Category Theory

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Chapter 1

Categories and Functors

Definition 1.1. *A* category $\mathcal{C}$ *consists of:*

1) a class of objects denoted by $\mathrm{Ob}(\mathcal{C})$.

2) for every $C_1, C_2 \in \mathrm{Ob}(\mathcal{C})$ *a set* $Hom_{\mathcal{C}}(C_1, C_2)$, *called the* set of morphisms from C_1 to C_2

3) for every $C_1, C_2, C_3 \in \mathrm{Ob}(\mathcal{C})$ *there is a map*

$$\begin{array}{rccl} \circ: & \mathrm{Hom}_{\mathcal{C}}(C_1, C_2) \times \mathrm{Hom}_{\mathcal{C}}(C_2, C_3) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(C_1, C_3) \\ & (f, g) & \longmapsto & g \circ f \text{ called the composite of } g \text{ and } f \end{array}$$

satisfying the following conditions:

1) *if* $(C_1, C_2) \neq (C_3, C_4)$, $\mathrm{Hom}_{\mathcal{C}}(C_1, C_2) \cap \mathrm{Hom}_{\mathcal{C}}(C_3, C_4) = \varnothing$;

2) *if* $h \in \mathrm{Hom}_{\mathcal{C}}(C_3, C_4)$, $h \circ (g \circ f) = (h \circ g) \circ f$;

3) *for every* $C \in \mathrm{Ob}(\mathcal{C})$, *there exists* $\mathrm{Id}_C \in \mathrm{Hom}_{\mathcal{C}}(C, C)$ *such that for every* $f \in \mathrm{Hom}_{\mathcal{C}}(C, C')$, $f \circ \mathrm{Id}_C = f = \mathrm{Id}_{C'} \circ f$.

We also write $f : C_1 \to C_2$ *or* $C_1 \xrightarrow{f} C_2$ *instead of* $f \in \mathrm{Hom}_{\mathcal{C}}(C_1, C_2)$.
Moreover if $C \in \mathrm{Ob}(\mathcal{C})$, *we will simply write* $C \in \mathcal{C}$.

Example 1.2. *Sets, together with functions between sets, form the category Sets. For every algebraic structure you can consider its category: take sets endowed with that algebraic structure as objects and take morphisms between two objects as morphisms. In this way, you obtain the category of groups, Grps, of rings, Rings, of right R-modules, Mod-R and so on.*

Definition 1.3. *A category is called* small *if the class of its objects is a set;* discrete *if, given two objects* C_1, C_2, *if* $C_1 = C_2$ *then* $\mathrm{Hom}_{\mathcal{C}}(C_1, C_2) = \{\mathrm{Id}_{C_1}\}$, *if* $C_1 \neq C_2$ *then* $\mathrm{Hom}_{\mathcal{C}}(C_1, C_2) = \varnothing$. *Let* $\mathcal{C}$ *be a category.*

The opposite *category of a category* $\mathcal{C}$ *is the category* $\mathcal{C}^{\mathrm{op}}$ *where* $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = Ob(\mathcal{C})$ *and* $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(C_1, C_2) = \mathrm{Hom}_{\mathcal{C}}(C_2, C_1)$.

Definition 1.4. *A* subcategory $\mathcal{D}$ *of a category* $\mathcal{C}$ *is a category such that* $\mathrm{Ob}(\mathcal{D}) \subseteq \mathrm{Ob}(\mathcal{C})$ *and for every* $D_1, D_2 \in \mathcal{D}$, $\mathrm{Hom}_{\mathcal{D}}(D_1, D_2) \subseteq \mathrm{Hom}_{\mathcal{C}}(D_1, D_2)$. *When the inclusion is an equality,* $\mathcal{D}$ *is called* full subcategory *of* $\mathcal{C}$.

Definition 1.5. *Let* $\mathcal{C}$ *be a category. A morphism* $C_1 \xrightarrow{f} C_2$ *is an* isomorphism *if there exists a morphism* $C_2 \xrightarrow{g} C_1$ *such that* $f \circ g = \mathrm{Id}_{C_2}$ *and* $g \circ f = \mathrm{Id}_{C_1}$.

Remark 1.6. *Let* $f : C_1 \to C_2$ *be an isomorphism in a category* $\mathcal{C}$ *and let* $g, g' : C_2 \to C_1$ *be such that* $f \circ g = \mathrm{Id}_{C_2} = f \circ g'$ *and* $g \circ f = \mathrm{Id}_{C_1} = g' \circ f$. *Then we have*

$$g' = g' \circ \mathrm{Id}_{C_2} = g' \circ (f \circ g) = (g' \circ f) \circ g = \mathrm{Id}_{C_1} \circ g = g.$$

Hence there exists a ***unique morphism*** $g : C_2 \to C_1$ *be such that* $f \circ g = \mathrm{Id}_{C_2}$ *and* $g \circ f = \mathrm{Id}_{C_1}$. *This unique morphism will be denoted by* f^{-1}.

Definition 1.7. *Let* $A, B \in \mathcal{C}$ *and* $f : A \longrightarrow B$, *then*

- f *is a* monomorphism *if, for every* $g_1, g_2 : C \longrightarrow A$ *such that* $f \circ g_1 = f \circ g_2$, *we have* $g_1 = g_2$;
- f *is an* epimorphism *if, for every* $g_1, g_2 : B \longrightarrow C$ *such that* $g_1 \circ f = g_2 \circ f$, *we have* $g_1 = g_2$.

Proposition 1.8. *Let* $A, B \in \mathcal{C}$ *and let* $f : A \longrightarrow B$. *If* f *is an isomorphism then* f *is a monomorphism and an epimorphism.*

Proof. Since f is an isomorphism, there exists a morphism f^{-1} which is a two-sided inverse of f. First we prove that f is a monomorphism. Let $g_1, g_2 : C \longrightarrow A$ be a morphism such that $f \circ g_1 = f \circ g_2$. Then, by composing to the left with f^{-1} we get $f^{-1} \circ f \circ g_1 = f^{-1} \circ f \circ g_2$ and thus $g_1 = g_2$, i.e. f is a monomorphism. Now we want to prove that f is an epimorphism. Let $g_1, g_2 : B \longrightarrow C$ such that $g_1 \circ f = g_2 \circ f$. By composing to the right with f^{-1} we get $g_1 \circ f \circ f^{-1} = g_2 \circ f \circ f^{-1}$ from which follows $g_1 = g_2$, i.e. f is an epimorphism. □

Exercise 1.9. *Let* $f : A \longrightarrow B$ *and* $g : B \longrightarrow C$ *be morphisms in a category* $\mathcal{C}$. *Then*

- *if both* f *and* g *are monomorphisms, also* $g \circ f$ *is a monomorphism;*
- *if both* f *and* g *are epimorphisms, also* $g \circ f$ *is an epimorphism.*

Remark 1.10. *The converse of Proposition 1.8 doesn't hold in general, such as in the case of the inclusion* $\mathbb{Z} \to \mathbb{Q}$ *in the category of rings. In fact, let* $\mathcal{C}$ *be the category of rings, let*

$$i : \mathbb{Z} \longrightarrow \mathbb{Q}$$

be the canonical inclusion and let $h_1, h_2 : \mathbb{Q} \longrightarrow A$ *be such that*

$$\mathbb{Z} \xrightarrow{i} \mathbb{Q} \underset{h_2}{\overset{h_1}{\rightrightarrows}} A$$

$h_1 \circ i = h_2 \circ i$. *We will prove that* $h_1 = h_2$. *Let* $m \in \mathbb{Z}$ *and let* $n \in \mathbb{N}$, $n \neq 0$. *Since* h_j *is a morphism of rings for* $j = 1, 2$, *we have that*

$$\begin{aligned} 1_A &= h_j(1) = h_j\left(\frac{n}{n}\right) = h_j(n)\, h_j\left(\frac{1}{n}\right) \text{ and also} \\ 1_A &= h_j(1) = h_j\left(\frac{n}{n}\right) = h_j\left(\frac{1}{n}\right)\ h_j(n) \end{aligned}$$

so that

$$h_j\left(\frac{1}{n}\right) = h_j(n)^{-1}.$$

Therefore we get

$$h_1\left(\frac{m}{n}\right) = mh_1\left(\frac{1}{n}\right) = mh_1(n)^{-1} = mh_2(n)^{-1} = mh_2\left(\frac{1}{n}\right) = h_2\left(\frac{m}{n}\right)$$

that is $h_1 = h_2$ *so that* i *is an epimorphism. Now, let* $g_1, g_2 : R \longrightarrow \mathbb{Z}$

$$R \overset{g_1}{\underset{g_2}{\rightrightarrows}} \mathbb{Z} \xrightarrow{i} \mathbb{Q}$$

be such that $i \circ g_1 = i \circ g_2$. *Then* $g_1 = g_2$ *i.e.* i *is also a monomorphism. Note that* i *is not an isomorphism: a non-zero group morphism*

$$f : \mathbb{Q} \longrightarrow \mathbb{Z}$$

does not exists since $\mathbb{Q}$ *is divisible but* $\mathbb{Z}$ *is not. In fact, assume there exists a group morphism*

$$f : D \longrightarrow \mathbb{Z}$$

where D *is divisible. By definition of divisible group, for every* $n \in \mathbb{N}$, $nD = D$. *Since* f *is a group morphism,* $f(D) \subseteq \mathbb{Z}$ *and thus* $f(D) = t\mathbb{Z}$ *for some* $t \in \mathbb{N} \setminus \{0\}$. *Since* f *is a group morphism and* D *is divisible we have that*

$$nf(D) = f(nD) = f(D) = t\mathbb{Z}$$

and therefore

$$nt\mathbb{Z} = t\mathbb{Z}.$$

In particular, for every $n \in \mathbb{N}$, *there exists* $y_n \in \mathbb{Z}$ *such that*

$$t = nty_n.$$

For $n = 2$ *we get*

$$t = 2ty_2$$

and thus

$$1 = 2y_2$$

contradiction since 2 *is not invertible in* $\mathbb{Z}$.

Proposition 1.11. *Let A be a ring and let $f : L \to M$ be a morphism in Mod-A. Then*

1) *f is injective $\Leftrightarrow$ f is a monomorphism in Mod-A.*

2) *f is surjective $\Leftrightarrow$ f is an epimorphism in Mod-A.*

3) *f is an isomorphism $\Leftrightarrow$ f is an isomorphism in Mod-A. $\Leftrightarrow$ f is both a monomorphism and an epimorphism in Mod-A.*

Proof. 1) $\Rightarrow$. It is trivial.

1) $\Leftarrow$. Let $x \in L$ such that $x \neq 0$. Let us consider the morphism in Mod-A

$$h_x : A_A \to L_A \text{ defined by setting } h_x(a) = xa \text{ for every } a \in A.$$

Then

$$h_x(1) = x \neq 0 = \mathbf{0}(x)$$

where $\mathbf{0}$ denotes the zero morphism from A to M. Since f is a monomorphism in Mod-A, we get

$$f \circ h_x \neq f \circ \mathbf{0}.$$

It is easy to see that this implies

$$(f \circ h_x)(1) \neq 0.$$

Since $(f \circ h_x)(1) = f(x)$ we conclude.

2) $\Rightarrow$. It is trivial.

2) $\Leftarrow$. Let $p : M \to M/\mathrm{Im}(f)$ be the canonical projection. We have to prove that $M = \mathrm{Im}(f)$ i.e. that $p = \mathbf{0}$ where $\mathbf{0} : M \to M/\mathrm{Im}(f)$ is the zero morphism.

Since $p \circ f = \mathbf{0} \circ f$ and since f is an epimorphism in Mod-A, we get that $p = \mathbf{0}$.

3) It follows easily from 1) and 2). $\square$

Notations 1.12. *Let A be a ring. In view of the foregoing, from now on*

- *an injective homomorphism f of right (left) A-modules will also be called a monomorphism. We will also say that f is mono, for short.*
- *a surjective homomorphism of right (left) A-modules will also be called an epimorphism. We will also say that f is mono, for short.*
- *a bijective homomorphism of right (left) A-modules will also be called an isomorphism. We will also say that f is iso, for short.*

Definition 1.13. *If $\mathcal{C}$ is a category, then we define a category $\mathcal{C}^{op}$ having the same objects of $\mathcal{C}$ and setting*

$$\mathrm{Hom}_{\mathcal{C}^{op}}(C, C') = \mathrm{Hom}_{\mathcal{C}}(C', C), \textit{ for every } C, C' \in \mathcal{C}.$$

If $f \in \mathrm{Hom}_{\mathcal{C}^{op}}(C, C') = \mathrm{Hom}_{\mathcal{C}}(C', C)$, $g \in \mathrm{Hom}_{\mathcal{C}^{op}}(C', C'') = \mathrm{Hom}_{\mathcal{C}}(C'', C')$

$$g \circ_{\mathcal{C}^{op}} f \overset{def}{=} f \circ g.$$

Definition 1.14. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be categories. A* covariant functor $F : \mathcal{C} \to \mathcal{D}$ *between* $\mathcal{C}$ *and* $\mathcal{D}$ *consists of*

1) *a collection of objects of* $\mathcal{D}$

$$(F(C))_{C\in\mathcal{C}}$$

2) *a collection of morphisms in* $\mathcal{D}$

$$(F(f) : F(C_1) \longrightarrow F(C_2))_{f\in\mathrm{Hom}_{\mathcal{C}}(C_1,C_2)} \text{ for every } C_1, C_2 \in \mathcal{C}$$

such that

$$F(\mathrm{Id}_C) = \mathrm{Id}_{F(C)} \text{ and } F(g\circ f) = F(g)\circ F(f)$$

for every morphism $f \in \mathrm{Hom}_{\mathcal{C}}(C_1, C_2)$ *and* $g \in \mathrm{Hom}_{\mathcal{C}}(C_2, C_3)$*.*

Definition 1.15. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be categories. A* contravariant functor $F : \mathcal{C} \to \mathcal{D}$ *between* $\mathcal{C}$ *and* $\mathcal{D}$ *consists of*

1) *a collection of objects of* $\mathcal{D}$ $(F(C))_{C\in\mathcal{C}}$

2) *a collection of morphisms in* $\mathcal{D}$

$$(F(f) : F(C_2) \longrightarrow F(C_1))_{f\in\mathrm{Hom}_{\mathcal{C}}(C_1,C_2)} \text{ for every } C_1, C_2 \in \mathcal{C}$$

such that

$$F(\mathrm{Id}_C) = \mathrm{Id}_{F(C)} \text{ and } F(g\circ f) = F(f)\circ F(g).$$

for every morphism $f \in \mathrm{Hom}_{\mathcal{C}}(C_1, C_2)$ *and* $g \in \mathrm{Hom}_{\mathcal{C}}(C_2, C_3)$*.*

Proposition 1.16. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be categories. A contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ *is exactly a covariant functor* $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ *(or* $F : \mathcal{C} \to \mathcal{D}^{\mathrm{op}}$*).*

Examples 1.17.

