On Cremona geometry of plane curves

(joint work with Alberto Calabri)

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Cremona transformations

I work over the field $\mathbb{C}$ of complex numbers.

Definition

A Cremona transformation (CT) of $\mathbb{P}^r$ is a birational map

$$\omega : \mathbb{P}^r \dashrightarrow \mathbb{P}^r$$

i.e., an automorphism on a dense Zariski open subset of $\mathbb{P}^r$.

Equivalently CT are $\mathbb{C}$–isomorphisms of $\mathbb{C}(x_1, \ldots, x_r)$.

CTs of $\mathbb{P}^r$ form the Cremona group $\mathrm{Cr}(r)$.

I will mainly consider the plane case $r = 2$.

The degree

In homogeneous coordinates $[x_0, \dots, x_r]$ of $\mathbb{P}^r$, one has

$$\omega : [\underline{x}] \dashrightarrow [f_0(\underline{x}), \dots, f_r(\underline{x})] \text{ where } \underline{x} = (x_0, \dots, x_r)$$

and the $f_i(\underline{x})$'s are coprime, linearly independent, homogeneous polynomials of the same degree d, called the degree $\deg(\omega)$ of ω.

For $r = 2$ a CT and its inverse have the same degree.

The base components free linear system of hypersurfaces with equations

$$\lambda_0 f_0(\underline{x}) + \ldots + \lambda_r f_r(\underline{x}) = 0$$

is usually called a homaloidal system.

Linear maps

CTs of degree 1 are linear maps, i.e., the automorphisms of $\mathbb{P}^r$, and fill up the linear projective group $\mathrm{PGL}(r+1, \mathbb{C})$.

One has $\mathrm{Cr}(1) = \mathrm{PGL}(2, \mathbb{C})$.

In general

For $r > 1$, in $\mathrm{Cr}(r)$ there are CT transformations of all degrees. The study of families of CT of given degree is very interesting (see recent work for the case $r = 2$ by Bisi–Calabri–Mella and Blanc–Calabri).

The Noether–Castelnuovo's Theorem

The standard quadratic transformation of $\mathbb{P}^2$

It is defined (up to linear transformation) as

$$\tau : [x_0, x_1, x_2] \dashrightarrow [x_1x_2, x_0x_2, x_0x_1].$$

Theorem (Noether 1872–Castelnuovo 1901)

$\mathrm{Cr}(2)$ is generated by $\mathrm{PGL}(3, \mathbb{C})$ and by τ.

By contrast:

Theorem (Dantoni, 1949)

If $r > 2$, for every positive integer d, CT of degree at most d generate a proper soubgroup of $\mathrm{Cr}(r)$.

Cremona geometry

One may study properties of objects in $\mathbb{P}^r$ which are Cremona invariant (CI), i.e., invariant under the action of $\mathrm{Cr}(r)$.

E.g., for $r = 2$, we may study properties of (linear systems of) plane curves, which are CI: the dimension of a linear system is CI.

The degree of a linear system $\mathcal{L}$, i.e., the degree of its curves, is not CI

Set
$$\mathcal{L} = \mathcal{L}_d(m_1, \ldots, m_h), \text{ with } m_1 \geqslant \ldots \geqslant m_h \geqslant 1$$
to say that $\mathcal{L}$ has degree d and (proper or infinitely near) base points $p_1, \ldots, p_h$ with multipliticities at least $m_1, \ldots, m_h$.

If $h \geqslant 3$ and p_1, p_2, p_3 are distinct, we may assume that p_1, p_2, p_3 coincide with $[1, 0, 0], [0, 1, 0], [0, 0, 1]$ respectively. By acting with the standard quadratic transformation τ, $\mathcal{L}$ becomes $\mathcal{L}_{2d-m_1-m_2-m_3}(d-m_2-m_3, d-m_1-m_3, d-m_1-m_2, m_4, \ldots, m_h)$.

The General Problem of plane Cremona classification

Classify (linear systems of) plane curves up to CTs.

