall A \subset \Omega.$$

If Ω a Borel set then $\mu : \mathcal{B}(\Omega) \to [0, +\infty]$ is a Borel measure.

Lemma 1.3.1 (Caratheodory criterion) *Let $(\Omega, \operatorname{dist})$ be a metric space and μ a measure such that $\operatorname{dist}(A, B) > 0 \Longrightarrow \mu(A \cap B) = \mu(A) + \mu(B)$. Then $\mathcal{B}(\Omega) \subset \mathfrak{M}(\mu)$.*

1.3.2 Hausdorff measure

We introduce the definition and some important properties of the $\mathcal{H}^s$ Hausdorff measure on $\mathbb{R}^N$. This notion of measure doesn't depend on N, and it allow us to characterize s-dimentional objects in $\mathbb{R}^N$.

Definition 1.3.2 (Hausdorff measure) *Let $A \subset \mathbb{R}^N$, $0 \le s < +\infty$ e $0 < \delta \le +\infty$, we define the following measures:*

(i) $$\mathcal{H}^s_\delta(A) := \inf \left\{ \sum_{j=1}^{+\infty} \alpha(s) \left[\frac{\operatorname{diam}(C_j)}{2} \right]^s : A \subset \bigcup_{j=1}^{+\infty} C_j,\ \operatorname{diam}(C_j) < \delta \right\}.$$

(ii) $$\mathcal{H}^s(A) := \sup_{\delta > 0} \mathcal{H}^s_\delta(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A).$$

Remark 1.3.3 *The constant $\alpha(s)$ is defined in such a way that, for any $N \in \mathbb{N}$, the $\mathcal{H}^N$ measure of an N-dimensional compact hypersurface coincides with its classical area. It is defined as*

$$\alpha(s) := \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2} + 1)},$$

where $\Gamma(s) := \int_0^{+\infty} e^{-x} x^{s-1}\, dx$ for every real number $0 < s < +\infty$, is the well-known gamma function. It is possible to prove that

$$\mathcal{L}^N(B_r(x)) = \alpha(N) r^N$$

for every ball $B_r(x) \subset \mathbb{R}^N$ of center x and radius $r > 0$.

By the Caratheodory criterion it is possible to show that $\mathcal{H}^s$ is Borel regular. Some fundamental properties of the Hausdorff measure are presented in the following.

Proposition 1.3.4 *Let $A \subset \mathbb{R}^N$ be a set.*

(i) $s > N \Rightarrow \mathcal{H}^s(A) = 0$

(ii) $s < N \Rightarrow \mathcal{H}^s(A) = +\infty \qquad \forall A$ *open*

(iii) $s = N \Rightarrow \mathcal{H}^s(A) = \mathcal{L}^N(A)$

(iv) Let $0 \leq s \leq N$ *such that* $0 < \mathcal{H}^s(A) < +\infty$ *then*

- *for every $t < s$ we have $\mathcal{H}^t(A) = +\infty$*
- *for every $r > s$ we have $\mathcal{H}^r(A) = 0$.*

Definition 1.3.5 (Hausdorff dimension) *Let $A \subset \mathbb{R}^N$, the real number*

$$\dim_{\mathcal{H}}(A) := \inf\Big\{0 \leq s < +\infty : \mathcal{H}^s(A) = 0\Big\} = \sup\Big\{0 \leq s < +\infty : \mathcal{H}^s(A) = +\infty\Big\}$$

is called the Hausdorff dimension of A.

A typical example of a set with non-integer Hausdorff dimension is the Cantor third middle set $\mathcal{C}$ which has $\dim_{\mathcal{H}}(\mathcal{C}) = \log 2/\log 3$.

A set $M \subset \mathbb{R}^N$ is said to be $\mathcal{H}^K$_rettificable if there exists a family $M_j \subset \mathbb{R}^N$ with $M_j \in C^1$ and $\mathcal{H}^K(M_j) < +\infty$ such that

$$\mathcal{H}^K\left(M \setminus \bigcup_j M_j\right) = 0.$$

The assumption $M_j \in C^1$ means that M_j is locally the graph of a C^1 function, that is

$$M_j = f_j(E_j) \qquad f_j : E_j \subset \mathbb{R}^K \to \mathbb{R}^N \qquad f_j \in C^1(E_j).$$

Hence M is $\mathcal{H}^K$_rettificabile if and only if there exists a family $f_j : E_j \subset \mathbb{R}^K \to \mathbb{R}^N$ with $f_j \in C^1(E_j)$ such that

$$\mathcal{H}^K\left(M \setminus \bigcup_j f_j(E_j)\right) = 0.$$

1.3.3 Lipschitz functions, Coarea formula

The well-known results of the Calculus for Lebesgue measure can be generalized to the Hausdorff measure. The C^1 functions of the Calculus are replaced with functions for which the graph is an Hausdorff-rectifiable set, the Lipschitz functions.

Definition 1.3.6 *A function $f : \mathbb{R}^K \to \mathbb{R}^N$ is called a Lipschitz function if there exists a positive constant C such that*

$$|f(x) - f(y)| \le C|x - y| \quad \forall x \ne y.$$

If f is Lipschitz we refer to the number

$$L = \operatorname{Lip}(f) := \sup_{x \ne y} \frac{|f(x) - f(y)|}{|x - y|}$$

as the Lipschitz constant of f and we say that f is L-Lipschitz.

Let $f : \mathbb{R}^K \to \mathbb{R}^N$ be a Lipschitz function then we have

$$\mathcal{H}^K(f(A)) \le [\operatorname{Lip}(f)]^K \mathcal{H}^K(A)$$

for every $\mathcal{H}^K$-measurable set. In particular, $\mathcal{H}^K(A) = 0$ implies $\mathcal{H}^k(f(A)) = 0$.

Let $f : E \subset \mathbb{R}^K \to \mathbb{R}$ be an L-Lipschitz functions, then there always exists an L-Lipschitz function $\tilde{f} : \mathbb{R}^K \to \mathbb{R}$ such that $\tilde{f}|_E = f$. Both of these are suitable:

$$\tilde{f}(y) := \inf_{x \in E}\{f(x) + L \operatorname{dist}(x, y)\}$$

$$\tilde{f}(y) := \inf_{x \in E}\{f(x) - L \operatorname{dist}(x, y)\}.$$

Coarea formula

In the sequel we always consider $K \le N$. For every linear map $L : \mathbb{R}^N \to \mathbb{R}^K$ we define the Jacobian of L as

$$JL(x) := \sqrt{\det(L \circ L^T)}.$$

Let $f : \mathbb{R}^N \to \mathbb{R}^K$ be a Lipschitz function, by Rademacher Theorem it is differentiable $\mathcal{L}^N$-a.e., we denote with $\nabla f(x)$ the gradient matrix of f (a $K \times N$ matrix), which can be regarded as a linear map from $\mathbb{R}^N$ to $\mathbb{R}^K$. We denote by $Jf(x)$ the Jacobian of $\nabla f(x)$.

Theorem 1.3.7 (Coarea formula) *Let $K \le N$ and $f : \mathbb{R}^N \to \mathbb{R}^K$ a Lipschitz function, then, for every $y \in \mathbb{R}^K$ and every $\mathcal{H}^N$-measurable set $A \subset \mathbb{R}^N$, the set $f^{-1}(y) \cap A$ is $\mathcal{H}^{N-K}$-measurable, and*

$$\int_A Jf(x)\, d\mathcal{H}^N(x) = \int_{\mathbb{R}^K} \mathcal{H}^{N-K}(f^{-1}(y) \cap A)\, d\mathcal{H}^K(y)$$

holds. Moreover, if $K = N$ and f is one-to-one we have $\int_A Jf(x)\, d\mathcal{H}^N(x) = \mathcal{H}^N(f(A))$.

Corollary 1.3.8 (Fleming-Rishel coarea formula) *Let $f : \mathbb{R}^N \to \mathbb{R}$ a Lipschitz function, then,*

$$\int_{\mathbb{R}^N} |\nabla f(x)|\, d\mathcal{H}^N = \int_{-}^{+\infty} \mathcal{H}^{N-1}(\{f = t\})\, dt$$

holds.

Proof: $Jf(x) = |\nabla f(x)|$. ■

Proposition 1.3.9 (Change of variables) *Let $K \leq N$ and $f : \mathbb{R}^N \to \mathbb{R}^K$, $g : \mathbb{R}^N \to \mathbb{R}$ Lipschitz functions; then*

$$\int_{\mathbb{R}^N} g(x) Jf(x)\, d\mathcal{H}^N = \int_{\mathbb{R}^K} \left[\int_{f^{-1}(y)} g(z)\, d\mathcal{H}^{N-K}(z) \right] d\mathcal{H}^N(y)$$

holds.

1.3.4 Total variation of a measure

Let μ be a measure, the total variation of μ over the set Ω is the number (possibly $+\infty$)

$$|\mu|(\Omega) = \sup \left\{ \sum_i |\mu(B_i)| : \Omega = \bigcup_i B_i \right\}.$$

When Ω is a topolgical space and $\mu \in \mathcal{M}(\Omega, \mathbb{R}^N)$, the space of vector-valued measures on Ω, it results

$$|\mu|(\Omega) = \sup \left\{ \int_\Omega \varphi\, d\mu : \varphi \in C(\Omega, \mathbb{R}^N),\ \|\varphi\|_\infty \leq 1 \right\} \tag{1.6}$$

Definition 1.3.10 *Let $\mathcal{M}_f(\Omega, \mathbb{R}^N)$ the space of the vector-valued measures on Ω such that $|\mu|(\Omega) < +\infty$.*

Theorem 1.3.11 (Radon-Nikodym) *Let $\mu \in \mathcal{M}_f(\Omega, \mathbb{R}^N)$ and $\lambda \in \mathcal{M}(\Omega)$ with $\lambda \geq 0$ and σ-finite. Then there exist two measures μ^a and μ^s such that $\mu = \mu^a + \mu^s$ and satisfying*

- *μ^a is absolutely continuous with respect to λ, $\lambda \gg \mu^a$, that is*

$$\lambda(B) = 0 \Rightarrow |\mu^a|(B) = 0$$

- *λ and μ^s are mutually singular, $\lambda \perp \mu^s$, that is*

$$\exists E \text{ such that } \lambda(E) = 0 \quad |\mu^s|(\Omega \setminus E) = 0$$

Moreover, there exists $f \in L^1(\Omega, \lambda, \mathbb{R}^N)$ such that

$$\mu^a = f\lambda$$

and we have

$$|\mu|(\Omega) = \int_\Omega |f| d\lambda + |\mu^s|(\Omega).$$

The function $f \in L^1(\Omega, \lambda, \mathbb{R}^N)$ such that $\mu^a = f\lambda$ is denoted by

$$f =: \frac{d\mu}{d\lambda}$$

and it is the density of μ with respect to λ. This notation comes from the Besicovitch Theorem, which states that λ-a.e. $x \in \Omega$ the following limit

$$f(x) = \lim_{r \to 0} \frac{\mu^a(B_r(x))}{\lambda(B_r(x))} = \lim_{r \to 0} \frac{\mu(B_r(x))}{\lambda(B_r(x))} =: \frac{d\mu}{d\lambda}(x)$$

exists and is finite.

1.3.5 Other important results

We recall here some very useful results for the demonstration of the Γ convergence.

Theorem 1.3.12 *Given λ and μ_h a family of positive bounded measures on an open set Ω such that*

(i) $\lambda(A) \leq \liminf_h \mu_h(A) \qquad \forall A \subset \Omega$ *open*

(ii) $\limsup_h \mu_h(\Omega) \leq \lambda(\Omega)$.

Then the family μ_h weakly converge to λ, that is*

$$\lim_{h\to+\infty} \int_\Omega f \, d\mu_h = \int_\Omega f \, d\lambda \qquad \forall f \in C_c(\Omega),$$

and in particular $\mu_h(B) \to \lambda(B)$ for every λ-measurable set such that $\lambda(\partial B) = 0$.

Lemma 1.3.13 *Let $\mu : \mathcal{B}(\Omega) \to [0, +\infty]$ be a σ-finite measure, and let $f_h \subset L^1(\Omega)$ a family of non-negative functions. Then*

$$\int_\Omega \sup_h f_h(x) \, d\mu(x)$$
$$= \sup \left\{ \sum_{i=1}^{n} \int_{A_i} f_i(x) \, d\mu(x) : \; A_i \subset \Omega \text{ open}, \; A_h \cap A_k = \emptyset \; \forall h \neq k, \; n \in \mathbb{N} \right\}.$$

1.4 Functions of bounded variation

Let $u \in W^{1,p}(\Omega)$ with $1 < p < +\infty$ then $u \in L^p(\Omega)$ and $\nabla u \in L^p(\Omega; \mathbb{R}^N)$, moreover

$$\|\nabla u\|_p = \sup \left\{ \int_\Omega \nabla u \cdot \varphi \, dx : \varphi \in L^{p'} \; \|\varphi\|_{p'} \leq 1 \right\}.$$

Since C_c^1 is dense in $L^{p'}$ by divergence theorem we get

$$\|\nabla u\|_p = \sup \left\{ \int_\Omega u \operatorname{div} \varphi \, dx : \varphi \in C_c^1 \; \|\varphi\|_{p'} \leq 1 \right\}.$$

Let $\nabla u \in L^p(\Omega; \mathbb{R}^N)$ then this sup is finite, hence the linear functional

$$T_u(\varphi) = \int_\Omega u \operatorname{div} \varphi \, dx \qquad \varphi \in C_c^1$$

is continuous, that is $|T_u(\varphi)| \leq C\|\varphi\|_{p'}$. By Hahn-Banach Theorem it can be extended to a continuous linear functional $T_u : L^{p'} \to \mathbb{R}$. Since $1 < p < +\infty$ the dual space of $L^{p'}$ is L^p, then the functional T_u is identified as an element of L^p.

Remark 1.4.1 *Let $p = 1$. The clousure of $C_c^1(\Omega, \mathbb{R}^N)$ with respect to the norm $\|.\|_\infty$ is exactly $C_0(\Omega, \mathbb{R}^N) = \{\varphi \in C(\bar{\Omega}, \mathbb{R}^N) : \varphi|_{\partial\Omega} = 0\}$. Then $T_u \in C_0(\Omega, \mathbb{R}^N)$, and by Riesz Theorem this space can be identified with the space $\mathcal{M}_f(\Omega, \mathbb{R}^N)$.*

EXAMPLE: Let us consider $u = \chi_E$ such that $\partial E \in C^1$ and $\mathcal{H}^{N-1}(\partial E \cap \Omega) < +\infty$. By Gauss-Green formulas we have

$$\int_\Omega u \operatorname{div} \varphi \, dx = \int_{\partial E \cap \Omega} \varphi \cdot \nu_E \, d\mathcal{H}^{N-1} = T_u(\varphi)$$

so T_u is represented by the measure $\mu := \nu_E \llcorner (\partial E \cap \Omega)$.

Definition 1.4.2 *An $u \in L^1(\Omega)$ is said to be a bounded variation function if*

$$|Du|(\Omega) := \sup \left\{ \int_\Omega u \operatorname{div} \varphi \, dx : \varphi \in C^1_c(\Omega, \mathbb{R}^N) \ \|\varphi\|_\infty \leq 1 \right\} < +\infty.$$

In this case we write $u \in BV(\Omega)$.

We observe that, by (1.6), the gradient of a function $u \in BV(\Omega)$ is a finite total variation measure.

Definition 1.4.3 *Let $E \subset \Omega$ be a set such that $\chi_E \in BV(\Omega)$, then we say that E has a finite perimeter, and we write $E \in FP(\Omega)$. We denote the perimeter of E in Ω as*

$$p(E, \Omega) := |D\chi_E|(\Omega).$$

Proposition 1.4.4 (Characterization of $BV(\Omega)$) *Let $u \in L^1(\Omega)$ then the following statements are equivalent*

(i) $u \in BV(\Omega)$

(ii) $\exists \mu \in \mathcal{M}_f(\Omega, \mathbb{R}^N)$ *such that*

$$\int_\Omega u \operatorname{div} \varphi \, dx = -\int_\Omega \varphi \, d\mu \qquad \forall \varphi \in C^1_c(\Omega, \mathbb{R}^N),$$

moreover $|Du|(\Omega) = |\mu|(\Omega)$.

(iii) $\exists (u_j)_j \subset C^\infty(\Omega)$ *such that*

$$u_j \xrightarrow{L^1} u \qquad \sup_j \int_\Omega |\nabla u_j| < \infty$$

moreover

$$|Du|(\Omega) = \inf \left\{ \liminf_j \int_\Omega |\nabla f_j| : f_j \subset C^\infty(\Omega) \ f_j \xrightarrow{var} u \right\}$$

Definition 1.4.5 *Given a family of functions $u_j \subset L^1(\Omega, \mathbb{R}^N)$, we say that they converge in variation to a u, and we write*

$$u_j \xrightarrow{var} u$$

if both the following conditions are verified

(i) $u_j \xrightarrow{L^1} u$

(ii) $|Du_j|(\Omega) \overset{*}{\rightharpoonup} |Du|(\Omega)$.

SKETCH OF THE PROOF OF 1.4.4: we see that $|Du|(\Omega)$ is lower semicontinuous with respect the $L^1(\Omega)$ convergence, in fact, if $f_j \to u$ then the functional $u \mapsto \int_\Omega u \operatorname{div} \varphi$ is continuous for every $\varphi \in C_c^1(\Omega, \mathbb{R}^N)$. Then $|Du|(\Omega)$ is lower semicontinuous as an upper bound of lower continuous functionals.
Let us consider $\rho \in L^1(\mathbb{R}^N)$ a classical convolution kernel, so $\rho \geq 0$ with $\operatorname{spt} \rho \subset B_1(0)$ and $\int_{\mathbb{R}^N} \rho = 1$. We want to prove that convergence in variation by the u_ϵ functions obtained by the mollification $\rho_\epsilon(x) := \epsilon^{-N}\rho(x/\epsilon)$.
Let

$$u_\epsilon(x) := u * \rho_\epsilon(x) = \int u(y)\rho_\epsilon(x-y)\,dy$$

then we have

$$\begin{aligned}\nabla_x u_\epsilon(x) = u * \nabla_x \rho_\epsilon(x) &= \int u(y)\nabla_x \rho_\epsilon(x-y)\,dy \\ &= -\int u(y)\nabla_y \rho_\epsilon(x-y)\,dy = \int \rho_\epsilon(x-y)\,dDu(y).\end{aligned}$$

By integrating we obtain

$$\begin{aligned}\int_\Omega |\nabla u_\epsilon|\,dx = \int_\Omega \left| \int \rho_\epsilon(x-y)\,dDu(y)\right| dx &\leq \int_\Omega \int \rho_\epsilon(x-y)\,d|Du|(y)\,dx \\ = \int\int_\Omega \rho_\epsilon(x-y)\,dx\,d|Du|(y) &\leq \int_\Omega 1\,d|Du|(y) = |Du|(\Omega)\end{aligned}$$

hence, by taking the superior limit and taking into account the semicontinuity we have

$$|Du|(\Omega) \leq \liminf_{\epsilon\to 0} \int_\Omega |\nabla u_\epsilon|\,dx \leq \limsup_{\epsilon \to 0} \int_\Omega |\nabla u_\epsilon|\,dx \leq |Du|(\Omega)$$

and we are done.

The finite perimeter sets are strictly related to the $BV(\Omega)$ functions. This relationship becomes from the following theorem.

Proposition 1.4.6 (Coarea formula for $BV(\Omega)$) *Let $u \in BV(\Omega)$. For every $t \in \mathbb{R}$ we define the set $E_t^u := \{x \in \Omega : u(x) > t\}$. Then for a.e. $t \in \mathbb{R}$ the set E_t^u has finite perimeter in Ω, moreover*

$$|Du|(B) = \int_{\mathbb{R}} |D\chi_{E_t^u}|(B)\,dt.$$

We recall, by defintion $|D\chi_{E_t^u}|(B) = p(E_t^u, \Omega)$.

Proof: Without loss of generality $u \geq 0$. Let $x \in \Omega$. Since the functions $\chi_{[0,u(x)]}(t)$ and $\chi_{E_t^u}(x)$, up to a null-measure set, equals, we have

$$u(x) = \int_0^{u(x)} dt = \int_{\mathbb{R}} \chi_{[0,u(x)]}(t)\,dt = \int_{\mathbb{R}} \chi_{E_t^u}(x)\,dt.$$

Fixed a $\varphi \in C_c^1(\Omega, \mathbb{R}^N)$ with $\|\varphi\|_\infty \leq 1$, thanks to the hypotesis $u \in BV(\Omega)$

$$\begin{aligned}\int_\Omega u \operatorname{div}\varphi(x)\,dx &= \int_\Omega \operatorname{div}\varphi(x) \left[\int_{\mathbb{R}} \chi_{E_t^u}(x)\,dt\right] dx \\ &= \int_\Omega \left[\int_{\mathbb{R}} \chi_{E_t^u}(x) \operatorname{div}\varphi(x)\,dt\right] dx < +\infty.\end{aligned}$$

By Fubini-Tonelli Theorem

$$\int_\Omega u \operatorname{div} \varphi(x)\, dx = \int_\Omega \left[\int_\mathbb{R} \chi_{E_t^u}(x) \operatorname{div} \varphi(x)\, dx \right] dt$$

so

$$\begin{aligned} |Du|(\Omega) &= \sup_{\substack{\varphi \in C_c^1(\Omega) \\ \|\varphi\|_\infty \leq 1}} \int_\Omega \left[\int_\mathbb{R} \chi_{E_t^u}(x) \operatorname{div} \varphi(x)\, dx \right] dt \\ &= \int_\mathbb{R} \left[\sup_{\substack{\varphi \in C_c^1(\Omega) \\ \|\varphi\|_\infty \leq 1}} \int_\Omega \chi_{E_t^u}(x) \operatorname{div} \varphi(x)\, dx \right] dt < +\infty \end{aligned}$$

again, by Fubini-Tonelli

$$|D\chi_{E_t^u}|(\Omega) = \sup_{\substack{\varphi \in C_c^1(\Omega) \\ \|\varphi\|_\infty \leq 1}} \int_\Omega \chi_{E_t^u}(x) \operatorname{div} \varphi(x)\, dx < \infty \qquad \text{a.e. } t \in \mathbb{R}$$

Easily we can prove the result for every $B \in \mathcal{B}(\Omega)$. ■

For a finite perimeter set $E \subset \Omega$ there exists a notion of set boundary $\mathcal{F}E$ such that $|D\chi_E| = \mathcal{H}^{N-1} \llcorner \mathcal{F}E$. In order to define it we need the following density definitions.