Let ${}_AL_R$ *be an A-R-bimodule. Then we can consider the following functors.*

1) *The covariant functor* $\mathrm{Hom}_R({}_AL_R, -) : Mod\text{-}R \to Mod\text{-}A$ *defined by setting*

$$\mathrm{Hom}_R({}_AL_R, -)(M_R) = \mathrm{Hom}_R({}_AL_R, M_R) \text{ and } \mathrm{Hom}_R({}_AL_R, -)(f) = \mathrm{Hom}_R({}_AL_R, f)$$

for every $M_R \in Mod\text{-}R$ *and* f *morphism in* $Mod\text{-}R$*.*

2) *The covariant functor* $- \otimes_A {}_AL_R : Mod\text{-}A \to Mod\text{-}R$ *defined by setting*

$$(- \otimes_A {}_AL_R)(M_A) = M_A \otimes_A {}_AL_R \text{ and } (- \otimes_A {}_AL_R)(f) = f \otimes_A {}_AL_R$$

for every $M_A \in Mod\text{-}A$ *and* f *morphism in* $Mod\text{-}A$*.*

3) *The contravariant functor* $\mathrm{Hom}_R(-, {}_AL_R) : Mod\text{-}R \to A\text{-}Mod$ *defined by setting*

$$\mathrm{Hom}_R(-, {}_AL_R)(M_R) = \mathrm{Hom}_R(M_R, {}_AL_R) \text{ and } \mathrm{Hom}_R(-, {}_AL_R)(f) = \mathrm{Hom}_R(f, {}_AL_R)$$

for every $M_R \in Mod\text{-}R$ *and* f *morphism in* $Mod\text{-}R$.

Example 1.18. *More generally, let* $\mathcal{C}$ *be a category and let* $A \in \mathcal{C}$. *Let us define the functor* $h^A = \mathrm{Hom}_{\mathcal{C}}(A, \bullet) : \mathcal{C} \to Sets$ *mapping the object* C *to the set* $\mathrm{Hom}_{\mathcal{C}}(A, C)$ *and the morphism* $C_1 \xrightarrow{f} C_2$ *to the map*

$$h^A(f) = \mathrm{Hom}_{\mathcal{C}}(A, f) : \begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(A, C_1) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(A, C_2) \\ \left(A \xrightarrow{\xi} C_1\right) & \longmapsto & \left(A \xrightarrow{\xi} C_1 \xrightarrow{f} C_2\right) \end{array}.$$

Then h^A *is a functor. In fact:*

- $h^A(\mathrm{Id}_C)(\xi) = \mathrm{Id}_C \circ \xi = \xi$ *for every* $\xi : A \to C$ *so that* $h^A(\mathrm{Id}_C) = \mathrm{Id}_{h^A(C)}$;
- $h^A(g \circ f)(\xi) = g \circ f \circ \xi = h^A(g)(f \circ \xi) = \left(h^A(g) \circ h^A(f)\right)(\xi)$, *thus* $h^A(g \circ f) = h^A(g) \circ h^A(f)$.

Similarly, we can define a contravariant functor $h_A = \mathrm{Hom}_{\mathcal{C}}(\bullet, A) : \mathcal{C} \to Sets$ *which maps an object* $C \in \mathcal{C}$ *to the set* $\mathrm{Hom}_{\mathcal{C}}(C, A)$ *and a morphism* $f : C_1 \to C_2$ *to the map*

$$h_A(f) = \mathrm{Hom}_{\mathcal{C}}(f, A) : \begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(C_2, A) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(C_1, A) \\ \left(C_2 \xrightarrow{\zeta} A\right) & \longmapsto & \left(C_1 \xrightarrow{f} C_2 \xrightarrow{\zeta} A\right) \end{array}.$$

Lemma 1.19. *Let* $F : \mathcal{C}_1 \to \mathcal{C}_2$ *and* $G : \mathcal{C}_2 \to \mathcal{C}_3$, *be functors. For every* $C \in \mathcal{C}_1$ *we set*

$$GF(C) = G(F(C))$$

and for every morphism $f : C_1 \to C_2$ *we set*

$$GF(f) = G(F(f)).$$

This gives rise to a functor $GF = G \circ F : \mathcal{C}_1 \to \mathcal{C}_3$ *which is*

1) *covariant whenever both* F *and* G *are covariant,*

2) *covariant whenever both* F *and* G *are contravariant,*

3) *contravariant whenever* F *is covariant and* G *is contravariant,*

4) *contravariant whenever* F *is contravariant and* G *is covariant.*

Proof. Exercise. □

Notation 1.20. ***From now on, if not otherwise specified, the world functor will mean covariant functor.***

Definitions 1.21. *Given two functors* $\mathcal{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{D}$, *a* functorial morphism *(or* natural transformation*)* $\alpha : F \to G$ *is a collection of morphims in* $\mathcal{D}$ $\left(F(C) \xrightarrow{\alpha_C} G(C)\right)_{C\in\mathcal{C}}$ *such that, for every* $C_1 \xrightarrow{f} C_2$,

$$\alpha_{C_2} \circ F(f) = G(f) \circ \alpha_{C_1}$$

i.e. the following diagram

$$\begin{array}{ccc} F(C_1) & \xrightarrow{\alpha_{C_1}} & G(C_1) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(C_2) & \xrightarrow[\alpha_{C_2}]{} & G(C_2) \end{array}$$

is commutative. α *is called a* functorial isomorphism *(or* natural equivalence*) if, for every* $C \in \mathcal{C}$, α_C *is an isomorphism in* $\mathcal{D}$. *In this case the functors are called* isomorphic *and we write* $F \cong G$.

Exercise 1.22. *Let* $F, G, H : \mathcal{C} \to \mathcal{D}$ *be functors and let* $\alpha : F \to G$ *and* $\beta : G \to H$ *be functorial morphisms. Show that the collection*

$$\beta \circ \alpha = (\beta_C \circ \alpha_C)_{C\in\mathcal{C}}$$

is a functorial morphsm from H *to* F.

Exercise 1.23. *Let* $\alpha : F \to G$ *be a functorial isomorphism. Show that the collection* $\beta = \left((\alpha_C)^{-1}\right)_{C\in\mathcal{C}}$ *is a functorial isomorphism from* G *to* F.

Notation 1.24. *Let* $\alpha : F \to G$ *be a functorial isomorphism. Then the functorial isomorphism* β *in Exercise 1.23 will be denoted by* α^{-1}.

Example 1.25. *Let* $\mathcal{C}$ *be a category and let* $t : A_1 \to A_2$ *be a morphism in* $\mathcal{C}$. *We will define a functorial morphism* $h^t = \mathrm{Hom}_{\mathcal{C}}(t, \bullet) : h^{A_2} = \mathrm{Hom}_{\mathcal{C}}(A_2, \bullet) \to h^{A_1} = \mathrm{Hom}_{\mathcal{C}}(A_1, \bullet)$ *by setting, for every* $C \in \mathcal{C}$

$$[\mathrm{Hom}_{\mathcal{C}}(t, \bullet)]_C = \mathrm{Hom}_{\mathcal{C}}(t, C) : h^{A_2}(C) = \mathrm{Hom}_{\mathcal{C}}(A_2, C) \to h^{A_1}(C) = \mathrm{Hom}_{\mathcal{C}}(A_1, C)$$
$$(a : A_2 \to C) \mapsto (a \circ t : A_1 \to C)$$

Let us check that $\mathrm{Hom}_{\mathcal{C}}(t, \bullet)$ *is a functorial morphism. For every* $C \in \mathcal{C}$, *we will set*

$$[\mathrm{Hom}_{\mathcal{C}}(t, \bullet)]_C = \mathrm{Hom}_{\mathcal{C}}(t, C).$$

Let $f : C_1 \to C_2$ *be a morphism in* $\mathcal{C}$. *We have to prove that*

$$h^{A_1}(f) \circ \mathrm{Hom}_{\mathcal{C}}(t, C_1) = \mathrm{Hom}_{\mathcal{C}}(t, C_2) \circ h^{A_2}(f).$$

Let $a \in \mathrm{Hom}_{\mathcal{C}}(A_2, C_1)$. *We compute*

$$\left[h^{A_1}(f) \circ \mathrm{Hom}_{\mathcal{C}}(t, C_1)\right](a) = h^{A_1}(f)(\mathrm{Hom}_{\mathcal{C}}(t, C_1)(a)) = h^{A_1}(f)(a \circ t) = f \circ (a \circ t) =$$
$$= (f \circ a) \circ t = \left[h^{A_2}(f)(a)\right] \circ t = \mathrm{Hom}_{\mathcal{C}}(t, C_2)\left(h^{A_2}(f)(a)\right) = \left[\mathrm{Hom}_{\mathcal{C}}(t, C_2) \circ h^{A_2}(f)\right](a).$$

Exercise 1.26. *Let $\mathcal{C}$ be a category and let $t : A_1 \to A_2$ be a morphism in $\mathcal{C}$. Show that $\mathrm{Hom}_{\mathcal{C}}(t, \bullet) : h^{A_2} = \mathrm{Hom}_{\mathcal{C}}(A_2, \bullet) \to h^{A_1} = \mathrm{Hom}_{\mathcal{C}}(A_1, \bullet)$ is a functorial isomorphism if and only if t is an isomorphism in $\mathcal{C}$.*

Exercise 1.27. *Let $\mathcal{C}$ be a category and let $t : A_1 \to A_2$ be a morphism in $\mathcal{C}$. Check that $h_t = \mathrm{Hom}_{\mathcal{C}}(\bullet, t) : h_{A_1} \to h_{A_2}$, defined by setting, for every $C \in \mathcal{C}$*

$$[\mathrm{Hom}_{\mathcal{C}}(\bullet, t)]_C = \mathrm{Hom}_{\mathcal{C}}(C, t) : h_{A_1}(C) = \mathrm{Hom}_{\mathcal{C}}(C, A_1) \to h_{A_2}(C) = \mathrm{Hom}_{\mathcal{C}}(C, A_2)$$
$$(a : C \to A_1) \mapsto (t \circ a : C \to A_2)$$

is a funtorial morphism.

Definitions 1.28. *Let $\mathcal{C}$ e $\mathcal{D}$ be categories and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Let $C_1, C_2 \in \mathcal{C}$ and consider the map*

$$\begin{array}{rcl} F_{C_2}^{C_1} : \mathrm{Hom}_{\mathcal{C}}(C_1, C_2) & \to & \mathrm{Hom}_{\mathcal{D}}(F(C_1), F(C_2)) \\ f & \mapsto & F(f) \end{array}$$

The functor F is called

- faithful *if $F_{C2}^{C_1}$ is injective for every $C_1, C_2 \in \mathcal{C}$;*
- full *if $F_{C_2}^{C_1}$ is surjective for every $C_1, C_2 \in \mathcal{C}$.*

Examples 1.29. *Let $\mathcal{C}$ be a category and let $A \in \mathcal{C}$.*

- *The functor $h^A = \mathrm{Hom}_{\mathcal{C}}(A, \bullet) : \mathcal{C} \to Sets$ is faithful if and only if for every parallel pair $C_1 \underset{g}{\overset{f}{\rightrightarrows}} C_2$ where $f \neq g$ there exists $A \xrightarrow{\xi} C_1$such that $f \circ \xi \neq g \circ \xi$. In this case A is called a* generator *for $\mathcal{C}$.*
- *The functor $h_A = \mathrm{Hom}_{\mathcal{C}}(\bullet, A) : \mathcal{C} \to Sets$ is faithful if and only if for every parallel pair $C_1 \underset{g}{\overset{f}{\rightrightarrows}} C_2$ where $f \neq g$ there exists $C_2 \xrightarrow{\chi} A$ such that $\chi \circ f \neq \chi \circ g$. In this case A is called a* cogenerator *for $\mathcal{C}$.*

Lemma 1.30. *Let $T : \mathcal{C} \to \mathcal{D}$ be a functor and let $C_1 \xrightarrow{f} C_2$ be a morphism in $\mathcal{C}$.*

- *If f is an isomorphism in $\mathcal{C}$, then $T(f)$ is an isomorphism in $\mathcal{D}$.*
- *If T is a full and faithful functor and $T(f)$ is an isomorphism in $\mathcal{D}$, then f is an isomorphism in $\mathcal{C}$.*

Proof. If f is an isomorphism, there exists f^{-1} and we have

$$\begin{array}{rcl} T(f) \circ T\left(f^{-1}\right) & = & T\left(f \circ f^{-1}\right) = T(\mathrm{Id}_{C_2}) = \mathrm{Id}_{T(C_2)} \\ T\left(f^{-1}\right) \circ T(f) & = & T\left(f^{-1} \circ f\right) = T(\mathrm{Id}_{C_1}) = \mathrm{Id}_{T(C_1)} \end{array}$$

so that, we get

$$T\left(f^{-1}\right) = T(f)^{-1}.$$

Assume now that T is a full and faithful functor and $T(f)$ is an isomorphism in $\mathcal{D}$. Then there exists $h = T(f)^{-1}$. Since T is full there exists a g in $\mathcal{C}$ such that $h = T(g)$. Then we have

$$T(\mathrm{Id}_{C_1}) = \mathrm{Id}_{T(C_1)} = h \circ T(f) = T(g) \circ T(f) = T(g \circ f).$$

Since T is faithful, we get $\mathrm{Id}_{C_1} = g \circ f$. Similarly, one proves that $f \circ g = \mathrm{Id}_{C_2}$ and thus $g = f^{-1}$ □

Definitions 1.31. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. We say that*

- *F is an* equivalence *of categories if there is a functor $G : \mathcal{D} \to \mathcal{C}$ such that $FG \cong \mathrm{Id}_{\mathcal{D}}$ and $GF \cong \mathrm{Id}_{\mathcal{C}}$. In this case we also say that (F, G) is an* equivalence of categories*.*

- *F is an* isomorphism *of categories if there is a functor $G : \mathcal{D} \to \mathcal{C}$ such that $FG = \mathrm{Id}_{\mathcal{D}}$ and $GF = \mathrm{Id}_{\mathcal{C}}$. In this case we also say that (F, G) is an* isomorphism of categories *.*

Definitions 1.32. *Two categories $\mathcal{C}$ and $\mathcal{D}$ are called*

- equivalent *if there exist functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that (F, G) is an equivalence of categories.*

- isomorphic *if there exist functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that (F, G) is an isomorphism of categories*

Theorem 1.33. *Let $T : \mathcal{C} \to \mathcal{D}$ be a functor. Then T is an equivalence of categories if and only if T is full, faithful and, for every $D \in \mathcal{D}$, there exist $C \in \mathcal{C}$ and an isomorphism $T(C) \xrightarrow{\xi_D} D$.*

Proof. Assume first that T is an equivalence. Then there exist a functor $S : \mathcal{D} \longrightarrow \mathcal{C}$ and functorial isomorphisms $\alpha : ST \longrightarrow \mathrm{Id}_{\mathcal{C}}$ and $\beta : TS \longrightarrow \mathrm{Id}_{\mathcal{D}}$.

