Rank 2 aCM bundles on del Pezzo varieties (of dimension at least three)

Gianfranco Casnati

Dipartimento di Scienze Matematiche – Politecnico di Torino

Dedicated to Philippe on the occasion of his 60th birthday
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1. ___, D. Faenzi, F. Malaspina: *Rank two aCM bundles on del Pezzo threefolds with Picard number* 3. Preprint, arXiv:1306.6008 [math.AG], to appear in J. Algebra.
2. ___, D. Faenzi, F. Malaspina: *Rank two aCM bundles on the del Pezzo fourfold of degree* 6 *and its general hyperplane section*. In preparation.
3. ___: *On rank two bundles without intermediate cohomology*. In preparation.
4. ___, M. Filip: *Rank two aCM bundles on the del Pezzo threefold of degree* 7. In preparation.

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- All the schemes are defined over the field of complex number $\mathbb{C}$.
- If X and Y are schemes and $X \subseteq Y$ is closed, then $\mathcal{J}_{X|Y}$ denotes the ideal sheaf.
- $\mathbb{P}^N$ is the projective N–space over $\mathbb{C}$.
- A variety X is a smooth, closed, irreducible subscheme of $\mathbb{P}^N$ not contained in any hyperplane. In this case we set $\mathcal{O}_X(1) := \mathcal{O}_{\mathbb{P}^N}(1) \otimes \mathcal{O}_X$.
- If $\mathcal{F}$ is a coherent sheaf on a variety X, then we denote by $H^i\big(X, \mathcal{F}\big)$ the i^{th}– cohomology group of $\mathcal{F}$ and by $h^i\big(X, \mathcal{F}\big)$ its dimension over $\mathbb{C}$. Moreover $H^i_*\big(X, \mathcal{F}\big) := \bigoplus_{t \in \mathbb{Z}} H^i\big(X, \mathcal{F}(t)\big)$.
- A vector bundle is a locally free sheaf of finite rank.
- A line bundle is an invertible sheaf.

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Theorem (A.Grothendieck)

Every vector bundle $\mathcal{F}$ on $\mathbb{P}^1$ is decomposable, i.e. splits as the direct sum of line bundles.

Theorem (Horrocks)

Let $\mathcal{F}$ be a vector bundle on $\mathbb{P}^N$. Then $\mathcal{F}$ is decomposable i.e. splits as the direct sum of line bundles, if and only if it has no intermediate cohomology, i.e.

$$H^i_*(\mathbb{P}^N, \mathcal{F}) = 0, \qquad 1 \le i \le N-1.$$

Definition

A variety $F \subseteq \mathbb{P}^N$ is aCM if $H^i_*\big(\mathbb{P}^N, \mathcal{J}_{F|\mathbb{P}^N}\big) = 0$, $1 \le i \le \dim(F)$.
A vector bundle $\mathcal{F}$ on an aCM variety F is aCM if $H^i_*\big(\mathbb{P}^N, \mathcal{F}\big) = 0$, $1 \le i \le \dim(F) - 1$.

The aCM property is invariant up to twist by $\mathcal{O}_F(t)$. Thus we restrict to initialized $\mathcal{F}$, i.e. such that $h^0\big(F, \mathcal{F}(-1)\big) = 0$ and $h^0\big(F, \mathcal{F}\big) \ne 0$. E.g.

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Let Q be a smooth quadric. Besides $\mathcal{O}_Q$, there exist exactly two (if $n \geq 2$ is even), one (if $n \geq 3$ is odd) initialized, non–isomorphic, aCM bundles which do not split in the direct sum of vector bundles of lower rank.

The vector bundles above are called spinor bundles and have rank $2^{[(n-1)/2]}$.

Theorem (A. Knörrer, Inv. Math. 88 (1987))

A vector bundle $\mathcal{F}$ on a smooth quadric Q splits as the direct sum of twisted spinor bundles and line bundles, if and only if it is aCM.