Definition 1.4.7 *Given an $x \in \mathbb{R}^N$ we define the*

(i) upper density of E at x

$$\Theta^*(E, x) := \limsup_{r \searrow 0} \frac{|E \cap B_r(x)|}{|B_r(x)|}$$

(ii) lower density of E at x

$$\Theta_*(E, x) := \liminf_{r \searrow 0} \frac{|E \cap B_r(x)|}{|B_r(x)|}$$

When $\Theta^*(E,x) = \Theta_*(E,x)$ we denote the common value as $\Theta(E,x)$, it is called the density of E at x. Let $E^{(a)}$ be the set of the points such that

$$E^{(a)} := \{x \in \mathbb{R}^N : \Theta(E,x) = a\}.$$

Definition 1.4.8 *Given a set $E \subset \mathbb{R}^N$ we define $\partial^* E := \mathbb{R}^N \setminus (E^{(0)} \cup E^{(1)})$.*

Let $E \in FP(\Omega)$, then $D\chi_E \ll |D\chi_E|$ hence by Radon-Nykodym Theorem 1.3.11 there exists $\nu_E \in L^1(\Omega, |D\chi_E|)$ such that $D\chi_E = \nu_E |D\chi_E|$. Besicovitch Theorem implies that

$$\nu_E(x) = \lim_{r \searrow 0} \frac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))} \quad |D\chi_E|\text{-qo } x \in \Omega.$$

Definition 1.4.9 *The reduced boundary of a set $E \subset \mathbb{R}^N$ is the set*

$$\mathcal{F}E := \left\{ x \in \operatorname{spt} |D\chi_E| : \exists \lim_{r \searrow 0} \frac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))} := \nu_E(x),\ \nu_E(x) = 1 \right\}.$$

It is possible to prove that, if $E \in \mathrm{FP}\,(\mathbb{R}^N)$ then $\mathcal{F}E$ is an $\mathcal{H}^{N-1}$- rectifiable set, moreover ν_E is the classical versor of ∂E. Since it is also possible to show that

$$|D\chi_E| = \mathcal{H}^{N-1} \llcorner \mathcal{F}E \qquad \mathcal{H}^{N-1}(\partial^* E \setminus \mathcal{F}E) = 0$$

then we have that

$$|D\chi_E| = \mathcal{H}^{N-1} \llcorner \partial^* E.$$

1.4.1 Decomposition of the measure $|Du|(\Omega)$

Summarizing, we have seen that

- $u \in C^1 \Rightarrow Du = \nabla u L^n$
- $u = \chi_E \quad \partial E \in C^\infty \Rightarrow Du = \nu_E \mathcal{H}^{N-1} \llcorner \partial E$
- $u = \chi_E \in BV \Rightarrow Du = \nu_E \mathcal{H}^{N-1} \llcorner \partial^* E$

There exist a $BV(\Omega)$ funtion for wich the measure $|Du|(\Omega)$ is made only by the singular part. This is the case of the Cantor-Lebesgue function f. Let $\mathcal{C}$ the Cantor third middle set, then f is constant on the set $[0,1] \setminus \mathcal{C}$ and $f' = 0$ a.e. in $[0,1]$. Clearly

$$Df = D^s f$$

where the measure D^s is concentrated only on the set $\mathcal{C}$.

Definition 1.4.10 *Let $u \in L^1(\Omega)$ and $x \in \Omega$, we define*

(i) the upper approximate limit of u at x as

$$u^\vee(x) := \inf\{t \in \mathbb{R} : \Theta(\{u > t\}, x) = 0\}$$

(ii) the lower approximate limit of u at x as

$$u^\wedge(x) := \sup\{t \in \mathbb{R} : \Theta(\{u > t\}, x) = 1\}$$

When $u^\vee(x) = u^\wedge(x)$ we denote the common value as $\tilde{u}(x)$ and we say that u is approximately continuous at x.

Definition 1.4.11 *Let $u \in L^1(\Omega)$ we define the jump set of u as*

$$S_u := \{x \in \Omega : u^\wedge(x) < u^\vee(x)\}.$$

Let us consider $u \in BV(\Omega)$, by Radon-Nikodym Theorem we can consider $D^a u \ll \mathcal{L}^N$ and $D^s u \perp \mathcal{L}^N$ such that

$$Du = D^a u + D^s u$$

then we split the singular part by restricting to the jump set S_u

$$D^s u = D^s u \llcorner S_u + D^s u \llcorner (\Omega \setminus S_u).$$

We denote every component of this decomposition as $D^j u := D^s u \llcorner S_u$, called the jump part of $|Du|(\Omega)$, and $D^c u := D^s u \llcorner (\Omega \setminus S_u)$, called the Cantor part of $|Du|(\Omega)$. Tipically the Cantor part is concentrated on Cantor-like sets.
We have the following decomposition:

$$Du = D^a u + D^j u + D^c u.$$

Proposition 1.4.12 *Let $B \subset \Omega$ such that $\mathcal{H}^{N-1}(B) < +\infty$ then $|D^c u| = 0$.*

Proposition 1.4.13 *Let $u \in BV(\Omega)$ then the jump part of the $|Du|(\Omega)$ can be decomposed as*

$$D^j u = (u^\wedge - u^\vee)\nu_u \mathcal{H}^{N-1} \llcorner S_u$$

Finally we have for every $u \in BV(\Omega)$ the following decomposition:

$$\begin{aligned} Du &= D^a u + D^s u = \\ &= D^a u + D^j u + D^c u = \\ &= D^a u + (u^\wedge - u^\vee)\nu_u \mathcal{H}^{N-1} \llcorner S_u + D^c u = \\ &= \varphi \mathcal{L}^N + (u^\wedge - u^\vee)\nu_u \mathcal{H}^{N-1} \llcorner S_u + D^c u \end{aligned}$$

where $\varphi \in L^1(\Omega, \mathbb{R}^N)$ equals the approximate gradient of u at x, that is

$$\lim_{r \searrow 0} \frac{1}{\omega_N r^N} \int_{B_r(x)} \frac{|u(y) - u(x) - \varphi(x) \cdot (y - x)|}{|y - x|} \, dy = 0.$$

Our aim now is to investigate some properties of the jump set of a bounded variation function.

Definition 1.4.14 *Let $u \in L^1(\Omega)$. A point $x \in \Omega$ is said to be an approximate discontinuity point (a Lebesgue point) if $\exists z_x \in \mathbb{R}$ such that*

$$\lim_{r \searrow 0} \frac{1}{\omega_N r^N} \int_{B_r(x)} |u(y) - z_x| \, dy.$$

In this case we have

$$z_x = \lim_{r \searrow 0} \frac{1}{\omega_N r^N} \int_{B_r(x)} u(y) \, dy.$$

Remark 1.4.15 *When $u \in L^\infty(\Omega)$ this notion of discontinuity is equivalent to the one given before, that is $z_x = \tilde{u}(x)$.*

Definition 1.4.16 *Let $u \in L^1(\Omega)$. A point $x \in \Omega$ is a jump discontinuity point if $\exists a, b \in \mathbb{R}$ e $\nu \in S^{N-1}$ such that*

$$\lim_{r \searrow 0} \frac{1}{|B_r^+(x,\nu)|} \int_{B_r^+(x,\nu)} |u(y) - a| \, dy = \lim_{r \searrow 0} \frac{1}{|B_r^-(x,\nu)|} \int_{B_r^-(x,\nu)} |u(y) - b| \, dy = 0$$

where $B_r^\pm(x,\nu) := \{y \in B_r(x) : \pm y \cdot \nu \geq 0\}$.
Up to a change of sign of the type $(a, b, \nu) - (b, a, -\nu)$, the values a, b, ν are univocally determined, so we can define

$$u^+(x) := a \qquad u^-(x) := b.$$

Definition 1.4.17 *Let $u \in L^1(\Omega)$ we define the jump discontinuity set of u as*

$$J_u := \{x \in \Omega : u^+(x) \neq u^-(x)\}.$$

Let $u \in BV(\Omega)$, it is possible to prove that J_u is an $\mathcal{H}^{N-1}$-rectifiable set, hence the normal versor ν_u is well defined. Moreover $\mathcal{H}^{N-1}(S_u \setminus J_u) = 0$.
If $u \in L^\infty(\Omega)$ then

$$u^+(x) = u^\vee(x) \qquad u^-(x) = u^\wedge(x)$$

hence, we can write without ambiguity

$$D^j u = (u^+ - u^-)\nu_u \mathcal{H}^{N-1} \llcorner J_u.$$

1.4.2 $SBV(\Omega)$ functions, compactness theorems

Proposition 1.4.18 *Let $u \in BV(\Omega)$ and $\varphi : \mathbb{R} \to \mathbb{R}$ an L-Lipschitz function. Then $\varphi(u) \in BV(\Omega)$.*

Remark 1.4.19 *A $BV(\Omega)$ function is not necessarly differentiable a.e.*

Definition 1.4.20 *We define the measure $D^d u := D^a u + D^c u$. It follows that*

$$Du = D^j u + D^d u$$

Proposition 1.4.21 *Let $u \in BV(\Omega)$ and $\varphi \in C_b^1(\mathbb{R})$. Then $\varphi(u) \in BV(\Omega)$, moreover*

$$D\varphi(u) = \varphi'(\tilde{u})\nabla u \mathcal{L}^N + (\varphi(u^+) - \varphi(u^-))\nu_u \mathcal{H}^{N-1} \llcorner J_u + \varphi'(\tilde{u}) D^c u.$$

Definition 1.4.22 *Given a function $u \in BV(\Omega)$, then u is a special function of bounded variation, $u \in SBV(\Omega)$, if and only if the Cantor part of $|Du|(\Omega)$ is $|D^c u| = 0$.*

Proposition 1.4.23 *Let $u \in SBV(\Omega)$ and $\varphi \in C_b^1(\mathbb{R})$. Then $\varphi(u) \in SBV(\Omega)$, moreover we have*

$$|D\varphi(u) - \varphi'(u)\nabla u \mathcal{L}^N| \leq |\varphi(u^+) - \varphi(u^-)|\nu_u \mathcal{H}^{N-1} \llcorner J_u.$$

In order to prove compactness results for $SBV(\Omega)$ we want to exploit the known properties of L^1. It is possible to show that

Proposition 1.4.24 *Let $|\Omega| < +\infty$. The space $BV(\Omega)$, endowed with the norm*

$$\|u\|_{BV} := \|u\|_{L^1} + |Du|(\Omega)$$

is compactly embedded in $L^1(\Omega)$. So we have

(i) a continuous embedding, $\exists C > 0$ such that $\|u\|_{L^1} \leq C\|u\|_{BV}$ holds $\forall u \in BV(\Omega)$.

(ii) for every sequence $u_j \subset BV(\Omega)$ with $\|u\|_{BV(\Omega)} \leq M$ there exists $u_{j_k} \subset u_j$ converging in $L^1(\Omega)$ to an $L^1(\Omega)$ function.

Let $p > 1$. A family of functions $G \subset L^p(\Omega)$ is pre-compact if it is equi-bounded, that is $\|u\|_{L^p} \leq M$ for every $u \in G$.
But in the particular case $p = 1$ this is not true. In order to ensure compactness another hypotesis is needed, this is the equi-summability of the family, that is

$$\forall \epsilon > 0\, \exists \delta > 0 : A \subset \Omega,\ |A| < \delta \Rightarrow \int_A |g|\, dx < \infty \quad \forall g \in G.$$

This is also equivalent to the following condition

$$\exists M > 0,\, \exists \varphi : (0, +\infty) \to [0, +\infty],\ \lim_{t \to +\infty} \frac{\varphi(t)}{t} = +\infty : \int_\Omega \varphi(g)\, dx \leq M \quad \forall g \in G.$$

Definition 1.4.25 *Given a function $\psi : (0, +\infty) \to [0, +\infty]$ such that $\lim_{t \to +\infty} \frac{\psi(t)}{t} = +\infty$ we define the set*

$$X(\psi) := \{\phi \in C^1_b(\mathbb{R}) : |\phi(t) - \phi(s)| \leq \psi(|t - s|)\}.$$

Proposition 1.4.26 *Let $u \in SBV(\Omega)$, then*

$$\sup_{\phi \in X(\psi)} |D\phi(u) - \phi'(u)\nabla u \mathcal{L}^N|(\Omega) \leq \int_{J_u \cap \Omega} \psi(|u^+ - u^-|)\nu_u \mathcal{H}^{N-1}.$$

Proof: Let us consider $\phi \in X(\psi)$ and $u \in SBV(\Omega)$. Then

$$\begin{aligned} |D\phi(u) - \phi'(u)\nabla u \mathcal{L}^N|(\Omega) &\leq \int_{J_u \cap \Omega} |\phi(u^+) - \phi(u^-)|\nu_u \mathcal{H}^{N-1} \leq \\ &\leq \int_{J_u \cap \Omega} \psi(|u^+ - u^-|)\nu_u \mathcal{H}^{N-1} \end{aligned}$$

By taking the upper limit we get the desired inequality. ■

Conversely, we have the following result.

Proposition 1.4.27 *Let $u \in BV(\Omega)$ and $\lambda \in \mathcal{M}_f(\Omega, \mathbb{R}^N)$ such that $|\lambda|(S_u) = 0$ and*

$$\sup_{\phi \in X(\psi)} |D\phi(u) - \phi'(\tilde{u})\lambda|(\Omega) \leq \int_{J_u \cap \Omega} \psi(|u^+ - u^-|)\nu_u \mathcal{H}^{N-1} < +\infty.$$

Then $\lambda = D^d u = \nabla u \mathcal{L}^N + D^c u$.

Corollary 1.4.28 ($SBV(\Omega)$ characterization) *$u \in SBV(\Omega)$ if and only if $\exists g \in L^1(\Omega, \mathbb{R}^N)$ such that*

$$\sup_{\phi \in X(\psi)} |D\phi(u) - \phi'(\tilde{u})g\mathcal{L}^N|(\Omega) \leq \int_{J_u \cap \Omega} \psi(|u^+ - u^-|)\nu_u \mathcal{H}^{N-1} < +\infty.$$

Proof: [Proposition 1.4.27] Let us define the measure $\mu := D^d u - \lambda$, then clearly $|\mu|(S_u) = 0$, moreover μ and $(\phi(u^+) - \phi(u^-))\nu_u \mathcal{H}^{N-1} \llcorner S_u$ are mutually singular.

Hence we have

$$\sup_{\phi\in X(\psi)}\int_\Omega |\phi'(\tilde u)|\,d|\mu| = \sup_{\phi\in X(\psi)} |\phi'(\tilde u)D^d u - \phi'(\tilde u)\lambda|(\Omega) \leq$$
$$\leq \sup_{\phi\in X(\psi)} |D\phi(u) - \phi'(\tilde u)\lambda|(\Omega) \leq$$
$$\leq \int_{J_u\cap\Omega} \psi(|u^+ - u^-|)\nu_u \mathcal{H}^{N-1} \leq M.$$

Since $|\phi'|$ can arbitrarly grow we must have $|\mu| \equiv 0$, that is $|D^d u - \lambda| \equiv 0$, hence $D^d u \equiv \lambda$. ■

Theorem 1.4.29 (Closure in $SBV(\Omega)$) *Let $(u_h)_h \subset SBV(\Omega)$ be a sequence converging*

$$u_h \xrightarrow{L^1(\Omega)} u \in L^1(\Omega)$$

and assume there exist two functions $\varphi : (0,+\infty) \to [0,+\infty]$ and $\theta : [0,+\infty) \to [0,+\infty]$ with $\lim_{t\to+\infty} \frac{\varphi(t)}{t} = +\infty$ e $\lim_{t\to 0} \frac{\theta(t)}{t} = +\infty$ such that

$$\sup_{h\geq 1} \int_\Omega \varphi(|\nabla u_h|)\,dx + \int_{S_{u_h}} \theta(|u^+ - u^-|)\,d\mathcal{H}^{N-1} < +\infty.$$

Then $u \in SBV(\Omega)$.

Proof: We have $(|\nabla u_h|)_h \subset L^1(\Omega,\mathbb{R}^N)$ equi-summable. Moreover it is equi-bounded, in fact there exists $a \in \mathbb{R}$ such that $\varphi(t) \geq t + a$ for every $t \in \mathbb{R}$ hence

$$\int_\Omega \varphi(|\nabla u_h|)\,dx \geq \int_\Omega |\nabla u_h|\,dx + a|\Omega|,$$

by taking the sup we obtain $\|\nabla u_h\|_{L^1} \leq M_1$. Hence $(|\nabla u_h|)_h$ is pre-compact in $L^1(\Omega,\mathbb{R}^N)$, so there exists $g \in L^1(\Omega,\mathbb{R}^N)$ such that (up to subsequences)

$$\nabla u_h \xrightarrow{L^1(\Omega,\mathbb{R}^N)} g.$$

We want to show that, if $\phi \in X(\theta)$ then

$$\phi'(u_h)\nabla u_h \xrightarrow{\omega L^1(\Omega,\mathbb{R}^N)} \phi'(u) g.$$

Given $f \in C_c(\Omega,\mathbb{R}^N)$ we have

$$\int_\Omega \phi'(u_h) f \nabla u_h\,dx = \int_\Omega (\phi'(u_h) - \phi'(u)) f \nabla u_h\,dx + \int_\Omega \phi'(u) f \nabla u_h\,dx.$$

When $h \to \infty$ the second term converges to $\int_\Omega \phi'(u) f g\,dx$ (thanks to the boundness of ϕ'), while the first term converges to 0, in fact f has compact support. The gradients ∇u_h are equi-summables, and $\phi'(u_h)$ converges to $\phi'(u)$ in measure.
On the other hand, $u_h \overset{var}{\to} u$ in fact $u_h \xrightarrow{L^1(\Omega)} u$, so the equi-bounded measures

Du_h weakly* converges to Du. This implies $\phi(u_h) \stackrel{var}{\to} \phi(u)$, hence, in particular $D\phi(u_h) \stackrel{*}{\rightharpoonup} D\phi(u)$. The weak limit of these measures is

$$\nu_h := D\phi(u_h) - \phi(u_h)\nabla u_h \mathcal{L}^N \stackrel{*}{\rightharpoonup} D\phi(u) - \phi'(u) g \mathcal{L}^N := \nu.$$

For every $\phi \in X(\theta)$ we have

$$\begin{aligned} |D\phi(u_h) - \phi'(u_h)\nabla u_h \mathcal{L}^N| &\leq |\phi(u_h^+) - \phi(u_h^-)| \mathcal{H}^{N-1} \llcorner J_{u_h} \leq \\ &\leq \theta(|u_h^+ - u_h^-|) \mathcal{H}^{N-1} \llcorner J_{u_h} := \mu_h. \end{aligned}$$

Thanks to the hypoteses the measures μ_h are equi-bounded by a constant that depends only on θ, so (up to subsequences) there exists a subsequence that converges to a measure μ. This implies also the equi-boundness of the family $|\nu_h|$, which also converge to a measure σ. We have that $\sigma \leq \mu$.
Thanks to the semicontinuity of the total variation we have in particular

$$\left.\begin{array}{c} \nu_h \stackrel{*}{\rightharpoonup} \nu \\ |\nu_h| \stackrel{*}{\rightharpoonup} \sigma \end{array}\right\} \Rightarrow |\nu| \leq \sigma$$

hence $|\nu| \leq \mu \leq M_2$. More explicitely

$$|D\phi(u) - \phi'(u)\nabla g \mathcal{L}^N| \leq \mu \leq M_2.$$

By taking the supremum

$$\sup_{\phi \in X(\theta)} |D\phi(u) - \phi'(u)\nabla g \mathcal{L}^N|(\Omega) < +\infty$$

Thanks to the Proposition 1.4.27 we obtain $D^d u = g\mathcal{L}^N$. This implies $u \in SBV(\Omega)$. ■

Corollary 1.4.30 *$g = \nabla u$.*

Remark 1.4.31 *Let us also suppose that*

- *φ convex, then we have lower semicontinuity, with respect the $L^1(\Omega)$ convergence, of the functional*

$$\int_\Omega \varphi(|\nabla u|)\, dx \leq \liminf_h \int_\Omega \varphi(|\nabla u_h|)\, dx$$

- *θ concave, then we have lower semicontinuity with respect the $L^1(\Omega)$ convergence, of the functional*

$$\int_{S_u} \theta(|u^+ - u^-|)\, d\mathcal{H}^{N-1} \leq \liminf_h \int_{S_{u_h}} \theta(|u_h^+ - u_h^-|)\, d\mathcal{H}^{N-1}.$$

Theorem 1.4.32 (Compactness in $SBV(\Omega)$) *Let $u_h \subset SBV(\Omega)$ such that*

$$M := \sup_h \|u_h\|_\infty < +\infty$$

and φ, θ as in the closure theorem:

$$\sup_{h\geq 1}\int_\Omega \varphi(|\nabla u_h|)\,dx + \int_{S_{u_h}} \theta(|u^+ - u^-|)\,d\mathcal{H}^{N-1} < \infty.$$

Then there exists a subsequence u_{h_k} such that

$$u_{h_k} \xrightarrow{var} u \in SBV(\Omega).$$

Proof: Since $\sup_h \|u_h\|_\infty < +\infty$ the u_h are equibounded in $L^1(\Omega)$. The properties of θ ensure the existence of $b \in \mathbb{R}$ such that $\theta(t) \geq bt$ in $[0, 2M)$; on the other hand, for φ, we know there exist $a \in \mathbb{R}$ such that $\varphi(t) \geq t + a$ for every $t \in \mathbb{R}$. Since the jumps of u don't exceed $2M$ we have

$$\begin{aligned}|Du_h|(\Omega) &= \int_\Omega |\nabla u_h|\,dx + \int_{S_{u_h}} |u^+ - u^-|\,d\mathcal{H}^{N-1} \leq \\ &\leq \int_\Omega \varphi(|\nabla u_h|)\,dx - a|\Omega| + \frac{1}{b}\int_{S_{u_h}} \theta(|u^+ - u^-|)\,d\mathcal{H}^{N-1}\end{aligned}$$

hence

$$\sup_h |Du_h|(\Omega) < \infty.$$

Since $\|u_h\|_{BV} = \|u_h\|_{L^1} + |Du_h|(\Omega)$ we easily get that u_h is equi-bounded in $BV(\Omega)$. By the embedding results of the Proposition 1.4.24 we know there exists a subsequence such that

$$u_{h_k} \xrightarrow{L^1(\Omega)} u \in L^1(\Omega).$$

Thanks to the closure theorem we are done. ■

1.4.3 Regularity properties

Slicing Theorem

A very useful result concerning the one-dimensional sections of a $SBV(\Omega)$ function is presented here. It will be of great importance in the demonstration of the liminf inequality of the Γ-convergence theorems.