T **is faithful.** Let $f, f' \in \mathrm{Hom}_{\mathcal{C}}(C_1, C_2)$ with $T(f) = T(f')$. Then $ST(f) = ST(f')$. Since α is a functorial morphism we have

$$\alpha_{C_2} \circ ST(f) = f \circ \alpha_{C_1} \text{ and } \alpha_{C_2} \circ ST(f') = f' \circ \alpha_{C_1}$$

i.e. the diagram

$$\begin{array}{ccc} ST(C_1) & \xrightarrow{\alpha_{C_1}} & C_1 \\ {\scriptstyle ST(f)=ST(f')}\downarrow & & {\scriptstyle f'}\downarrow{\scriptstyle f} \\ ST(C_2) & \xrightarrow{\alpha_{C_2}} & C_2. \end{array}$$

is commutative. Since α is an isomorphism we deduce that

$$f = \alpha_{C_2} \circ ST(f) \circ \alpha_{C_1}^{-1} \overset{T(f)=T(f')}{=} \alpha_{C_2} \circ ST(f') \circ \alpha_{C_1}^{-1} = f'.$$

T is full. Let $T(C_1) \xrightarrow{h} T(C_2)$. We set

$$g = \alpha_{C_2} \circ S(h) \circ \alpha_{C_1}^{-1} \in \mathrm{Hom}_{\mathcal{C}}(C_1, C_2).$$

Since α is a functorial morphism we have

$$\alpha_{C_2} \circ ST(g) = g \circ \alpha_{C_1}$$

i.e. the diagram

$$\begin{array}{ccc} ST(C_1) & \xrightarrow{\alpha_{C_1}} & C_1 \\ {\scriptstyle S(h)=ST(g)}\downarrow & & \downarrow{\scriptstyle g} \\ ST(C_2) & \xrightarrow[\alpha_{C_2}]{} & C_2. \end{array}$$

is commutative. Since α a functorial isomorphism, we deduce that

$$ST(g) = \alpha_{C_2}^{-1} \circ g \circ \alpha_{C_1} \overset{\mathrm{def} g}{=} S(h).$$

Since T is an equivalence, so is S. Then, by the previous step, we have that S is faithful, so that we deduce that $h = T(g)$.

Now, for every $D \in \mathcal{D}$ we set $C = S(D) \in \mathcal{C}$ and $\xi_D = \beta_D : TS(D) \longrightarrow D$.

Conversely assume that T is full, faithful and, for every $D \in \mathcal{D}$, there exists $C \in \mathcal{C}$ and an isomorphism $T(C) \xrightarrow{\xi_D} D$.

Construction of $S\colon \mathcal{D} \to \mathcal{C}$. Let $D \in \mathcal{D}$, we set $S(D) = C$, where $C \in \mathcal{C}$ is such that there exists an isomorphism $T(C) \xrightarrow{\xi_D} D$. Here we applied the Axiom of Choice. Let $f : D_1 \longrightarrow D_2$ and consider the morphism

$$f' = \xi_{D_2}^{-1} \circ f \circ \xi_{D_1} : T(C_1) \longrightarrow T(C_2)$$

Since T is full, there exists a morphism $f'' : C_1 \longrightarrow C_2$ such that $T(f'') = f'$. Since T is faithful, f" is unique with respect to this proeperty. Thus we set $S(f) = f''$. Hence $S(f)$ is uniquely determined by

$$T(S(f)) = \xi_{D_2}^{-1} \circ f \circ \xi_{D_1} \tag{1.1}$$

S **is a functor.** Let $f : D_1 \longrightarrow D_2$ and $g : D_2 \longrightarrow D_3$ be morphisms in $\mathcal{D}$. We have

$$T(S(f)) \overset{(1.1)}{=} \xi_{D_2}^{-1} \circ f \circ \xi_{D_1} \text{ and } T(S(g)) \overset{(1.1)}{=} \xi_{D_3}^{-1} \circ g \circ \xi_{D_2}$$

i.e. the following diagram

$$\begin{array}{ccccc} D_1 & \xrightarrow{f} & D_2 & \xrightarrow{g} & D_3 \\ \uparrow{\scriptstyle \xi_{D_1}} & & \uparrow{\scriptstyle \xi_{D_2}} & & \uparrow{\scriptstyle \xi_{D_3}} \\ T(C_1) & \xrightarrow[TS(f)]{} & T(C_2) & \xrightarrow[TS(g)]{} & T(C_3), \end{array}$$

commutes. We deduce that

$$\begin{aligned} T\left(S\left(g\right)\circ S\left(f\right)\right) &= TS\left(g\right)\circ TS\left(f\right)=\left(\xi_{D_3}^{-1}\circ g\circ\xi_{D_2}\right)\circ\left(\xi_{D_2}^{-1}\circ f\circ\xi_{D_1}\right)= \\ &= \xi_{D_3}^{-1}\circ g\circ f\circ\xi_{D_1}\overset{(1.1)}{=}T\left(S\left(g\circ f\right)\right) \end{aligned}$$

so that

$$T\left(S\left(g\right)\circ S\left(f\right)\right)=T\left(S\left(g\circ f\right)\right).$$

We note that both $S\left(g\right)\circ S\left(f\right)$ and $S\left(g\circ f\right)$ are element of $\mathrm{Hom}_{\mathcal{D}}\left(T\left(C_1\right),T\left(C_2\right)\right)$. Thus, since T is faithful, we obtain that $S\left(g\right)\circ S\left(f\right)=S\left(g\circ f\right)$. Moreover, from

$$T\left(S\left(\mathrm{Id}_D\right)\right)=\xi_D^{-1}\circ\mathrm{Id}_D\circ\xi_D=\mathrm{Id}_{T(S(D))}=T\left(\mathrm{Id}_{S(D)}\right),$$

we deduce that $S\left(\mathrm{Id}_D\right)=\mathrm{Id}_{S(D)}$.

Construction of $\alpha\colon ST\to\mathrm{Id}_{\mathcal{C}}$. For every $C\in\mathcal{C}$ we need to construct an isomorphism $ST\left(C\right)\xrightarrow{\alpha_C}C$. By definition of $ST\left(C\right)$, there exists an isomorphism $TST\left(C\right)\xrightarrow{\xi_{T(C)}}T\left(C\right)$. Since T is full and faithful, there exists a unique morphism

$$ST\left(C\right)\xrightarrow{\alpha_C}C \text{ such that } \mathbf{T}\left(\alpha_C\right)=\xi_{T(C)}.$$

We will prove that $\left(\alpha_C\right)_{C\in\mathcal{C}}$ is a functorial isomorphism.

α is a functorial morphism. We have to prove that, for every morphism $h:C_1\to C_2$ in $\mathcal{C}$,

$$h\circ\alpha_{C_1}=\alpha_{C_2}\circ ST\left(h\right)$$

i.e. the following diagram

$$\begin{array}{ccc} ST\left(C_1\right) & \xrightarrow{\alpha_{C_1}} & C_1 \\ {\scriptstyle ST(h)}\downarrow & & \downarrow{\scriptstyle h} \\ ST\left(C_2\right) & \xrightarrow[\alpha_{C_2}]{} & C_2. \end{array}$$

is commutative. By applying T, we have

$$\begin{aligned} T\left(h\circ\alpha_{C_1}\right) &= T\left(h\right)\circ T\left(\alpha_{C_1}\right) \\ &= T\left(h\right)\circ\xi_{T(C_1)} \end{aligned}$$

and

$$\begin{aligned} T\left(\alpha_{C_2}\circ ST\left(h\right)\right) &= T\left(\alpha_{C_2}\right)\circ TST\left(h\right) \\ &= \xi_{T(C_2)}\circ TST\left(h\right). \end{aligned}$$

By definition of $ST\left(h\right)$, we have

$$T\left(ST\left(h\right)\right)\overset{(1.1)}{=}\xi_{T(C_2)}^{-1}\circ T\left(h\right)\circ\xi_{T(C_1)}$$

and thus we get

$$\begin{aligned} T\left(\alpha_{C_2}\circ ST\left(h\right)\right) &= \xi_{T(C_2)}\circ TST\left(h\right) \\ &= \xi_{T(C_2)}\circ \xi_{T(C_2)}^{-1}\circ T\left(h\right)\circ \xi_{T(C_1)} \\ &= T\left(h\right)\circ \xi_{T(C_1)} \\ &= T\left(h\circ \alpha_{C_1}\right) \end{aligned}$$

i.e.

$$T\left(h\circ \alpha_{C_1}\right) = T\left(\alpha_{C_2}\circ ST\left(h\right)\right).$$

Since T is faithful we conclude that α is a functorial morphism.

α **is a functorial isomorphism.** Since $\xi_{T(C)}$ is an isomorphism and $\xi_{T(C)} = T\left(\alpha_C\right)$, by applying Lemma 1.30 to the case "f" $= \alpha_C$, we get that α_C is an isomorphism.

Construction of $\beta\colon TS \to \mathrm{Id}_{\mathcal{D}}$**.** Let us consider

$$\beta = \left(\xi_D\right)_{D\in\mathcal{D}}.$$

β **is a functorial morphism.** Let $f : D_1 \to D_2$ be a morphism in $\mathcal{D}$. By definition of $S\left(f\right)$ we get that

$$\xi_{D_2}\circ TS\left(f\right) \overset{(1.1)}{=} \xi_{D_2}\circ\left(\xi_{D_2}^{-1}\circ f\circ \xi_{D_1}\right) = f\circ \xi_{D_1}$$

and hence we deduce that

$$\xi_{D_2}\circ TS\left(f\right) = f\circ \xi_{D_1}.$$

β **is a functorial isomorphism.** Since each ξ_D is an isomorphism, we deduce that β is a functorial isomorphism. $\square$

Chapter 2

Yoneda Lemma

Theorem 2.1 (Yoneda Lemma)**.** *Let* $F : \mathcal{C} \to \mathrm{Sets}$ *be a contravariant functor. Let* $A \in \mathcal{C}$ *and let us consider the contravariant functor*

$$h_A = \mathrm{Hom}_{\mathcal{C}}(\bullet, A) : \mathcal{C} \to Sets$$

introduced in Example 1.18. Let $\mathrm{Hom}(h_A, F)$ *be the collection of functorial morphisms from* h_A *to* F*. Set*

$$\begin{array}{rccc} \alpha_A^F : & \mathrm{Hom}(h_A, F) & \longrightarrow & F(A) \\ & \left(h_A \xrightarrow{\Gamma} F\right) & \longmapsto & \Gamma_A(\mathrm{Id}_A), \end{array}$$

α_A^F *is a bijection and it is natural in* A *and* F *where*

- α_A^F *natural in* A *means that* $\alpha_\bullet^F : \mathrm{Hom}(h_\bullet, F) \to F$ *is a functorial morphism between functors from* $\mathcal{C}$ *to* Sets*.*

- α_A^F *natural in* F *means that* $\alpha_A^\bullet : \mathrm{Hom}(h_A, \bullet) \to \bullet(A)$ *is a functorial morphism between functors from* $\mathrm{Hom}(\mathcal{C}, \mathrm{Sets})$ *to* Sets*.*

Proof. **Construction of** $\beta = \left(\alpha_A^F\right)^{-1} : F(A) \longrightarrow \mathrm{Hom}(h_A, F)$. Let $x \in F(A)$. For every object C in $\mathcal{C}$, we set

$$\begin{array}{rcl} \beta(x)_C & : & h_A(C) \to F(C) \\ \beta(x)_C(f) & = & F(f)(x) \text{ for every } f \in h_A(C) = \mathrm{Hom}_{\mathcal{C}}(C, A) \end{array}$$

$\beta(x)$ **is a functorial morphism for every** $x \in F(A)$. Let $x \in F(A)$. For every morphism $g : C_1 \to C_2$ in $\mathcal{C}$, we have to prove that

$$F(g) \circ \beta(x)_{C_2} \stackrel{?}{=} \beta(x)_{C_1} \circ h_A(g).$$

i.e. that the following diagram

$$\begin{array}{ccc} h_A(C_2) & \xrightarrow{\beta(x)_{C_2}} & F(C_2) \\ {\scriptstyle h_A(g)}\downarrow & & \downarrow{\scriptstyle F(g)} \\ h_A(C_1) & \xrightarrow[\beta(x)_{C_1}]{} & F(C_1) \end{array}$$

commutes. Let $f \in \mathrm{Hom}_{\mathcal{C}}(C_2, A)$. We compute

$$\begin{aligned}\left[F(g) \circ \beta(x)_{C_2}\right](f) = F(g)\left(\beta(x)_{C_2}(f)\right) = \\ = F(g)(F(f)(x)) = [F(f) \circ F(g)](x) = F(f \circ g)(x)\end{aligned}$$

and

$$\begin{aligned}\left[\beta(x)_{C_1} \circ h_A(g)\right](f) = \beta(x)_{C_1}(h_A(g)(f)) = \\ = \beta(x)_{C_1}(f \circ g) = F(f \circ g)(x)\end{aligned}$$

so that we get

$$\left[F(g) \circ \beta(x)_{C_2}\right](f) = \left[\beta(x)_{C_1} \circ h_A(g)\right](f)$$

Since this holds for every $f \in \mathrm{Hom}_{\mathcal{C}}(C_2, A)$, we deduce that $F(g) \circ \beta(x)_{C_2} = \beta(x)_{C_1} \circ h_A(g)$.

$\beta \circ \alpha_{\mathbf{A}}^{\mathbf{F}} = \mathrm{Id}_{\mathrm{Hom}(h_A, F)}$**.** Let $\Gamma : h_A \longrightarrow F$ be a functorial morphism. Then for every $f : C_1 \to C_2$ morphism $\mathcal{C}$ we have

$$\Gamma_{C_1} \circ h_A(f) = F(f) \circ \Gamma_{C_2}.$$

In particular, for every $f : C \to A$ we have

$$\Gamma_C \circ h_A(f) = F(f) \circ \Gamma_A. \tag{2.1}$$

Let us recall that $h_A(f)(t) = t \circ f$ for every $t \in \mathrm{Hom}_{\mathcal{C}}(A, A)$. Therefore we get

$$\begin{aligned}\Gamma_C(f) = \Gamma_C(\mathrm{Id}_A \circ f) = \Gamma_C(h_A(f)(\mathrm{Id}_A)) = \\ = [\Gamma_C \circ h_A(f)](\mathrm{Id}_A) \overset{(2.1)}{=} [F(f) \circ \Gamma_A](\mathrm{Id}_A) = F(f)(\Gamma_A(\mathrm{Id}_A))\end{aligned}$$

which yields

$$\Gamma_C(f) = F(f)(\Gamma_A(\mathrm{Id}_A)) \tag{2.2}$$

We have to prove that

$$\begin{aligned}\left(\beta \circ \alpha_A^F\right)(\Gamma) \overset{?}{=} \mathrm{Id}_{\mathrm{Hom}(h_A, F)}(\Gamma) \text{ i.e.} \\ \beta(\Gamma_A(\mathrm{Id}_A)) \overset{\mathrm{def}\alpha_A^F}{=} \beta\left(\alpha_A^F(\Gamma)\right) \overset{?}{=} \Gamma \text{ for every } \Gamma \in \mathrm{Hom}(h_A, F).\end{aligned}$$

For every $C \in \mathcal{C}$ and $f : C \to A$, we compute

$$\beta(\Gamma_A(\mathrm{Id}_A))_C(f) \overset{\mathrm{def}\beta}{=} F(f)(\Gamma_A(\mathrm{Id}_A)) \overset{(2.2)}{=} \Gamma_C(f).$$

Hence we deduce that $\left(\beta \circ \alpha_A^F\right)(\Gamma) = \mathrm{Id}_{\mathrm{Hom}(h_A, F)}(\Gamma)$.

$\alpha_{\mathbf{A}}^{\mathbf{F}} \circ \beta = \mathrm{Id}_{F(A)}$. Let $x \in F(A)$. We have

$$\alpha_A^F(\beta(x)) \overset{\mathrm{def}\alpha_A^F}{=} \beta(x)_A(\mathrm{Id}_A) \overset{\mathrm{def}\beta}{=} F(\mathrm{Id}_A)(x) \overset{F \mathrm{isafunct}}{=} \mathrm{Id}_{F(A)}(x) = x.$$

$\alpha_{\mathbf{A}}^{\mathbf{F}}$ **is natural in** A **i.e.** $\alpha_{\bullet}^{F}$: $\mathrm{Hom}\,(h_\bullet, F) \to F$ is a functorial morphism between functors from $\mathcal{C}$ to Sets.