Cremona degree

Definition

The Cremona degree of a linear system $\mathcal{L}$ of plane curves is the minimal degree of a linear system in the Cremona orbit of $\mathcal{L}$, i.e., in the orbit of $\mathcal{L}$ via the $\mathrm{Cr}(2)$ action. Such minimal degree systems are called Cremona minimal models of $\mathcal{L}$.

A classical example: pencils of rational plane curves

Let Λ be a pencil whose general element is an irreducible rational curve. Then there exists a CT which maps Λ to the pencil of lines through a fixed point. Thus pencils of rational plane curves form a unique Cremona orbit and have Cremona degree 1.

Theorem (Jung, 1888)

Let $\mathcal{L} = \mathcal{L}_d(m_1, \ldots, m_h)$. If $d \geqslant m_1 + \cdots + m_\mu$ with $\mu = \min\{3, h\}$, then $\mathcal{L}$ is Cremona minimal.

Cremona contractible curves

The problem of determining the Cremona degree of irreducible plane curves and of classifying Cremona minimal models has been open for more than one century, with contributions by vv.aa., among them I like to mention Marletta (1911) and Iitaka (1980–90's).

A solution has been given by Mella–Polastri and Calabri–C independently in 2010. The approach of the latter authors is more explicit and algorithmic in essence.

The first step in this circle of ideas is the characterization of irreducible plane curves which are Cremona equivalent to a line, or equivalently to a point, i.e., plane curves of Cremona degree 0.

Remark

The standard quadratic transformation τ contracts the fundamental line $\lambda_i = \{x_i = 0\}$ to the fundamental point ξ_i, for $i = 0, 1, 2$.

Definition

A (reduced but not necessarily irreducible) plane curve C is Cremona contractible, shortly Cr-contractible, if C has Cremona degree 0.

Adjoint systems

A basic tool for studying a reduced plane curve C is its sequence of adjoint linear systems $\mathrm{ad}_m(C)$ ($m \geqslant 1$ is the index of $\mathrm{ad}_m(C)$).

Definition

Let $C = \mathcal{L}_d(m_1, \ldots, m_h)$, where $m_h \geqslant 2$ and all (proper or infinitely near) singular points of C have been listed. Then

$$\mathrm{ad}_1(C) := \mathrm{ad}(C) := \mathcal{L}_{d-3}(m_1 - 1, \ldots, m_h - 1)$$

is the (first) adjoint system of $\mathcal{L}$. If $m > 1$, one inductively sets

$$\mathrm{ad}_m(C) := \mathrm{ad}(\mathrm{ad}_{m-1}(C)).$$

Alternatively, let $f \colon S \to \mathbb{P}^2$ be a birational morphism such that the strict transform $\tilde{C}$ of C on S is smooth. Then

$$\mathrm{ad}_m(C) := f_*(|\tilde{C} + mK_S|), \quad m \geqslant 1,$$

where K_S is a canonical divisor on S.

Note: taking this viewpoint, adjoint systems make sense for linear systems and even for non–reduced curves.

Two basic facts

Adjuntion extinguishes

Namely, $\mathrm{ad}_m(C)$ is empty for $m \gg 0$. This is the case if $m > \frac{d}{3}$, but it may happen even for lower values of m.

Cremona invariance of dimension of adjoint systems

The dimension $\dim(\mathrm{ad}_m(C))$ is invariant under the action of $\mathrm{Cr}(2)$.

Remark

In particular, if C is Cr–contractible, then $\mathrm{ad}_m(C) = \emptyset$ for all $m \geqslant 1$.

Cr–contractibility for irreducible curves

Theorem (Ferretti, 1902)

An irreducible plane curve C is Cr–contractible if and only if $\mathrm{ad}_m(C) = \emptyset$ for all $m \geqslant 1$.