Given $\Omega \subset \mathbb{R}^N$ and $\nu \in S^{N-1}$ we define

- $\pi_\nu := \nu^\perp$
- $\Omega_\nu^x := \{t \in \mathbb{R} : x + t\nu \in \Omega\} \qquad \forall x \in \pi_\nu$
- $\Omega_\nu := \{x \in \pi_\nu : \Omega_\nu^x \neq \emptyset\}$
- $u_\nu^x(t) := u(x + t\nu) \qquad \forall x \in \Omega_\nu \quad \forall t \in \Omega_\nu^x.$

Theorem 1.4.33 (Slicing) *Let $u \in L^\infty(\Omega)$ be a function such that*

$$(i) \qquad u_\nu^x \in SBV(\Omega_\nu^x) \text{ for } \mathcal{H}^{N-1} - a.e. x \in \Omega_\nu$$

and

$$(ii) \qquad \int_{\Omega_\nu} \left[\int_{\Omega_\nu^x} |\nabla u_\nu^x| \, dt + \mathcal{H}^0(S_{u_\nu^x}) \right] d\mathcal{H}^{N-1}(x) < +\infty$$

for any choise of $\nu \in S^{N-1}$. *Then* $u \in SBV(\Omega)$ *and* $\mathcal{H}^{N-1}(S_u) < +\infty$. *Conversely, let* $u \in SBV(\Omega) \cap L^\infty(\Omega)$ *such that* $\mathcal{H}^{N-1}(S_u) < +\infty$, *then the conditions (i) and (ii) are satisfied for every* $\nu \in S^{N-1}$. *We also have*

$$(iii) \qquad \langle \nabla u(x + t\nu), \nu \rangle = \nabla u_\nu^x(t) \quad a.e.\ t \in \Omega_\nu^x$$

for $\mathcal{H}^{N-1}$*-a.e.* $x \in \Omega_\nu$, *and there exists a Borel function* $\nu_u : S_u \to S^{N-1}$ *depending only on* S_u *such that*

$$(iv) \qquad \int_{S_u} |\langle \nu_u, \nu \rangle| \, d\mathcal{H}^{N-1} = \int_{\Omega_\nu} \mathcal{H}^0(S_u) \, d\mathcal{H}^{N-1}(x).$$

Another important result is the following, which gives some conditions to make a Sobolev function an $SBV(\Omega)$ function.

Theorem 1.4.34 *Let* $\Omega \subset \mathbb{R}^N$ *be an open and bounded set, and* $K \subset \mathbb{R}^N$ *such that* $\mathcal{H}^{N-1}(K \cap \Omega) < +\infty$. *Then*

$$u \in W^{1,1}(\Omega \setminus K) \cap L^\infty(\Omega) \Longrightarrow u \in SBV(\Omega) \text{ and } \mathcal{H}^{N-1}(S_u \setminus K) = 0.$$

Minkowski content of S_u

For every set $A \subset \mathbb{R}^N$ and $\rho > 0$ we denote the open tubular neighbourhood of A with radius ρ the set

$$(A)_\rho := \{x \in \mathbb{R}^N : \operatorname{dist}(x, A) < \rho\} \subset \mathbb{R}^N.$$

We define the Minkowski $(N-1)$-dimensional lower and upper content of the set A, respectively, as

$$\mathcal{M}_*(A) := \liminf_{\rho \searrow 0} \frac{\mathcal{H}^N((A)_\rho)}{2\rho} \quad \mathcal{M}^*(A) := \limsup_{\rho \searrow 0} \frac{\mathcal{H}^N((A)_\rho)}{2\rho}. \tag{1.7}$$

Not always these quantities coincide; when it happens, the common value is called the Minkowski content of the set A and it is denoted by

$$\mathcal{M}(A) := \lim_{\rho \searrow 0} \frac{\mathcal{H}^N((A)_\rho)}{2\rho}. \tag{1.8}$$

Federer showed (H. FEDERER: Geometric Measure Theory, Section 3.2.39) that for any compact subset A of a C^1 hypersurface the Minkowski content exists, moreover $\mathcal{M}(A) = \mathcal{H}^{N-1}(A)$.

Lemma 1.4.35 *Let* $u \in BV(\Omega)$. *Then the set* S_u *is* $\mathcal{H}^{N-1}$*-rectifiable. So, it can be covered up to at most a set of* $\mathcal{H}^{N-1}$ *measure zero, by* C^1 *hypersurfaces.*

Corollary 1.4.36 *By inner approximation it is possible to show that*

$$\mathcal{H}^{N-1}(B) \leq \mathcal{M}_*(B)$$

holds for any Borel set $B \subset S_u$.

1.4.4 $GSBV(\Omega)$ functions

When dealing with functions for which the Hessian operator (also in approximate sense) is deifned, some conditions on the gradient must be required.

Definition 1.4.37 *A function u belongs to $GSBV(\Omega)$ if and only if $-n \vee u \wedge n \in SBV_{loc}(\Omega)$ for every $n \in \mathbb{N}$.*

Given a function $u \in GSBV(\Omega)$ its jump set exists and it is defined by

$$S_u := \bigcup_{n=1}^{+\infty} S_{-n\vee u\wedge n}$$

moreover it is a countably $(N-1)$-rectifiable set. Also an approximate gradient can be defined, in particular we have that ∇u exists a.e. and it is represented by

$$\nabla u = \nabla(-n \vee u \wedge n) \text{ a.e. on } \{|u| \leq n\}.$$

Definition 1.4.38 *We define the space of generalized functions of bounded variation for which the gradient is also in $GSBV(\Omega)$*

$$GSBV^2(\Omega) := \Big\{u \in GSBV(\Omega) : \nabla u \in [GSBV(\Omega)]^N\Big\}.$$

The following compactness result is foundamental in the proof of Γ-convergence of the Blake-Zisserman functional.

Theorem 1.4.39 *Let $u_h \subset GSBV^2(\Omega)$ be a sequence such that*

$$\|u_h\|_{L^2(\Omega)}, \qquad \mathcal{H}^{N-1}(S_{u_h} \cup S_{\nabla u}), \qquad \int_\Omega |\nabla^2 u|^2 \, dx$$

are uniformly bounded. Then there exist a subsequence u_{h_k} and $u \in GSBV^2(\Omega) \cap L^2(\Omega)$ such that

- $u_{h_k} \to u$ *strongly in $L^1(\Omega)$,*
- $\nabla u_{h_k} \to \nabla u$ *a.e. in Ω*
- $\nabla^2 u_{h_k} \to \nabla^2 u$ *weakly in $L^2(\Omega, M^{N\times N})$.*

Chapter 2

The Mumford-Shah model

In the variational approach to the signal segmentation problem proposed by Mumford and Shah in [5] we are given $\Omega \subset \mathbb{R}^N$ a bounded open set and $g \in L^\infty(\Omega)$. In the case of images N=2 and Ω is a rectangle, and a function $g : \Omega \to [0,1]$ representing the grey levels of a picture. The aim is to find a pair (u, K) where $K \subset \overline{\Omega}$ is a closed set representing the countours reconstucted from the discontinuities of g and u a smooth representation of g outside K.
One can find such a pair by minimizing the energy functional

$$\mathcal{MS}(u,K) = \int_{\Omega \setminus K} |\nabla u|^2 \, dx + \mu \int_\Omega |u - g|^2 \, dx + \alpha \mathcal{H}^{N-1}(K \cap \Omega) \tag{2.1}$$

among all possible pairs (u, K) with $K \subset \overline{\Omega}$ closed and $u \in C^1(\Omega \setminus K)$, where $\mathcal{H}^{N-1}$ is the Hausdorff $(N-1)$-dimensional measure and $\alpha, \mu > 0$ are positive parameters.
Due to the process of minimization we obtain that the term involving g (the image to be segmented) forces u to be close to g according to the parameter μ which rules the *closeness*, the term involving $|\nabla u|^2$ forces u to be as smooth as possible over the set $\Omega \setminus K$ in order to cancel the discontinuities due to noise and small irregularities, while the term $\alpha \mathcal{H}^{N-1}(K \cap \Omega)$ penalizes large sets K and the parameter α controls the level of the penalization. The interesting feature of the functional is that, due to the presence of the term $\mathcal{H}^{N-1}(K \cap \Omega)$ whenever g has sharp discontinuities (as it is likely to happen on the edges of the objects in the picture) it is more convenient to insert a contour instead of having a big gradient of u.
The minimization of the Mumford-Shah functional (and the Blake-Zisserman one, see Chapter 3) is an example of a large class of variational problems called "free discontinuity problems". This terminology refers to the fact that the corresponding functionals are characterized by a competition between volume energies, concentrated on N-dimensional sets, and surface energies, concentrated on $(N-1)$-dimensional sets, whose supports are not fixed a priori. Indeed, as in the case of the two functionals above, the sets where the lower dimensional energy concentrate are the most relevant unknown of the problem.

2.1 Weak formulation of the problem

In order to study the existence of solutions for the problem

$$\min\{\mathcal{MS}(u,K) : \; u \in C^1(\Omega \setminus K) \quad K \subset \overline{\Omega} \text{ closed}\} \tag{2.2}$$

by meanings of classical direct methods of the Calculus of Variations, since the space $C^1(\Omega \setminus K)$ has no good compactness properties, it's necessary to consider a weaker class of functions which gives some semicontinuity and coerciveness properties to the functional.

Remark 2.1.1 *The problem (2.2) is equivalent to the following*

$$\min\{\mathcal{MS}(u,K) : \; u \in W^{1,2}_{loc}(\Omega \setminus K) \quad K \subset \overline{\Omega} \text{ closed}\}. \tag{2.3}$$

Proof: Since $C^1(\Omega \setminus K) \subset W^{1,2}_{loc}(\Omega \setminus K)$ if (2.3) has no solutions neither (2.2) has. Let (u, K) be a solution of (2.3) then by meanings of null first variation of the functional u is weak solution of the equation

$$\Delta u = \alpha(u - g)$$

obtained with additive perturbations $u + \epsilon\varphi$, where $\varphi \in C^1_0(\Omega \setminus K)$, as $\epsilon \to 0$. By well known regularity properties for the solutions of elliptic equations $u \in L^\infty_{loc}(\Omega \setminus K)$ and $u \in W^{2,p}_{loc}(\Omega \setminus K)$ for all $p < +\infty$. By Sobolev embedding theorems $u \in C^{1,\alpha}(\Omega \setminus K)$ for any $\alpha < 1$.
If (2.2) has no solutions neither (2.3) has, in fact, by absurd, if (u, K) is a solution for (2.3) then by the same argument above we get $u \in C^1(\Omega \setminus K)$ then the strong problem admits one competitor. ■

Now we want to test some lower semicontinuity and compactness properties of the weaker functional. Let (u_h, K_h) be a minimizing sequence of $\mathcal{MS}$, we would like to show that, up to a subsequence, it converges in a suitable sense to an admissible pair (u, K) and that

$$\mathcal{MS}(u,K) \leq \liminf_h \mathcal{MS}(u_h, K_h).$$

There is no loss of generality if we consider Tu_h instead of any u_h, where Tu_h is obtained from u_h by truncation at $\pm\|g\|_\infty$, in fact still there holds

$$\mathcal{MS}(Tu_h, K_h) \leq \mathcal{MS}(u_h, K_h).$$

On the other hand, we know by Blaschke Theorem that the family of nonempty sets in $\overline{\Omega}$ is a compact metric space endowed with the Hausdorff distance

$$\delta(K_1, K_2) := \inf\{r > 0 : (K_1)_r \subset K_2, \; (K_2)_r \subset K_1\}.$$

We that this notion of distance induces the convergence of Kuratowsky on compact sets; then we can assume, possibly up to subsequences, $K_h \to K$. For any ball $B \subset (\Omega \setminus K)$ there exists h_0 such that for any $h > h_0$ is $B \cap K_h \neq \emptyset$ hence for any $A \subset\subset (\Omega \setminus K)$ is $u_h \in W^{1,2}(A)$ for h large enough. Thanks to the boundeness of $(u_h)_h$, by compactness results in Sobolev spaces they weakly converge to some $u \in W^{1,2}_{loc}(\Omega \setminus K)$.

Problems occours when we consider the piece of the functional involving $\mathcal{H}^{N-1}$ because in general the map

$$K \to \mathcal{H}^{N-1}(K)$$

is not lower semicontinuous with respect to the convergence induced by the Hausdorff distance. For a simple counterexample we fix $N = 2$ and we consider $K := [0,1] \times \{0\}$ and $K_h := \{(k/h, 0) : k = 0, \ldots, h\}$; easily $K_h \to K$ and $\mathcal{H}^1(K) = 1$, but $\mathcal{H}^1(K_h) = 0$ for every h.

To overcome the failure of the direct methods De Giorgi has proposed a weaker formulation in $SBV(\Omega)$ of the Mumford-Shah problem. The key idea is to deal with a simpler object, just depending on the function u, and then to recover the set of contours K by taking the discontinuity set S_u. Then we consider the functional

$$H(u) := \begin{cases} F(u) & u \in L^\infty(\Omega) \cap SBV(\Omega) \\ \\ +\infty & u \in L^\infty(\Omega) \setminus SBV(\Omega) \end{cases} \tag{2.4}$$

where

$$F(u) := \int_\Omega |\nabla u|^2 \, dx + \mu \int_\Omega |u - g|^2 \, dx + \alpha \mathcal{H}^{N-1}(S_u \cap \Omega).$$

A remark is needed: actually, the natural domain of the functional H is $L^2(\Omega)$ and the convergence of this space is the one we are going to consider along the whole Section. Anyway, we will see in Section 2.1.1 that, for purpose of minimization the requirement of $u \in L^\infty(\Omega)$ is not restrictive, so we consider the functional H restricted to $L^\infty(\Omega)$ right from here.
Lower semicontinuity and compactness results (in the $L^2(\Omega)$ convergence) of Ambrosio (see Section 1.4) show that this functional is lower semicontinuous and it always admits minima (see Section 2.1.1). Another important result, due to Carriero, De Giorgi and Leaci (see [9]), is that every minimum of H belongs to the class of piecewise C^1 functions, that is

$$PC^1(\Omega) := \left\{ u \in L^\infty(\Omega) : \, u \in C^1(\Omega \setminus \overline{S_u}) \quad \mathcal{H}^{N-1}((\overline{S_u} \setminus S_u) \cap \Omega) = 0 \right\}. \tag{2.5}$$

The property $\mathcal{H}^{N-1}((\overline{S_u} \setminus S_u) \cap \Omega) = 0$ is a set regularity property of the minima of H. It is proved using a local density property which is satisfyied by the minima of H, expressed in the following.

Proposition 2.1.2 *There exist three constants $\theta, \gamma, r_0 > 0$ depending only on N, α, μ and $\|g\|_\infty$ such that if $u \in SBV(\Omega)$ is a minimizer for H, given $x \in S_u$ such that $B_r(x) \subset \Omega$ and $r < r_0$, then*

$$\theta r^{N-1} \leq \mathcal{H}^{N-1}(S_u \cap B_r(x)) \leq \gamma r^{N-1}$$

holds.

The study of H can be done by viewing it as a relaxation of the functional $F(u)$ on the class of $PC^1(\Omega)$ functions. In fact, the relaxation of F, which is given for every $u \in L^\infty(\Omega)$ by

$$\overline{F}(u) := \inf \left\{ \liminf_j F(u_j) : \, u_j \subset PC^1(\Omega) \quad u_j \to u \text{ in } L^2(\Omega) \right\},$$

equals H.

Proposition 2.1.3 *We have the integral representation of $\overline{F}$, in the sense that $\overline{F} = H$.*

Proof: H is lower semicontinuous, hence for every $u \in L^\infty(\Omega)$ and $u_j \to u$ the inequality $H(u) \leq \liminf_j H(u_j)$ holds, in particular $H(u) \leq \overline{F}(u)$.
Let u be such that $H(u) < +\infty$ (otherwise there is nothing to prove) then $u \in SBV(\Omega) \cap L^\infty(\Omega)$. Consider u_j minimizers of the family of minimum problems

$$H_j^u(v) := \begin{cases} \displaystyle\int_\Omega |\nabla v|^2\, dx + j \int_\Omega (v-u)^2\, dx + \mathcal{H}^{N-1}(S_v \cap \Omega) & L^\infty(\Omega) \cap SBV(\Omega) \\ +\infty & L^\infty(\Omega) \setminus SBV(\Omega) \end{cases}$$

then $u_j \subset PC^1(\Omega)$, trivially $u_j \to u$ in $L^2(\Omega)$, and by the minimality we have

$$\begin{aligned} F(u_j) - \int_\Omega (u_j - g)^2 &= \int_\Omega |\nabla u_j|^2\, dx + \mathcal{H}^{N-1}(S_{u_j} \cap \Omega) \\ &\leq \int_\Omega |\nabla u_j|^2\, dx + j \int_\Omega (u_j - u)^2 + \mathcal{H}^{N-1}(S_{u_j} \cap \Omega) \\ &\leq \int_\Omega |\nabla u|^2\, dx + j \int_\Omega (u-u)^2 + \mathcal{H}^{N-1}(S_u \cap \Omega) \\ &= H(u) - \int_\Omega (u-g)^2. \end{aligned}$$

It follows that

$$\overline{F}(u) \leq \liminf_j F(u_j) \leq \liminf_j \left[H(u) + \int_\Omega (u_j - g)^2 - \int_\Omega (u-g)^2 \right] = H(u)$$

hence $H(u) = \overline{F}(u)$. ■

The set regularity requirement in (2.5) is necessary to ensure $u \in SBV(\Omega)$. Why $SBV(\Omega)$ and not $BV(\Omega)$?
For the purpose of image segmentation, in order to have non trivial solutions we can see that the space $BV(\Omega)$ is too large. Given any function $g \in L^2(\Omega)$, one can construct a sequence $u_h \subset BV(\Omega)$ converging to g such that for any h the derivative of u_h is made up only by the Cantor part. Therefore the infimum of H on $BV(\Omega)$ is trivially zero and the set K is empty.

Finally we see the equivalence for the strong problem and the weaker one.

Proposition 2.1.4 (Equivalence strong/weak formulation) *If $g \in L^\infty(\Omega)$ then to solve the minimum problem (3.2) is equivalent to solve the (2.2) one.*

Proof: Given a pair (u, K) competitor for the problem (2.2), we have $u \in W^{1,1}(\Omega \setminus K) \cap L^\infty(\Omega \setminus K)$ and $K \subset \Omega$ closed. Thanks to the Theorem 1.4.34 is actually $u \in SBV(\Omega)$ and $\mathcal{H}^{N-1}(S_u \setminus K) = 0$ so the pair (u, K) is also a competitor for the problem (3.2), thus

$$\min H(u) \leq \inf \mathcal{MS}(u, K).$$

To show that the opposite inequality holds is not trivial. Does a pair $(u, \overline{S_u})$ with $u \in SBV(\Omega)$ give also a competitor for the problem (2.2)? It's not obviously due to the fact that in general the set S_u of a $SBV(\Omega)$ can be not regular or even dense in Ω. Luckily certain pathological behaviours of BV functions cannot occur when dealing with minimizers of H. As said above

$$\mathcal{H}^{N-1}((\overline{S_u} \setminus S_u) \cap \Omega) = 0;$$

since $u \in SBV(\Omega)$ we have $|D^s u|(\Omega \setminus S_u) = 0$, it follows that $u \in W^{1,1}(\Omega \setminus \overline{S_u})$. Then $u \in W^{1,2}(\Omega \setminus \overline{S_u})$ and $H(u) = \mathcal{MS}(u, \overline{S_u})$. ■

2.1.1 Existence of minimizers

Simply applying compactness and lower semicontinuity results presented in the previous chapter we immediately get the existence of minimizers for the relaxed functional.