First of all let us prove that $\mathrm{Hom}\,(h_\bullet, F)$ **is a contravariant functor from** C **to** *Sets*. For every $A \in \mathcal{C}$, let us set

$$\mathrm{Hom}\,(h_\bullet, F)\,(A) = \mathrm{Hom}\,(h_A, F)$$

and for every $u : A \longrightarrow B$ let

$$\begin{gathered}\mathrm{Hom}\,(h_\bullet, F)\,(u) = \mathrm{Hom}\,(h_u, F) : \mathrm{Hom}\,(h_B, F) \to \mathrm{Hom}\,(h_A, F)\\ (\Gamma : h_B \to F) \mapsto (\Gamma \circ h_u : h_A \to F)\end{gathered}$$

where $h_u = \mathrm{Hom}_{\mathcal{C}}\,(\bullet, u) : h_A \to h_B$ was defined in Exercise 1.27 by setting, for every $C \in \mathcal{C}$:

$$\begin{gathered}h_{u_C} = [\mathrm{Hom}_{\mathcal{C}}\,(\bullet, u)]_C = \mathrm{Hom}_{\mathcal{C}}\,(C, u) : h_A\,(C) = \mathrm{Hom}_{\mathcal{C}}\,(C, A) \to h_B\,(C) = \mathrm{Hom}_{\mathcal{C}}\,(C, B)\\ (a : C \to A) \mapsto (u \circ a : C \to B)\end{gathered}$$

We have

$$\mathrm{Hom}\,(h_\bullet, F)\,(\mathrm{Id}_A) = \mathrm{Hom}\,(h_{\mathrm{Id}_A}, F) = \mathrm{Hom}\,(\mathrm{Id}_{h_A}, F) = \mathrm{Id}_{\mathrm{Hom}(h_A, F)} = \mathrm{Id}_{\mathrm{Hom}(h_\bullet, F)(A)}.$$

Let now $u : A \longrightarrow B$ and $v : B \longrightarrow D$ be morphisms in $\mathcal{C}$. We have to prove that

$$\mathrm{Hom}\,(h_\bullet, F)\,(v \circ u) \stackrel{?}{=} \mathrm{Hom}\,(h_\bullet, F)\,(u) \circ \mathrm{Hom}\,(h_\bullet, F)\,(v)$$

i.e.

$$\mathrm{Hom}\,(h_{v\circ u}, F) \stackrel{?}{=} \mathrm{Hom}\,(h_u, F) \circ \mathrm{Hom}\,(h_v, F)\,.$$

Let $\Gamma \in \mathrm{Hom}\,(h_D, F)$. We compute

$$\begin{aligned}[\mathrm{Hom}\,(h_u, F) \circ \mathrm{Hom}\,(h_v, F)]\,(\Gamma) &= \mathrm{Hom}\,(h_u, F)\,[\mathrm{Hom}\,(h_v, F)\,(\Gamma)] = \mathrm{Hom}\,(h_u, F)\,(\Gamma \circ h_v) =\\ &= \Gamma \circ h_v \circ h_u\end{aligned}$$

Let $C \in \mathcal{C}$. Now for every $a : C \to A$ we compute

$$(h_v \circ h_u)\,(a) = h_v\,(h_u\,(a)) = h_v\,(u \circ a) = v \circ (u \circ a) = (v \circ u) \circ a = h_{v\circ u}\,(a)$$

so that we get

$$[\mathrm{Hom}\,(h_u, F) \circ \mathrm{Hom}\,(h_v, F)]\,(\Gamma) = \Gamma \circ h_{v\circ u} = \mathrm{Hom}\,(h_{v\circ u}, F)\,(\Gamma)\,.$$

Having established that $\mathrm{Hom}\,(h_\bullet, F)$ is a contravariant functor from $\mathcal{C}$ to Sets, let us prove that $\alpha_{\bullet}^{F} : \mathrm{Hom}\,(h_\bullet, F) \to F$ is a functorial morphism. Let $u : A \longrightarrow B$ be a morphism in $\mathcal{C}$. We have to prove that

$$F\,(u) \circ \alpha_B^F \stackrel{?}{=} \alpha_A^F \circ \mathrm{Hom}\,(h_u, F)$$

i.e. that the following diagram

$$\begin{array}{ccc} \operatorname{Hom}(h_B, F) & \xrightarrow{\alpha_B^F} & F(B) \\ {\scriptstyle \operatorname{Hom}(h_u,F)}\downarrow & & \downarrow{\scriptstyle F(u)} \\ \operatorname{Hom}(h_A, F) & \xrightarrow[\alpha_A^F]{} & F(A) \end{array}$$

commutes. Let $\Gamma : h_B \to F$ be a functorial morphism. Then we have

$$F(u) \circ \Gamma_B = \Gamma_A \circ h_B(u)$$

so that we get

$$\begin{aligned} \left[F(u) \circ \alpha_B^F\right](\Gamma) &= F(u)\left(\alpha_B^F(\Gamma)\right) \overset{\text{def}\alpha_B^F}{=} F(u)(\Gamma_B(\mathrm{Id}_B)) = [F(u) \circ \Gamma_B](\mathrm{Id}_B) = \\ &= [\Gamma_A \circ h_B(u)](\mathrm{Id}_B) = \Gamma(h_B(u)(\mathrm{Id}_B)) = \Gamma_A(\mathrm{Id}_B \circ u) = \Gamma_A(u) \end{aligned}$$

and

$$\begin{aligned} \left[\alpha_A^F \circ \operatorname{Hom}(h_u, F)\right](\Gamma) &= \alpha_A^F(\operatorname{Hom}(h_u, F)(\Gamma)) = \alpha_A^F(\Gamma \circ h_u) \\ &\overset{\text{def}\alpha_A^F}{=} (\Gamma \circ h_u)_A(\mathrm{Id}_A) = \Gamma_A(h_{uA}(\mathrm{Id}_A)) = \Gamma_A(u \circ \mathrm{Id}_A) = \Gamma_A(u). \end{aligned}$$

α_A^F **is natural in** F**.** Let $\psi\colon F \to G$ be a functorial morphism, we have to prove that

$$\psi_A \circ \alpha_A^F = \alpha_A^G \circ \operatorname{Hom}(h_u, \psi)$$

i.e. that the following diagram

$$\begin{array}{ccc} \operatorname{Hom}(h_A, F) & \xrightarrow{\alpha_A^F} & F(A) \\ {\scriptstyle \operatorname{Hom}(h_A,\psi)}\downarrow & & \downarrow{\scriptstyle \psi_A} \\ \operatorname{Hom}(h_A, G) & \xrightarrow[\alpha_A^G]{} & G(A). \end{array}$$

commutes. Let $\Gamma \in \operatorname{Hom}(h_A, F)$, we have

$$\left[\psi_A \circ \alpha_A^F\right](\Gamma) = \psi_A\left(\alpha_A^F(\Gamma)\right) \overset{\text{def}\alpha_A^F}{=} \psi_A(\Gamma_A(\mathrm{Id}_A))$$

and

$$\begin{aligned} \left[\alpha_A^G \circ \operatorname{Hom}(h_u, \psi)\right](\Gamma) &= \alpha_A^G(\operatorname{Hom}(h_A, \psi)(\Gamma)) = \alpha_A^G(\psi \circ \Gamma) \\ &\overset{\text{def}\alpha_A^G}{=} (\psi \circ \Gamma)_A(\mathrm{Id}_A) \\ &= \psi_A(\Gamma_A(\mathrm{Id}_A)) \end{aligned}$$

so that the diagram commutes and α_A^F is natural in F. □

Corollary 2.2. *Let $\mathcal{C}$ be a category and let $A, B \in \mathcal{C}$. The map*

$$\begin{aligned} \chi : \mathrm{Hom}_{\mathcal{C}}(A,B) &\longrightarrow \mathrm{Hom}(h_A, h_B) \\ t &\mapsto h_t \end{aligned}$$

is bijective.

Proof. By Theorem 2.1 applied to $F = h_B$, we know that

$$\begin{array}{rccc} \alpha_A^{h_B} : & \mathrm{Hom}(h_A, h_B) & \longrightarrow & h_B(A) = \mathrm{Hom}_{\mathcal{C}}(A,B) \\ & \left(h_A \xrightarrow{\Gamma} h_B\right) & \longmapsto & \Gamma_A(\mathrm{Id}_A), \end{array}$$

$\alpha_A^{h_B}$ is a bijection and it is natural in A and B. For every $t \in \mathrm{Hom}_{\mathcal{C}}(A,B)$, let us compute

$$\left(\alpha_A^{h_B} \circ \chi\right)(t) = \alpha_A^{h_B}(\chi(t)) = \alpha_A^{h_B}(h_t) = h_{t_A}(\mathrm{Id}_A) = t \circ \mathrm{Id}_A = t = \mathrm{Id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}(t).$$

We deduce that

$$\alpha_A^{h_B} \circ \chi = \mathrm{Id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}.$$

Since $\alpha_A^{h_B}$ is bijective, we obtain that also χ is bijective. □

Corollary 2.3. *Let $t : A \to B$ be a morphism in $\mathcal{C}$. Then t is an isomorphism if and only if h_t is a functorial isomorphism.*

Proof. Assume that $\phi = h_t$ is a functorial isomorphism. By Corollary 2.2, there exists a morphism $u : B \to A$ in $\mathcal{C}$ such that

$$\phi^{-1} = h_u.$$

Using the notations of Corollary 2.2, we have

$$\begin{aligned} \chi(\mathrm{Id}_B) &= h_{\mathrm{Id}_B} = \mathrm{Id}_{h_B} = \phi \circ \phi^{-1} = h_t \circ h_u = h_{t \circ u} = \chi(t \circ u) \\ \chi(\mathrm{Id}_A) &= h_{\mathrm{Id}_A} = \mathrm{Id}_{h_A} = \phi^{-1} \circ \phi = h_u \circ h_t = h_{u \circ t} = \chi(u \circ t). \end{aligned}$$

In view of Corollary 2.2, we deduce that

$$\mathrm{Id}_B = t \circ u \text{ and } \mathrm{Id}_A = u \circ t.$$

Conversely assume that there exists $u : B \to A$ in $\mathcal{C}$ such that

$$\mathrm{Id}_B = t \circ u \text{ and } \mathrm{Id}_A = u \circ t.$$

Then, given $f : C \to A$ and $g : C \to B$ we have

$$(h_t \circ h_u)(f) = t \circ (u \circ f) = (t \circ u) \circ f = f$$

and

$$(h_u \circ h_t)(g) = u \circ (t \circ g) = (u \circ t) \circ g = g.$$

□

Corollary 2.4. *Let* $A, B \in \mathcal{C}$*, then* $A \cong B$ *if and only if* $h_A \cong h_B$*.*

Proof. Assume that $h_A \cong h_B$. Then there exists a functorial morphism $\phi : h_A \to h_B$ such that ϕ_C is an isomorphism for every $C \in \mathcal{C}$. By Corollary 2.2, there exists a morphism $t : A \to B$ such that $\phi = h_t$. By Corollary 2.3 we get that t is an isomorphism. The converse follows directly from Corollary 2.2. □

In a similar way one can prove the following results.

Theorem 2.5 (Covariant Yoneda Lemma)**.** *Let* $F : \mathcal{C} \to \text{Sets}$ *be a covariant functor. Let* $A \in \mathcal{C}$ *and let us consider the covariant functor*

$$h^A = \text{Hom}_{\mathcal{C}}(A, \bullet) : \mathcal{C} \to Sets$$

introduced in Example 1.18. Let $\text{Hom}\left(h^A, F\right)$ *be the collection of functorial morphisms from* h_A *to* F*. Set*

$$\begin{array}{rccc} \Phi_A : & F(A) & \longrightarrow & \text{Hom}\left(h^A, F\right) \\ & t & \longmapsto & \Phi_A(t) : h^A \to F, \end{array}$$

where

$$\begin{array}{rcl} \Phi_A(t)_X & : & \text{Hom}_{\mathcal{C}}(A, X) \to F(X) \\ f & \longmapsto & F(f)(t) \end{array}$$

Φ_A *is a bijection and it is natural in* A *i.e.*
$\Phi_\bullet : F \to \text{Hom}(h^\bullet, F)$ *is a functorial morphism between functors from* $\mathcal{C}$ *to* Sets*.*

Corollary 2.6. *Let* $\mathcal{C}$ *be a category and let* $A, B \in \mathcal{C}$*. The map*

$$\begin{array}{c} \xi : \text{Hom}_{\mathcal{C}}(A, B) \longrightarrow \text{Hom}\left(h^B, h^A\right) \\ t \mapsto h^t \end{array}$$

is bijective

Corollary 2.7. *Let* $t : A \to B$ *be a morphism in* $\mathcal{C}$*. Then* t *is an isomorphism if and only if* h^t *is a functorial isomorphism.*

Corollary 2.8. *Let* $A, B \in \mathcal{C}$*, then* $A \cong B$ *if and only if* $h^A \cong h^B$*.*

Chapter 3

Abelian categories

3.1 Kernel

Lemma 3.1. *Let $\mathcal{C}$ be a category and let $f : A \longrightarrow B$ and $g : B \longrightarrow D$ be morphisms in $\mathcal{C}$ and let $h = g \circ f$. Then*

- *f is a monomorphism whenever h is a monomorphism,*
- *g is an epimorphism whenever h is an epimorphism.*

Proof. Let C be an object of $\mathcal{C}$, $\lambda_1, \lambda_2 : C \longrightarrow A$ and $\xi_1, \xi_2 : D \longrightarrow C$ be morphisms in $\mathcal{C}$. Assume that

$$f \circ \lambda_1 = f \circ \lambda_2.$$

Then we have

$$h \circ \lambda_1 = g \circ f \circ \lambda_1 = g \circ f \circ \lambda_2 = h \circ \lambda_2.$$

We deduce that $\lambda_1 = \lambda_2$, whenever h is a monomorphisms.

Now, assume that $\xi_1 \circ g = \xi_2 \circ g$. Then we have

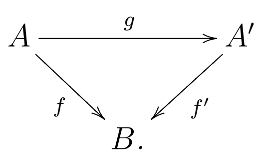
$$\xi_1 \circ h = \xi_1 \circ g \circ f = \xi_2 \circ g \circ f = \xi_2 \circ h$$

We deduce that $\xi_1 = \xi_2$, whenever h is an epimorphisms. $\square$

Definition 3.2. *Let $\mathcal{C}$ be a category. Two morphisms $f : A \longrightarrow B$ and $f' : A' \longrightarrow B$ in $\mathcal{C}$ are called* equivalent*, denoted by $f \sim f'$, if there exists an isomorphism $g : A \longrightarrow A'$ in $\mathcal{C}$ such that*

$$f' \circ g = f$$

i.e. the following diagram



is commutative.

Proposition 3.3. *In the setting of Definition 3.2 we have*

1) *The relation $\sim$ is an equivalence relation whose equivalent classes will be denoted by $[\ ]$.*

2) *If $f \sim f'$ then f is a monomorphism if and only if f' is a monomorphism.*

3) *If $f \sim f'$ then f is an epimorphism if and only if f' is an epimorphism.*

Proof. 1) it is trivial.

