Projective geometry of some special Fano manifolds

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Table of contents

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Note that the above is slightly different from the usual definition of index. A prime Fano manifold has *high index* if $i \geq \frac{n+1}{2}$. Our goal is to try to understand prime Fanos of high index.

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Remark

Assume that the HCF holds. If $X \subset \mathbb{P}^N$ is a prime Fano manifold of index $i \geq \frac{n+3}{2}$, then X is a complete intersection if and only if $n \geq 2c+1$.

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Let G be a simple algebraic group acting on the linear space V by an irreducible representation of highest weight. $X \subset \mathbb{P}(V^*)$ is *rational-homogeneous* (for G) if it is (the projectivization of) a closed orbit for a maximal vector of V.

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Two ... familiar examples, right ?

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Secant variety of X: $SX =$ closure of the locus of secants to $X \subset \mathbb{P}^N$, $\dim SX = 2n + 1 - \delta$, $\delta \geq 0$ is the *secant defect*.
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X is a *local quadratic entry locus variety* (LQEL) if $\delta > 0$ and for any $x, x' \in X$ general points there is a quadric $Q^{\delta}_{xx'}$, $x, x' \in Q^{\delta}_{xx'} \subseteq X$.

Clearly, LQEL manifolds are CC.

Theorem (Russo, —— - Russo)

A LQEL manifold is rational, Fano and has $b_2 \leq 2$. Those with $b_2 = 2$ are classified, while the others are prime Fanos with $i = \frac{n+\delta}{2}$, or the Veronese variety $v_2(\mathbb{P}^n)$. When $\delta \geq 3$, the variety of lines $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ is also LQEL, of secant defect $\delta - 2$.

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We believe that a prime Fano DD is LQEL. To prove this, it would be enough to show that $\delta \geq 2$ and $k \geq n - c - 1$. The last inequality would prove that DD manifolds also satisfy the HC.

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Step 1.
The inequality in the hypothesis ensures that X is covered by lines. Moreover, the manifold $Y =: \mathbb{P}(\mathcal{P}_X)$ is also Fano, the projection $\phi : Y \to TX$ being a Mori contraction. Let F be its general fiber, which is a Fano manifold of dimension at least 2.

The inequality in the hypothesis tells exactly that $i(F) > \frac{\dim(F)+1}{2}$, so F is covered by lines.

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As the point u was general in $T_x(X)$, this means exactly that $T\mathcal{L}_x = \mathbb{P}^{n-1}$. So we have a fortiori $S\mathcal{L}_x = \mathbb{P}^{n-1}$.

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This follows from the previous claim, since
$\dim(V_e) = \dim(C(x) \cap C(e))$.

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- Philippe & The Organizers (for the joyful opportunity)!