Monomial Togliatti Systems of Cubics

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CURVES OF MAXIMUM GENUS IN RANGE A AND STICK-FIGURES

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ABSTRACT. In this paper we show the existence of smooth connected space curves not contained in a surface of degree less than m, with genus maximal with respect to the degree, in half of the so-called range A. The main tool is a technique of deformation of stick-figures due to G. Fløystad.

0. INTRODUCTION

A classical problem, which goes back to Halphen [H], is to determine, for given integers d and m, the maximal genus $G(d,m)$ of a smooth projective curve of degree d not contained in a surface of degree $< m$. This problem is actually very natural, and it is in fact one of the cornerstones of the numerical classification of space curves.

The problem of determining $G(d,m)$ depends on the size of d with respect to m. Following a long tradition, we distinguish four ranges:

Range $\varnothing$. If $d < \frac{m^2+4m+6}{6}$, then every curve X of degree d satisfies $h^0(\mathbf{P}^3, \mathcal{I}_X(m-1)) \neq 0$.

Range A. If m^2+4m+6 $\leq d <$ m^2+4m+6, then $G(d,m) \leq G_A(d,m)$, where


CONGRATS

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Joint work with E. Mezzetti, G. Ottaviani
Joint work with M. Michałek

- E. Mezzetti, R.M. Miró-Roig and G. Ottaviani. *Laplace equations and the Weak Lefschetz Property*, Canadian Journal of Mathematics, **65** (2013), 634-654.
- M. Michałek and R.M. Miró-Roig. *Smooth monomial Togliatti systems of cubics*, arXiv:1310.2529.

GOAL: To establish a close relationship between a priori two unrelated problems:

(1) The existence of homogeneous artinian ideals $I \subset k[x_0, x_1, \cdots, x_n]$ which fail the Weak Lefschetz Property;

(2) The existence of (smooth) projective varieties $X \subset \mathbb{P}^n$ satisfying at least one Laplace equation of order $s \geq 2$.

- An n-dimensional projective variety $X \subset \mathbb{P}^N$ satisfies δ independent Laplace equations of order s if its s-osculating space at a general point $p \in X$ has dimension $\binom{n+s}{s} - 1 - \delta$.
- A homogeneous artinian ideal $I \subset k[x_0, x_1, \cdots, x_n]$ has the Weak Lefschetz Property (WLP) if $\exists L \in k[x_0, x_1, \cdots, x_n]$ such that, for all integers j, the multiplication map

$$\times L : (k[x_0, x_1, \cdots, x_n]/I)_j \to (k[x_0, x_1, \cdots, x_n]/I)_{j+1}$$

has maximal rank, i.e. it is injective or surjective.

BRENNER-KAID's EXAMPLE vs TOGLIATTI's EXAMPLE

- Brenner-Kaid (2009): $(x^3, y^3, z^3, f(x,y,z)) \subset k[x,y,z]$ with $deg(f) = 3$ fails WLP iff $f \in (x^3, y^3, z^3, xyz)$. In addition, (x^3, y^3, z^3, xyz) is the only monomial artinian ideal generated by 4 cubics that fails WLP.

- Togliatti (1929): The only non-trivial smooth surface $X \subset \mathbb{P}^5$ obtained by projecting the Veronese surface $V(2,3) \subset \mathbb{P}^9$ and satisfying a Laplace equation of order 2 is the image of $\mathbb{P}^2$ via the linear system

$$\langle x^2y, xy^2, x^2z, xz^2, y^2z, yz^2 \rangle \subset |\mathcal{O}_{\mathbb{P}^2}(3)|.$$

Remark:

The linear system of cubics given by Brenner-Kaid's example $\langle x^3, y^3, z^3, xyz\rangle$ is apolar to the linear system of cubics $\langle x^2y, xy^2, x^2z, xz^2, y^2z, yz^2\rangle$ given in Togliatti's example.

Question:

Is there a relationship between artinian ideals $I \subset k[x_0, \cdots, x_n]$ generated by r forms of degree d that fails WLP and projections of the Veronese variety $V(n, d)$ satisfying at least a Laplace equation of order $d - 1$?