2) Since $f \sim f'$ there exists an isomorphism $g : A \longrightarrow A'$ such that

$$f' \circ g = f.$$

Assume that f is a monomorphism. Then, by Proposition 1.8 and exercise 1.9, $f' = f \circ g^{-1}$ is a monomorphism. Conversely, assume that f' is a monomorphism. Then, by Proposition 1.8 and exercise 1.9, $f = f' \circ g$ is a monomorphism.

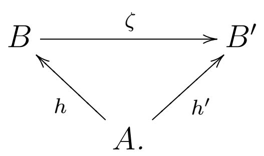
3) Similar to 2). □

Definition 3.4. *Let $\mathcal{C}$ be a category and let $C \in \mathcal{C}$. A* subobject *of C is an equivalence class $[i : A \longrightarrow C]$ where i is a monomorphism. We will however make an abuse of notation (and language) by denoting a subobject by (A, i) where $i : A \longrightarrow C$ is some representing monomorphism.*

Definition 3.5. *Let $\mathcal{C}$ be a category. Let $h : A \longrightarrow B$ and $h' : A \longrightarrow B'$ are called* coequivalent *, denoted by $h \overset{\circ}{\sim} h'$, if there exists an isomorphism $\zeta : B \longrightarrow B'$ such that*

$$h' = \zeta \circ h$$

i.e. the following diagram



is commutative

Proposition 3.6. *In the setting of Definition 3.5 we have*

1) *The relation $\overset{\circ}{\sim}$ is an equivalence relation whose equivalent classes will be denoted by $\langle\ \rangle$.*

2) *If $h \overset{\circ}{\sim} h'$ then h is a monomorphism if and only if h' is a monomorphism.*

3) *If $h \overset{\circ}{\sim} h'$ then h is an epimorphism if and only if h' is an epimorphism.*

Proof. Dual to Proposition 3.3. □

Definition 3.7. *Let $\mathcal{C}$ be a category and let $C \in \mathcal{C}$. A* quotient *of C is an equivalence class $\langle p : C \longrightarrow B \rangle$ where p is an epimorphism. We will however make an abuse of notation (and language) by denoting a quotient by (B, p) where $p : C \longrightarrow B$ is some representing epimorphism.*

Definitions 3.8. *Let $\mathcal{C}$ be a category.*

- *An object $X \in \mathcal{C}$ is called* initial object *of $\mathcal{C}$ if $|\mathrm{Hom}_{\mathcal{C}}(X, C)| = 1$ for every $C \in \mathcal{C}$.*
- *An object $Z \in \mathcal{C}$ is called* final object *of $\mathcal{C}$ if $|\mathrm{Hom}_{\mathcal{C}}(C, Z)| = 1$ for every $C \in \mathcal{C}$.*
- *If $X = Z$ is both initial and final object of $\mathcal{C}$ then it is called* zero *of the category $\mathcal{C}$.*

Example 3.9. *In Mod-R, $\{0\}$ is both initial and final object.*

Example 3.10. *In Rings, $\mathbb{Z}$ is initial object. In fact, for any ring R there exists a unique ring morphism*

$$f : \mathbb{Z} \longrightarrow R$$

determined by $f(n) = n \cdot f(1_{\mathbb{Z}}) = n \cdot 1_R$.

Lemma 3.11. *If an initial (final) object in a category $\mathcal{C}$ exists, then is unique up to isomorphism.*

Proof. Assume that X, X' are initial objects for the category $\mathcal{C}$. Then, for every $C \in \mathcal{C}$ there exists a unique morphism $h_C : X \longrightarrow C$ and a unique morphism $k_C : X' \longrightarrow C$. In particular there exists a unique morphism $h_{X'} : X \longrightarrow X'$ and a unique morphism $k_X : X' \longrightarrow X$. Then we get

$$\mathrm{Id}_X = h_X = k_X \circ h_{X'}$$

and

$$\mathrm{Id}_{X'} = k_{X'} = h_{X'} \circ k_X.$$

□

Definition 3.12. *A category $\mathcal{C}$ is called* preadditive *if*

1) *for every $A, B \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is an abelian group whose neutral element will be denoted by 0_B^A or simply by 0;*

2) *the composition of maps*

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) &\longrightarrow \mathrm{Hom}_{\mathcal{C}}(A, C) \\ (f, g) &\mapsto g \circ f \end{aligned}$$

   is a group morphism, i.e.

$$\begin{aligned} (g_1 + g_2) \circ f &= g_1 \circ f + g_2 \circ f \\ g \circ (f_1 + f_2) &= g \circ f_1 + g \circ f_2. \end{aligned}$$

Lemma 3.13. *Let $\mathcal{C}$ be a preadditive category and let X be an object of $\mathcal{C}$. Then the following are equivalent*

(a) X is an initial (final) object in $\mathcal{C}$

(b) $\mathrm{Hom}_{\mathcal{C}}(X, C) = \{0_C^X\}$ ($\mathrm{Hom}_{\mathcal{C}}(C, X) = \{0_X^C\}$) for every $C \in \mathcal{C}$.

If X is a zero object for $\mathcal{C}$ we will write $X = 0_{\mathcal{C}}$.

Lemma 3.14. *Let $\mathcal{C}$ be a preadditive category. Then, for every morphism $f : A \longrightarrow B$, we have*

$$f \circ 0_A^C = 0_B^C \text{ and } 0_C^B \circ f = 0_C^A \text{ for every } C \in \mathcal{C}$$

Proof. We have

$$f \circ 0_A^C = f \circ \left(0_A^C + 0_A^C\right) = f \circ 0_A^C + f \circ 0_A^C.$$

Since $\mathrm{Hom}_{\mathcal{C}}(C, B)$ is a group, we deduce that

$$f \circ 0_A^C = 0_B^C.$$

The other statement as an analogous proof. □

Notation 3.15. *Let $\mathcal{C}$ be a preadditive category and let $A, B \in \mathcal{C}$. From now on, we will simply write 0 instead of 0_B^A whenever there is no risk of confusion.*

Proposition 3.16. *Let $\mathcal{C}$ be a preadditive category and let $f : A \longrightarrow B$ be a morphism in $\mathcal{C}$. Then*

1) f is a monomorphism if and only if for every $g : C \longrightarrow A$ such that $f \circ g = 0$ we have $g = 0$

2) f is an epimorphism if and only if for every $h : B \longrightarrow D$ such that $h \circ f = 0$ we have $h = 0$.

Proof. 1) Assume that f is a monomorphism and that there exists g such that $f \circ g = 0$. In view of Lemma 3.14, we have:

$$f \circ g = 0 = f \circ 0.$$

Since f is a monomorphism we get that $g = 0$. Conversely, assume that for every g such that $f \circ g = 0$ we have $g = 0$. Let g_1, g_2 such that $f \circ g_1 = f \circ g_2$. Then we have $f \circ (g_1 - g_2) = 0$ and hence, in view of our assumptions, we get that $g_1 - g_2 = 0$, i.e. $g_1 = g_2$, so that f is a monomorphism.

2) Similar to 1). □

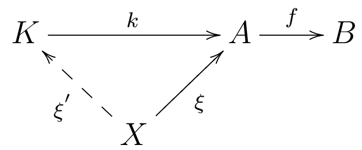
Definition 3.17. *Let $\mathcal{C}$ be a preadditive category and let $f : A \longrightarrow B$ be a morphism in $\mathcal{C}$. A* kernel *of f, if it exists, is a pair (K, k) where $k : K \longrightarrow A$ satisfies:*

1) $f \circ k = 0$

2) universal property of the kernel: if $\xi : X \longrightarrow A$ is a morphism in $\mathcal{C}$ such that $f \circ \xi = 0$, there exists a morphism $\xi' : X \longrightarrow K$ such that

$$\xi = k \circ \xi'$$

i.e. the following diagram



is commutative. Moreover, such ξ' is unique with respect to this property.

Proposition 3.18. *Let $\mathcal{C}$ be a preadditive category. If (K, k) is a kernel of $f : A \longrightarrow B$, then k is a monomorphism.*

Proof. Let (K, k) be a kernel of f. Let $g : X \longrightarrow K$ be a morphism such that $k \circ g = 0$. We have to prove that $g = 0$. We have

$$f \circ k \circ g \overset{(3.14)}{=} 0$$

so that there exists a unique $\xi' : X \longrightarrow K$ such that $k \circ \xi' = k \circ g$. Since $k \circ g = 0 = k \circ 0$ and ξ' is unique with respect to the property $k \circ \xi' = k \circ g$, we deduce that $\xi' = g = 0$ and thus k is a monomorphism. □

Proposition 3.19. *Let $\mathcal{C}$ be a preadditive category. Assume that (K, k) is a kernel of $f : A \longrightarrow B$. Then given a pair (K', k') where $k' : K' \to A$, we have that*

(K', k') is a kernel of $f : A \longrightarrow B$ if and only if the morphisms k and k' are equivalent.

Proof. Assume that (K', k') is a kernel of $f : A \longrightarrow B$. Since (K, k) is a kernel of f and $f \circ k' = 0$, there exists a unique morphism $\gamma : K' \to K$ such that

$$k' = k \circ \gamma.$$

Since (K', k') is a kernel of f and $f \circ k = 0$, there is a unique morphism $\gamma' : K \to K'$ such that

$$k = k' \circ \gamma'.$$

Therefore we obtain

$$k \circ \mathrm{Id}_K = k = k' \circ \gamma' = k \circ \gamma \circ \gamma'$$

and

$$k' \circ \mathrm{Id}_{K'} = k' = k \circ \gamma = k' \circ \gamma' \circ \gamma.$$

Since both (K, k) and (K, k') are kernels of f, by Proposition 3.18, both k and k' are monomorphisms so that we deduce that

$$\gamma \circ \gamma' = \mathrm{Id}_K \text{ and } \gamma' \circ \gamma = \mathrm{Id}_{K'}$$

i.e. k is equivalent to k'.

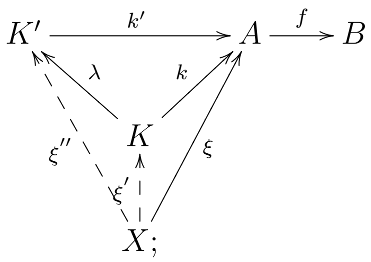
Conversely assume that k and k' are equivalent, i.e. there exists an isomorphism $\lambda : K \to K'$ such that $k = k' \circ \lambda$. Since (K, k) is a kernel of f, we have

$$0 = f \circ k = f \circ k' \circ \lambda$$

and since λ is an isomorphism, we deduce that

$$f \circ k' = 0.$$

Now let $\xi : X \longrightarrow A$ such that $f \circ \xi = 0$. Then there exists a unique morphism $\xi' : X \longrightarrow K$ such that $\xi = k \circ \xi'$. We have to prove that there exists a morphism $\xi'' : X \longrightarrow K'$ such that $\xi = k' \circ \xi''$ and such ξ'' is unique with respect to this property.



We have

$$\xi = k \circ \xi' = k \circ \mathrm{Id}_K \circ \xi' = k \circ \lambda^{-1} \circ \lambda \circ \xi' = k' \circ \xi''$$

where $\xi'' = \lambda \circ \xi'$. We now have to prove that ξ'' is unique. Assume that $\overline{\xi''} : X \to K'$ is another morphism such that

$$\xi = k' \circ \overline{\xi''}.$$

Then we have

$$\xi = k' \circ \overline{\xi''} = k' \circ \mathrm{Id}_{K'} \circ \overline{\xi''} = k' \circ \lambda \circ \lambda^{-1} \circ \overline{\xi''} = k \circ \lambda^{-1} \circ \overline{\xi''}$$

and since

$$\xi = k \circ \xi'$$

where ξ' is unique with respect to the property $\xi = k \circ \xi'$, we deduce that

$$\xi' = \lambda^{-1} \circ \overline{\xi''}$$

and thus

$$\xi'' = \lambda \circ \xi' = \lambda \circ \lambda^{-1} \circ \overline{\xi''} = \overline{\xi''}.$$

□

Notation 3.20. *Let (K, k) be a kernel of a morphism $f : A \to B$. Then, in view of Proposition 3.19, k is a monomorphism. Hence k is a representative monomorphism of a subobject of A which will be denoted by* $\mathrm{Ker}\,(f)$. *We will also write $(K, k) =$* $\mathrm{Ker}\,(f)$ *to mean that k is a representative of the equivalence class* $\mathrm{Ker}\,(f)$.

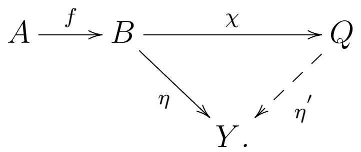
Definition 3.21. *Let $\mathcal{C}$ be a preadditive category and let $f : A \longrightarrow B$ be a morphism in $\mathcal{C}$. A* cokernel *of f, if it exists, is a pair (Q, χ) where $\chi : B \longrightarrow Q$ satisfies:*

1) *$\chi \circ f = 0$*

2) *universal property of the cokernel: if $\eta : B \longrightarrow Y$ is a morphism in $\mathcal{C}$ such that $\eta \circ f = 0 = 0$, there exists a morphism $\eta' : Q \longrightarrow Y$ such that*

$$\eta = \eta' \circ \chi$$

i.e. the following diagram



is commutative. Moreover, such η' is unique with respect to this property.

Proposition 3.22. *If (Q, χ) is a cokernel of $f : A \longrightarrow B$, then χ is an epimorphism.*

Proof. Let $g : Q \longrightarrow Y$ be such that $g \circ \chi = 0$.We have to prove that $g = 0$. We have

$$g \circ \chi \circ f \overset{(3.14)}{=} 0$$

so that there exists a unique morphism $\eta' : Q \to Y$ such that

$$g \circ \chi = \eta' \circ \chi.$$

Since $g \circ \chi = 0 = 0 \circ \chi$ and η' is unique with respect to the property that $g \circ \chi = \eta' \circ \chi$, we deduce that $\eta' = g = 0$ and thus χ is an epimorphism. □

Proposition 3.23. *Assume that (Q, χ) is a cokernel of $f : A \longrightarrow B$. Then given a pair (Q', χ') where $\chi' : B \to Q'$, we have that*

(Q', χ') is a cokernel of $f : A \longrightarrow B$ if and only if the morphisms χ and χ' are coequivalent.

Proof. Assume that (Q', χ') is a cokernel of f. Since (Q, χ) is a cokernel of f and $\chi' \circ f = 0$, there exists a unique morphism $\sigma : Q \to Q'$ such that

$$\chi' = \sigma \circ \chi.$$

Since (Q', χ') is a cokernel of f and $\eta \circ f = 0$, there is a unique morphism $\sigma' : Q' \to Q$ such that

$$\chi = \sigma' \circ \chi'$$

Therefore we obtain

$$\mathrm{Id}_Q \circ \chi = \chi = \sigma' \circ \chi' = \sigma' \circ \sigma \circ \chi$$

and

$$\mathrm{Id}_{Q'} \circ \chi' = \chi' = \sigma \circ \chi = \sigma \circ \sigma' \circ \chi'.$$

Since both (Q, χ) and (Q', χ') are cokernel of f, by Proposition 3.22, both χ and χ' are epimorphisms so that we deduce that

$$\sigma' \circ \sigma = \mathrm{Id}_Q \quad \text{and} \quad \sigma \circ \sigma' = \mathrm{Id}_{Q'}$$

i.e. χ and χ' are equivalent morphisms.