Theorem (Kumar–Murthy, 1982)

An irreducible plane curve C is Cr-contractible if and only if

$$(*) \qquad \mathrm{ad}_1(C) = \mathrm{ad}_2(C) = \emptyset.$$

Consider $(S, \tilde{C})$ as above. Condition $(*)$ is equivalent to

$$P_2(S, \tilde{C}) := h^0(S, \mathcal{O}_S(2\tilde{C} + 2K_S)) = 0$$

where $P_2(S, \tilde{C})$ is the second log plurigenus of the pair $(S, \tilde{C})$. Thus:

The Kumar and Murthy Theorem can be considered as a log analogue of Castelnuovo's rationality.

Kodaira dimension of pairs

Let $(S, \tilde{C})$ be a pair, i.e. $\tilde{C}$ is a smooth curve on a smooth irreducible projective surface S.

Definition

The log m-plurigenus of the pair $(S, \tilde{C})$ is

$$P_m(S, \tilde{C}) = h^0(S, \mathcal{O}_S(m\tilde{C} + mK_S)).$$

The pair $(S, \tilde{C})$ has log Kodaira dimension $\mathrm{kod}(S, \tilde{C}) = -\infty$ if

$$P_m(S, \tilde{C}) = 0, \qquad \text{for all } m \geqslant 1.$$

Otherwise, if $\varphi_{|m\tilde{C}+mK_S|}$ is the rational map determined by the linear system $|m\tilde{C} + mK_S|$, whenever it is not empty,

$$\mathrm{kod}(S, \tilde{C}) = \max \left\{ \dim(\mathrm{im}\left(\varphi_{|m\tilde{C}+mK_S|}\right)) \right\}.$$

Since $\tilde{C}$ is effective, $P_m(S, \tilde{C}) = 0$ implies that $\mathrm{ad}_m(C) = \emptyset$.

Kodaira dimension of plane curves

Definition

If C is a plane curve, and if $(S,\tilde{C})$ is a resolution of the singularities of C, we define the log m-plurigenus of C as

$$P_m(C) := P_m(S,\tilde{C})$$

and the log Kodaira dimension of C as

$$\mathrm{kod}(C) := \mathrm{kod}(S,\tilde{C}).$$

The definition does not depend on the resolution $(S,\tilde{C})$.

Cremona invariance of log plurigenera

If C is a plane curve, the log plurigenera $P_m(C)$ and the log Kodaira dimension are invariant under the $\mathrm{Cr}(2)$–action.

Consequently, if C is Cr–contractible, then $\mathrm{kod}(C) = -\infty$.

Equivalences for irreducible plane curves

For an irreducible plane curve C the following conditions are equivalent:

1. C is Cr-contractible,
2. $\mathrm{kod}(C) = -\infty$,
3. $\mathrm{ad}_m(C) = \emptyset$ for all $m > 0$,
4. $\mathrm{ad}_m(C) = \emptyset$ for $m = 1, 2$.

The last condition may be replaced by the following

5. $P_2(C) = 0$.

The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) hold, as we saw, even for reduced (but not necessarily irreducible) plane curves, while (4) $\Rightarrow$ (1) is the Kumar–Murthy Theorem.

Next we deal with reducible, but still reduced, plane curves.

Plane curves with 2 irreducible components

Iitaka studied pairs $(S, \tilde{C})$ according to their log Kodaira dimension and their log plurigenera, with $\tilde{C}$ reduced, but not necessarily irreducible.

Theorem (Iitaka, 1982–1988)

Let C be a reduced plane curve with two irreducible components. Then C is Cr-contractible if and only if

$$\mathrm{ad}_1(C) = \mathrm{ad}_2(C) = \emptyset. \tag{4}$$

In particular, Iitaka proved that, if (4) holds, there exists a Cremona transformation mapping C to the union of two distict lines, that in turn can be contracted to a point via standard quadratic transformations.

The following question naturally arises:

Question

Is it possible to generalize the equivalence of conditions (1),...,(4) to reduced (not necessarily irreducible) curves?