$X \subset \mathbb{P}^N$ projective variety of dim n, $p \in X$ a smooth point. Choose affine coordinates around p and a local parametrization of X of the form $\phi(t_1, ..., t_n)$ where $p = \phi(0, ..., 0)$.
The tangent space to X at p is the k-vector space generated by the n partial derivatives of ϕ at p.

$$\dim T_p X = n.$$

We define the s-th osculating space $T_p^{(s)} X$ to be the span of all partial derivatives of ϕ of order $\leq s$.

$$\dim T_p^{(s)} X \leq \binom{n+s}{s} - 1.$$

We say that X satisfies $\delta > 0$ Laplace equations of order s if strict inequality holds for all smooth points p of X, and $\dim T_p^{(s)} X = \binom{n+s}{s} - 1 - \delta$ for general p.

We will also consider the projective sth osculating space $\mathbb{T}_p^{(s)} X$, embedded in $\mathbb{P}^N$.

Remarks.

(1) A non-degenerate curve $X \subset \mathbb{P}^N$ does not satisfy any Laplace equation.

(2) If $N < \binom{n+s}{s} - 1$, then X satisfies at least one Laplace equation of order s.

(3) If $X \subset \mathbb{P}^N$ is a rational variety of dimension n, $\exists \mathbb{P}^n \dashrightarrow X$ a birational map given by $F_0, \cdots, F_N \in k[x_0, \cdots, x_n]_d$ and for a general point $p \in X$, the projective s-th osculating space $\mathbb{T}_p^{(s)} X$ of X at p is generated by the s-th partial derivates of $F_0, F_1, \cdots, F_N$ at p.

Problems:

(1) To classify all rational surfaces $X \subset \mathbb{P}^N$, $N \geq 5$, which satisfy at least a Laplace equation of order 2.

(2) To classify all rational surfaces $X \subset \mathbb{P}^N$, $N \geq \binom{2+s}{s} - 1$, which satisfy at least a Laplace equation of order s.

(3) To classify all n-dimensional rational varieties $X \subset \mathbb{P}^N$, $N \geq \binom{n+s}{s} - 1$, which satisfy a Laplace equation of order s.

Macaulay-Matlis duality

• $V = k^{n+1}$, $R = \oplus_{i\geq 0} Sym^i V^* \cong k[x_0, x_1, \cdots, x_n]$, $\mathcal{D} = \oplus_{i\geq 0} Sym^i V \cong k[y_0, y_1, \cdots, y_n]$.

• There are products

$$\begin{array}{ccc} Sym^j V^* \otimes Sym^i V & \longrightarrow & Sym^{i-j} V \\ F \otimes D & \mapsto & F \cdot D \end{array}$$

making $\mathcal{D}$ into a graded R-module. If $F(x_0, x_1, \cdots, x_n) \in R$ and $D(y_0, y_1, \cdots, y_n) \in \mathcal{D}$, then

$$F \cdot D = F(\partial/\partial y_0, \partial/\partial y_1, \cdots, \partial/\partial y_n)D.$$

• If $I \subset R$ is a homogeneous ideal, we define the Macaulay's inverse system I^{-1} for I as

$$I^{-1} := \{D \in \mathcal{D}, F \cdot D = 0 \text{ for all } F \in I\} \subset \mathcal{D}.$$

• If $M \subset \mathcal{D}$ is a graded R-submodule, then

$$Ann(M) := \{F \in R, F \cdot D = 0 \text{ for all } D \in M\} \subset R.$$

• The pairing $R_i \times \mathcal{D}_i \longrightarrow k \qquad (F, D) \mapsto F \cdot D$ is exact; it is called the apolarity or Macaulay-Matlis duality action of R on $\mathcal{D}$. If $F \cdot D = 0$ and $deg(F) = deg(D)$, then F and D are said to be apolar to each other.

• For any integer i, we have $h_{R/I}(i) = \dim_k(R/I)_i = \dim_k(I^{-1})_i$.