Conversely assume that χ and χ' are coequivalent i.e. there exists an isomorphism $\lambda : Q \longrightarrow Q'$ such that $\chi' = \lambda \circ \chi$. Since (Q, χ) is a cokernel of $f : A \longrightarrow B$, we have

$$\chi' \circ f = \lambda \circ \chi \circ f = \lambda \circ 0 = 0.$$

Now let $\eta : B \longrightarrow Y$ such that $\eta \circ f = 0$. We have to prove that there exists $\eta'' : Q' \longrightarrow Y$ such that $\eta = \eta'' \circ \chi'$ and η'' is unique with respect to this property. Since (Q, χ) is a cokernel of f and $\eta \circ f = 0$, there exists a unique $\eta' : Q \longrightarrow Y$ such that $\eta = \eta' \circ \chi$. We have

$$\eta = \eta' \circ \chi = \eta' \circ \mathrm{Id}_Q \circ \chi = \eta' \circ \lambda^{-1} \circ \lambda \circ \chi = \eta' \circ \lambda^{-1} \circ \chi' = \eta'' \circ \chi'$$

where $\eta'' = \eta' \circ \lambda^{-1}$. We prove that such morphism η'' is unique. Assume that there exists another morphism $\overline{\eta''} : Q' \longrightarrow Y$ such that $\eta = \overline{\eta''} \circ \chi'$. We have

$$\eta = \overline{\eta''} \circ \chi' = \overline{\eta''} \circ \mathrm{Id}_{Q'} \circ \chi' = \overline{\eta''} \circ \lambda \circ \lambda^{-1} \circ \chi' = \overline{\eta''} \circ \lambda \circ \chi$$

and since

$$\eta = \eta' \circ \chi$$

where η' is unique with respect to the property $\eta = \eta' \circ \chi$, we deduce that

$$\eta' = \overline{\eta''} \circ \lambda$$

and thus

$$\overline{\eta''} = \eta' \circ \lambda^{-1} = \eta''.$$

□

Notation 3.24. *Let (Q, χ) be a cokernel of a morphism $f : A \to B$. Then, in view of Proposition 3.23, χ is an epimorphism. Hence χ is a representative epimorphism of a quotient of B which will be denote by* $\mathrm{Coker}\,(f)$. *We will also write $(Q, \chi) =$* $\mathrm{Coker}\,(f)$ *to mean that χ is a representative of the equivalent class* $\mathrm{Coker}\,(f)$.

Theorem 3.25. *Let $\mathcal{C}$ be a preadditive category with $0_{\mathcal{C}}$ and let $f : A \longrightarrow B$ be a morphism uin $\mathcal{C}$.*

1) Then f is a monomorphism if and only if $\mathrm{Ker}\,(f) = \left(0_{\mathcal{C}}, 0_A^{0_{\mathcal{C}}}\right)$.

2) Then f is an epimorphism if and only if $\mathrm{Coker}\,(f) = \left(0_{\mathcal{C}}, 0_{0_{\mathcal{C}}}^B\right)$.

Proof. 1) Assume that $\mathrm{Ker}\,(f)$ exists. Suppose that f is a monomorphism. Since $\mathcal{C}$ is a preadditive category, by Lemma 3.14, we have $f \circ 0_A^{0_\mathcal{C}} = 0_B^{0_\mathcal{C}}$. Let now $\xi : X \longrightarrow A$ such that $f \circ \xi = 0_B^X$. Since we also have $f \circ 0_A^X = 0_B^X = f \circ \xi$ and f is a monomorphism, we deduce that $\xi = 0_A^X$ and hence we get

$$\xi = 0_A^X = 0_A^{0_\mathcal{C}} \circ 0_{0_\mathcal{C}}^X.$$

Since $0_\mathcal{C}$ is a final object, the unique $\xi' : X \to 0^{0_\mathcal{C}}$ we can choose is $0_{0_\mathcal{C}}^X$. Conversely suppose that $\left(0_\mathcal{C}, 0_A^{0_\mathcal{C}}\right) = \mathrm{Ker}\ (f)$. Let $\xi : X \longrightarrow A$ such that $f \circ \xi = 0_B^X$. . Then there exists a unique morphism $\xi' : X \longrightarrow 0_\mathcal{C}$ such that $0_A^{0_\mathcal{C}} \circ \xi' = \xi$, i.e. $\xi = 0_A^X$.

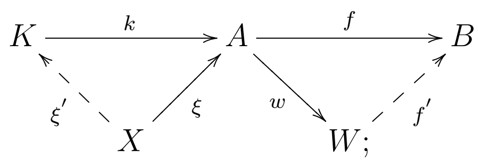
2) Similar to 1). □

Proposition 3.26. *Let $\mathcal{C}$ be a preadditive category with $0_\mathcal{C}$ and assume that for every morphism in $\mathcal{C}$ there exist both kernel and cokernel. Then, if $f : A \longrightarrow B$ is a morphism in $\mathcal{C}$ and $(K, k) = \mathrm{Ker}\,(f)$ and $(Q, \chi) = \mathrm{Coker}\,(f)$ we have*

1) $(K, k) = \mathrm{KerCoker}\,(k)$,

2) $(Q, \chi) = \mathrm{CokerKer}\,(\chi)$.

Proof. 1) Let us set $(W, w) = \mathrm{Coker}\,(k)$. We have to prove that $(K, k) = \mathrm{Ker}\,(w)$. Note that, by definition of, w we have $w \circ k = 0$. Let $\xi : X \longrightarrow A$ be a morphism such that $w \circ \xi = 0$. We have to prove that there exists $\xi' : X \longrightarrow K$ such that $\xi = k \circ \xi'$ and such ξ' is unique with respect to this property.



Since $(W, w) = \mathrm{Coker}\,(k)$ and $f \circ k = 0$, there exists a unique morphism $f' : W \longrightarrow B$ such that $f' \circ w = f$, then

$$f \circ \xi = f' \circ w \circ \xi = f' \circ 0 = 0.$$

Since $(K, k) = \mathrm{Ker}\,(f)$ and $f \circ \xi = 0$, there exists a unique morphism $\xi' : X \longrightarrow K$ such that

$$k \circ \xi' = \xi.$$

2) Similar to 1). □

Lemma 3.27. *Let $\mathcal{C}$ be a preadditive category with $0_\mathcal{C}$ and let $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{C}$.*

1) If g is a monomorphism and $\mathrm{Ker}\,(f)$ *exists, then* $\mathrm{Ker}\,(f) = \mathrm{Ker}\,(g \circ f)$.

2) If f is an epimorphism and $\mathrm{Coker}\,(g)$ *exists, then* $\mathrm{Coker}\,(g) = \mathrm{Coker}\,(g \circ f)$.

Proof. 1) Let $(K,k) = \mathrm{Ker}\,(f)$. We prove that $(K,k) = \mathrm{Ker}\,(g \circ f)$. By definition of k we have $f \circ k = 0$ so that we get $g \circ f \circ k = 0$. Now, let $\xi : X \longrightarrow A$ $g \circ f \circ \xi = 0$. Since g is a monomorphism we get that $f \circ \xi = 0$ and hence, since $(K,k) = \mathrm{Ker}\,(f)$, there exists a unique morphism $\xi' : X \longrightarrow K'$ such that $\xi = k' \circ \xi'$.

2) Let $(Q,\chi) = \mathrm{Coker}\,(g)$ and let us prove that $(Q,\chi) = \mathrm{Coker}\,(g \circ f)$. By definition of χ we have that $0 = \chi \circ g = 0$ so that we get $\chi \circ g \circ f = 0$.Now let $\eta : C \longrightarrow Y$ such that $\eta \circ g \circ f = 0$. Since f is an epimorphism, we get that $\eta \circ g = 0$. Since $(Q,\chi) = \mathrm{Coker}\,(f)$, there exists a unique $\eta' : Q \longrightarrow Y$ such that $\eta = \eta' \circ \chi$. □

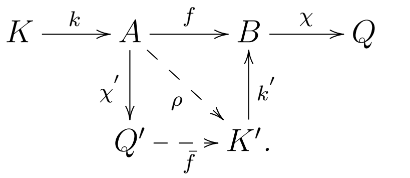
Lemma 3.28. *Let $\mathcal{C}$ be a preadditive category with $0_{\mathcal{C}}$, let $f : A \longrightarrow B$ be a morphism and assume that there exist $(K,k) = \mathrm{Ker}\,(f)$ and $(Q,\chi) = \mathrm{Coker}\,(f)$. Let $\alpha : L \longrightarrow K$ and $\beta : Q \longrightarrow P$ be isomorphisms. Then*

1) $(L, k \circ \alpha) = \mathrm{Ker}\,(f)$

2) $(P, \beta \circ \chi) = \mathrm{Coker}\,(f)$.

Proof. It follows by Propositions 3.19 and 3.23. □

Remark 3.29. *Let $\mathcal{C}$ be a preadditive category with $0_{\mathcal{C}}$, kernels and cokernels. Let $f : A \longrightarrow B$ be a morphism in $\mathcal{C}$ and let $(K,k) = \mathrm{Ker}\,(f)$ and $(Q,\chi) = \mathrm{Coker}\,(f)$. Let $(Q',\chi') = \mathrm{Coker}\,(k)$ and $(K',k') = \mathrm{Ker}\,(\chi)$:*



Since $(K',k') = \mathrm{Ker}\,(\chi)$ and $\chi \circ f = 0$, there exists a unique morphism $\rho : A \longrightarrow K'$ such that $k' \circ \rho = f$; then $0 = f \circ k = k' \circ \rho \circ k$ and since k' is a monomorphism we have $\rho \circ k = 0$. As $(Q',\chi') = \mathrm{Coker}\,(k)$ there exists a unique morphism $\overline{f} : Q' \longrightarrow K'$ such that $\overline{f} \circ \chi' = \rho$. Finally we have

$$f = k' \circ \rho = k' \circ \overline{f} \circ \chi'.$$

In general, $\overline{f}$ is not an isomorphism.

Definition 3.30. *We say that a preadditive category $\mathcal{C}$ with $0_{\mathcal{C}}$, kernels and cokernels satisfies the* Ab property *if, for every morphism f, $\overline{f}$ as in Remark 3.29 is an isomorphism.*

Definition 3.31. *A preadditive category $\mathcal{C}$ with $0_{\mathcal{C}}$, kernels and cokernels satisfying the Ab property is called* preabelian category.

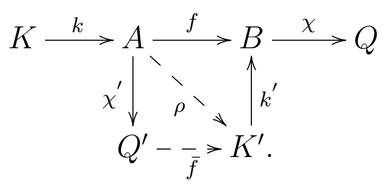
Theorem 3.32. *Let $\mathcal{C}$ be a preadditive category with $0_{\mathcal{C}}$, kernels and cokernels. Then $\mathcal{C}$ is preabelian, i.e. $\mathcal{C}$ satisfies the property* Ab, *if and only if for every morphism $f : A \to B$ there exist a kernel (X,ξ) and a cokernel (X,η) such that $f = \xi \circ \eta$. In this case*

$$(X,\xi) = \mathrm{KerCoker}\,(f) \text{ and } (X,\eta) = \mathrm{CokerKer}\,(f).$$

Proof. Let $(K,k) = \mathrm{Ker}\,(f)$ and $(Q,\chi) = \mathrm{Coker}\,(f)$, $(K',k') = \mathrm{Ker}\,(\chi)$, $(Q',\chi') = \mathrm{Coker}\,(k)$ and $\overline{f} : Q' \to K'$ as in Remark 3.29 so that

$$f = k' \circ \overline{f} \circ \chi'$$

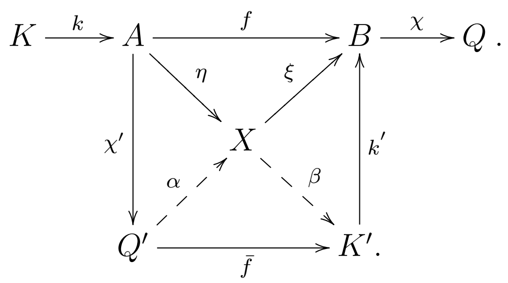
i.e. the following diagram



is commutative.

Assume first that $\mathcal{C}$ satisfies property Ab i.e. that $\overline{f}$ is an isomorphism. Then, by Lemma 3.28, we have that $\left(Q', \overline{f} \circ \chi'\right) = \mathrm{Coker}\,(k)$. Thus $f = k' \circ \left(\overline{f} \circ \chi'\right)$ where (K',k') is a kernel and $\left(K', \overline{f} \circ k'\right)$ is a cokernel.

Conversely, assume that for every morphism f, there exist $(X,\xi) = \mathrm{Ker}\,(w)$ and $(X,\eta) = \mathrm{Coker}\,(\zeta)$ such that $f = \xi \circ \eta$. Then we can consider the following diagram



Since ξ is a kernel, ξ is a monomorphism so that, by Lemma 3.27, we have

$$\mathrm{Ker}\,(\eta) = \mathrm{Ker}\,(\xi \circ \eta) = \mathrm{Ker}\,(f) = (K,k)\,.$$

Since η is a cokernel, then by Proposition 3.26 we have

$$(X,\eta) = \mathrm{CokerKer}\,(\eta) = \mathrm{Coker}\,(k)\,.$$

Since $(Q',\chi') = \mathrm{Coker}\,(k)$ and cokernels are unique up to isomorphism, there exists an isomorphism $\alpha : Q' \longrightarrow X$ such that

$$\alpha \circ \chi' = \eta.$$

Since η is a cokernel, η is an epimorphism so that, by Lemma 3.27, we have

$$\mathrm{Coker}\,(\xi) = \mathrm{Coker}\,(\xi \circ \eta) = \mathrm{Coker}\,(f) = (Q,\chi)\,.$$

Since ξ is a kernel, by Proposition 3.26,

$$(X,\xi) = \mathrm{KerCoker}\,(\xi) = \mathrm{Ker}\,(\chi)\,.$$

Since $\text{Ker}\,(\chi) = (K', k')$, then there exists an isomorphism $\beta : X \longrightarrow K'$ such that

$$k' \circ \beta = \xi.$$

Then we have

$$f = k' \circ \overline{f} \circ \chi'$$

and

$$f = \xi \circ \eta = k' \circ \beta \circ \alpha \circ \chi'$$

and since k' is a kernel and thus a monomorphism and χ' is a cokernel and so an epimorphism, we deduce that

$$\overline{f} = \beta \circ \alpha$$

where α and β are isomorphism. Therefore $\overline{f}$ is also an isomorphism. □

Lemma 3.33. *Consider the morphisms* $Z \xrightarrow{0_A^Z} A \xrightarrow{\text{Id}_A} A$ *and* $B \xrightarrow{\text{Id}_B} B \xrightarrow{0_W^B} W$ *in a preadditive category* $\mathcal{C}$ *with* $0_\mathcal{C}$*, kernels and cokernels. We have*

$$(A, \text{Id}_A) = \text{Coker}\left(0_A^Z\right) \quad \textit{and} \quad (B, \text{Id}_B) = \text{Ker}\left(0_W^B\right).$$

Proof. Clearly $\text{Id}_A \circ 0_A^Z = 0_A^Z$. Now, let $\eta : A \longrightarrow Y$ such that $\eta \circ 0_A^Z = 0_Y^Z$. Clearly $\eta = \eta \circ \text{Id}_A$. Let $\eta' : A \longrightarrow Y$ such that $\eta = \eta' \circ \text{Id}_A$. Then $\eta' = \eta$. Thus $(A, \text{Id}_A) = \text{Coker}\left(0_A^Z\right)$.