Pompilj's example

Iitaka's Theorem cannot be extended to three components:

Example (Pompilj, 1945)

Let C_1, C_2 be rational plane quartics and C_3 be a line with

	deg	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}
C_1	4	2	2	2	1	1	1	1	1	1	1
C_2	4	1	1	1	2	2	2	1	1	1	1
C_3	1	0	0	0	0	0	0	1	1	1	1
C	9	3	3	3	3	3	3	3	3	3	3

E.g., let $C_1 : x^2y^2 + 2x^2z^2 + 3y^2z^2 + 6xyz(x + y + z) = 0$, $C_3 : x + y + z = 0$, C_2 be the symmetric to C_1 w.r.t. C_3.

Setting $C = C_1 + C_2 + C_3$, one has $\mathrm{ad}_1(C) = \mathrm{ad}_2(C) = \emptyset$, but C is not Cr-contractible because $\mathrm{ad}_3(C) \neq \emptyset$! Hence

$$(4)\quad \mathrm{ad}_m(C) = \emptyset \ \text{ for } \ m = 1, 2 \not\Rightarrow \ (3)\quad \mathrm{ad}_m(C) = \emptyset, \ \forall \ m > 0.$$

One more example

Recall that for an irreducible plane curve C the following two conditions are equivalent:

2. $\mathrm{kod}(C) = -\infty$,
3. $\mathrm{ad}_m(C) = \emptyset$ for all $m > 0$.

When C is reduced (not necessarily irreducible), (2) $\Rightarrow$ (3) still holds. But the next example shows that (3) $\not\Rightarrow$ (2) for reducible curves.

A union of $d \geqslant 9$ distinct lines with a $(d-3)$–tuple point

I make the case $d = 9$, the case $d > 9$ is similar.

Let $C = \ell_1 + \cdots + \ell_9 \in \mathcal{L}_9(6, 2^{21})$, where $\ell_1, \ldots, \ell_6$ are lines through a point P_0 and ℓ_7, ℓ_8, ℓ_9 are general. Let P_1, P_2, P_3 be the vertices of the triangle whose sides are ℓ_7, ℓ_8, ℓ_9. Then $\mathrm{ad}_m(C) = \emptyset$ for all $m > 0$, but

$$\mathrm{ad}_3(2C) = \{\ell_1 + \cdots + \ell_6 + \ell'_1 + \ell'_2 + \ell'_3\} \in \mathcal{L}_9(9, 1^{21}),$$

where ℓ'_i is the line through P_0 and P_i, $i = 1, 2, 3$. Hence $P_3(C) > 0$.

The problem of Cr–contractibility

Summing up, if C is reduced but not irreducible

$$(3) \quad \mathrm{ad}_m(C) = \emptyset, \text{ for all } m > 0$$

is not sufficient for Cr–contractibility.

Problem

Asks whether for a reduced plane curve C one has

$$(1)\ C \text{ is Cr-contractible} \Leftrightarrow (2)\ \mathrm{kod}(C) = -\infty$$

I will address this problem in the special case C is a reduced union of lines, which presents some aspects of general interest.

The idea is to first classify reduced unions C of lines with vanishing adjoint linear systems and then to study the Kodaira dimension.

Reduced unions of "many" lines

Theorem (Calabri–C)

Let C be the union of $d \geqslant 12$ distinct lines. Then:

1. $\mathrm{ad}_m(C) = \emptyset$ for all $m > 0$ if and only if C has a point of multiplicity $m \geqslant d-3$;
2. $\mathrm{kod}(C) = -\infty$ if and only if C has a point of multiplicity $m \geqslant d-2$;
3. (1) $\mathrm{kod}(C) = -\infty \Leftrightarrow$ (2) C is Cr–contractible.

Remarks

A posteriori, it follows that for C a union of $d \geqslant 12$ distinct lines with $\mathrm{ad}_m(C) = \emptyset$ for all $m > 0$, one has

$$P_3(C) = 0 \Leftrightarrow \mathrm{kod}(C) = -\infty,$$

and

$$(3)\ \ \mathrm{ad}_m(C) = \emptyset,\ \forall\ m > 0 \Leftrightarrow (4)\ \ \mathrm{ad}_m(C) = \emptyset,\ \text{ for } m = 1, 2.$$

Reduced unions of few lines

The case of a reduced union C of $d \leqslant 11$ lines is also interesting but the classification is more complicated, since it requires the analysis of several dozens of possible configurations.