Theorem

We have a bijective correspondence

$$\begin{array}{ccc} \{Homogeneous\ ideals\ I \subset R\} & \rightleftharpoons & \{Graded\ R-submodules\ of\ \mathcal{D}\} \\ I & \to & I^{-1} \\ Ann(M) & \leftarrow & M \end{array}.$$

Moreover, I^{-1} is a finitely generated R-module if and only if R/I is an artinian ring.

When considering only monomial ideals, we can simplify by regarding the inverse system in the same polynomial ring R, and in any degree, d, the inverse system I_d^{-1} is spanned by the monomials in R_d not in I_d.

Example

If $I = (x^4, y^4, z^4, x^3y, x^3z, xy^3, xz^3, y^3z, yz^3) \subset k[x, y, z]$, then $I^{-1} = (x^2y^2, x^2yz, x^2z^2, xy^2z, xyz^2, y^2z^2)$.

• Let I be an artinian ideal generated by r forms of degree d: $F_1, \cdots, F_r \in R = k[x_0, x_1, \cdots, x_n]$. Let $I^{-1} \subset \mathcal{D}$ be its Macaulay inverse system. Associated to $(I^{-1})_d$ there is a rational map

$$\varphi_{(I^{-1})_d} : \mathbb{P}^n \dashrightarrow \mathbb{P}^{\binom{n+d}{d}-r-1}.$$

$\overline{Im(\varphi_{(I^{-1})_d})} \subset \mathbb{P}^{\binom{n+d}{d}-r-1}$ is the projection of $V(n,d)$ from the linear system $\langle F_1, \cdots, F_r \rangle \subset |\mathcal{O}_{\mathbb{P}^n}(d)|$. Let us call it $X_{n,(I^{-1})_d}$.

• Associated to I_d there is a morphism

$$\varphi_{I_d} : \mathbb{P}^n \longrightarrow \mathbb{P}^{r-1}.$$

φ_{I_d} is regular because I is artinian. Its image $Im(\varphi_{I_d}) \subset \mathbb{P}^{r-1}$ is the projection of the n-dimensional Veronese variety $V(n,d)$ from the linear system $\langle (I^{-1})_d \rangle \subset |\mathcal{O}_{\mathbb{P}^n}(d)|$. Let us call it X_{n,I_d}.

• The varieties X_{n,I_d} and $X_{n,(I^{-1})_d}$ are called apolar.

Thm (M-MR-O) Let $I \subset R$ be an artinian ideal generated by r forms $F_1, \ldots, F_r$ of degree d, $r \leq \binom{n+d-1}{n-1}$. Then TFAE:

(1) The ideal I fails the WLP in degree $d-1$,

(2) The homogeneous forms $F_1, \ldots, F_r$ become k-linearly dependent on a general hyperplane H of $\mathbb{P}^n$,

(3) The n-dimensional variety $X_{n,(I^{-1})_d}$ satisfies at least one Laplace equation of order $d-1$.

Remarks:

- The assumption $r \leq \binom{n+d-1}{n-1}$ ensures that the Laplace equations obtained in (3) are not trivial. In the particular case $n = 2$, this assumption gives $r \leq d+1$.
- For $n = 2$, $d = 3$ and $I = (x_0^3, x_1^3, x_2^3, x_0x_1x_2) \subset k[x_0, x_1, x_2]$, we recover Togliatti's example.

Definition: With notation as above, we will say that I^{-1} (or I) defines a Togliatti system if it satisfies the three equivalent conditions in the above Theorem.

EXAMPLE.

Let $d = 2q + 1$ be an odd number and $n = 2$. Let $l_1, \dots, l_d$ be general linear forms in $k[x, y, z]$. The ideal $(l_1^d, \dots, l_d^d, l_1 l_2 \cdots l_d)$ is generated by $d + 1$ forms of degree d and it fails the WLP in degree $d - 1$ because $l_1^d, \dots, l_d^d, l_1 l_2 \cdots l_d$ become dependent on a general line $L \subset \mathbb{P}^2$.