Clearly $0_W^B \circ \text{Id}_B = 0_W^B$. Let $\lambda : X \longrightarrow B$ be a morphism such that $0_W^B \circ \lambda = 0_W^X$. Then, of course, we have $\text{Id}_B \circ \lambda = \lambda$ and thus $(B, \text{Id}_B) = \text{Ker}\left(0_W^B\right)$. □

Proposition 3.34. *Let* $\mathcal{C}$ *be a preabelian category and let* $f : A \longrightarrow B$*. Then*

1) *f is an isomorphism if and only if f is a monomorphism and an epimorphism;*

2) *f is a monomorphism if and only if* $(A, f) = \text{KerCoker}\,(f)$*;*

3) *f is an epimorphism if and only if* $(B, f) = \text{CokerKer}\,(f)$*.*

Proof. 1) In view of Proposition 1.8, we already know that an isomorphism is both a monomorphism and an epimorphism .

Conversely, let f be a monomorphism and an epimorphism. Then, by Theorem 3.25, we have $\text{Ker}\,(f) = \left(0_\mathcal{C}, 0_A^{0_\mathcal{C}}\right)$ and $\text{Coker}\,(f) = \left(0_\mathcal{C}, 0_{0_\mathcal{C}}^B\right)$. Since by Lemma 3.33 $\text{Ker}\left(0_{0_\mathcal{C}}^B\right) = (B, \text{Id}_B)$ and $\text{Coker}\left(0_A^{0_\mathcal{C}}\right) = (A, \text{Id}_A)$, the decomposition of Remark 3.29 is given by $f = \text{Id}_B \circ \overline{f} \circ \text{Id}_A = \overline{f}$ which is an isomorphism since $\mathcal{C}$ is preabelian. Thus f is an isomorphism.

2) Assume that f is a monomorphism. By Theorem 3.32,

$$f = \xi \circ \eta.$$

where $(X, \xi) = \text{KerCoker}\,(f)$ and $(X, \eta) = \text{CokerKer}\,(f)$.

Since f is a monomorphism, by Lemma 3.1, also η is a monomorphism. Since (X, η) is a cokernel, by Proposition 3.22, η is an epimorphism. Therefore, by 1), η is an isomorphism so that, in view of 1) in Lemma 3.28, we get

$$\mathrm{KerCoker}(f) = (A, \xi \circ \eta) = (A, f).$$

Conversely if $(A, f) = \mathrm{KerCoker}(f)$, then, by Proposition 3.18, f is a monomorphism

3) It is analogous to 2) and it is left as an exercise to the reader. □

Definitions 3.35. *Let $\mathcal{C}$ be a preadditive category $\mathcal{C}$ with $0_{\mathcal{C}}$, kernels and cokernels*

- *The* image *of a morphism f, that will be denoted by* $\mathrm{Im}(f)$*, is defined by setting*

$$\mathrm{Im}(f) = \mathrm{KerCoker}(f).$$

- *The* coimage *of a morphism f, that will be denoted by* $\mathrm{Coim}(f)$*, is defined by setting*

$$\mathrm{Coim}(f) = \mathrm{CokerKer}(f).$$

Corollary 3.36. *Let $\mathcal{C}$ be a preabelian category and let $f : A \to B$ be a morphism in $\mathcal{C}$. Then*

$$\mathrm{Im}(f) \cong \mathrm{Coim}(f).$$

Moreover

1) *f is a monomorphism if and only if* $\mathrm{Im}(f) = (A, f)$.

2) *f is an epimorphism if and only if* $\mathrm{Coim}(f) = (B, f)$.

Proof. The first statement follows by the property Ab. 1) and 2) are obtained by applying Proposition 3.34. □

3.2 Products, Coproducts and Biproducts

Definition 3.37. *Let $(C_i)_{i \in I}$ be a family of objects in the category $\mathcal{C}$. A* product *of such a family in $\mathcal{C}$ is an ordered pair $\left(P, (\pi_i)_{i \in I}\right)$ where*

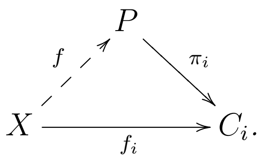
1) $P \in \mathcal{C}$

2) $\pi_i : P \longrightarrow C_i$ is a morphism in $\mathcal{C}$ for every $i \in I$

3) if $(f_i)_{i \in I}$ is a family of morphisms in $\mathcal{C}$ where $f_i : X \longrightarrow C_i$, then there exists a unique morphism $f : X \longrightarrow P$ such that

$$\pi_i \circ f = f_i$$

for every $i \in I$, i.e. the following diagrams

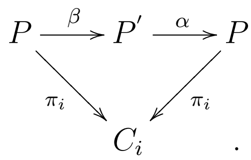


are commutative.

Theorem 3.38. *Let $\left(P, (\pi_i)_{i\in I}\right)$ and $\left(P', (\pi'_i)_{i\in I}\right)$ be products in the category $\mathcal{C}$ of the family $(C_i)_{i\in I}$. Then there exists a morphism $\alpha : P' \longrightarrow P$ such that $\pi_i \circ \alpha = \pi'_i$ for every $i \in I$. Moreover this morphism is unique with respect to this property and it is an isomorphism.*

Proof. Apply definition of product to $\left(P, (\pi_i)_{i\in I}\right)$ and "f_i" $= \pi'_i$. Then there exists a unique morphism $\alpha : P' \longrightarrow P$ such that $\pi_i \circ \alpha = \pi'_i$ for every $i \in I$. Now, apply definition of product to $\left(P', (\pi'_i)_{i\in I}\right)$ and "f_i" $= \pi_i$. Then there exists a unique morphism $\beta : P \longrightarrow P'$ such that $\pi'_i \circ \beta = \pi_i$. Then we have

$$\pi_i \circ \alpha \circ \beta = \pi_i \text{ and } \pi'_i \circ \beta \circ \alpha = \pi'_i.$$



By definition of product there exists a unique morphism $f : P \longrightarrow P$ such that $\pi_i \circ f = \pi_i$. Since

$$\pi_i \circ \mathrm{Id}_P = \pi_i = \pi_i \circ (\alpha \circ \beta),$$

we get $\alpha \circ \beta = \mathrm{Id}_P$. Similarly, there exists a unique morphism $f : P' \longrightarrow P'$ such that $\pi'_i \circ f = \pi'_i$. Since

$$\pi'_i \circ \mathrm{Id}_{P'} = \pi'_i = \pi'_i \circ (\beta \circ \alpha),$$

we deduce that $\beta \circ \alpha = \mathrm{Id}_{P'}$. Therefore α is an isomorphism. □

Notation 3.39. *In the following, we denote a product of the family $(C_i)_{i\in I}$ in $\mathcal{C}$ by $\left(\prod_{i\in I} C_i, (\pi_i)_{i\in I}\right)$. The unique morphism f is denoted by $\Delta(f_i)_{i\in I}$ and it is called* diagonal morphism *of the family of morphisms $(f_i)_{i\in I}$.*

Notation 3.40. *Let $\mathcal{C}$ be a preadditive category $\mathcal{C}$ and assume that the product $\left(\prod_{i\in I} C_i, (\pi_i)_{i\in I}\right)$ of the family $(C_i)_{i\in I}$ exists. For every $j \in I$ consider the family of morphisms $(\delta_{ji})_{i\in I}$ where $\delta_{ji} = \mathrm{Id}_{C_j}$ if $j = i$ and $\delta_{ji} = 0_{C_i}^{C_j}$ if $j \neq i$. We denote by $e_j : C_j \longrightarrow \prod_{i\in I} C_i$ the diagonal morphism of the family of morphisms $(\delta_{ji})_{i\in I}$. This means that*

$$\pi_i \circ e_j = 0_{C_i}^{C_j} : C_j \longrightarrow C_i \qquad \text{if } i \neq j$$
$$\pi_i \circ e_j = \mathrm{Id}_{C_j} : C_j \longrightarrow C_j \qquad \text{if } i = j.$$

Proposition 3.41. *Let $\mathcal{C}$ be a preadditive category $\mathcal{C}$. If the product $\left(\prod_{i\in I} C_i, (\pi_i)_{i\in I}\right)$ of the family $(C_i)_{i\in I}$ exists, then every π_i is an epimorphism.*

Proof. Let us fix a $j \in I$ and let $g, h : C_j \longrightarrow X$ be such that

$$g \circ \pi_j = h \circ \pi_j. \tag{3.1}$$

We get

$$g = g \circ \mathrm{Id}_{C_j} = g \circ \pi_j \circ e_j \overset{(3.1)}{=} h \circ \pi_j \circ e_j = h \circ \mathrm{Id}_{C_j} = h$$

and thus π_j is an epimorphism. □

Exercise 3.42. *Let $\mathcal{C}$ be a preadditive category $\mathcal{C}$ with $0_{\mathcal{C}}$ and assume that the product $\left(\prod_{i\in I} C_i, (\pi_i)_{i\in I}\right)$ of the family $(C_i)_{i\in I}$ exists. Let*

$$\alpha, \beta : X \to \prod_{i\in I} C_i$$

be morphisms in $\mathcal{C}$. Show that

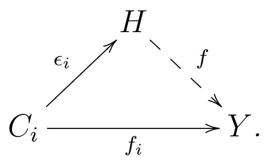
$$\alpha = \beta \iff \pi_i \circ \alpha = \pi_i \circ \beta \textit{ for every } \iota \in I.$$

Definition 3.43. *Let $(C_j)_{j\in I}$ be a family of objects in a category $\mathcal{C}$. A* coproduct *of such a family in $\mathcal{C}$ is an ordered pair $\left(H, (\varepsilon_i)_{i\in I}\right)$ where*

1) $H \in \mathcal{C}$

2) $\varepsilon_i : C_i \longrightarrow H$ is a morphism in $\mathcal{C}$ for every $i \in I$

3) if $(f_i)_{i\in I}$ is a family of morphisms in $\mathcal{C}$ where $f_i : C_i \longrightarrow Y$, then there exists a unique morphism $f : H \longrightarrow Y$ such that $f \circ \varepsilon_i = f_i$ for every $i \in I$, i.e. the following diagrams



are commutative.

Theorem 3.44. *Let $\left(H, (\varepsilon_i)_{i\in I}\right)$ and $\left(H', (\varepsilon'_i)_{i\in I}\right)$ be coproducts of a family $(C_i)_{i\in I}$ of objects in a category $\mathcal{C}$. Then there exists a morphism $\alpha : H \longrightarrow H'$ such that $\alpha \circ \varepsilon_i = \varepsilon'_i$ for every $i \in I$. Moreover this morphism is unique withrespect to this property and it is an isomorphism.*

Proof. Apply definition of coproduct to $\left(H, (\varepsilon_i)_{i\in I}\right)$ and "f_i" $= \varepsilon'_i$. Then there exists a unique morphism $\alpha : H \longrightarrow H'$ such that $\alpha \circ \varepsilon_i = \varepsilon'_i$ for every $i \in I$. Now, apply definition of coproduct to $\left(H', (\varepsilon'_i)_{i\in I}\right)$ and "f_i" $= \varepsilon_i$. Then there exists a unique morphism $\beta : H' \longrightarrow H$ such that $\beta \circ \varepsilon'_i = \varepsilon_i$. Then we have

$$\beta \circ \alpha \circ \varepsilon_i = \varepsilon_i \text{ and } \alpha \circ \beta \circ \varepsilon'_i = \varepsilon'_i.$$

By definition of coproduct there exists a unique morphism $f : H \longrightarrow H$ such that $f \circ \varepsilon_i = \varepsilon_i$. Since

$$\mathrm{Id}_H \circ \varepsilon_i = \varepsilon_i = (\beta \circ \alpha) \circ \varepsilon_i,$$

we get $\beta \circ \alpha = \mathrm{Id}_H$. Similarly, there exists a unique morphism $f : H' \longrightarrow H'$ such that $f \circ = \varepsilon_i'$. Since

$$\mathrm{Id}_{H'} \circ \varepsilon_i' = \varepsilon_i' = (\alpha \circ \beta) \circ \varepsilon_i',$$

we deduce that $\alpha \circ \beta = \mathrm{Id}_{H'}$. Therefore α is an isomorphism. □

Remark 3.45. *A coproduct of a family $(C_i)_{i \in I}$ in $\mathcal{C}$ is a product of the family $(C_i)_{i \in I}$ in $\mathcal{C}^o$.*

Notation 3.46. *We denote by $\left(\coprod_{i \in I} C_i, (\varepsilon_i)_{i \in I}\right)$ the coproduct of the family $(C_i)_{i \in I}$ in $\mathcal{C}$ and by $\nabla (f_i)_{i \in I}$ the unique morphism f and it is called* codiagonal morphism.

Notation 3.47. *Let $\mathcal{C}$ be a preadditive category $\mathcal{C}$ with $0_{\mathcal{C}}$ and assume that the coproduct $\left(\coprod_{i \in I} C_i, (\varepsilon_i)_{i \in I}\right)$ of the family $(C_i)_{i \in I}$ exists. For every $j \in I$ consider the family of morphisms $(\delta_{ji})_{i \in I}$ where $\delta_{ji} = \mathrm{Id}_{C_j}$ if $j = i$ and $\delta_{ji} = 0_{C_i}^{C_j}$ if $j \neq k$. We denote by $p_j : \coprod_{i \in I} C_i \longrightarrow C_j$ the codiagonal morphism of the family of morphisms $(\delta_{ji})_{i \in I}$. This means that*

$$\begin{aligned} p_j \circ \varepsilon_i &= 0_{C_j}^{C_i} : C_i \longrightarrow C_j \qquad \text{if } i \neq j \\ p_j \circ \varepsilon_i &= \mathrm{Id}_{C_j} : C_j \longrightarrow C_j \qquad \text{if } i = j. \end{aligned}$$

Proposition 3.48. *Let $\mathcal{C}$ be a preadditive category $\mathcal{C}$. If the coproduct $\left(\coprod_{i \in I} C_i, (\varepsilon_i)_{i \in I}\right)$ of the family $(C_i)_{i \in I}$ exists, then every ε_i is a monomorphism.*

Proof. Let us fix a $j \in I$ and let $g, h : X \longrightarrow C_j$ be such that

$$\varepsilon_j \circ g = \varepsilon_j \circ h. \tag{3.2}$$

We get

$$g = \mathrm{Id}_{C_j} \circ g = p_j \circ \varepsilon_j \circ g \overset{(3.2)}{=} p_j \circ \varepsilon_j \circ h = \mathrm{Id}_{C_j} \circ h = h$$

and thus ε_j is an monomorphism. □

Exercise 3.49. *Let $\mathcal{C}$ be a preadditive category $\mathcal{C}$ and assume that the coproduct $\left(\coprod_{i \in I} C_i, (\varepsilon_i)_{i \in I}\right)$ of the family $(C_i)_{i \in I}$ exists. Let*

$$\alpha, \beta : \coprod_{i \in I} C_i \to X$$

be morphisms in $\mathcal{C}$. Show that

$$\alpha = \beta \Longleftrightarrow \alpha \circ \varepsilon_i = \beta \circ \varepsilon_i \text{ for every } \iota \in I.$$

Definition 3.50. *Let $\mathcal{C}$ be a preadditive category with $0_{\mathcal{C}}$, let $I = \{1, \ldots, n\}$ and let $(C_i)_{i\in I}$ be a family of objects in $\mathcal{C}$. A* biproduct *of such a family in $\mathcal{C}$ is a triple $\left(Q, (\varepsilon_i)_{i\in I}, (\pi_i)_{i\in I}\right)$ where*

1) $Q \in \mathcal{C}$

2) $\varepsilon_i : C_i \longrightarrow Q$ and $\pi_i : Q \longrightarrow C_i$ are morphisms in $\mathcal{C}$ for every $i \in I$ such that

$$\pi_k \circ \varepsilon_j = \delta_{jk} \qquad \sum_{k\in I} \varepsilon_k \circ \pi_k = \mathrm{Id}_Q$$

where $\delta_{jk} = \mathrm{Id}_{C_j}$ if $j = k$ and $\delta_{jk} = 0_{C_k}^{C_j}$ if $j \neq k$.