Calabri and I performed this analysis for $d \leqslant 8$ and $d = 11$. The remaining cases are work in progress.

Remark

For all cases with $d \leqslant 11$ we met so far, one still has

$$(1)\ \ C \text{ is Cr–contractible} \Leftrightarrow (2)\ \ \mathrm{kod}(C) = -\infty.$$

Few lines are more complicated

Dual of the 9 flexes of a smooth cubic

Let $p_1, \ldots, p_9$ be the flexes of a smooth plane cubic. A line through two flexes passes through a third flex and there are 12 such lines.

The dual configuration C consists of 9 lines with 12 triple points (and no node). Then

$$\mathrm{ad}_1(C) = \mathrm{ad}_2(C) = \emptyset, \text{ but } \mathrm{ad}_3(C) \neq \emptyset,$$

in particular C is not Cr–contractible.

Hence, for C a union of $d \leqslant 11$ distinct lines, it is not always true that

$$(3) \quad \mathrm{ad}_m(C) = \emptyset, \ \forall \ m > 0 \Leftrightarrow (4) \quad \mathrm{ad}_m(C) = \emptyset, \ \text{ for } \ m = 1, 2.$$

Minimal and contractible pairs

Consider pairs (S, D) with S a smooth, irreducible, projective, rational surface and D an effective, non-zero, reduced divisor.

The pair (S, D) is said to be minimal if there is no (-1)-curve E on S such that $E \cdot D \leqslant 1$.

By contracting all (-1)-curves offending minimality, any non-minimal pair can be made minimal without changing D. In particular, in this process, the number of connected components of D stays the same.

A pair (S, D) is said to be contractible if (S, D) is birationally equivalent to $(\mathbb{P}^2, C)$, where C is Cr-contractible.

A contractibility criterion

Definition

A pair (S, D) is said to be connected if D is connected.

The following result extends Ferretti's theorem:

Theorem (Calabri–C)

Let (S, D) be a minimal connected pair, such that $\mathrm{ad}_m(C) = \emptyset$ for all $m > 0$. Then (S, D) is contractible.

Remark

The converse does not hold, i.e., there are pairs (S, D) such that $\mathrm{ad}_m(C) \neq \emptyset$, for some $m > 0$, and nonetheless (S, D) is contractible. This is the case of $S = \mathbb{P}^2$ and D any Cr-contractible curve of degree $d \geqslant 3$.

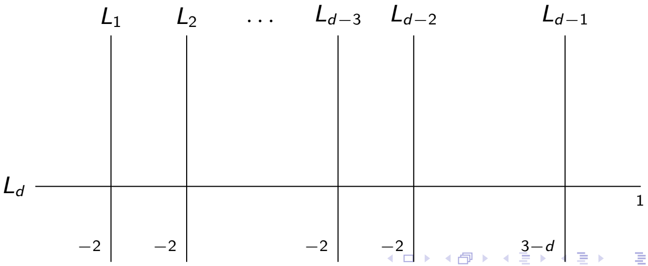
An application

Theorem

Let C be the union of $d \geqslant 4$ distinct lines with a point P_0 of multiplicity $d-2$ and $2d-3$ nodes. Then C is Cr–contractible.