Proposition 2.1.5 (Existence of minimizers) *If $g \in L^\infty(\Omega)$ then $H(u)$ always admits minima in the class of $SBV(\Omega)$ functions; moreover, and for minima we have $u \in L^\infty(\Omega)$.*

Proof: Because of its non-negativity the functional is bounded from below, let $a = \inf_{u \in SBV(\Omega)} F(u)$ and $u_h \subset SBV(\Omega)$ be a minimizing sequence, that is

$$H(u_h) \leq a + \frac{1}{h}.$$

Since $g \in L^\infty(\Omega)$ there is no loss of generality if we consider $u_h \subset L^\infty(\Omega)$ equi-bounded. Let $M := \|g\|_\infty$, if we replace every u_h with its truncaded $u_h^M := -M \vee u \wedge M$ we get

- $S_{u_h^M} \subset S_{u_h}$
- $\int_\Omega |\nabla u_h^M| \, dx \leq \int_\Omega |\nabla u_h| \, dx$
- $\int_\Omega (u_h^M - g)^2 \, dx \leq \int_\Omega (u_h - g)^2 \, dx$

so we easily obtain $H(u_h^M) \leq H(u_h)$ for every h. A simple estimate leads to the equiboundeness of the functional over the sequence

$$\begin{aligned} a + 1 \geq H(u_h) = & \\ &= \int_\Omega |\nabla u_h|^2 \, dx + \alpha \int_\Omega |u_h - g|^2 \, dx + \mu \mathcal{H}^{N-1}(S_u) \geq \\ &\geq \int_\Omega |\nabla u_h|^2 \, dx + \mu \mathcal{H}^{N-1}(S_u) = \\ &= \int_\Omega \varphi(|\nabla u_h|) \, dx + \int_{S_u} \theta(|u^+ - u^-|) \, d\mathcal{H}^{N-1} \end{aligned}$$

where $\varphi(t) := t^2$ e $\theta(t) := \mu$. By the compactness Theorem 1.4.32 we know that there exist a subsequence

$$u_{h_k} \longrightarrow u \in SBV(\Omega) \qquad \text{in } L^1(\Omega)$$

and since $\|u_h\|_\infty \leq M$ the convergence is also in $L^2(\Omega)$ in fact we have

$$\int_\Omega |u_h - u|^2 \, dx \leq 2M \int_\Omega |u_h - u| \, dx.$$

The function φ is convex, θ is concave, thanks to the semicontinuity of the $L^2(\Omega)$ norm we have that every term of the functional is lower semicontinuous (and in particular $\int_\Omega |u_h - g|$ is continuous) hence

$$\begin{aligned} a &= \lim_{h \to +\infty} a + \frac{1}{h} \\ &\geq \liminf_h H(u_h) \\ &= \liminf_h \left[\int_\Omega |\nabla u_h|^2 \, dx + \alpha \int_\Omega |u_h - g|^2 \, dx + \mu \mathcal{H}^{N-1}(S_{u_h}) \right] \\ &\geq \int_\Omega |\nabla u|^2 \, dx + \alpha \int_\Omega |u - g|^2 \, dx + \mu \mathcal{H}^{N-1}(S_u) \\ &= H(u) \geq a. \end{aligned}$$

and $H(u) = a$ is a minimum. ∎

2.2 Approximation by elliptic functionals via Γ-convergence

In this section we will show a way proposed by L. Ambrosio and V. M. Tortorelli in [3] to solve the problem of finding effective algorithms for computing minimizers of the Mumford-Shah functional (3.2) in $SBV(\Omega)$, by meanings of Γ-convergence.
The matter is, because of the term $\mathcal{H}^{N-1}(S_u)$, the functional is not differentiable, so the classical methods based on the gradient descent fails. Ambrosio and Tortorelli have proved the existence of a family of elliptic (and so differentiable) functionals Γ-converging to the functional H.
The key idea is to introduce an extra function variable s as a substitute of the set variable S_u and to give a family of functionals $\mathcal{F}_\epsilon(u,s)$ such that, if (u_ϵ, s_ϵ) minimizes $\mathcal{F}_\epsilon$, then (possibly up to subsequences) u_ϵ is closer and closer to a minimizer u of H and s_ϵ is different from 1 only in a small neighbourhood of S_u which shrinks as $\epsilon \to 0$.
So we need to redefine the functional H introducing the formal variable s, let us consider the functional $\mathcal{F}$ defined by

$$\mathcal{F}(u,s) := \begin{cases} F(u) & (u,s) \in \overline{X}(\Omega) \times \overline{Y}(\Omega) \\ \\ +\infty & \text{otherwise in } Z(\Omega) \end{cases} \tag{2.6}$$

where

$$F(u) := \int_\Omega |\nabla u|^2 \, dx + \alpha \mathcal{H}^{N-1}(S_u \cap \Omega) + \mu \int_\Omega (u-g)^2 \, dx$$

and

$$Z(\Omega) := L^\infty(\Omega) \times \mathrm{L}^\infty(\Omega; [0,1])$$

$$\overline{X}(\Omega) := L^\infty(\Omega) \cap SBV(\Omega)$$
$$\overline{Y}(\Omega) := \{s \in L^\infty(\Omega) : \, s \equiv 1\}.$$

We want to prove a Γ-convergence result by using the following family of elliptic functionals

$$\mathcal{F}_\epsilon(u,s) := \begin{cases} F_\epsilon(u,s) & (u,s) \in D(\Omega) \\ \\ +\infty & \text{otherwise in } Z(\Omega) \end{cases} \tag{2.7}$$

where

$$F_\epsilon(u,s) := \int_\Omega (s^2 + o_\epsilon)|\nabla u|^2 \, dx + \alpha \mathcal{G}(s) + \mu \int_\Omega (u-g)^2 \, dx$$
$$\mathcal{G}(s) := \int_\Omega \epsilon |\nabla s|^2 + \frac{1}{4\epsilon}(s-1)^2 \, dx. \tag{2.8}$$

and

$$D(\Omega) := \Big\{ (u,s) \in Z(\Omega) : \, u \in W^{1,2}_{loc}(\Omega), \; s \in W^{1,2}(\Omega) \Big\}.$$

The term (2.8), the so called *Ambrosio-Tortorelli component* of the functional, resembles the first known example (due to Modica-Mortola) of a sequence of quadratic elliptic functionals converging to an area-like functional.
We are going to prove the following Γ-convergence's result.

Theorem 2.2.1 *If o_ϵ is a non-negative infinitesimal faster than ϵ then*

$$\mathcal{F} = \Gamma\text{-}\lim_\epsilon \mathcal{F}_\epsilon.$$

Moreover, if (u_ϵ, s_ϵ) minimizes $\mathcal{F}_\epsilon$ then it is compact in $[L^2(\Omega)]^2$ and any of its limit point, as $\epsilon \to 0$, corresponds to a pair $(u,1)$ with u minimizer of $\mathcal{F}$.

Some remarks. First we observe that the infinitesimal o_ϵ ensures the C^1 regularity of the minimizers of $\mathcal{F}_\epsilon$ in fact, even when s is 0 the term $|\nabla u|$ cannot expode. Another fact is that, thanks to the stability under continuous perturbations of Γ-convergence, the term $\mu \int_\Omega (u-g)^2 \, dx$ can be replaced with any other additive perturbation which is continuous with respect to the $L^2(\Omega)$ topology.
In the following Sections of this Chapter we present the three fundamental steps for the proof of Γ-convergence: the liminf and limsup inequalities, compactness results for sequences of minimizers.

2.2.1 The liminf inequality

In this section we will see how to obtain a lower bound in the sense of Γ-convergence for the behaviour of the family $\mathcal{F}_\epsilon$. For the sake of simplicity we define

$$\mathcal{F}_-(u,s) := \inf \Big\{ \liminf_\epsilon \mathcal{F}_\epsilon(u_\epsilon, s_\epsilon) : \, (u_\epsilon, s_\epsilon) \to (u,s) \text{ in } [L^2(\Omega)]^2 \Big\} \tag{2.9}$$

moreover, we can remove the term $\int |u-g|^2$ in the functionals along the whole section because of its continuity in the $L^2(\Omega)$ topology and thanks to the superaddtivity of the liminf.

Theorem 2.2.2 (liminf inequality) *Given a pair $(u,s) \in Z(\Omega)$ then the inequality*

$$\mathcal{F}(u,s) \leq \mathcal{F}_-(u,s) \tag{2.10}$$

holds, and whenever $\mathcal{F}_-(u,s) < +\infty$ then $u \in \overline{X}(\Omega)$ and $s \equiv 1$ almost everywhere.

Proof: There is no loss of generality by assuming $\alpha = 1$. First we observe that if $s \not\equiv 1$ then both sides of inequality are trivially equal to $+\infty$, so we can assume $s \equiv 1$ almost everywhere. The key idea in the proof of the theorem is to test the inequality separately near regular points for the function u and near jump points. So a localization of the functional is needed, we introduce the following notation (with the obvious meanings)

$$\mathcal{F}(u,s;A) \qquad \mathcal{F}_\epsilon(u,s;A) \qquad \mathcal{F}_-(u,s;A)$$

whenever A is an open set in $\mathbb{R}^N$.
First we show a result in the one-dimensional case, then a generalization in dimension $N \geq 1$ is proved by a slicing argument. To avoid confusion and to emphasize we are in dimension $N = 1$, we use a different notation introducing the functionals

$$G(u,s;I) \qquad G_\epsilon(u,s;I) \qquad G_-(u,s;I).$$

Let $I_r^x \subset \mathbb{R}$ be the open interval with center $x \in \mathbb{R}$ and radius $r > 0$. The key result of this section is presented in the following Lemma, which gives the required estimates in the one dimensional case.

Lemma 2.2.3 *Let $\Omega \subset \mathbb{R}$ be an open set and $x \in \Omega$. If $u \in L^\infty(I_r^x)$ with $\eta > 0$ such that $I_\eta^x \subset \Omega$, then*

(i) whenever $u \notin W^{1,2}(I_{\eta/2}^x)$, we have

$$1 \leq \inf \left\{ \liminf_\epsilon G_\epsilon(u_\epsilon, s_\epsilon; I_\eta^x) : (u_\epsilon, s_\epsilon) \to (u,1) \right\}$$

(ii) otherwise, if $u \in W^{1,2}(I_\eta^x)$, we have

$$\int_{x-\eta}^{x+\eta} |u'|^2 \, dt \leq \inf \left\{ \liminf_\epsilon G_\epsilon(u_\epsilon, s_\epsilon; I_\eta^x) : (u_\epsilon, s_\epsilon) \to (u,1) \right\}$$

Observe that, in the case $\alpha \neq 1$, in (i) there is α instead of 1.

PROOF OF 2.2.3: Let $(u_\epsilon, s_\epsilon) \subset D(\Omega)$ such that $(u_\epsilon, s_\epsilon) \to (u,1)$. We can assume $\liminf_\epsilon G_\epsilon(u_\epsilon, s_\epsilon; I_\eta^x) < +\infty$ (otherwise there is nothing to prove). There exists a subsequence $(u_{\epsilon_\sigma}, s_{\epsilon_\sigma})$ such that

$$\lim_\sigma G_{\epsilon_\sigma}(u_{\epsilon_\sigma}, s_{\epsilon_\sigma}; I_\eta^x) = \liminf_\epsilon G_\epsilon(u_\epsilon, s_\epsilon; I_\eta^x) < +\infty$$

so there is no loss of generality in assuming

$$\lim_\epsilon G_\epsilon(u_\epsilon, s_\epsilon; I_\eta^x) = \lim_\epsilon \left[\int_{I_\eta^x} (s^2 + o_\epsilon)|\nabla u_\epsilon|^2 \, dx + \int_{I_\eta^x} \epsilon |\nabla s_\epsilon|^2 + \frac{1}{4\epsilon}(s_\epsilon - 1)^2 \, dx \right] < +\infty. \tag{2.11}$$

(i) Let us consider the case $u \notin W^{1,2}(I^x_{\eta/2})$. Trivially by (2.11) we have $\lim_\epsilon G_\epsilon(u_\epsilon, s_\epsilon; I^x_{\eta/2})$ finite. We observe that if

$$\lim_\epsilon \inf_{I^x_{\eta/2}} s_\epsilon > 0,$$

then $u_\epsilon \to u$ in $W^{1,2}(I^x_{\eta/2})$ and by closure $u \in W^{1,2}(I^x_{\eta/2})$, absurd. So, we have that

$$\lim_\epsilon \inf_{I^x_{\eta/2}} s_\epsilon = 0.$$

Again by (2.11) we also have $1/(4\epsilon)(s_\epsilon - 1)^2$ bounded in $L^1(I^x_\eta)$, hence it is possible to find three sequences $y'_\epsilon, x_\epsilon, y''_\epsilon$ with

$$x - \frac{\eta}{2} < x_\epsilon < x + \frac{\eta}{2}$$

$$x - \eta < y'_\epsilon < x_\epsilon < y''_\epsilon < x + \eta$$

such that

$$\lim_\epsilon s_\epsilon(y'_\epsilon) = \lim_\epsilon s_\epsilon(y''_\epsilon) = 1, \qquad \lim_\epsilon s_\epsilon(x_\epsilon) = 0.$$

Simply by applying the inequality $a^2 + b^2 \geq 2ab$ we obtain

$$\begin{aligned}
G_\epsilon(u_\epsilon, s_\epsilon; I^x_\eta) &\geq \int_{I^x_{\eta/2}} \epsilon|\nabla s_\epsilon|^2 + \frac{1}{4\epsilon}|s_\epsilon - 1|^2\, dx \\
&\geq \int_{I^x_{\eta/2}} |s_\epsilon - 1||\nabla s_\epsilon|\, dx \\
&\geq \int_{y'_\epsilon}^{x_\epsilon} |s_\epsilon - 1||\nabla s_\epsilon|\, dx + \int_{x_\epsilon}^{y''_\epsilon} |s_\epsilon - 1||\nabla s_\epsilon|\, dx \\
&= \int_{y'_\epsilon}^{x_\epsilon} |s_\epsilon - 1||\nabla s_\epsilon|\, dx + \int_{x_\epsilon}^{y''_\epsilon} |1 - s_\epsilon||\nabla s_\epsilon|\, dx \\
&\geq \int_{y'_\epsilon}^{x_\epsilon} (s_\epsilon - 1)\nabla s_\epsilon\, dx + \int_{x_\epsilon}^{y''_\epsilon} (1 - s_\epsilon)\nabla s_\epsilon\, dx \\
&= \int_{s_\epsilon(y'_\epsilon)}^{s_\epsilon(x_\epsilon)} (t-1)\, dt + \int_{s_\epsilon(y''_\epsilon)}^{s_\epsilon(x_\epsilon)} (t-1)\, dt
\end{aligned} \tag{2.12}$$

and passing to the liminf as $\epsilon \to 0$ we obtain the desired inequality.
(ii) Now let $u \in W^{1,2}(I^x_{\eta/2})$. The condition (2.11) easily implies $s_\epsilon \to 1$ almost everywhere, but it is not enough to uniformly control neither s_ϵ nor the ∇s_ϵ in any $L^p(I^x_\eta)$. This does not allow the direct application of a semicontinuity argument.
By using Coarea formula Ambrosio and Tortorelli have proved a result (see [2]) which gives, under suitable hypotesis, an uniform bound over compact sets.

Lemma 2.2.4 *Let $I \subset \mathbb{R}$ an open interval. For evey $\delta > 0$ and every sequence $w_h \subset C^1(I)$ such that $w_h \to 0$ almost everywhere in I and $\int_I |\nabla w_h| \leq M$, there exists a finite set $J \subset I$ such that*

$$\limsup_h \left(\max_K |w_h| \right) < \delta$$

for every compact set $K \subset I \setminus J$. It follows that $\max_K |w_h| < \delta$ holds up to a finite number of indices h.

We define $w_\epsilon := (s_\epsilon - 1)^2$, by density we can assume $w_\epsilon \subset C^1(I^x_\eta)$, moreover the inequality $a^2 + b^2 \geq 2ab$ easily implies $\int_{I^x_\eta} |\nabla w_h| \leq M$. Then there exists a finite set $J \subset I^x_\eta$ such that for every $\delta > 0$, possibly up to a finite number of indices, $\max_K |s_\epsilon - 1|^2 < \delta$, hence

$$\min_K s_\epsilon \geq 1 - \sqrt{\delta}$$

for every compact set $K \subset I^x_\eta$. Then we have

$$\begin{aligned}\liminf_\epsilon \int_{I^x_\eta} s_\epsilon^2 |\nabla u_\epsilon|^2\, dx &\geq \liminf_\epsilon \int_K s_\epsilon^2 |\nabla u_\epsilon|^2\, dx \geq (1-\sqrt{\delta})^2 \liminf_\epsilon \int_K |\nabla u_\epsilon|^2\, dx \\ &\geq (1-\sqrt{\delta})^2 \liminf_\epsilon \int_{K\setminus\partial K} |\nabla u_\epsilon|^2\, dx \geq (1-\sqrt{\delta})^2 \int_{K\setminus\partial K} |\nabla u|^2\, dx\end{aligned}$$

and the thesis follows by letting $K \uparrow I^x_\eta$ and $\delta \downarrow 0$, in fact since $u \in W^{1,2}(I^x_\eta)$ and $K \subset I^x_\eta$ compact, then $\int_{\partial K} |\nabla u|^2\, dx = 0$. □

By using the superadditivity of the infimum as a set function (see (2.20) for a similar procedure) this result can be easily generalized for every open bounded set $I \subset \mathbb{R}$.

Remark 2.2.5 *The important consequence of this result is that, fixed an interval $I \subset \mathbb{R}$ and a function u defined on it, the existence of $(u_\epsilon, s_\epsilon) \to (u, 1)$ such that the lower limit $G_\epsilon(u_\epsilon, s_\epsilon; I)$ is finite, yields the existence of a finite set $J \subset I$ such that u is absolutely continuous on $I \setminus J$. In particular u is actually in $SBV(I)$ and*

$$G(u, 1; I) \leq G_-(u, 1; I) \tag{2.13}$$

holds.

Now let $N > 1$, we consider $u \in L^\infty(\Omega)$ such that $\mathcal{F}_-(u, 1; \Omega) < +\infty$ and an open set $A \subset \Omega$: then there exists a sequence $(u_\delta, s_\delta) \to (u, 1)$ such that

$$\lim_{\delta\to 0} \mathcal{F}_\delta(u_\delta, s_\delta; A) = \mathcal{F}_-(u, 1; A) < +\infty.$$

We recall the notation used in Theorem 1.4.33. Fixed $\nu \in S^{N-1}$, by integrating on A_ν and applying the Fatou Lemma we have

$$\begin{aligned}&\int_{A_\nu} \liminf_\delta G_\delta(u^\nu_{\delta x}, s^\nu_{\delta x}; A^\nu_x)\, d\mathcal{H}^{N-1}(x) \\ &\quad\leq \liminf_\delta \int_{A_\nu} G_\delta(u^\nu_{\delta x}, s^\nu_{\delta x}; A^\nu_x)\, d\mathcal{H}^{N-1}(x) \\ &\quad= \liminf_\delta \int_{A_\nu} \left[\int_{A^\nu_x} (s^\nu_{\delta x} + o_\delta)|\nabla u^\nu_{\delta x}|^2 + \delta|\nabla s^\nu_{\delta x}|^2 + \frac{1}{4\delta}(s^\nu_{\delta x} - 1)^2\, dt\right] d\mathcal{H}^{N-1}(x) \qquad (2.14) \\ &\quad\leq \liminf_\delta \int_A (s_\delta + o_\delta)|\nabla u_\delta|^2 + \delta|\nabla s_\delta|^2 + \frac{1}{4\delta}(s_\delta - 1)^2\, dy \qquad (2.15) \\ &\quad= \liminf_\delta \mathcal{F}_\delta(u_\delta, s_\delta; A) \\ &\quad\leq \mathcal{F}_-(u, 1; A). \qquad (2.16)\end{aligned}$$

The inequality between (2.14) and (2.15) is justified by Fubini's Theorem and the following estimate (let v play the role of u_δ and s_δ)

$$\int_{A_\nu}\left[\int_{A_x^\nu}|\nabla v_x^\nu|^2\,dt\right]d\mathcal{H}^{N-1}(x)=\int_{A_\nu}\left[\int_{A_x^\nu}|\partial_t v(x+t\nu)|^2\,dt\right]d\mathcal{H}^{N-1}(x)$$
$$=\int_A|\partial_\nu v|^2\,dy=\int_A|\langle\nabla v,\nu\rangle|\,dy\leq\int_A|\nabla v|^2\,dy.$$

Since the estimate (2.16) implies in particular

$$\liminf_\delta G_\delta(u_{\delta x}^\nu,s_{\delta x}^\nu;A_x^\nu)<+\infty \qquad \mathcal{H}^{N-1}\text{-a.e. } x\in A_\nu,$$

also the infimum is finite over all the sequences converging to $(u_x^\nu,1)$

$$G_-(u_x^\nu,1;A_x^\nu)<+\infty \qquad \mathcal{H}^{N-1}\text{-a.e. } x\in A_\nu.$$

Then, since it is easy to check that $u_{\delta x}^\nu\to u_x^\nu$ for $\mathcal{H}^{N-1}$-a.e. $x\in A_\nu$, by Lemma 2.2.3 we have $u_x^\nu\in SBV(A_x^\nu)\cap L^\infty(A_x^\nu)$ and

$$\int_{A_x^\nu}|\nabla u_x^\nu|^2\,dt+\mathcal{H}^0(S_{u_x^\nu}\cap A_x^\nu)=G(u_x^\nu,1;A_x^\nu)\leq G_-(u_x^\nu,1;A_x^\nu). \tag{2.17}$$

Thanks to (iii) of the Theorem 1.4.33 (in dimension 1 the hypoteses are verified), by integrating (2.17) on A_ν and using (2.16), we get

$$\int_A|\langle\nabla u,\nu\rangle|^2\,dy+\int_{S_u\cap A}|\langle\nu_u,\nu\rangle|\,d\mathcal{H}^{N-1} \tag{2.18}$$
$$=\int_{A_\nu}\left[\int_{A_x^\nu}|\nabla u_x^\nu|^2\,dt+\mathcal{H}^0(S_{u_x^\nu}\cap A_x^\nu)\right]d\mathcal{H}^{N-1}(x)$$
$$\leq\int_{A_\nu}G_-(u_x^\nu,1;A_x^\nu)\,d\mathcal{H}^{N-1}(x)$$
$$\leq\int_{A_\nu}\liminf_\delta G_\delta(u_{\delta x}^\nu,s_{\delta x}^\nu;A_x^\nu)\,d\mathcal{H}^{N-1}(x)$$
$$\leq\mathcal{F}_-(u,1;A). \tag{2.19}$$

Using the arbitrariness of ν and A and the slicing Theorem 1.4.33 we get $u\in SBV(\Omega)\cap L^\infty(\Omega)$ because almost every slice is essentially bounded.
Let us consider $D\subset S^{N-1}$ a dense subset, we define the functions and the measure

$$f_\nu:=|\langle\nu,\nu_u\rangle|\chi_{S_u} \qquad \mu:=\mathcal{H}^{N-1}\llcorner(S_u).$$

Since $\sup_{v\in D}f_\nu=\chi_{S_u}$, by Lemma 1.3.13 we get

$$\mathcal{H}^{N-1}(S_u)=\int_\Omega\sup_{v\in D}|\langle\nu,\nu_u\rangle|\chi_{S_u}\,d\mu=$$
$$\sup\left\{\sum_{i=0}^{+\infty}\int_{S_u\cap A_i}|\langle\nu_i,\nu_u\rangle|\,d\mathcal{H}^{N-1}:\,\Omega=\bigcup_{i=0}^{+\infty}A_i,\ A_i\text{ pairwise disjoint, }\nu_i\in S^{N-1}\right\}$$

This implies that the quantity $\mathcal{F}(u,1;\Omega)$ is the least upper bound in the lattice of measures of the family (2.18) as ν varies in S^{N-1}, hence by using the inequality (2.19) and the superadditivity of $\mathcal{F}_-$ as a set function we get

$$\begin{aligned}
\mathcal{F}(u,1;\Omega) &= \int_\Omega |\nabla u|^2\,dy + \mathcal{H}^{N-1}(S_u\cap\Omega) \\
&= \sup\Bigg\{\sum_{i=0}^{+\infty}\int_{A_i}|\langle\nabla u,\nu_i\rangle|^2\,dy + \int_{S_u\cap A_i}|\langle\nu_u,\nu_i\rangle|\,d\mathcal{H}^{N-1}: \\
&\qquad \Omega=\bigcup_{i=0}^{+\infty}A_i,\ A_i \text{ pairwise disjoint},\ \nu_i\in S^{N-1}\Bigg\} \\
&\le \sup\left\{\sum_{i=0}^{+\infty}\mathcal{F}_-(u,1;A_i)\,:\,\Omega=\bigcup_{i=0}^{+\infty}A_i\right\} \\
&\le \sup\left\{\mathcal{F}_-\left(u,1;\bigcup_{i=0}^{+\infty}A_i\right)\,:\,\Omega=\bigcup_{i=0}^{+\infty}A_i\right\} \\
&= \mathcal{F}_-(u,1;\Omega). \qquad (2.20)
\end{aligned}$$

and the proof is complete. ■

2.2.2 The limsup inequality

The limsup inequality is proved by first assuming that S_u satisfyies the regularity property that its Minkowski content equals its Hausdorff measure.
Then, using the fact that every minimizer of the MS has the property of the Minkowsky content, we apply a diagonal argument on sequences of minimizers of a suitable family of minimum problems involving MS functionals.