Theorem (G. Ottaviani, Ann. Mat. Pura Appl. 155 (1989))

A vector bundle $\mathcal{F}$ on a smooth quadric Q splits as the direct sum of bundles $\mathcal{O}_Q(\alpha)$, if and only if both $\mathcal{F}$ and $\mathcal{F} \otimes \mathcal{S}$ are aCM for the spinor bundles $\mathcal{S}$ on Q.

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The zero locus $(s)_0$ of a non–zero section s of a vector bundle $\mathcal{F}$ of rank 2 on F is either empty or it has codimension at most 2.

Let D be the codimension 1 part of $(s)_0$ (if any). Then there exists a section of $\mathcal{F}(-D)$ vanishing on the (possibly empty) codimension 2 part C of $(s)_0$.

The corresponding Koszul complex yields the exact sequence

$$0 \longrightarrow \mathcal{O}_F \longrightarrow \mathcal{F}(-D) \longrightarrow \mathcal{J}_{C|F}(c_1(\mathcal{F}) - 2D) \longrightarrow 0.$$

Moreover $\deg(C) = \deg(c_2(\mathcal{F}(-D)) = \deg(c_2(\mathcal{F}) - c_1(\mathcal{F})D + D^2)$.

The Hartshorne–Serre correspondence allows us to reverse the above result.

One can hope to study $\mathcal{F}$ by dealing with C and conversely.
This is possible only if something can be said on the vanishing of the cohomology of $\mathcal{F}(-D)$.
E.g., if $\mathrm{Pic}(F)$ is generated by $\mathcal{O}_F(1)$, then $D = 0$, because $\mathcal{F}$ is initialized. If this is the case

$$\omega_C \cong \omega_X \otimes \mathcal{O}_C(c_1).$$

Theorem (C. Madonna, Rend. Sem. Mat. Torino 56 (1998))

Let $F \subseteq \mathbb{P}^N$ be a smooth aCM variety with $\dim(F) \geq 2$ with $\mathrm{Pic}(F) \cong \mathbb{Z}\mathcal{O}_F(1)$. Assume that $\omega_F = \mathcal{O}_F(\alpha)$ and set $\gamma(F) := \alpha + \dim(F)$. If $\mathcal{F}$ is an indecomposable, initialized, aCM bundle of rank 2 on F with $c_1(\mathcal{F}) = \mathcal{O}_F(c_1)$, then

$$1 - \gamma(F) \leq c_1 \leq 1 + \gamma(F).$$

If $\mathrm{Pic}(F)$ is not principal, the description is difficult: e.g., see the papers 1 and 2.

Easy facts

- If $c_1 = 1 - \gamma(F)$ (resp. $2 - \gamma(F)$), then C is a linear space (resp. a quadric) of dimension $\dim(F) - 2$.
- If $C \subseteq F$ is a linear space (resp. a quadric) of dimension $\dim(F) - 2$, then there exists an initialized aCM bundle $\mathcal{F}$ of rank 2 with $c_1 = 1 - \gamma(F)$ (resp. $2 - \gamma(F)$) with a section vanishing exacly along C.
- $c_1 = 1 + \gamma(F)$ if and only if $\mathcal{F}$ is an Ulrich bundle, i.e. it has a linear resolution as sheaf on $\mathbb{P}^N$. In this case a general section of $\mathcal{F}$ vanishes on a smooth, irreducible, linearly normal scheme of pure codimension 2 not contained in a hyperplane of $\mathbb{P}^N$.

Known fact

- If $F \subseteq \mathbb{P}^N$ is a smooth quadric, then $\gamma(F) = 0$. Only $c_1 = 1$ is admissible. If $N = 4, 5$ (resp. $N \geq 6$) such a value is attained by the spinor bundle (resp. is not attained).

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If $F \subseteq \mathbb{P}^4$ is a smooth hypersurface with $\deg(F) \leq 5$, then $\gamma(F) \geq 0$ and there exist indecomposable, aCM bundle of rank 2 on F: indeed each such an F contains a line. Other examples are given in C. Madonna, Rev. Mat. Complut. 13 (2000) and L. Chiantini–C. Madonna, Matematiche, 55 (2000).