- For $d = 3$ we recover Togliatti example.
- A similar construction in even degree produces ideals which do satisfy the WLP.

TORIC CASE

Proposition

Let $I \subset R := k[x_0, x_1, \cdots, x_n]$ be an artinian monomial ideal. Then R/I has the WLP if and only if $x_0 + x_1 + \cdots + x_n$ is a Lefschetz element for R/I.

$$\mathcal{L}_{n,d} := |\mathcal{O}_{\mathbb{P}^n}(d)| \text{ and } n_d := \dim(\mathcal{L}_{n,d}) = \binom{n+d}{n} - 1.$$

Definition

- A linear subspace $\mathcal{L} \subset \mathcal{L}_{n,d}$ is called a monomial linear system if it can be generated by monomials.
- $\mathcal{L} \subset \mathcal{L}_{n,d}$ is called a monomial Togliatti's system if, in addition, its apolar system $\mathcal{L}'$ generates an artinian ideal which fails WLP in degree $d-1$, or equivalently, $X = \overline{Im(\varphi_{\mathcal{L}})}$ ($\varphi_{\mathcal{L}} : \mathbb{P}^n \dashrightarrow \mathbb{P}^{\dim \mathcal{L}}$ the rational map associated to $\mathcal{L}$) satisfies a Laplace equation of order $d-1$.

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- A monomial Togliatti's system $\mathcal{L} \subset \mathcal{L}_{n,d}$ is said to be smooth if, in addition, X is a smooth variety.
- A monomial Togliatti's system $\mathcal{L} \subset \mathcal{L}_{n,d}$ is said to be minimal if its apolar system $\mathcal{L}'$ is generated by monomials $m_1, \cdots, m_r$ and there is no a proper subset $m_{i_1}, \cdots, m_{i_{r-1}}$ defining a monomial Togliatti system

- Togliatti: For $n = 2$, the only smooth minimal monomial Togliatti system of cubics is $I = (x_0^3, x_1^3, x_2^3, x_0x_1x_2)$.

Theorem (Michałek - MR)

Let $I \subset R := k[x_0, x_1, \cdots, x_n]$ be a minimal smooth monomial Togliatti system of quadrics and $n \geq 3$. Then, there is a bipartition of $n + 1$: $n + 1 = a_1 + a_2$ with $n - 1 \geq a_1 \geq a_2 \geq 2$, such that, up to permutation of the coordinates

$$I = (x_0, \cdots, x_{a_1 - 1})^2 + (x_{a_1}, \cdots, x_n)^2.$$

GOAL:

To classify ALL smooth minimal monomial Togliatti systems of cubics $I \subset R := k[x_0, x_1, \cdots, x_n]$, $n \geq 2$.

ILARDI's CONJECTURE

Conjecture

The only smooth minimal monomial Togliatti system $\mathcal{L} \subset \mathcal{L}_{n,3}$ of cubics of dimension $n(n+1)-1$ is

$$\mathcal{L} = |\{x_i^2 x_j\}_{0 \leq i \neq j \leq n}| \subset \mathcal{L}_{n,3}.$$

(1) $\mathcal{L} = |\{x_i^2 x_j\}_{0 \leq i \neq j \leq n}| \subset \mathcal{L}_{n,3}$ is a smooth minimal monomial Togliatti system of cubics of dimension $n(n+1)-1$.

(2) $\mathcal{M} := |\{x_0^2 x_2, x_0 x_2^2, x_0^2 x_3, x_0 x_3^2, x_1^2 x_2, x_1 x_2^2, x_1^2 x_3, x_1 x_3^2, x_2^2 x_3, x_2 x_3^2, x_0 x_1 x_2, x_0 x_1 x_3\}| \subset \mathcal{L}_{3,3}$ is a smooth minimal monomial Togliatti system of cubics of dimension 11 (n=3).

Problem:

To classify all smooth minimal monomial Togliatti systems of cubics $I \subset R := k[x_0, x_1, \cdots, x_n]$, $n \geq 2$.