Theorem 2.2.6 (limsup inequality) *For every $(u,s)\in Z(\Omega)$ there exists a sequence $(u_\epsilon,s_\epsilon)\subset D(\Omega)$ converging to (u,s) in $[L^2(\Omega)]^2$ such that*

$$\limsup_{\epsilon\to0}\mathcal{F}_\epsilon(u_\epsilon,s_\epsilon)\le\mathcal{F}(u,s).$$

Proof: When the right hand side of the inequality equals $+\infty$ there is nothing to prove, so we can assume $u\in\overline{X}(\Omega)$, $s\equiv1$ and $\mathcal{H}^{N-1}(S_u)<+\infty$. The variable function s would be the characteristic function of the set S_u, so we expect a right behaviour of (u_ϵ,s_ϵ) when we choose $u_\epsilon\equiv u$ and $s_\epsilon\equiv1$ outside a small neighbour of S_u, which shrinks as $\epsilon\to0$.
STEP 1: We are going to prove the existence of the recovery sequence first in the case when u satisfies an additional regularity condition of the jump set. We assume that $\mathcal{M}^*(S_u\cap\Omega)\le\mathcal{H}^{N-1}(S_u\cap\Omega)$, in such a way that (see Corollary 1.4.36) we get

$$\lim_{\rho\searrow0}\frac{\mathcal{H}^N(S_u\cap\Omega)}{2\rho}=\mathcal{H}^{N-1}(S_u\cap\Omega)=:L. \qquad (2.21)$$

We are looking for some infinitesimals $a_\epsilon,b_\epsilon,\eta_\epsilon$ suitable for the construction of the functions

$$u_\epsilon:=\begin{cases}u & \Omega\setminus(S_u)_{b_\epsilon}\\ \text{Sobolev} & \text{elsewhere}\end{cases}\qquad s_\epsilon:=\begin{cases}0 & (S_u)_{b_\epsilon}\\ 1-\eta_\epsilon & \Omega\setminus(S_u)_{b_\epsilon+a_\epsilon}\\ \text{Sobolev} & \text{elsewhere}\end{cases}$$

Without loss of generality we can assume $\alpha = \mu = 1$, now it's posible to write the explicit form of the functionals $\mathcal{F}_\epsilon$ on the functions defined above.

$$\begin{aligned}\mathcal{F}_\epsilon(u_\epsilon, s_\epsilon) &= \int_\Omega (s_\epsilon^2 + o_\epsilon)|\nabla u_\epsilon|^2 + \int_\Omega \epsilon|\nabla s_\epsilon|^2 + \frac{1}{4\epsilon}(s_\epsilon - 1)^2 + \int_\Omega (u_\epsilon - g)^2 \\ &= \int_{\Omega\setminus(S_u)_{b_\epsilon+a_\epsilon}} (s_\epsilon^2 + o_\epsilon)|\nabla u|^2 + \int_{\Omega\cap(S_u)_{b_\epsilon+a_\epsilon}} o_\epsilon|\nabla u_\epsilon|^2 \\ &\quad + \int_{(S_u)_{b_\epsilon+a_\epsilon}\setminus(S_u)_{b_\epsilon}} \epsilon|\nabla s_\epsilon|^2 + \frac{1}{4\epsilon}(s_\epsilon - 1)^2 + \int_{\Omega\cap(S_u)_{b_\epsilon}} \frac{1}{4\epsilon} \\ &\quad + \int_{\Omega\cap(S_u)_{b_\epsilon+a_\epsilon}} \frac{1}{4\epsilon}\eta_\epsilon^2 + \int_\Omega (u_\epsilon - g)^2.\end{aligned}$$

If we set

$$\mathcal{A}_\epsilon(s_\epsilon) := \int_{(S_u)_{b_\epsilon+a_\epsilon}\setminus(S_u)_{b_\epsilon}} \epsilon|\nabla s_\epsilon|^2 + \frac{1}{4\epsilon}(s_\epsilon - 1)^2$$

this becomes

$$\begin{aligned}\mathcal{F}_\epsilon(u_\epsilon, s_\epsilon) = &\int_{\Omega\setminus(S_u)_{b_\epsilon+a_\epsilon}} (s_\epsilon^2 + o_\epsilon)|\nabla u|^2 + \int_{\Omega\cap(S_u)_{b_\epsilon+a_\epsilon}} o_\epsilon|\nabla u_\epsilon|^2 + \mathcal{A}_\epsilon(s_\epsilon) \\ &+ \frac{1}{4\epsilon}\mathcal{H}^N(\Omega\cap(S_u)_{b_\epsilon}) + \frac{1}{4\epsilon}\eta_\epsilon^2\mathcal{H}^N(\Omega\cap(S_u)_{b_\epsilon+a_\epsilon}) + \int_\Omega (u_\epsilon - g)^2.\end{aligned}$$

We easily see that only with the hypotesis on $a_\epsilon, b_\epsilon, \eta_\epsilon$ to be infinitesimals we get

$$\int_{\Omega\setminus(S_u)_{b_\epsilon+a_\epsilon}} (s_\epsilon^2 + o_\epsilon)|\nabla u|^2 \longrightarrow \int_\Omega |\nabla u|^2$$

$$\int_\Omega (u_\epsilon - g)^2 \longrightarrow \int_\Omega (u - g)^2$$

as $\epsilon \to 0$. Our aim is to prove that the upper bound of $\mathcal{A}_\epsilon(s_\epsilon)$ doesn't exceed L while the other terms are infinitesimals.
If b_ϵ is an infinitesimal faster than ϵ we have

$$\lim_{\epsilon\to 0} \frac{1}{4\epsilon}\mathcal{H}^N(\Omega\cap(S_u)_{b_\epsilon}) = \lim_{\epsilon\to 0} \frac{b_\epsilon}{2\epsilon}\frac{\mathcal{H}^N(\Omega\cap(S_u)_{b_\epsilon})}{2b_\epsilon} = 0,$$

moreover, we can build the function u_ϵ, in the open tubular neighbourood with radius b_ϵ, such that $|\nabla u_\epsilon| \leq |c/2b_\epsilon|$, so if b_ϵ is also an infinitesimal intermediate between ϵ and o_ϵ (for example $\sqrt{\epsilon o_\epsilon}$) we have

$$\lim_{\epsilon\to 0} \int_{\Omega\cap(S_u)_{b_\epsilon+a_\epsilon}} o_\epsilon|\nabla u_\epsilon|^2 \leq \lim_{\epsilon\to 0} \frac{c^2}{2}\frac{o_\epsilon}{b_\epsilon}\frac{\mathcal{H}^N(\Omega\cap(S_u)_{b_\epsilon})}{2b_\epsilon} = 0.$$

For the last term we see that

$$\lim_{\epsilon\to 0} \frac{\eta_\epsilon^2}{4\epsilon}\mathcal{H}^N(\Omega\cap(S_u)_{b_\epsilon+a_\epsilon}) = \lim_{\epsilon\to 0} \left[\frac{\eta_\epsilon^2 a_\epsilon}{2\epsilon} + \frac{\eta_\epsilon^2}{2\epsilon}\sqrt{\epsilon o_\epsilon}\right]\frac{\mathcal{H}^N(\Omega\cap(S_u)_{b_\epsilon+a_\epsilon})}{2(b_\epsilon + a_\epsilon)} = 0$$

if η_ϵ is an infinitesimal faster than $\sqrt{\epsilon}$. In order to have the estimate

$$\limsup_{\epsilon\to 0} \mathcal{A}_\epsilon(s_\epsilon) \leq L$$

we first observe that we can restrict our choice of s_ϵ among the functions which are constants over the points having the same distance from S_u, this leads also to a reduction in 1-dimension of the problem. Let $\tau : \Omega \to [0, +\infty) : y \mapsto \operatorname{dist}(y, \overline{S_u})$, then we assume s_ϵ to be of the form $s_\epsilon(y) := \sigma_\epsilon(\tau(y))$ for some $\sigma_\epsilon : [0, +\infty) \to [0, 1]$.
Taking into account that τ is Lipshitz and $|\nabla \tau| = 1$ a.e., hence by Coarea Formula (Proposition 1.3.9) with $f = \tau$ and $g = \epsilon|\nabla s_\epsilon|^2 + (s_\epsilon - 1)^2/4\epsilon$ and the chain rule for the composition with Lipshitz functions, we have

$$\begin{aligned}
\mathcal{A}_\epsilon(s_\epsilon) &= \int_{(S_u)_{b_\epsilon+a_\epsilon}\setminus(S_u)_{b_\epsilon}} \left[\epsilon|\nabla s_\epsilon|^2 + \frac{1}{4\epsilon}(s_\epsilon - 1)^2\right] \cdot |\nabla \tau| \, d\mathcal{H}^N(y) \\
&= \int_{(S_u)_{b_\epsilon+a_\epsilon}\setminus(S_u)_{b_\epsilon}} \left[\epsilon|\nabla s_\epsilon|^2 + \frac{1}{4\epsilon}(s_\epsilon - 1)^2\right] \cdot 1 \, d\mathcal{H}^N(y) \\
&= \int_{b_\epsilon}^{b_\epsilon+a_\epsilon} \left[\int_{\tau^{-1}\{t\}} \epsilon|\sigma_\epsilon'[\tau(z)]|^2|\nabla\tau(z)|^2 + \frac{1}{4\epsilon}[\sigma_\epsilon(\tau(z)) - 1]^2 \, d\mathcal{H}^{N-1}(z)\right] dt \\
&= \int_{b_\epsilon}^{b_\epsilon+a_\epsilon} \left[\int_{\tau^{-1}\{t\}} \epsilon|\sigma_\epsilon'[\tau(z)]|^2 + \frac{1}{4\epsilon}[\sigma_\epsilon(\tau(z)) - 1]^2 \, d\mathcal{H}^{N-1}(z)\right] dt.
\end{aligned}$$

The last integrand does depend only on t in fact $z \in \tau^{-1}\{t\}$ if and only if $\tau(z) = t$ so that

$$\begin{aligned}
\mathcal{A}_\epsilon(s_\epsilon) &= \int_{b_\epsilon}^{b_\epsilon+a_\epsilon} \left[\int_{\tau^{-1}\{t\}} \epsilon|\sigma_\epsilon'(t)|^2 + \frac{1}{4\epsilon}[\sigma_\epsilon(t) - 1]^2 \, d\mathcal{H}^{N-1}(z)\right] dt \\
&= \int_{b_\epsilon}^{b_\epsilon+a_\epsilon} \epsilon|\sigma_\epsilon'(t)|^2 + \frac{1}{4\epsilon}[\sigma_\epsilon(t) - 1]^2 \left[\int_{\tau^{-1}\{t\}} d\mathcal{H}^{N-1}(z)\right] dt \\
&= \int_{b_\epsilon}^{b_\epsilon+a_\epsilon} \left(\epsilon|\sigma_\epsilon'(t)|^2 + \frac{1}{4\epsilon}[\sigma_\epsilon(t) - 1]^2\right) \mathcal{H}^{N-1}(\{x \in \Omega : \tau(x) = t\}) \, dt.
\end{aligned}$$

The inequalities (2.12), which are sufficient to give a lower bound of the functional, are based on the estimate $a^2 + b^2 \geq 2ab$. Since a recovery sequence actually realize a limit, we are interested in a function s_ϵ such that $a^2 + b^2 = 2ab$, that is, $a = b$. So let σ_ϵ be a solution of the Cauchy problem

$$(P_\epsilon) \qquad \left\{\sigma' = \frac{1-\sigma}{2\epsilon}; \qquad \sigma(b_\epsilon) = 0\right\},$$

then we can compute explicitly $\sigma_\epsilon(t) = 1 - e^{\frac{b_\epsilon - t}{2\epsilon}}$. By construction a_ϵ must be such that $\sigma_\epsilon(b_\epsilon + a_\epsilon) = 1 - \eta_\epsilon$, hence $a_\epsilon = -2\epsilon \log \eta_\epsilon$, we choose η_ϵ such that a_ϵ is infinitesimal.
Consider the function

$$h(t) := \mathcal{H}^N(\{x \in \Omega : \tau(x) < t\})$$

which is differentiable, with

$$h'(t) := \mathcal{H}^{N-1}(\{x \in \Omega : \tau(x) = t\}).$$

With an integration by parts we obtain

$$\begin{aligned}
\mathcal{A}_\epsilon(s_\epsilon) &= \frac{1}{2\epsilon}\int_{b_\epsilon}^{b_\epsilon+a_\epsilon}[1-\sigma_\epsilon(t)]^2\,dt \\
&= \frac{1}{2\epsilon}\int_{b_\epsilon}^{b_\epsilon+a_\epsilon} e^{\frac{b_\epsilon-t}{\epsilon}}\,h'(t)\,dt \\
&= \frac{1}{2\epsilon}\left\{e^{\frac{-a_\epsilon}{\epsilon}}h(b_\epsilon+a_\epsilon)-h(b_\epsilon)+\frac{1}{\epsilon}\int_{b_\epsilon}^{b_\epsilon+a_\epsilon} e^{\frac{b_\epsilon-t}{\epsilon}}\,h(t)\,dt\right\} \\
&= \frac{\eta_\epsilon^2}{2\epsilon}h(b_\epsilon+a_\epsilon)-\frac{h(b_\epsilon)}{2\epsilon}+\frac{1}{2\epsilon^2}\int_{b_\epsilon}^{b_\epsilon+a_\epsilon} e^{\frac{b_\epsilon-t}{\epsilon}}\,h(t)\,dt \\
&= \frac{1}{2\epsilon^2}\int_{b_\epsilon}^{b_\epsilon+a_\epsilon} e^{\frac{b_\epsilon-t}{\epsilon}}\,h(t)\,dt+o(\epsilon)
\end{aligned}$$

in fact in the same way as just done we get

$$\lim_{\epsilon\to 0}\frac{h(b_\epsilon)}{2\epsilon}=\lim_{\epsilon\to 0}\frac{\mathcal{H}^N((S_u)_{b_\epsilon}\cap\Omega)}{2b_\epsilon}\frac{b_\epsilon}{\epsilon}=0$$

and

$$\lim_{\epsilon\to 0}\frac{\eta_\epsilon^2}{2\epsilon}h(b_\epsilon+a_\epsilon)=\lim_{\epsilon\to 0}\frac{\eta_\epsilon^2}{2\epsilon}\mathcal{H}^N(\Omega\cap(S_u)_{b_\epsilon+a_\epsilon})=0.$$

From the hypotesis (2.21) we know that there exists a sequence δ_ϵ decreasing to zero such that $h(t)/2t \le L+\delta_\epsilon$ for all $t\in(0,b_\epsilon+a_\epsilon)$, then again with an integration by parts

$$\begin{aligned}
\limsup_{\epsilon\to 0}\mathcal{A}_\epsilon(s_\epsilon) &= \limsup_{\epsilon\to 0}\frac{1}{2\epsilon^2}\int_{b_\epsilon}^{b_\epsilon+a_\epsilon} e^{\frac{b_\epsilon-t}{\epsilon}}\,\frac{h(t)}{2t}2t\,dt \\
&\le \limsup_{\epsilon\to 0}\frac{L+\delta_\epsilon}{\epsilon^2}\int_{b_\epsilon}^{b_\epsilon+a_\epsilon} t\,e^{\frac{b_\epsilon-t}{\epsilon}}\,dt \\
&= \limsup_{\epsilon\to 0}\frac{L+\delta_\epsilon}{\epsilon^2}\left\{-\epsilon\left[e^{-\frac{a_\epsilon}{\epsilon}}(b_\epsilon+a_\epsilon)-b_\epsilon\right]+\epsilon\int_{b_\epsilon}^{b_\epsilon+a_\epsilon} e^{\frac{b_\epsilon-t}{\epsilon}}\,dt\right\} \\
&= \limsup_{\epsilon\to 0}\frac{L+\delta_\epsilon}{\epsilon}\int_{b_\epsilon}^{b_\epsilon+a_\epsilon} e^{\frac{b_\epsilon-t}{\epsilon}}\,dt+o(\epsilon) \\
&= \limsup_{\epsilon\to 0}\frac{L+\delta_\epsilon}{\epsilon}\left[-\epsilon(e^{-\frac{a_\epsilon}{\epsilon}}-1)\right] \\
&= \limsup_{\epsilon\to 0}(L+\delta_\epsilon)=L.
\end{aligned}$$

STEP 2: By regularity properties of local density for minimizers of the Mumford-Shah (see Theorem 2.1.2) it is possible to prove the following Theorem.

Theorem 2.2.7 *Let $u\in\overline{X}(\Omega)$ be a minimizer of the functional $\mathcal{F}$ with $g\in L^\infty(\Omega)$ then*

$$\lim_{\rho\searrow 0}\frac{\mathcal{H}^N((K)_\rho\cap\Omega)}{2\rho}=\mathcal{H}^{N-1}(K\cap\Omega).$$

for every compact set $K\subset(\overline{S_u}\cap\Omega)$.