Proof. Let $\ell_1, \dots, \ell_{d-2}$ be the lines passing through P_0 and ℓ_{d-1}, ℓ_d be the other two lines. Set $P_{i,j} = \ell_i \cap \ell_j$ for $i \neq j$.
Blow up P_0, $P_{1,d-1}$, $P_{2,d-1}$, ..., $P_{d-2,d-1}$.
Setting L_i the strict transform of ℓ_i, $i = 1, \dots, d$, on the blown–up surface S, then $D = L_1 \cup \cdots \cup L_d$ is connected and the hypothesis of the criterion are fulfilled. $\odot$

The following picture shows D with the self–intersection of its components:



Another application (I)

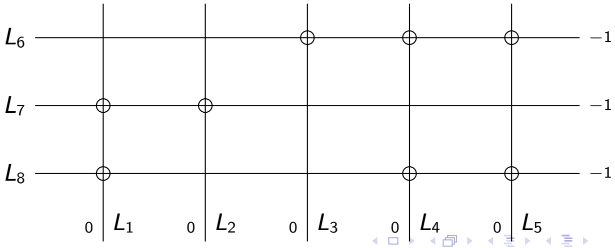
Recall that a union of $d \geqslant 9$ lines with a point of multiplicity exactly $d-3$, and $3(d-2)$ nodes, is not Cr-contractible.

By contrast:

Theorem

A union of $d \leqslant 8$ lines with a point P_0 of multiplicity $d-3$ and $3(d-2)$ nodes is Cr-contractible.

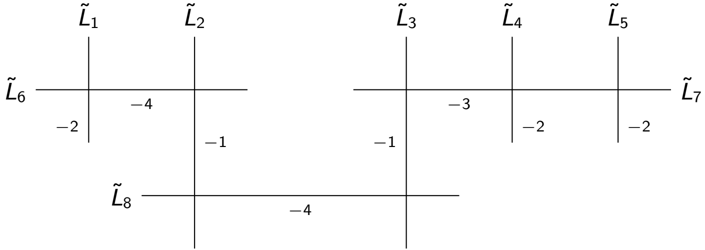
Proof. It suffices to make the case $d = 8$. Let C be the union of 8 distinct lines with a point P_0 of multiplicity 5 and 18 nodes. Let $\ell_1, \ldots, \ell_5$ be the lines passing through P_0. Blow up P_0, $P_{6,7} = \ell_6 \cap \ell_7$, $P_{6,8}$, $P_{7,8}$ and call L_i the strict transform of ℓ_i, for $i = 1, \ldots, 8$.



Another application (II)

Then blow up the encircled intersection points, i.e., $P_{17} = L_1 \cap L_7$, P_{18}, P_{27}, P_{36}, P_{46}, P_{48}, P_{56}, P_{58}.

Setting $\tilde{L}_i$ the strict transform of L_i, $i = 1, \ldots, 8$, on the new blown–up surface S, then $D = \tilde{L}_1 \cup \cdots \cup \tilde{L}_8$ is connected:



One verifies that $\mathrm{ad}_m(D) = \emptyset$ for all $m > 0$. Therefore, (S, D) is contractible by the above criterion.

The Coolidge–Nagata Conjecture

A reduced curve C is said to be cuspidal if it has only unibranched singularities.

Conjecture (Coolidge (1928)–Nagata (1960))

Every rational cuspidal plane curve is Cr–contractible.

The higher dimensional case

Problem (the divisorial case)

Figure out the Cr–classification of reduced hypersurfaces in $\mathbb{P}^r$, for $r > 2$. In particular, give conditions for X to be Cr–equivalent to a plane or Cr–contractible (the two concepts are no longer equivalent).

1. Mella–Polastri (2010) gave a criterion for an irreducible surface X in $\mathbb{P}^3$ to be Cr–equivalent to a plane. Unfortunately this is not effective: it requires to visit all good models of the pair $(\mathbb{P}^3, X)$ to check it.
2. Angelini–Mella (2015) recently proved that all irreducible ruled surfaces X in $\mathbb{P}^3$ are Cr–equivalent to a scroll.
3. Unlike in the planar case, I would not expect that

$$X \text{ Cr-contractible} \Leftrightarrow \ \mathrm{kod}(\mathbb{P}^3, X) = -\infty$$

   (the arrow $\Rightarrow$ holds). It would be very interesting to find an irreducible surface for which $\Leftarrow$ fails.
4. Mella (2014) proved that two irreducible cones in $\mathbb{P}^r$ are Cr–equivalent if their general hyperplane sections are birational. Besides this, very little is known in the divisorial case for $r > 3$.
5. The non–divisorial case is somehow trivial (cfr. Mella–Polastri (2009), Cueto–Mella–Ranestad–Zwiernik–C. (2014), Calabri–C (2014)).