If $F \subseteq \mathbb{P}^4$ is a general (hence smooth) hypersurface with $\deg(F) = 6$, then there are no indecomposable, aCM bundle of rank 2 on F (see L. Chiantini–C. Madonna, Internat. J. Math., 15 (2004)). When $\deg(F) \geq 5$ indecomposable, aCM bundle of rank 2 on F satisfy $H^i\big(F, \mathcal{F} \otimes \mathcal{F}^\vee\big) = 0$, $i = 1, 2$ (see K. Mohan Kumar–A.P. Rao–G.V. Ravindra, Comment. Math. Helvetici, 82 (2007)).

If $F \subseteq \mathbb{P}^N$, $N \geq 6$, is a smooth hypersurface, then there are no indecomposable, aCM bundle of rank 2 on F (see A. Beauville, Michigan Math. J., 48 (2000) and H. Kleppe, J. Algebra, 53 (1978)).

If $F \subseteq \mathbb{P}^5$ is a general hypersurface, then there are no indecomposable, aCM bundle of rank 2 on F (see K. Mohan Kumar–A.P. Rao–G.V. Ravindra, Comment. Math. Helvetici, 82 (2007)).

If $F \subseteq \mathbb{P}^N$ is a variety with $\dim(F) \geq 4$ which is the complete intersection of general hypersurfaces of sufficiently high degrees, then there are no indecomposable, aCM bundle of rank 2 on F (see J Biswas–G.V. Ravindra, Math. Z., 265 (2010)).

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Let $F \subseteq \mathbb{P}^N$ be a del Pezzo n–fold, i.e. a smooth n–fold with $\omega_F \cong \mathcal{O}_F(1-n)$ with $n \geq 2$.
Notice that each general hyperplane section H of F is still del Pezzo (of dimension $n-1$).
We have $3 \leq d := \deg(F) \leq 9$ and $N = d + n - 2$.
In what follows $\mathcal{F}$ will denote an indecomposable, initialized, aCM bundle on F of rank 2. It is easy to check that $\mathcal{F} \otimes \mathcal{O}_H$ is an initialized aCM bundle on H of rank 2 not isomorphic to $\mathcal{O}_H(a) \oplus \mathcal{O}_H(b)$

If $n = 2$ we have the following.

- The blow up of $\mathbb{P}^2$ at $9-d$ general points, $3 \leq d \leq 9$, embedded in $\mathbb{P}^d$ via the system of cubics through such points.
- The 2–tuple embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^8$.

If $n \geq 3$, then $d \neq 9$ and we have the following.

- If $d = 3$, then F is a smooth cubic hypersurface: $\mathrm{Pic}(F) \cong \mathbb{Z}\mathcal{O}_F(1)$.
- If $d = 4$, then F is a smooth complete intersection of two quadric hypersurfaces: $\mathrm{Pic}(F) \cong \mathbb{Z}\mathcal{O}_F(1)$.
- If $d = 5$, then F is a general linear sections of the grassmannian of lines in $\mathbb{P}^4$, thus $n \leq 6$: $\mathrm{Pic}(F) \cong \mathbb{Z}\mathcal{O}_F(1)$.
- If $d = 6$, then F is either the image of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^7$, or $\mathbb{P}^2 \times \mathbb{P}^2 \subseteq \mathbb{P}^8$, or a general hyperplane section of the latter, thus $n \leq 4$: in the first case $\mathrm{Pic}(F)$ is $\mathbb{Z}^{\oplus 3}$, in the second and third $\mathbb{Z}^{\oplus 2}$.
- If $d = 7$, then F is the blow up of $\mathbb{P}^3$ at a point p embedded in $\mathbb{P}^8$ via the quadrics through p, thus $n = 3$: $\mathrm{Pic}(F) \cong \mathbb{Z}^{\oplus 2}$.
- If $d = 8$, then F the 2–uple embedding of $\mathbb{P}^3$ in $\mathbb{P}^8$, thus $n = 3$: $\mathrm{Pic}(F) \cong \mathbb{Z}\mathcal{O}_{\mathbb{P}^3}(2)$.