So, by solving some minimum problem involving the MS functional we get solutions for which the regularity condition (2.21) is satisfied, the idea is then to use these solutions to approximate u.
Now let $u \in \overline{X}(\Omega)$ such that $\mathcal{F}(u,1) < +\infty$. It is possible to extend u in a bounded domain $\Omega' \supset\supset \Omega$ in such a way that

$$\int_{\Omega'} |\nabla u|^2 + \mathcal{H}^{N-1}(S_u \cap \Omega') < +\infty \tag{2.22}$$

$$\mathcal{H}^{N-1}(S_u \cap \partial\Omega) = 0. \tag{2.23}$$

This can be done if Ω is a Lipshitz domain satisfying the following *reflection condition*:
there is a neighbourood U of $\partial\Omega$ and a map $\varphi : U \cap \Omega \to U \setminus \overline{\Omega}$ Lipschitz continuous with its inverse, such that $\lim_{y\to x} \varphi(y) = x$ for any $x \in \partial\Omega$.
The extention is made by setting $u(y) := u(\varphi^{-1}(y))$ for every $y \in U \setminus \overline{\Omega}$.
We recall that the reflection condition is fulfilled by every C^2 domain and by hypercubes of $\mathbb{R}^N$.
We consider the sequence u_ϵ of minimizers for the family of problems

$$\int_{\Omega'} |\nabla v|^2 + \mathcal{H}^{N-1}(S_v \cap \Omega') + \frac{1}{\epsilon}\int_{\Omega'} (v-u)^2 \qquad (P_\epsilon)$$

then because of the last term we have $u_\epsilon \to u$ in $\mathrm{L}^2(\Omega)$, moreover, since $(\overline{S_{u_\epsilon}} \cap \overline{\Omega}) \subset (\overline{S_{u_\epsilon}} \cap \Omega')$ Theorem 2.2.7 implies

$$\begin{aligned}\limsup_{\rho\searrow 0} \frac{\mathcal{H}^N((S_{u_\epsilon})_\rho \cap \Omega)}{2\rho} &\le \lim_{\rho\searrow 0} \frac{\mathcal{H}^N((\overline{S_{u_\epsilon}} \cap \overline{\Omega})_\rho \cap \Omega)}{2\rho} \\ &= \mathcal{H}^{N-1}(\overline{S_{u_\epsilon}} \cap \overline{\Omega}) = \mathcal{H}^{N-1}(S_{u_\epsilon} \cap \Omega).\end{aligned}$$

For every $BV(\Omega)$ function the Hausdorff measure of S_u is less or equal than its lower Minkowski content (see Corollary 1.4.36), hence

$$\lim_{\rho\searrow 0} \frac{\mathcal{H}^N((S_{u_\epsilon})_\rho \cap \Omega)}{2\rho} = \mathcal{H}^{N-1}(S_{u_\epsilon} \cap \Omega).$$

Applying the results proved in the Step 1 we know that there exists, for every ϵ, a sequence $(u_\epsilon^{\sigma_\epsilon}, s_\epsilon^{\sigma_\epsilon}) \in D(\Omega)$ converging to $(u_\epsilon, 1)$ in $[\mathrm{L}^2(\Omega)]^2$ such that

$$\limsup_{\sigma_\epsilon\to 0} \mathcal{F}_{\sigma_\epsilon}(u_\epsilon^{\sigma_\epsilon}, s_\epsilon^{\sigma_\epsilon}) \le \mathcal{F}(u_\epsilon, 1).$$

Moreover, by the liminf inequality, we have

$$\lim_{\sigma_\epsilon\to 0} \mathcal{F}_{\sigma_\epsilon}(u_\epsilon^{\sigma_\epsilon}, s_\epsilon^{\sigma_\epsilon}) = \mathcal{F}(u_\epsilon, 1).$$

Let us define the following finite positive measures

$$\mu_\epsilon(B) := \int_\Omega |\nabla u_\epsilon|^2\,dx + \mathcal{H}^{N-1}(S_{u_\epsilon} \cap B)$$

$$\mu(B) := \int_\Omega |\nabla u|^2\,dx + \mathcal{H}^{N-1}(S_u \cap B);$$

because of the minimality of u_ϵ we easily get

$$\limsup_{\epsilon\to 0} \mu_\epsilon(\Omega') \le \mu(\Omega')$$

on the other hand, by the semicontinuity of $\mathcal{F}$, we also have

$$\mu(A) \le \liminf_{\epsilon\to 0} \mu_\epsilon(A) \qquad \forall A \subset \Omega' \text{ open.}$$

By Theorem 1.3.12 the measures μ_ϵ weakly* converge to μ, moreover, by the condition (2.23) which gives exactly that $\mu(\partial\Omega) = 0$, we get $\mu_\epsilon(\Omega) \to \mu(\Omega)$, that is

$$\mathcal{F}(u_\epsilon, 1) \longrightarrow \mathcal{F}(u, 1). \tag{2.24}$$

Finally we take the diagonal subsequence $(u_\epsilon^\epsilon, s_\epsilon^e)_\epsilon \subset (u_\epsilon^{\sigma_\epsilon}, s_\epsilon^{\sigma_\epsilon})_{\epsilon,\sigma_\epsilon}$; clearly we have that $(u_\epsilon^\epsilon, s_\epsilon^e) \to (u, 1)$ and thanks to (2.24) we obtain

$$\begin{aligned}\limsup_{\epsilon\to 0} \mathcal{F}_\epsilon(u_\epsilon^\epsilon, s_\epsilon^\epsilon) &\le \limsup_{\epsilon\to 0} \limsup_{\sigma_\epsilon\to 0} \mathcal{F}_{\sigma_\epsilon}(u_\epsilon^{\sigma_\epsilon}, s_\epsilon^{\sigma_\epsilon}) \\ &= \limsup_{\epsilon\to 0} \mathcal{F}(u_\epsilon, 1) = \mathcal{F}(u, 1)\end{aligned}$$

the requested estimate. ■

2.2.3 Compactness of sequences of minimizers

By well-known results of semicontinuity and compactness in Sobolev spaces and by regularity theory of the weak Laplace equation it is not difficult to prove the existence of minimizers for the functional $\mathcal{F}_\epsilon$ in $D(\Omega)$. Let (u_ϵ, s_ϵ) be a sequence of minimizer for the family $\mathcal{F}_\epsilon$, then we have

$$\begin{aligned}\frac{1}{4\epsilon}\int_\Omega (1-s_\epsilon)^2 \le \mathcal{F}_\epsilon(u_\epsilon, s_\epsilon) = \min \mathcal{F}_\epsilon \le \mathcal{F}_\epsilon(1,1) \\ = \mu\int_\Omega (1-g)^2 \le \mu|\Omega| + \mu\int_\Omega g^2 < +\infty\end{aligned}$$

hence $s_\epsilon \to 1$ in $L^2(\Omega)$, and in particular, possibly up to subsequences, the convergence is also almost everywhere. We want to use this fact to prove also the convergence of u_ϵ, this can be done by a suitable change of variables.
First we observe that by the minimality of (u_ϵ, s_ϵ) it is not a restriction to assume $\|u_\epsilon\|_\infty \le \|g\|_\infty$. In fact, by considering the truncation $Tu_\epsilon := -\|g\|_\infty \vee u_\epsilon \wedge \|g\|_\infty$ it results $\mathcal{F}_\epsilon(Tu_\epsilon, s_\epsilon) \le \mathcal{F}_\epsilon(u_\epsilon, s_\epsilon)$.
Now we apply the well-known estimate $2ab \le a^2 + b^2$ to the Ambrosio-Tortorelli functional with $a = \sqrt{\epsilon}|\nabla s_\epsilon|$ and $b = 1/(2\sqrt{\epsilon})|\nabla s_\epsilon||1-s_\epsilon|$ obtaining

$$|\nabla s_\epsilon||1-s_\epsilon| \le \epsilon|\nabla s_\epsilon|^2 + \frac{1}{4\epsilon}(1-s_\epsilon)^2. \tag{2.25}$$

The function $(1-s_\epsilon)\nabla s_\epsilon$, up to the constant 2, is exactly the gradient of $(2s_\epsilon - s_\epsilon^2)$, we define $v_\epsilon := (2s_\epsilon - s_\epsilon^2)u_\epsilon$ to obtain the following estimate

$$\begin{aligned}|\nabla v_\epsilon| &= |u_\epsilon \nabla(2s_\epsilon - s_\epsilon^2) + (2s_\epsilon - s_\epsilon^2)\nabla u_\epsilon| \\ &\le 2|u_\epsilon||\nabla s_\epsilon||1-s_\epsilon| + |2-s_\epsilon||s_\epsilon||\nabla u_\epsilon| \\ &\le c_1\left[\epsilon|\nabla s_\epsilon| + \frac{1}{4\epsilon}(1-s_\epsilon)^2\right] + c_2 + s_\epsilon^2|\nabla u_\epsilon|^2. \end{aligned}\tag{2.26}$$

Hence, by integrating on Ω we see that there exist two constants $k_1, k_2 > 0$ depending only on $\|g\|_\infty$ and $|\Omega|$ such that

$$\int_\Omega |\nabla v_\epsilon| \leq k_1 + k_2 \mathcal{F}_\epsilon(u_\epsilon, s_\epsilon)$$

so, by compactness proerties of $BV(\Omega)$ functions, possibly up to subsequences, is $v_\epsilon \to v \in L^\infty(\Omega)$ and the convergence is almost everywhere. Easily this implies $u_\epsilon \to v$ almost everywhere, and by boundedness, this convergence is also in $L^2(\Omega)$.

Chapter 3

The Blake-Zisserman model

The model proposed by Mumford-Shah for image segmentation has some drawbacks. The first and more easily observable is that, as a first order model, it is unable to detect crease discontinuities (discontinuities of the gradient), which are actually of a great relevance, for example, in surface reconstruction problems from image data; another fact is that it yelds to the phenomenon of the *over-segmentation of steep gradients*, where steep grandient surfaces are approximated by step functions (see e.g. kawhol).
Another relevant defect emerges from regularity studies of the set S_u. The minimiization of the $\mathcal{MS}$ functional yelds to an areas-competition between the regions of smoothness of u and so, this turns out, to the point of view of the edges, to a minimal connection problem of points in the plane. Hence we have that the jump set of u is made up by at most a countable union of C^1 arcs with finite length. Every end-point of each arc may be of three types: a crack tip (ending point with no others arcs joined), a double junction or a triple junction. In the last case the three arcs form three congruent angles of $2/3\pi$.
Some questions are still open. For example Mumford and Shah conjectured that the 3rd-points don't accumulate, some results are found in this direction but we are far from an exaustive treatment of the subject. Another problem, too far to be solved, is to know the asimptothic behaviour of u near the end-points of the crack tips.

To overcome these defects of the first order model, Blake and Zisserman introduced a second order functional (see [6]) which can be written as

$$\begin{aligned} \mathcal{BZ}(u, K_0, K_1) := \int_{\Omega\setminus(K_0\cup K_1)} &|\nabla^2 u|^2 + \mu(u-g)^2\, dx \\ &+ \alpha\mathcal{H}^{N-1}(K_0\cap\Omega) + \beta\mathcal{H}^{N-1}((K_1\setminus K_0)\cap\Omega) \end{aligned} \tag{3.1}$$

where α, β, μ are positive parameters, $\Omega \subset \mathbb{R}^N$ an open set and $g \in L^\infty(\Omega)$. The minimization takes place over the functions $u \in C^2(\Omega\setminus(K_0\cup K_1))$ approximately continuous on $\Omega \setminus K_0$, and K_0, K_1 unknown sets such that $K_0 \cup K_1$ is closed. Here K_0 represents the set of jump points and $K_1 \setminus K_0$ the set of crease points of u.

3.1 Weak formulation of the problem

Again, in order to prove the existence of minimizers, a weak formulation of the problem is needed. A suitable relaxation of $\mathcal{BZ}$ leads to the functional

$$H(u) := \begin{cases} F(u) & u \in L^2(\Omega) \cap GSBV^2(\Omega) \\ \\ +\infty & u \in L^2(\Omega) \setminus GSBV^2(\Omega) \end{cases} \tag{3.2}$$

where

$$\begin{aligned} F(u) &:= \int_\Omega |\nabla^2 u|^2 + \mu(u-g)^2\,dx + \alpha\mathcal{H}^{N-1}(S_u) + \beta\mathcal{H}^{N-1}(S_{\nabla u} \setminus S_u) \\ &= \int_\Omega |\nabla^2 u|^2 + \mu(u-g)^2\,dx + (\alpha-\beta)\mathcal{H}^{N-1}(S_u) + \beta\mathcal{H}^{N-1}(S_{\nabla u} \cup S_u). \end{aligned}$$

Carriero, Leaci and Tomarelli has been proved (see [7]) for any $N \geq 1$ the existence of minimizers for this functional when the condition

$$\beta \leq \alpha \leq 2\beta, \tag{3.3}$$

which ensures the semicontinuity of H with respect to the $L^1(\Omega)$ convergence, is satisfied. Moreover the same authors showed in [8] that, in the particular case $N = 2$, any weak minimizer of H actually provides a triplet (u, K_0, K_1) minimizer of $\mathcal{BZ}$ by taking a suitable representative of the function and the closure of S_u and $S_{\nabla u}$.

3.2 Approximation by elliptic functionals via Γ-convergence

The aim of this section is to present a generalization to the second order model of the Ambrosio-Tortorelli approach to the approximation, due to Ambrosio, Faina and March (see [4]).
In order to give a suitable family of elliptic functionals approximating the fuctional H they introduced two extra variables as a sostitute of the set variables S_u and $S_{\nabla u}$. Now, the control of the values of these variables is made by the introduction of two Ambrosio-Tortorelli components in the functional.
Let us consider the functional $\mathcal{F}$ defined by

$$\mathcal{F}(u,s,z) := \begin{cases} F(u) & (u,s,z) \in \overline{X}(\Omega) \times \overline{Y}(\Omega) \times \overline{Y}(\Omega) \\ \\ +\infty & \text{otherwise in } Z(\Omega) \end{cases} \tag{3.4}$$

where

$$\begin{aligned} Z(\Omega) &:= L^2(\Omega) \times L^\infty(\Omega;[0,1]) \times L^\infty(\Omega;[0,1]). \\ \overline{X}(\Omega) &:= L^2(\Omega) \cap GSBV^2(\Omega) \\ \overline{Y}(\Omega) &:= \{s \in L^\infty(\Omega) :\ s \equiv 1\}. \end{aligned}$$

Γ-convergence results can be achieved by using the following family of elliptic functionals

$$\mathcal{F}_\epsilon(u,s,z) := \begin{cases} F_\epsilon(u,s,z) & (u,s,z) \in D(\Omega) \\ \\ +\infty & \text{otherwise in } Z(\Omega) \end{cases} \tag{3.5}$$

where the functional F_ϵ is defined by

$$F_\epsilon(u,s,z) := \int_\Omega z^2|\nabla^2 u|^2\,dx + \xi_\epsilon \int_\Omega (s^2+o_\epsilon)|\nabla u|^2\,dx$$
$$+ (\alpha-\beta)\mathcal{G}(s) + \beta\mathcal{G}(z) + \mu\int_\Omega (u-g)^2\,dx \tag{3.6}$$

with $\mathcal{G}$ as in (2.8) and o_ϵ, ξ_ϵ infinitesimals, and

$$D(\Omega) := \Big\{(u,s,z) \in X(\Omega) :\ u,s,z \in W^{1,2}(\Omega),\ z\nabla u \in W^{1,1}(\Omega;\mathbb{R}^N)\Big\}$$

Some remarks about the choise of the domain $D(\Omega)$. If $u \in D(\Omega)$ then the Hessian operator does exist a.e. in $\{z > 0\}$ in fact it can berived from the formula

$$\nabla(z\nabla u) = \nabla z \otimes \nabla u + z\nabla^2 u$$

and it can be extended to 0 in $\{z = 0\}$. Moreover, in this domain are ensured the lower semicontinuity with respect the $[L^1(\Omega)]^3$ convergence and the compactness for the sublevels of the functionals $\mathcal{F}_\epsilon$ (see [4], pagg 1181-1182).

Γ-convergence results are proved with respect the $L^1(\Omega)$ convergence; the liminf inequality is proved in any space dimension N and it does not require any additional remarks, while the limsup is proved by assuming $u \in L^\infty(\Omega)$, $|\nabla u| \in L^2(\Omega)$ and $S_u, S_{\nabla u}$ satisfying some regularity condition (content of Minkowski properties) which are in general fulfilled in computer vision applications.

3.2.1 The liminf inequality

The liminf inequality cannot be obtained by means of the slicing tecnique and consequent reduction to a one-dimensional problem used in the previous chapter. Such a reduction yelds the operator norm of the Hessian matrix in the Γ-limit instead of the euclidean norm.
The second derivatives are estimated by adapting a global tecnique proposed by Ambrosio in [1] and relying on a compactness theorem (Theorem 1.4.39)in the space $GSBV^2(\Omega)$ due to Carriero, Leaci and Tomarelli.
Conversely, the jump part of the functional is estimated by using a slicing argument, taking into account that the space $GSBV(\Omega)$ is a vector space under suitable energy condition.

Theorem 3.2.1 (liminf inequality) *Assume that $o_\epsilon > 0$ and*

$$\lim_{\epsilon\searrow 0} \frac{\xi_\epsilon}{e} = +\infty. \tag{3.7}$$

Then, for every triple $(u,s,z) \in Z(\Omega)$ and every sequence $(u_\epsilon, s_\epsilon, z_\epsilon) \subset D(\Omega)$ converging

$$(u_\epsilon, s_\epsilon, z_\epsilon) \longrightarrow (u,s,z) \qquad \textit{in } [L^1(\Omega)]^3$$

we have

$$\mathcal{F}(u,s,z) \le \liminf_\epsilon \mathcal{F}_\epsilon(u_\epsilon, s_\epsilon, z_\epsilon).$$

3.2.2 The limsup inequality

In order to adapt the constructive part of the Ambrosio-Tortorelli proof of the previous chapter to this second order problem, a suitable estimate of the component $\int |\nabla u|^2$ deriving from the finiteness of F is proved.

Proposition 3.2.2 *Let $A, B \subset \mathbb{R}^N$ be open sets and $r > 0$ such that $(A)_{2r} \subset\subset B$. Then*

$$\int_A |\nabla u|^2 \, dx \leq 16N \left[\frac{1}{r^2} \int_B u^2 \, dx + 2r^2 \int_B |\nabla^2 u|^2 \, dx \right] \tag{3.8}$$

for every $u \in W^{2,2}_{loc}(B)$.

We observe that this estimate returns finite values when $u \in L^\infty(\Omega)$.
It is possible to prove the limsup inequality under the assumptions that $u \in L^\infty(\Omega)$, $|\nabla u| \in L^2(\Omega)$ and that, for the sets $S_u, S_{\nabla u}$ the $\mathcal{H}^{N-1}$ measure and the Minkowski content coincide.

Theorem 3.2.3 (limsup inequality) *Assume that $o_\epsilon > 0$, ξ_ϵ satisfies (3.7) and $\xi_\epsilon o_\epsilon = o(\epsilon)$. Then, for every triple $(u, s, z) \in Z(\Omega)$ such that $u \in L^\infty(\Omega)$, $|\nabla u| \in L^2(\Omega)$ and*

$$\mathcal{M}^*(S_u) \leq \mathcal{H}^{N-1}(S_u) \qquad \mathcal{M}^*(S_u \cup S_{\nabla u}) \leq \mathcal{H}^{N-1}(S_u \cup S_{\nabla u})$$

there exists a sequence $(u_\epsilon, s_\epsilon, z_\epsilon) \subset D(\Omega)$ converging

$$(u_\epsilon, s_\epsilon, z_\epsilon) \longrightarrow (u, s, z) \qquad \text{in } [L^1(\Omega)]^3$$

such that

$$\limsup_\epsilon \mathcal{F}_\epsilon(u_\epsilon, s_\epsilon, z_\epsilon) \leq \mathcal{F}(u, s, z).$$

3.2.3 Compactness of sequences of minimizers

The assumptions made in the Theorem 3.2.1 on the infinitesimals o_ϵ, ξ_ϵ are sufficient to prove the equicoercivity for the family of functionals $\mathcal{F}_\epsilon$.

Theorem 3.2.4 *Let o_ϵ, ξ_ϵ be as inTheorem 3.2.1 and consider a family $(u_\epsilon, s_\epsilon, z_\epsilon) \subset D(\Omega)$ such that*

$$\sup_{\epsilon > 0} \mathcal{F}_\epsilon(u_\epsilon, s_\epsilon, z_\epsilon) < +\infty.$$

Then the family $(u_\epsilon, s_\epsilon, z_\epsilon)$ is relatively compact (admits some convergent subsequences) in the $[L^1(\Omega)]^3$ topology. Moreover, every limit point is of the form $(u, 1, 1)$ with $u \in \overline{X}(\Omega)$.

Chapter 4

Numerical implementation of the Blake-Zisserman model

We consider an application of the Blake-Zisserman model related to a 2D image. Then $N = 2$ and $\Omega \subset \mathbb{R}^2$ a rectangle. By the properties of Γ-convergence and by the regularity results described in the previous Chapter, we know that, for ϵ small enough, a minimizer of the functional

$$\begin{aligned} F_\epsilon(u, s, z) = & \int_\Omega z^2 |\nabla^2 u|^2 \, dx + \xi_\epsilon \int_\Omega (s^2 + o_\epsilon) |\nabla u|^2 \, dx \\ & + (\alpha - \beta) \int_\Omega \epsilon |\nabla s|^2 + \frac{1}{4\epsilon}(s-1)^2 \, dx + \beta \int_\Omega \epsilon |\nabla z|^2 + \frac{1}{4\epsilon}(z-1)^2 \, dx \\ & + \mu \int_\Omega (u - g)^2 \, dx \end{aligned} \tag{4.1}$$

is sufficiently close to a minimizer of the $\mathcal{BZ}$. The elliptic functional F_ϵ is differentiable, furthermore we easily observe that it is quadratic with respect to the single variables

$$F_\epsilon(., s, z) \qquad F_\epsilon(u, ., z) \qquad F_\epsilon(u, s, .). \tag{4.2}$$

Hence, after a discretizazion of the energy, we will obtain three symmetrical linear systems associated to the partial gradients in the variables u, s, z.