Historical note (I)

- Instances of quadratic transformations had been studied by Poncelet (1822), Plücker (1830), Steiner (1832), Magnus (1832), who made the wrong assertion that the only birational transformation of $\mathbb{P}^2$ are linear and quadratic! This mistake was repeated by Schiapparelli (1861–2) and even by Cremona (1861), who corrected it one year later, becoming aware of the fact that the composition of two general standard quadratic transformation has degree 4, corresponding to the homaloidal net $\mathcal{L}_4(2^3, 1^3)$.
- Meanwhile De Jonquières (1859) had independently introduced and studied (though he published this only much later) the higher degree transformations, later named after him, corresponding to homaloidal nets of the form $\mathcal{L}_d(d-1, 1^{2d-2})$.
- The so–called Noether–Castelnuovo theorem was independently stated in 1869 by Clifford, Noether and Rosanes. Clifford only examined transformations of degree $d \leqslant 8$.
- Noether's idea was based on the correct remark that a homaloidal net $\mathcal{L}_d(m_1, m_2, m_3, \ldots)$ of degree $d \geqslant 2$ is such that $d < m_1 + m_2 + m_3$. Noether deduced from this that the degree of a homaloidal net can be lowered by applying a standard quadratic transformation. This is not correct. A first partial correction was made by Noether in 1872.
- Noether's proof was believed to be correct till 1901, when C. Segre remarked that a delicate case escaped Noether's analysis. Segre exhibited an infinite family of homaloidal nets whose degree cannot be lowered with a quadratic transformation. Coolidge reports in his book of 1931 that Noether cried when Segre's objection was communicated to him.
- Segre's objection affected a series of results about Cremona classifications of linear systems of plane curves, by various authors (Bertini, Castelnuovo, Del Pezzo, Enriques, Guccia, Jung, Martinetti, Segre himself).
- The gap was soon fixed by Castelnuovo in 1901 using adjoint linear systems and De Jonquière's transformations, which, in turn, are products of linear and quadratic transformations (C. Serge).
- In 1902 Castelnuovo's student Ferretti, using Castelnuovo's techniques, fixed the aforementioned results by vv. aa.
- Castelnuovo's proof has been re–exposed, with little improvements, by various authors, e.g., Alexander and Nencini (1916), Franciosi (1917), Chisini (1921), Calabri.

Historical note (II)

- Adjoint systems have been introduced by Brill–Noether (1873) for the study of linear series on curves. S. Kantor (1885) used them for studying Cremona geometry of linear systems of plane curves. This approach was taken up by Castelnuovo (1891) and has become a standard technique in birational geometry.
- The first time the statement of Ferretti's theorem appears in the literature is in a paper by Castelnuovo–Enriques (1900) on the classification of rational double planes. No proof is given, but only a two–lines–sketch of it, which is in any case affected by Segre's criticism. Moreover the authors make the wrong assertion that the result holds for reduced, not necessarily irreducible, curves.
- The first correct proof is due to Ferretti (1902), using Castelnuovo's approach for fixing the gap in Noether's proof indicated by Segre.
- Another, conceptually similar, but quicker, proof, is due to Marletta (1907), who also contributed (in 1911) to the study of Cremona minimal models of irreducible plane curves.
- Different proofs are due to Franciosi (1918) and Enriques–Chisini (1924). The latter authors however wrongly insist on asserting that their proof applies to reduced, not necessarily irreducible, curves.
- In Coolidge's book of 1931 a proof is given which is again affected by C. Segre's criticism of 30 years before! Though his book is dedicated "To the Italian Geometers, passed away or living", Coolidge does not give credit to anybody for this result! Thus, incredibly enough Ferretti's theorem has been attributed in the literature to Coolidge! In particular, Kumar–Murthy (1982) reproduce Coolidge's wrong proof!