Remark

Recall the following facts.

- *If* $c_1 = 0$ *(resp.* 1*), then* C *is a linear space (resp. a quadric) of dimension* $n - 2$.
- *If* $c_1 = 2$*, then a general section of* $\mathcal{F}$ *vanishes on a smooth, irreducible, linearly normal scheme* C *of pure codimension* 2 *with* $\omega_C \cong \mathcal{O}_C(2 + 1 - n)$*. In particular:*
  - *if* $n = 3$*, then* C *is an elliptic normal curve;*
  - *if* $n \geq 4$*, then* C *is a del Pezzo* $(n - 2)$*–fold;*

  in both the cases $\deg(C) = d + 2$.
- *If* $C \subseteq F$ *is a linear space (resp. a quadric, an ellptic normal curve, if* $n = 3$*, or a del Pezzo variety, if* $n \geq 4$*) of dimension* $n - 2$*, then there exists an initialized aCM bundle* $\mathcal{F}$ *of rank* 2 *with* $c_1 = 0$ *(resp.* 1*,* 2*) with a section vanishing exacly along* C.

$3 \leq d \leq 5$

In this case $\mathrm{Pic}(F) \cong \mathbb{Z}\mathcal{O}_F(1)$, hence $0 \leq c_1 \leq 2$.
If $n = 3$ the three cases $c_1 = 0, 1, 2$ actually occur, because each such a threefold contains lines, conic and elliptic normal curves of degree $d + 2$ (see E. Arrondo–L. Costa, Comm. Algebra 28 (2000)).

$d = 3$

Due to the aforementioned results for hypersurfaces:

- if $n \geq 5$, there are no indecomposable, initialized, aCM bundle of rank 2;
- if $n = 4$, the general cubic does not carry indecomposable, initialized, aCM bundles of rank 2;
- if $n = 4$, there are cubics endowed with indecomposable, initialized, aCM bundles of rank 2 with $c_1 = 0, 1$.

$d = 5$

- The restriction to F of the tautological bundle is an indecomposable, initialized, aCM bundles of rank 2 with $c_1 = 1$ for each n;
- if $n = 4$, there are indecomposable, initialized, aCM bundle on F of rank 2 with $c_1 = 0$, because F contains planes;
- if $n = 5$, there are no indecomposable, initialized, aCM bundle on F of rank 2 with $c_1 = 0$, because the existence of a linear space of dimension 3 inside F forces F to be singular;
- If $n = 4, 5$, the problem of the existence of indecomposable, initialized, aCM bundle on F of rank 2 with $c_1 = 2$ is open;
- if $n = 6$, there are no indecomposable, initialized, aCM bundle on F of rank 2 with c_1 either 0 or 2, because there are neither $\mathbb{P}^4$ nor del Pezzo fourfolds of degree 7 inside F.

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$d = 6$

$\mathrm{Pic}(F)$ is not principal, thus we do not have an immediate restriction on $c_1(\mathcal{F})$.
The classification of indecomposable, initialized, aCM bundles $\mathcal{F}$ of rank 2 on F is as follows (see 1 and 2).

- If $F \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, then $\mathcal{F}$ exists if and only if $c_1(\mathcal{F}) \in \mathbb{Z}^{\oplus 3}$ is, up to permutations:

$$(0,0,0), \quad (0,0,1), \quad (1,2,2), \quad (2,2,2), \quad (1,2,3).$$

- If $F \cong \mathbb{P}^2 \times \mathbb{P}^2$ or its general hyperplane section, then $\mathcal{F}$ exists if and only if $c_1(\mathcal{F}) \in \mathbb{Z}^{\oplus 2}$ is, up to permutations:

$$(0,0), \quad (0,1), \quad (1,2), \quad (2,2).$$

$d = 7$

Again Pic(F) is not principal: the same methods used in the case $d = 7$ can be applied in this case (see M. Filip, in preparation).