4.1 Discretization of the problem

We define the discrete representation of the rectangle Ω with a square lattice of coordinates with step $t > 0$

$$Q := \Big\{ (it, jt) : \, i, j = 1, 2, \ldots, n \Big\}$$

for some $n \in \mathbb{N}$ which represents the number of the pixels of the image. For notational simplicity, according to the representation of the matrices on the most numerical computing environments, we consider the origin of the lattice up-left. We denote the discrete representations of u, s, z at every point (it, jt) as

$$u^t_{i,j} := u(it, jt) \qquad s^t_{i,j} := s(it, jt) \qquad z^t_{i,j} := z(it, jt)$$

in such a way that the discrete variables

$$u^t := (u^t_{i,j})_{i,j} \qquad s^t := (s^t_{i,j})_{i,j} \qquad z^t := (z^t_{i,j})_{i,j}$$

are representable as elements of the space of matrices $M^{n\times n}$ which takes values in $\mathbb{R}$. Clearly this identification doesn't depend on t.
In order to simplify the exposition of the procedure in the sequel, we chose to split the functional F_ϵ in three pieces as follows

$$F^1_\epsilon(u, s, z) := \int_\Omega z^2|\nabla^2 u|^2\,dx + \xi_\epsilon \int_\Omega (s^2 + o_\epsilon)|\nabla u|^2\,dx$$

$$F^2_\epsilon(s, z) := (\alpha - \beta)\int_\Omega \epsilon|\nabla s|^2 + \frac{1}{4\epsilon}(s-1)^2\,dx + \beta\int_\Omega \epsilon|\nabla z|^2 + \frac{1}{4\epsilon}(z-1)^2\,dx$$

$$F^3_\epsilon(u) := \mu\int_\Omega (u-g)^2\,dx.$$

so that we easily get $F_\epsilon = F^1_\epsilon + F^2_\epsilon + F^3_\epsilon$. In order to approximate the differential operators appearing in the functional we use the following finite difference formulas

$$\partial_x u^t_{i,j} := \frac{\partial u}{\partial x}(it, jt) \cong \frac{1}{t}(u^t_{i+1,j} - u^t_{i,j})$$

$$\partial_y u^t_{i,j} := \frac{\partial u}{\partial y}(it, jt) \cong \frac{1}{t}(u^t_{i,j+1} - u^t_{i,j})$$

$$\partial^2_x u^t_{i,j} := \frac{\partial^2 u}{\partial x^2}(it, jt) \cong \frac{1}{t^2}(u^t_{i+1,j} - 2u^t_{i,j} + u^t_{i-1,j})$$

$$\partial^2_y u^t_{i,j} := \frac{\partial^2 u}{\partial y^2}(it, jt) \cong \frac{1}{t^2}(u^t_{i,j+1} - 2u^t_{i,j} + u^t_{i,j-1})$$

$$\partial^2_{xy} u^t_{i,j} := \frac{\partial^2 u}{\partial x \partial y}(it, jt) \cong \frac{1}{t^2}(u^t_{i+1,j+1} - u^t_{i,j+1} - u^t_{i+1,j} + u^t_{i,j})$$

in such a way, after some calculations, we obtain

$$\begin{aligned}
|\nabla u^t_{i,j}|^2 &:= (\partial_x u^t_{i,j})^2 + (\partial_y u^t_{i,j})^2 \\
&= \frac{1}{t^2}\Big[(u^t_{i+1,j})^2 + 2(u^t_{i,j})^2 + (u^t_{i,j+1})^2 - 2u^t_{i+1,j}u^t_{i,j} - 2u^t_{i,j+1}u^t_{i,j}\Big] \\
|\nabla^2 u^t_{i,j}|^2 &:= (\partial^2_x u^t_{i,j})^2 + 2(\partial^2_{xy} u^t_{i,j})^2 + (\partial^2_y u^t_{i,j})^2 \\
&= \frac{1}{t^4}\Big[3(u^t_{i+1,j})^2 + 10(u^t_{i,j})^2 + (u^t_{i-1,j})^2 + 3(u^t_{i,j+1})^2 + (u^t_{i,j-1})^2 \\
&+ 2(u^t_{i+1,j+1})^2 - 8u^t_{i,j}u^t_{i+1,j} - 4u^t_{i-1,j}u^t_{i,j} + 2u^t_{i+1,j}u^t_{i-1,j} \\
&+ 4u^t_{i,j+1}u^t_{i+1,j} - 8u^t_{i,j}u^t_{i,j+1} + 2u^t_{i,j-1}u^t_{i,j+1} - 4u^t_{i,j-1}u^t_{i,j} \\
&- 4u^t_{i,j+1}u^t_{i+1,j+1} - 4u^t_{i+1,j}u^t_{i+1,j+1} + 4u^t_{i,j}u^t_{i+1,j+1}\Big].
\end{aligned}$$

Analogous formulas can be obtained for $|\nabla s_{i,j}|^2$ and $|\nabla z_{i,j}|^2$. Now we can compute the discrete energies related to $F^1_\epsilon, F^2_\epsilon, F^3_\epsilon$, we obtain

$$
\begin{aligned}
F^1_{\epsilon,t}(u^t,s^t,z^t) :=& \\
&\sum_{i,j} t^2 (z^t_{i,j})^2 |\nabla^2 u^t_{i,j}|^2 + t^2 \xi_\epsilon [(s^t_{i,j})^2 + o_\epsilon] |\nabla u^t_{i,j}|^2 \\
=&\sum_{i,j} \frac{1}{t^2} (z^t_{i,j})^2 \Big[3(u^t_{i+1,j})^2 + 10(u^t_{i,j})^2 + (u^t_{i-1,j})^2 + 3(u^t_{i,j+1})^2 \\
&+ (u^t_{i,j+1})^2 + 2(u^t_{i+1,j+1})^2 - 8u^t_{i,j}u^t_{i+1,j} - 4u^t_{i-1,j}u^t_{i,j} + 2u^t_{i+1,j}u^t_{i-1,j} \\
&+ 4u^t_{i,j+1}u^t_{i+1,j} - 8u^t_{i,j}u^t_{i,j+1} + 2u^t_{i,j-1}u^t_{i,j+1} - 4u^t_{i,j-1}u^t_{i,j} - 4u^t_{i,j+1}u^t_{i+1,j+1} \\
&- 4u^t_{i+1,j}u^t_{i+1,j+1} + 4u^t_{i,j}u^t_{i+1,j+1} \Big] + \xi_\epsilon \Big[(s^t_{i,j})^2 + o_\epsilon \Big] \\
&\cdot \Big[(u^t_{i+1,j})^2 + 2(u^t_{i,j})^2 + (u^t_{i,j+1})^2 - 2u^t_{i+1,j}u^t_{i,j} - 2u^t_{i,j+1}u^t_{i,j} \Big]
\end{aligned}
$$

$$
\begin{aligned}
F^2_{\epsilon,t}(s^t,z^t) :=& \\
&(\alpha-\beta) \sum_{i,j} t^2 \left[\epsilon |\nabla s^t_{i,j}|^2 + \frac{1}{4\epsilon}(s^t_{i,j}-1)^2 \right] \\
&+ \beta \sum_{i,j} t^2 \left[\epsilon |\nabla z^t_{i,j}|^2 + \frac{1}{4\epsilon}(z^t_{i,j}-1)^2 \right] \\
=&(\alpha-\beta) \sum_{i,j} \epsilon \Big[(s^t_{i+1,j})^2 + 2(s^t_{i,j})^2 + 2(s^t_{i,j+1})^2 - 2s^t_{i+1,j}s^t_{i,j} - 2s^t_{i,j+1}s^t_{i,j} \Big] \\
&+ \frac{t^2}{4\epsilon} \Big[(s^t_{i,j})^2 - s^t_{i,j} + 1 \Big] \\
&+ \beta \sum_{i,j} \epsilon \Big[(z^t_{i+1,j})^2 + 2(z^t_{i,j})^2 + 2(z^t_{i,j+1})^2 - 2z^t_{i+1,j}z^t_{i,j} - 2z^t_{i,j+1}z^t_{i,j} \Big] \\
&+ \frac{t^2}{4\epsilon} \Big[(z^t_{i,j})^2 - z^t_{i,j} + 1 \Big]
\end{aligned}
$$

$$
F^3_{\epsilon,t}(u^t) := \mu \sum_{i,j} t^2 (u^t_{i,j} - g^t_{i,j})^2 = \mu \sum_{i,j} t^2 \Big[(u^t_{i,j})^2 - 2u^t_{i,j}g^t_{i,j} + g^t_{i,j})^2 \Big].
$$

4.2 Minimization of the discrete energy

The computation of minimizers for the discrete energy $F_{\epsilon,t} := F^1_{\epsilon,t} + F^2_{\epsilon,t} + F^3_{\epsilon,t}$ is not trivial. Global properties of the discrete energy $F_{\epsilon,t}$ which ensure that some iterative method applied to the system of non-linear equations

$$
\nabla F_{\epsilon,t}(u^t, s^t, z^t) = 0 \tag{4.3}
$$

have some properties of convergence are difficult to find. The empirical approach we are going to use is not new in the field of images segmentation. First we recall that the energy functional to minimize is quadratic in its three groups of variables u^t, s^t, z^t, so the partial gradients in these three groups of variables will be symmetric. We compute the explicit form of them by writing

$$\frac{\partial F_{\epsilon,t}}{\partial u_{h,k}}(u^t,s^t,z^t)=0, \quad \frac{\partial F_{\epsilon,t}}{\partial s_{h,k}}(u^t,s^t,z^t)=0, \quad \frac{\partial F_{\epsilon,t}}{\partial z_{h,k}}(u^t,s^t,z^t)=0$$

in explicit form for every $h,k\in\{1,2,\dots,n\}$. Fixed a pair (h,k), it appears in some terms of the energy $F_{\epsilon,t}$, exactly when it coincides with the pairs $(i,j-1),(i-1,j),(i,j),(i+1,j),(i,j+1),(i+1,j+1)$. By taking account of this, after some calculations we obtain

- $\displaystyle\frac{\partial F^1_{\epsilon,t}}{\partial u_{h,k}}(u^t,s^t,z^t)=$

$$\begin{aligned}
&u^t_{h,k-2}\Big\{\frac{1}{t^2}\Big[2(z^t_{h,k-1})^2\Big]\Big\}\\
&+u^t_{h-1,k-1}\Big\{\frac{1}{t^2}\Big[4(z^t_{h-1,k-1})^2\Big]\Big\}\\
&+u^t_{h,k-1}\Big\{\frac{1}{t^2}\Big[-4(z^t_{h,k})^2-8(z^t_{h,k-1})^2-4(z^t_{h-1,k-1})^2\Big]-\xi_\epsilon\Big[2(s^t_{h,k-1})^2+2o_\epsilon\Big]\Big\}\\
&+u^t_{h+1,k-1}\Big\{\frac{1}{t^2}\Big[4(z^t_{h,k-1})^2\Big]\Big\}\\
&+u^t_{h-2,k}\Big\{\frac{1}{t^2}\Big[2(z^t_{h-1,k})^2\Big]\Big\}\\
&+u^t_{h-1,k}\Big\{\frac{1}{t^2}\Big[-4(z^t_{h,k})^2-8(z^t_{h-1,k})^2-4(z^t_{h-1,k-1})^2\Big]-\xi_\epsilon\Big[2(s^t_{h-1,k})^2+2o_\epsilon\Big]\Big\}\\
&+u^t_{h,k}\Big\{\frac{1}{t^2}\Big[2(z^t_{h+1,k})^2+2(z^t_{h,k+1})^2+20(z^t_{h,k})^2+6(z^t_{h-1,k})^2+6(z^t_{h,k-1})^2+4(z^t_{h-1,k-1})^2\Big]\\
&\quad+\ \xi_\epsilon\Big[4(s^t_{h,k})^2+2(s^t_{h-1,k})^2+2(s^t_{h,k-1})^2+8o_\epsilon\Big]\Big\}\\
&+u^t_{h+1,k}\Big\{\frac{1}{t^2}\Big[-4(z^t_{h+1,k})^2-8(z^t_{h,k})^2-4(z^t_{h,k-1})^2\Big]-\xi_\epsilon\Big[2(s^t_{h,k})^2+2o_\epsilon\Big]\Big\}\\
&+u^t_{h+2,k}\Big\{\frac{1}{t^2}\Big[2(z^t_{h+1,k})^2\Big]\Big\}\\
&+u^t_{h-1,k+1}\Big\{\frac{1}{t^2}\Big[4(z^t_{h-1,k})^2\Big]\Big\}\\
&+u^t_{h,k+1}\Big\{\frac{1}{t^2}\Big[-4(z^t_{h,k+1})^2-8(z^t_{h,k})^2-4(z^t_{h-1,k})^2\Big]-\xi_\epsilon\Big[2(s^t_{h,k})^2+2o_\epsilon\Big]\Big\}\\
&+u^t_{h+1,k+1}\Big\{\frac{1}{t^2}\Big[4(z^t_{h,k})^2\Big]\Big\}\\
&+u^t_{h,k+2}\Big\{\frac{1}{t^2}\Big[2(z^t_{h,k+1})^2\Big]\Big\}
\end{aligned}$$

- $\displaystyle\frac{\partial F^2_{\epsilon,t}}{\partial u_{h,k}}(u^t,s^t,z^t)=0$

- $\displaystyle\frac{\partial F^3_{\epsilon,t}}{\partial u_{h,k}}(u^t,s^t,z^t)=u^t_{h,k}\Big\{2\mu t^2\Big\}-2\mu t^2 g_{h,k}$

- $\displaystyle\frac{\partial F^1_{\epsilon,t}}{\partial s_{h,k}}(u^t,s^t,z^t)=$

$$s^t_{h,k}\Big\{2\xi_\epsilon\Big[(u^t_{h+1,k})^2+2(u^t_{h,k})^2+(u^t_{h,k+1})^2-2u^t_{h+1,k}u^t_{h,k}-2u^t_{h,k+1}u^t_{h,k}\Big]\Big\}$$

- $\dfrac{\partial F^2_{\epsilon,t}}{\partial s_{h,k}}(u^t,s^t,z^t) =$

$$s^t_{h,k-1}\Big\{-2(\alpha-\beta)\epsilon\Big\} + s^t_{h-1,k}\Big\{-2(\alpha-\beta)\epsilon\Big\} + s^t_{h,k}\Big\{(\alpha-\beta)\Big[8\epsilon + \frac{t^2}{4\epsilon}\Big]\Big\}$$
$$+s^t_{h+1,k}\Big\{-2(\alpha-\beta)\epsilon\Big\} + s^t_{h,+1k}\Big\{-2(\alpha-\beta)\epsilon\Big\} - (\alpha-\beta)\frac{t^2}{4\epsilon}$$

- $\dfrac{\partial F^3_{\epsilon,t}}{\partial s_{h,k}}(u^t,s^t,z^t) = 0$

- $\dfrac{\partial F^1_{\epsilon,t}}{\partial z_{h,k}}(u^t,s^t,z^t) =$

$$z^t_{h,k}\Big\{\frac{2}{t^2}\Big[3(u^t_{h+1,k})^2 + 10(u^t_{h,k})^2 + (u^t_{h-1,k})^2 + 3(u^t_{h,k+1})^2 + (u^t_{h,k-1})^2$$
$$+ 2(u^t_{h+1,k+1})^2 - 8u^t_{h+1,k}u^t_{h,k} - 4u^t_{h-1,k}u^t_{h,k} + 2u^t_{h+1,k}u^t_{h-1,k}$$
$$+ 4u^t_{h,k+1}u^t_{h+1,k} - 8u^t_{h,k+1}u^t_{h,k} - 2u^t_{h,k-1}u^t_{h,k+1} - 4u^t_{h,k-1}u^t_{h,k}$$
$$- 4u^t_{h,k+1}u^t_{h+1,k+1} - 4u^t_{h+1,k}u^t_{h+1,k+1} + 4u^t_{h,k}u^t_{h+1,k+1}\Big]\Big\}$$

- $\dfrac{\partial F^2_{\epsilon,t}}{\partial z_{h,k}}(u^t,s^t,z^t) =$

$$z^t_{h,k-1}\Big\{-2\beta\epsilon\Big\} + z^t_{h-1,k}\Big\{-2\beta\epsilon\Big\} + z^t_{h,k}\Big\{\beta\Big[8\epsilon + \frac{t^2}{4\epsilon}\Big]\Big\} + z^t_{h+1,k}\Big\{-2\beta\epsilon\Big\}$$
$$+z^t_{h,+1k}\Big\{-2\beta\epsilon\Big\} - \beta\frac{t^2}{4\epsilon}$$

- $\dfrac{\partial F^3_{\epsilon,t}}{\partial z_{h,k}}(u^t,s^t,z^t) = 0$

Fixed the values $\overline{u}^t, \overline{s}^t, \overline{z}^t \in M^{n\times n}$ we consider the functions

$$\nabla F_{\epsilon,t}(u^t,\overline{s}^t,\overline{z}^t) =: g_1(u^t) \in M^{n\times n}$$
$$\nabla F_{\epsilon,t}(\overline{u}^t,s^t,\overline{z}^t) =: g_2(s^t) \in M^{n\times n}$$
$$\nabla F_{\epsilon,t}(\overline{u}^t,\overline{s}^t,z^t) =: g_3(z^t) \in M^{n\times n}$$

and we see that, after a vectorization of the variables, it is possible to rewrite the problems

$$g_1(u^t) = 0 \qquad g_2(s^t) = 0 \qquad g_3(z^t) = 0$$

as three square linear systems. We show the method only for $g_1(u^t)$, the others can be obtained for analogy. Let us consider the bijection from the space Q to $I := \{1, 2, \ldots, n^2\}$ given by

$$w : Q \to I : (h,k) \mapsto \left(\frac{k}{t} - 1\right)n + \frac{h}{t}$$

$$w^{-1} : I \to Q : i \mapsto t\Big(\operatorname{rem}(i-1,n)+1, \operatorname{floor}(i-1,n)+1\Big)$$

where $\operatorname{rem}(a,b)$ denotes the remainder after the division a/b and $\operatorname{floor}(a,b)$ the quotient. With this identification we can consider without ambiguity the matrix $u^t \in M^{n\times n}$ as a vector $\underline{u}^t \in \mathbb{R}^{n^2}$.
Our aim is to find an algorithm which gives a square matrix $A_1^t \in M^{n^2\times n^2}$ and a vector $v_1^t \in \mathbb{R}^{n^2}$ such that

$$w(g_1(u^t)) = A_1^t \underline{u}^t - v_1^t,$$

that is, it turns the problem to solve the equations $g_1(u^t) = 0$ to the problem of solve the linear sistem $A_1^t \underline{u}^t = v_1^t$.
Let us fix i a row index of the matrix A_1^t, the scalar product between this row and $\underline{u}^t$ must returns the equation corresponding to

$$\frac{\partial F_{\epsilon,t}}{\partial u_{h,k}}(u^t, \overline{s}^t, \overline{z}^t)$$

with $(h,k) = w^{-1}(i)$. Hence at the element in the column $j = w^{-1}(h, k-2)$ we put

$$A_i^t(i,j) = \frac{1}{t^2}\Big[2(z_{h,k-1}^t)^2\Big]$$

at the element in the column $j = w^{-1}(h-1, k-1)$ we put

$$A_i^t(i,j) = \frac{1}{t^2}\Big[4(z_{h-1,k-1}^t)^2\Big]$$

and so on. The vector v_1^t is trivially built by $v_1^t(i) = 2\mu t^2 g_{h,k}$.

The coefficients matrices of these three systems have some good properties. All of them are sparse and symmetrical with positive diagonal entries. A_1^t is clearly 13-diagonal and A_2^t, A_3^t are 5-diagonal. Moreover A_2^t and A_3^t are diagonally dominant hence positive definite.

The iteration is carried on as follows: at step k, fixed the parameters $\underline{s}_k^t, \underline{z}_k^t \in M^{n\times n}$ we implement a few steps of an algorithm for the solution of symmetrical sytems to the

$$A_1^t \underline{u}^t - v_1^t = 0.$$

We denote the returned iterate as $\underline{u}_{k+1}^t$. The next step is to implement 1 or 2 steps of an iterative method (for instance the Coniugate Gradient method) to the symmetrical and positve definite system

$$A_2^t \underline{s}^t - v_2^t = 0$$

with the parameters $\underline{u}_{k+1}^t$ and $\underline{z}_k^t$. Denoted the returned iterate as $\underline{s}_{k+1}^t$, the same procedure is applied to the system

$$A_3^t \underline{z}^t - v_3^t = 0$$

with parameters $\underline{u}_{k+1}^t$ and $\underline{s}_{k+1}^t$. Every iteration procedure is triggered with an initial datum which coincide to the last value of the unknown variable obtained with the previous step. In practice we solve the non linear system by an inexact block Gauss Seidel iterative method. At step $k = 1$ the inital values are chosen as follows: for $\underline{u}^t$ we choose exactly the image to be segmented $\underline{g}^t$, the initial values of $\underline{s}^t$ and $\underline{z}^t$ are set to the functions identically equal to 1.

4.3 Examples of segmentation

We show here some results obtained with the iterative procedure described in the previous Section. We denote with N the number of pixels of the image, the mesh size is set to $t = 1$ for all images. Every image takes grey values into the interval $[0, 255]$.

4.3.1 Noise removal

The algorithm is applied to an image v^t representing four squares with different grey intensities. The image is soiled with an artificial random noise of 12%. The size is $N = 100$, the parameters are set to $\alpha = 2$, $\beta = 1$, $\mu = 0.01$ and $\epsilon = 0.1$.

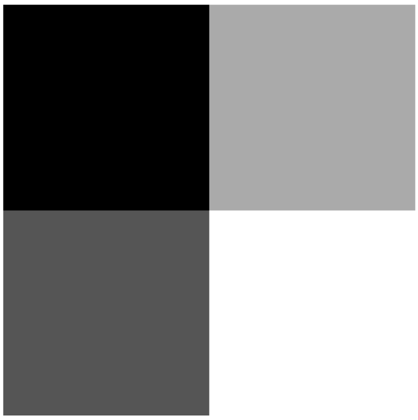

Figure 4.1: Original image.

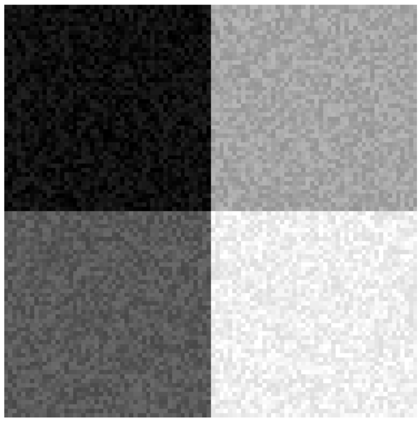

Figure 4.2: Noised image: g^t.

After only 18 iterations the obtained image is almost denoized. The Figure 4.3 represents the partial result, the comparison with the original image v^t gives $\max |u^t - v^t| = 1.0204$. With further 5 iterations the image is totally denoized, we have $\max |u^t - v^t| = 0.0027$.