- $n = 2$. The only smooth minimal monomial Togliatti system of cubics is $I = (a^3, b^3, c^3, abc) \subset k[a, b, c]$.
- $n = 3$.

Theorem

Let $I \subset k[a, b, c, d]$ be a smooth minimal monomial Togliatti system of cubics. Then, up to a permutation of the coordinates, I^{-1} is:

(1) $(a^2b, a^2c, a^2d, ab^2, ac^2, ad^2, b^2c, b^2d, bc^2, bd^2, c^2d, cd^2)$, *$X$ is of degree 23, in $\mathbb{P}^{11}$, it is isomorphic to $\mathbb{P}^3$ blown up in the 4 coordinate points; or*

(2) $(abc, abd, a^2c, a^2d, ac^2, ad^2, b^2c, b^2d, bc^2, bd^2, c^2d, cd^2)$, *$X$ is of degree 18, in $\mathbb{P}^{11}$, it is isomorphic to $\mathbb{P}^3$ blown up in the line $\{c = d = 0\}$ and in the two points $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$; or*

(3) $(abc, abd, acd, bcd, a^2c, ac^2, a^2d, ad^2, b^2c, bc^2, b^2d, bd^2)$, *$X$ is of degree 13, in $\mathbb{P}^{11}$, it is isomorphic to $\mathbb{P}^3$ blown up in the two lines $\{a = b = 0\}$ and $\{c = d = 0\}$.*

PROPOSITION (Mezzetti - MR - Ottaviani):

Consider a partition of $n+1$: $n+1 = a_1 + a_2 + \cdots + a_s$ with $n-1 \geq a_1 \geq a_2 \geq \cdots \geq a_s \geq 1$ and the monomial ideal

$$I = (x_0, \cdots, x_{a_1-1})^3 + \cdots + (x_{n+1-a_s}, \cdots, x_n)^3 + J \text{ where}$$

$$J := (x_i x_j x_k \mid 0 \leq i < j < k \leq n \text{ and}$$

$$\forall 1 \leq \lambda \leq s \quad \#(\{i,j,k\} \cap \{\sum_{\alpha \leq \lambda - 1} a_\alpha, \cdots, \sum_{\alpha \leq \lambda} a_\alpha - 1\}) \leq 1).$$

I is a smooth minimal monomial Togliatti system of cubics.

REMARK: $\mu(I) = \binom{a_1+2}{3} + \cdots + \binom{a_s+2}{3} + \sum_{1 \leq i < j < h \leq s} a_i a_j a_h$.

In particular, if $a_1 = a_2 = \cdots = a_{n+1} = 1$ or $a_1 = n-1$ and $a_2 = a_3 = 1$, they have dimension $n(n+1) - 1$ and we have a family of counterexamples to Ilardi's conjecture.

CONJECTURE (Mezzetti - MR - Ottaviani):

Up to permutation of the coordinates, the above ideals are the only smooth minimal monomial Togliatti system of cubics.

THEOREM (Michałek - MR):

Let I (or its inverse system I^{-1}) be a minimal smooth monomial Togliatti system of cubics. Then, up to a permutation of the coordinates, the pair (I, I^{-1}) is one of the ideals described in Proposition A. Moreover, $|I| \leq \binom{n+1}{3} + n + 1$ and if $|I| = \binom{n+1}{3} + n + 1$ then it corresponds to one of the following partitions:

- $n + 1 = (n - 1) + 1 + 1$,
- $n + 1 = 1 + 1 + \cdots + 1$,
- $4 = 2 + 2$.

OPEN PROBLEM:

To classify all smooth minimal monomial Togliatti systems $I \subset R := k[x_0, x_1, \cdots, x_n]$, $n \geq 2$, of forms of degree $d \geq 4$.

THEOREM (Mezzetti - MR):

Let $I \subset R := k[x_0, x_1, \cdots, x_n]$, $n \geq 2$, be a smooth minimal monomial Togliatti system of forms of degree $d \geq 4$. It holds

$$2n + 1 \leq \mu(I) \leq \binom{n+d-1}{n-1}.$$

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THANK YOU!