$d = 8$

Pic(F) is principal, hence $0 \le c_1 \le 2$. Notice that F does not contain curves of odd degree, thus only the two cases $c_1 = 1, 2$ are allowed. In the first case $\det(\mathcal{F}) \cong \mathcal{O}_F(1) \cong \mathcal{O}_{\mathbb{P}^3}(2)$, in the second $\det(\mathcal{F}) \cong \mathcal{O}_F(2) \cong \mathcal{O}_{\mathbb{P}^3}(4)$.
Only the second case occurs. $\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^3}(-1)$ is the null–correlation bundle (see 3).

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Lemma

Let $F \subseteq \mathbb{P}^N$ be a complete intersection with $n \geq 2r+1 \geq 3$. If $C \subseteq F$ is a complete intersection of dimension $n-r$ contained in a linear space of dimension $N-r-1$, then F is singular at some point of C.

F is the complete intersection of two quadrics. Due to the Remark and Lemma:

- if $n \geq 7$, there are no indecomposable, initialized, aCM bundle on F of rank 2;
- if $n = 5, 6$, the only possible indecomposable, initialized, aCM bundle on F of rank 2 must have $c_1 = 2$;
- if $n = 4$, there are indecomposable, initialized, aCM bundle on F of rank 2 with $c_1 = 0$, because F contains 64 planes (e.g. see M. Reid, PhD thesis).

Let $n = 4$. F is the complete intersection of a smooth quadric Q with a quadric Q' of rank 6. The equation Q' is the pfaffian of a antisymmetric matrix M of linear forms. Restricting M to Q, we obtain

$$\varphi\colon \mathcal{O}_Q(-1)^{\oplus 4} \rightarrowtail \mathcal{O}_Q^{\oplus 4}.$$

It is easy to check (see , 3) that

$$\mathcal{F} := \mathrm{coker}(\varphi)$$

is an initialized, aCM bundle of rank 2 on $F = Q \cap Q'$ with $c_1 = 1$.
$\mathcal{F}$ is indecomposable, because $h^0\big(F, \mathcal{F}\big) = 4$.

Conversely each indecomposable, initialized, aCM bundle of rank 2 on F arises in the above way.
Recall that there are exactly 7 quadrics of rank 6 in the ideal of F.

Let $n = 4$. Take the spinor bundle $\mathcal{S}$ on a smooth quadric Q through F. $\mathrm{rk}(\mathcal{S}) = 4$, $\wedge^2\mathcal{S}^\vee$ is globally generated. Thus there is

$$\varphi\colon \mathcal{S} \rightarrowtail \mathcal{S}^\vee$$

antisymmetric. $c_1(\mathcal{S}) = -2$, thus

$$\mathcal{F} := \mathrm{coker}(\varphi)$$

is supported on a smooth quadric section F of Q (see 3). $\mathcal{F}$ is an indecomposable, initialized, aCM bundle of rank 2 on F with $c_1 = 2$.

We do not know if there is such a bundle on each del Pezzo 4–fold of degree 4.

Let $n = 5, 6$. Indecomposable, initialized bundles of rank 2 on F with $c_2 = 2$ correspond to del Pezzo threefolds and fourfolds E of degree 6. The ideal of E is generated by 9 quadratic polynomials $f_1, \dots, f_9$. Thus the two generators f_1 and f_2 of the ideal of F satisfy

$$f_i = \sum_{h=1}^{9} a_i^h q_h.$$

In 3, using CoCoA and the above expressions for f_1 and f_2, we prove the following.

- If $E \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ one checks that the discriminant of the pencil of quadrics generated by f_1 and f_2 is a square. This implies that F would be singular (see M. Reid, PhD thesis).
- If $E \cong \mathbb{P}^2 \times \mathbb{P}^2$ or its hyperplane section, then the jacobian criterion for the equations of F again implies that F would be singular.