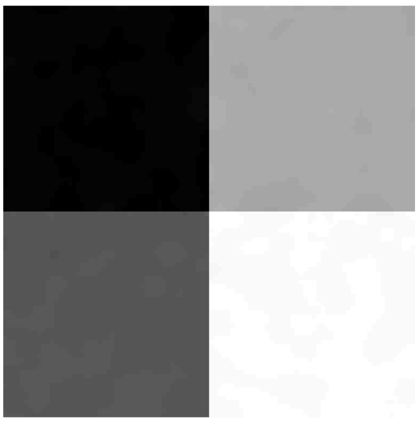

Figure 4.3: Partially denoised image: u^t.

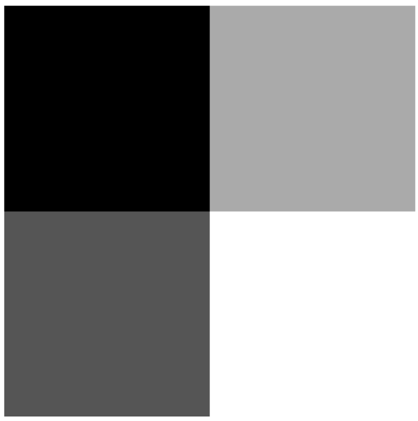

Figure 4.4: Final segmented image u^t.

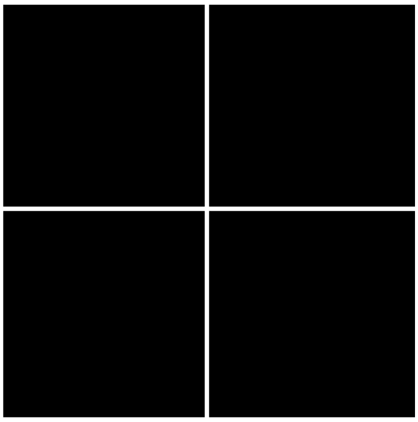

Figure 4.5: Edge detecting function: s^t.

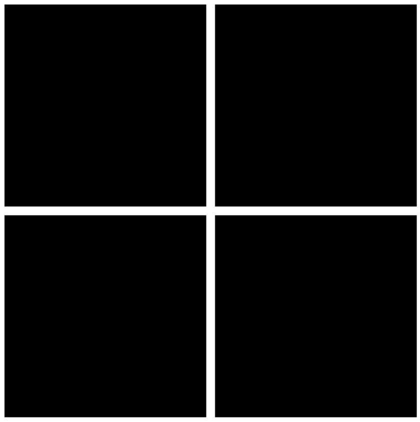

Figure 4.6: Edge-crease detecting functionl: z^t.

The edge detecting function returned values which are very close to 0, the edge points are all in a range of 0.0002. The values of the edge-crease detecting function are all in a range of 0.0006, for both edges and creases points.

4.3.2 Crease-detecting

In the following image, which represent a truncated piramid, in addition to the discontinuities of the grey level also discontinuities of the gradient appear. The size is $N = 300$, the functional parameters are $\alpha = 2$, $\beta = 1$, $\mu = 1$ and $\epsilon = 0.1$.

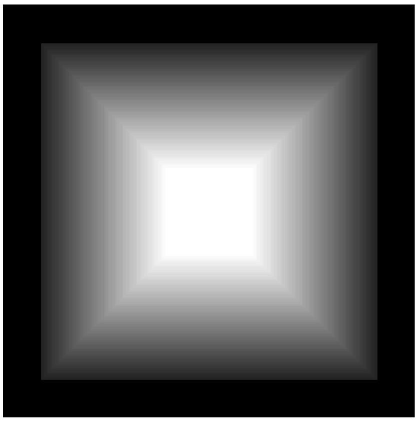

Figure 4.7: Original image: g^t.

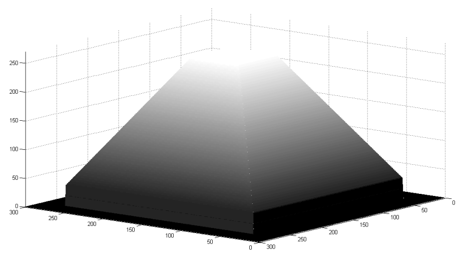

Figure 4.8: 3D representation of g^t.

The image is totally noise free, so the iteration procedure is not applied to the function u^t. This means that the parameter u^t is always set equal to g^t and the solution of the system g_1 is not implemented.

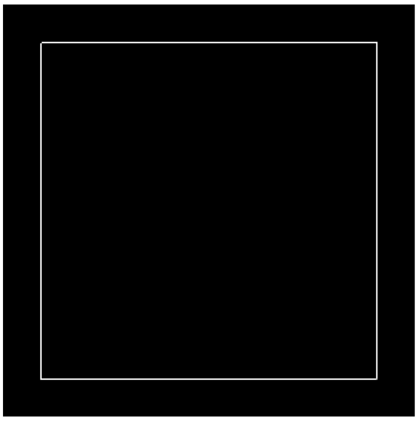

Figure 4.9: Edge detecting function: s^t.

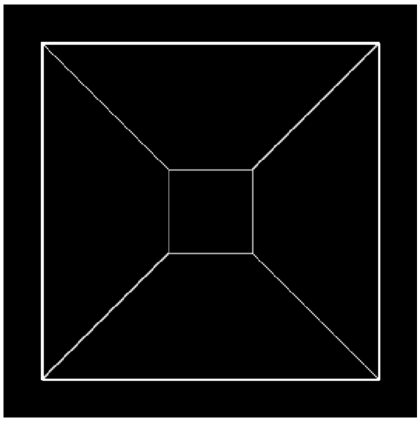

Figure 4.10: Edge-crease detecting functionl: z^t.

The procedure consisted in 6 outer iterations with 2 inner iterations in g_2 and 8 inner iterations in g_3. The follwing two graphs represents the values of z^t at the rows 70 and 150 respectively. We see that the crease values are slightly less accurate than the jump values.

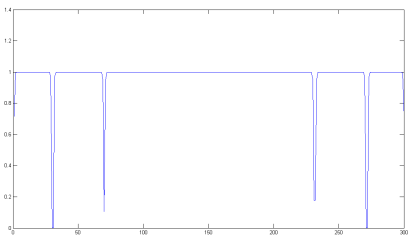

Figure 4.11: Function z^t at row 70.

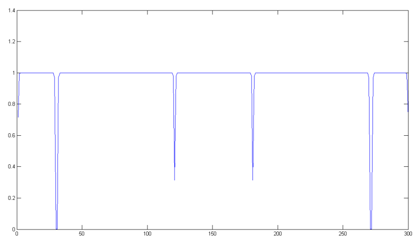

Figure 4.12: Function z^t at row 150.

4.3.3 Digital surface models (DSM)

The following examples are based on two small portion of the Digital Surface Models (DSM) of the Provincia Autonoma di Trento. In this case, the DSM is a representation of the terrain surface created from elevation data. Elevation data are obtained from a LIDAR (LIght Detection And Ranging) survey of the surface. By means of the LIDAR technique, one can obtain the distance from the instrument to a target illuminated with laser light. The elevation of the target can be computed knowing the elevation of the instrument. The data can be downloaded from the *Portale Cartografico Trentino* `http://www.territorio.provincia.tn.it`.
The DSM used in the next examples have a regular spatial resolution of 1mt × 1mt, details on the LIDAR survey and on the model can be found at `http://www.territorio.provincia.tn.it/portal/server.pt/community/lidar/847/lidar/23954`.

The first one is a particular of *Palazzo delle Albere, TN*, it is a regular grid of $N = 300$. The important feature of this image is the presence of the roof of the central building (Palazzo delle Albere). We want to dedect the roof's creases. The segmentation procedure is made with the following parameters: $\alpha = 2$, $\beta = 1$, $\mu = 1$ and $\epsilon = 0.01$.

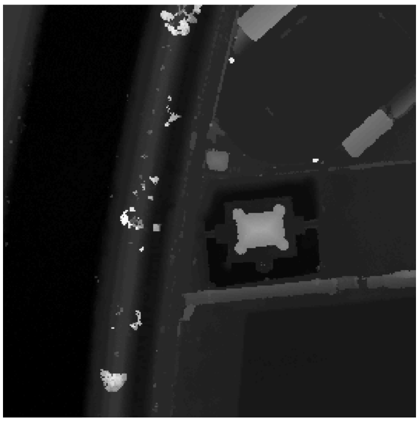

Figure 4.13: Original Digital Surface Model: g^t.

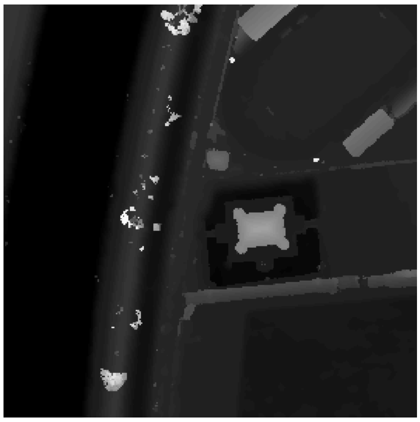

Figure 4.14: Segmented Digital Surface Model: u^t.

The procedure consisted in 18 outer iterations, with 80 inner iterations in the solution of the system g_1, 25 in the system g_2 and 80 in the system g_3.
In order to simplify the analysis of the behaviour of the algorithm we changed the colormap of the images.
Blue values are very close to 0, the deep red is 1.

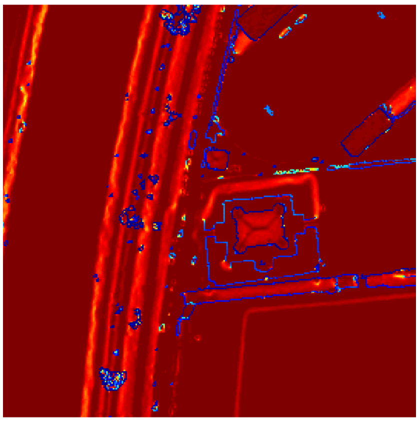

Figure 4.15: Edge detecting function: s^t.

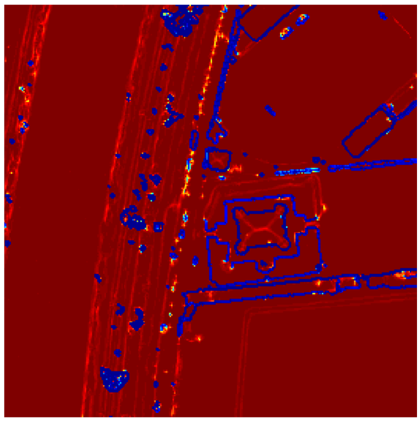

Figure 4.16: Edge-crease detecting functionl: z^t.

As we can see in the results of function z^t, convergence in crease points is very slow. This motived us to implement such a high number of inner iterations in the system g_3.
In Figure 4.26 we can see a particular of the roof in center of the Figure 4.16. The extracted figure is a 50×50 submatrix, in the Figure 4.27 are plotted the

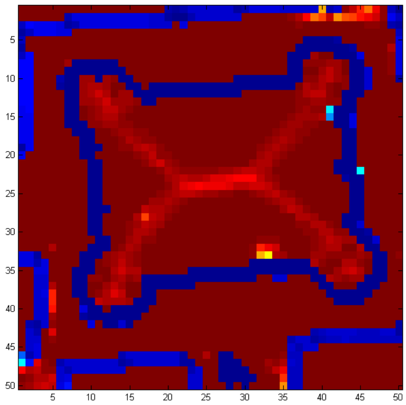

Figure 4.17: Particular of the roof.

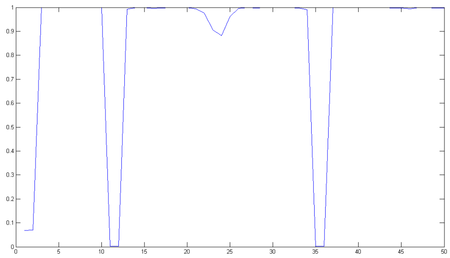

Figure 4.18: Column 25 values.

values of the function for all rows of the column 30, from the top to the bottom. We can see that the accuracy in the crease detection is very difficult.

We show here some 3D representations of the experiment's data. In figure 4.19 we can see the original DSM g^t, in the Figures 4.20 and 4.21 are represented the functions z^t and $1 - z^t$ respectively.

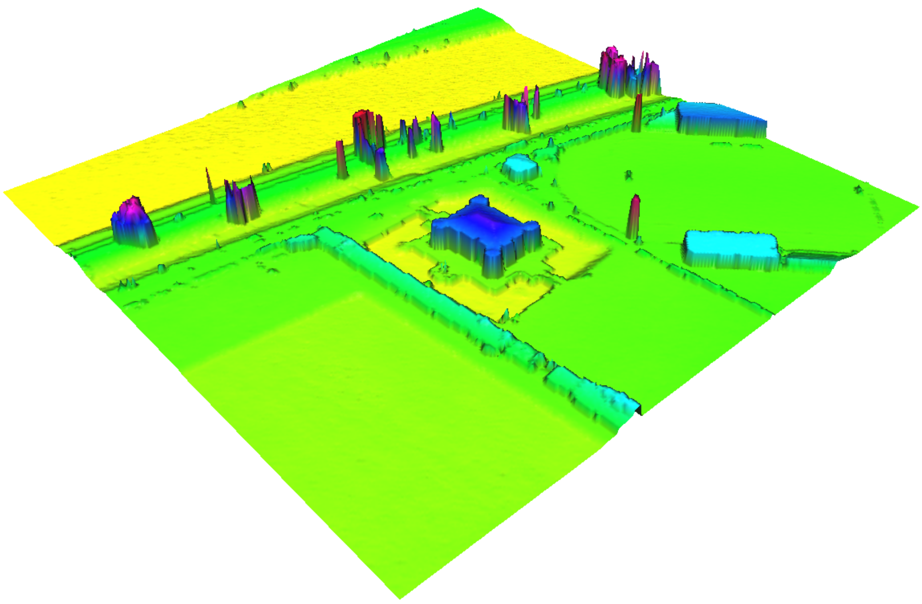

Figure 4.19: 3D representation of the original function g^t.

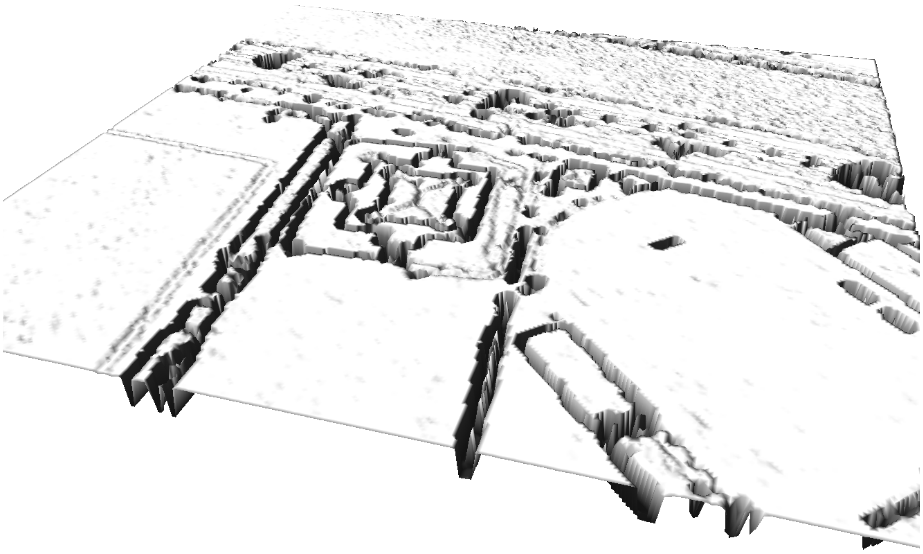

Figure 4.20: 3D representation of the function z^t.

Figure 4.21: 3D representation of the function $1 - z^t$.

The second one, bigger than the first, is a regular grid of $N = 600$. The following parameters are used: $\alpha = 2$, $\beta = 1$, $\mu = 0.1$ and $\epsilon = 0.01$. The procedure consisted in 9 outer iterations, with 10 inner iterations in the solution of the system g_1, 25 in the system g_2 and 50 in the system g_3.

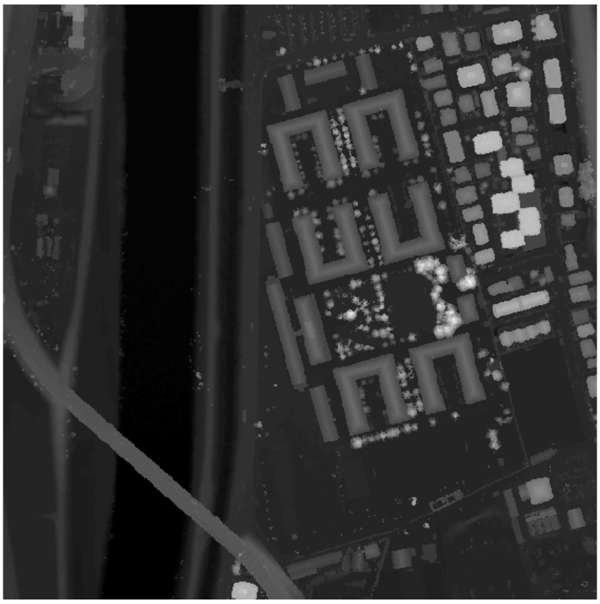

Figure 4.22: Original Digital Surface Model: g^t.

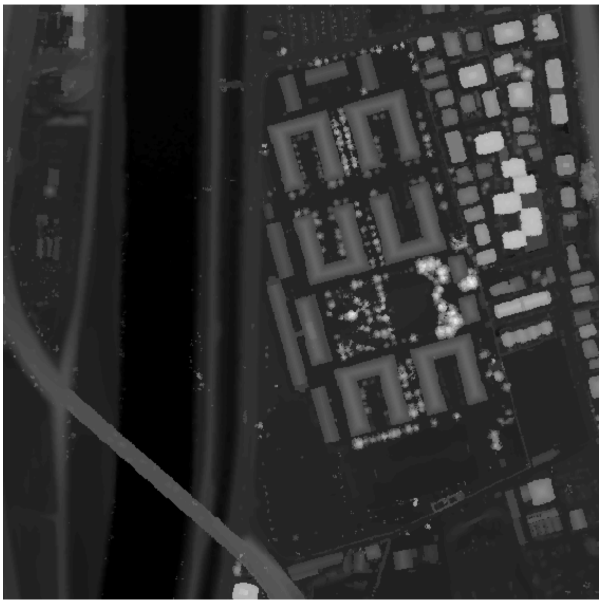

Figure 4.23: Segmented Digital Surface Model: u^t.

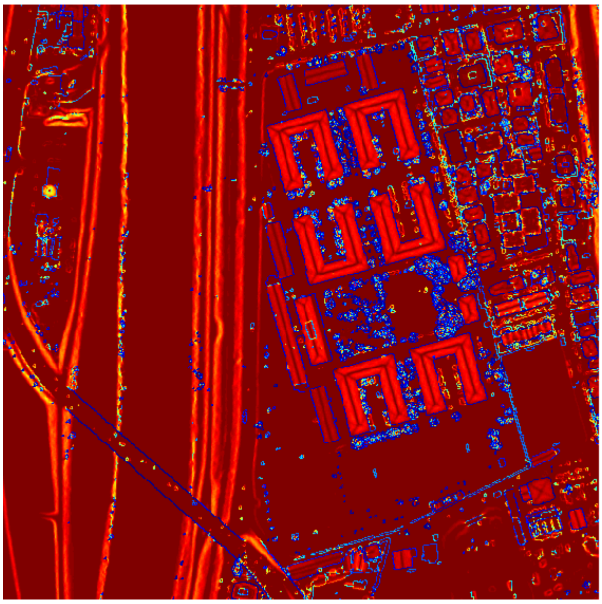

Figure 4.24: Edge detecting function: s^t.

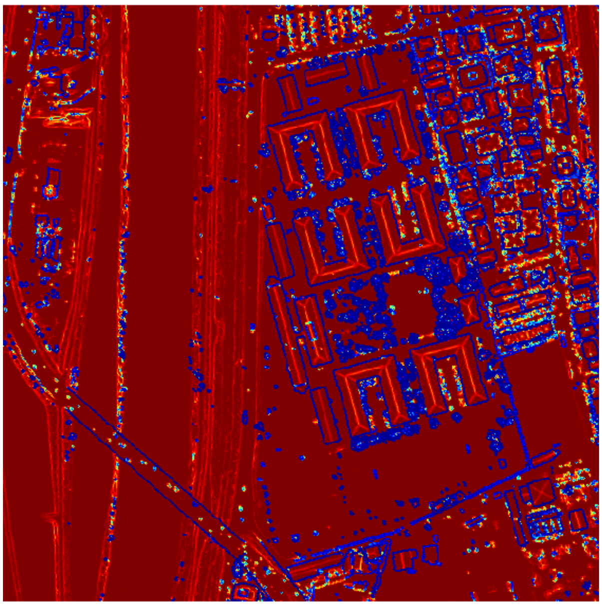

Figure 4.25: Edge-crease detecting functionl: z^t.

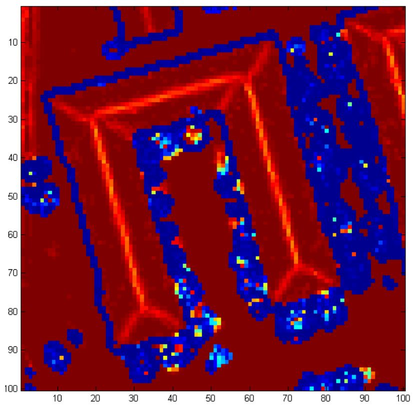

Figure 4.26: Particular of the roof.

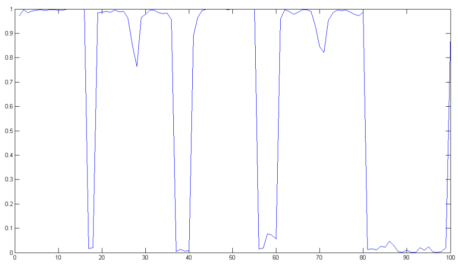

Figure 4.27: Column 25 values.

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