Lemma 3.51. *Let $\left(Q, (\varepsilon_i)_{i\in I}, (\pi_i)_{i\in I}\right)$ be a biproductof a family $(C_i)_{i\in I}$ of objects in $\mathcal{C}$ where $I = \{1, \ldots, n\}$. Then $\left(Q, (\varepsilon_i)_{i\in I}\right)$ is a coproduct of the family $(C_i)_{i\in I}$ and $\left(Q, (\pi_i)_{i\in I}\right)$ is a product of the family $(C_i)_{i\in I}$.*

Proof. Let us show that $\left(Q, (\varepsilon_i)_{i\in I}\right)$ is a coproduct of the family $(C_i)_{i\in I}$. Let $(f_i : C_i \to X)_{i\in I}$ be a family of morphism in $\mathcal{C}$. □

Theorem 3.52. *Let $\left(Q, (\varepsilon_i)_{i\in I}, (\pi_i)_{i\in I}\right)$ and $\left(Q', (\varepsilon'_i)_{i\in I}, (\pi'_i)_{i\in I}\right)$ be biproducts of a family $(C_i)_{i\in I}$ of objects in a preadditive category $\mathcal{C}$ where $I = \{1, \ldots, n\}$. Then there exists a morphism $\alpha : Q \longrightarrow Q'$ such that*

$$\alpha \circ \varepsilon_i = \varepsilon'_i \text{ for every } i \in I.$$

Moreover α is unique with respect to this property, and $\pi'_i \circ \alpha = \pi_i$ for every $i \in I$ and α is an isomorphism.

Proof. By Lemma 3.51, both $\left(Q, (\varepsilon_i)_{i\in I}\right)$ and $\left(Q', (\varepsilon'_i)_{i\in I}\right)$ are coproducts of the family $(C_i)_{i\in I}$. By Theorem 3.44, there is a morphism $\alpha : Q \longrightarrow Q'$ such that $\alpha \circ \varepsilon_i = \varepsilon'_i$ for every $i \in I$. Moreover this morphism is unique with respect to this property and it is an isomorphism. We have

$$\pi'_i \circ \alpha = \pi'_i \circ \alpha \circ \mathrm{Id}_Q = \pi'_i \circ \alpha \circ \sum_{j\in I} \varepsilon_j \circ \pi_j = \pi'_i \circ \sum_{j\in I} \alpha \circ \varepsilon_j \circ \pi_j = \sum_{j\in I} \pi'_i \circ \varepsilon'_j \circ \pi_j = \sum_{j\in I} \delta_{ij} \circ \pi_j = \pi_i.$$

□

Notation 3.53. *Let $I = \{1, \ldots, n\}$. In the following, we denote by $\left(\bigtimes_{i\in I} C_i, (\varepsilon_i)_{i\in I}, (\pi_i)_{i\in I}\right)$ the biproduct of the family $(C_i)_{i\in I}$ in $\mathcal{C}$.*

Theorem 3.54. *Let $\mathcal{C}$ be a preadditive category, let $I = \{1, \ldots, n\}$ and let $(C_i)_{i\in I}$ be a family of objects in $\mathcal{C}$. The following statements are equivalent:*

(a) there exists the product of the family $(C_i)_{i\in I}$ in $\mathcal{C}$;

(b) there exists the biproduct of the $(C_i)_{i\in I}$ family in $\mathcal{C}$;

(c) there exists the coproduct of the family $(C_i)_{i\in I}$ in $\mathcal{C}$.

Moreover, if one of the statements holds, every product is a biproduct, every coproduct is a biproduct and every biproduct is both product and coproduct in $\mathcal{C}$.

Proof. $(a) \Rightarrow (b)$. Consider the family of morphisms $(e_i)_{i\in I}$ of notation 3.40. We will prove that $\left(\prod_{i\in I} C_i, (e_i)_{i\in I}, (\pi_i)_{i\in I}\right)$ is the biproduct of the family $(C_i)_{i\in I}$ in $\mathcal{C}$. By construction, we have that

$$\pi_i \circ e_j = \delta_{ij}$$

so that the first property of the biproduct holds. Now we have to prove that $\sum_{k\in I} e_k \circ \pi_k = \mathrm{Id}_{\prod_{i\in I} C_i}$. In fact we have

$$\pi_i \circ \left(\sum_{k\in I} e_k \circ \pi_k\right) = \sum_{k\in I} \pi_i \circ e_k \circ \pi_k = \sum_{k\in I} \delta_{ik} \circ \pi_k = \pi_i \text{ for every } i \in I$$

Since we also have

$$\pi_i \circ \mathrm{Id}_{\prod_{i\in I} C_i} = \pi_i, \text{ for every } i \in I,$$

by the uniqueness of the morphism t such

$$\pi_i \circ t = \pi_i, \text{ for every } i \in I,$$

that we deduce that

$$\sum_{k\in I} e_k \circ \pi_k = \mathrm{Id}_{\prod_{i\in I} C_i}.$$

$(b) \Rightarrow (a)$. It follows by Lemma 3.51.

$(c) \Rightarrow (b)$. .Consider the family of morphisms $(p_i)_{i\in I}$ of notation 3.47 . We will prove that $\left(\coprod_{i\in I} C_i, (\varepsilon_i)_{i\in I}, (p_i)_{i\in I}\right)$ is the biproduct of the family $(C_i)_{i\in I}$ in $\mathcal{C}$. By construction, we have that

$$p_j \circ \varepsilon_i = \delta_{ij}$$

so that the first property of the biproduct holds. Now we have to prove that $\sum_{k\in I} \varepsilon_k \circ p_k = \mathrm{Id}_{\coprod_{i\in I} C_{ii}}$. In fact we have

$$\left(\sum_{k\in I} \varepsilon_k \circ p_k\right) \circ \varepsilon_i = \sum_{k\in I} \varepsilon_k \circ p_k \circ \varepsilon_i = \sum_{k\in I} \varepsilon_k \circ \delta_{ik} = \varepsilon_i, \text{ for every } i \in I,$$

Since we also have

$$\mathrm{Id}_{\coprod_{i\in I} C_i} \circ \varepsilon_i = \varepsilon_i, \text{ for every } i \in I,$$

by the uniqueness of the morphism t such

$$t \circ \varepsilon_i = \varepsilon_i, \text{ for every } i \in I,$$

that we deduce that

$$\sum_{k\in I} \varepsilon_k \circ p_k = \mathrm{Id}_{\coprod_{i\in I} C_{ii}}.$$

$(b) \Rightarrow (c)$. It follows by Lemma 3.51. □

Definition 3.55. *An* abelian category *is a preabelian category where every finite family of objects has a product.*

3.3 Exact sequences

Definition 3.56. *Let $\mathcal{C}$ be a preabelian category and let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be morphisms in $\mathcal{C}$. The sequence*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is called exact *if* $\mathrm{Ker}\,(g) = \mathrm{Im}\,(f)$.

Lemma 3.57. *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be an exact sequence in a preabelian category $\mathcal{C}$. Then*

1) $g \circ f = 0$;

2) f is a monomorphism $\Leftrightarrow$ $(A, f) = \mathrm{Ker}\,(g)$;

3) g is an epimorphism $\Leftrightarrow$ $(C, g) = \mathrm{Coker}\,(f)$.

Proof. 1) Let $(K, k) = \mathrm{Ker}\,(g)$ and let $(Q, \chi) = \mathrm{Coker}\,(f)$. Since the sequence is exact, we have

$$(K, k) = \mathrm{Ker}\,(g) = \mathrm{Im}\,(f) = \mathrm{KerCoker}\,(f) = \mathrm{Ker}\,(\chi).$$

and $\chi \circ f \overset{(Q,\chi)=\mathrm{Coker}(f)}{=} 0$ there exists a unique morphism $\xi : A \longrightarrow K$ such that $f = k \circ \xi$ and thus

$$g \circ f = g \circ k \circ \xi \overset{(K,k)=\mathrm{Ker}(g)}{=} 0 \circ \xi = 0$$

since $(K, k) = \mathrm{Ker}\,(g)$.

2) If f is a monomorphism, by Proposition 3.34, we have $(A, f) = \mathrm{KerCoker}\,(f) = \mathrm{Im}\,(f) = \mathrm{Ker}\,(g)$. The converse follows in vie of Proposition 3.18.

3) If g is an epimorphism, by Proposition 3.34, we have

$$\begin{aligned}(C, g) &= \mathrm{CokerKer}\,(g) = \mathrm{CokerIm}\,(f) \\ &= \mathrm{CokerKerCoker}\,(f) = \mathrm{Coker}\,(f)\end{aligned}$$

where the last equality holds by Proposition 3.34 since $\mathrm{Coker}\,(f)$ is an epimorphism. The converse follows in view of Proposition 3.22. □

Definition 3.58. *A* sequence *of morphisms*

$$0_{\mathcal{C}} \to C_1 \xrightarrow{f} C \xrightarrow{g} C_2 \to 0_{\mathcal{C}}$$

in a preabelian category $\mathcal{C}$ is called short exact *if we have*

1) $0_{\mathcal{C}} \to C_1 \xrightarrow{f} C$ *is exact, i.e.* $\mathrm{Im}\left(0_{C_1}^{0_{\mathcal{C}}}\right) = \mathrm{Ker}\,(f)$;

2) $C_1 \xrightarrow{f} C \xrightarrow{g} C_2$ *is exact, i.e.* $\mathrm{Im}\,(f) = \mathrm{Ker}\,(g)$;

3) $C \xrightarrow{g} C_2 \to 0_{\mathcal{C}}$ *is exact, i.e.* $\mathrm{Im}\,(g) = \mathrm{Ker}\left(0_{0_{\mathcal{C}}}^{C_2}\right)$.

Lemma 3.59. *Let $\mathcal{C}$ be a preabelian category. Then* $\mathrm{Ker}\left(0_B^A\right) = (A, \mathrm{Id}_A)$ *and* $\mathrm{Im}\left(0_B^A\right) = \left(0_{\mathcal{C}}, 0_B^{0_{\mathcal{C}}}\right)$.

Proof. The first equality follows by Lemma 3.33. Moreover we have

$$\mathrm{Im}\left(0_B^A\right) = \mathrm{KerCoker}\left(0_B^A\right) = \mathrm{Ker}\left(\mathrm{Id}_B\right) = \left(0_{\mathcal{C}}, 0_B^{0_{\mathcal{C}}}\right)$$

where the second equality follows by 3.33 and the third one by Theorem 3.25 □

Lemma 3.60. *Let $\mathcal{C}$ be a preabelian category and let $f : A \to B$ be a morphism in $\mathcal{C}$. Then the following are equivalent:*

(a) *f is an epimorphism;*

(b) $\mathrm{Im}\,(f) = \mathrm{Ker}\left(0_{0_{\mathcal{C}}}^B\right)$*;*

(c) $\mathrm{Im}\,(f) = (B, \mathrm{Id}_B)$*.*

Proof. $(a) \Rightarrow (b)$ In view of Theorem 3.25 f is an epimorphism if and only if $\mathrm{Coker}\,(f) = \left(0_{\mathcal{C}}, 0_{0_{\mathcal{C}}}^B\right)$. Thus if f is an epimorphism, we have $\mathrm{Im}\,(f) = \mathrm{KerCoker}\,(f) = \mathrm{Ker}\left(0_{0_{\mathcal{C}}}^B\right)$.

$(b) \Leftrightarrow (c)$ By Lemma 3.59, $\mathrm{Ker}\left(0_{0_{\mathcal{C}}}^B\right) = (B, \mathrm{Id}_B)$.

$(c) \Rightarrow (a)$ If $\mathrm{Im}\,(f) = \mathrm{Ker}\left(0_{0_{\mathcal{C}}}^B\right)$, we have

$$\mathrm{Coker}\,(f) \overset{3.26}{=} \mathrm{CokerKerCoker}\,(f) = \mathrm{CokerIm}\,(f) = \mathrm{Coker}\left(\mathrm{Id}_B\right) \overset{3.25}{=} \left(0_{\mathcal{C}}, 0_{0_{\mathcal{C}}}^{C_2}\right).$$

In view of Theorem 3.25, f is an epimorphism. □

Proposition 3.61. *A sequence of morphisms*

$$0_{\mathcal{C}} \to C_1 \xrightarrow{f} C \xrightarrow{g} C_2 \to 0_{\mathcal{C}}$$

in a preabelian category $\mathcal{C}$ is short exact if and only if

1) $\left(0_{\mathcal{C}}, 0_{C_1}^{0_{\mathcal{C}}}\right) = \mathrm{Ker}\,(f)$ *i.e. f is a monomorphism;*

2) $\mathrm{Im}\,(f) = \mathrm{Ker}\,(g)$;

3) $\mathrm{Im}\,(g) = (C_2, \mathrm{Id}_{C_2})$ *i.e. g is an epimorphism.*

Proof. By Lemma 3.59, we have $\mathrm{Im}\left(0_{C_1}^{0_{\mathcal{C}}}\right) = \left(0_{\mathcal{C}}, 0_{C_1}^{0_{\mathcal{C}}}\right)$ and $\mathrm{Ker}\left(0_{0_{\mathcal{C}}}^{C_2}\right) = (C_2, \mathrm{Id}_{C_2})$. The, by Theorem 3.25 f is a monomorphism if and only if $\mathrm{Ker}\,(f) = \left(0_{\mathcal{C}}, 0_A^{0_{\mathcal{C}}}\right)$ and by Lemma 3.60 g is an epimorphism if and only if $\mathrm{Im}\,(g) = (C_2, \mathrm{Id}_{C_2})$. □

Proposition 3.62. *A sequence of morphisms*

$$0_{\mathcal{C}} \to C_1 \xrightarrow{f} C \xrightarrow{g} C_2 \to 0_{\mathcal{C}}$$

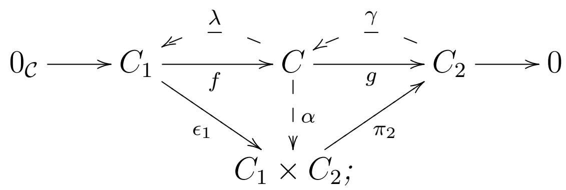
in a preabelian category $\mathcal{C}$ is short exact if and only if

1) $(C_1, f) = \mathrm{Ker}\,(g)$*;*

2) $(C_2, g) = \mathrm{Coker}\,(f)$.

Proof. Assume that the sequence is exact. Then, by Proposition 3.61, f is mono, g is epi. Then, since the sequence $C_1 \xrightarrow{f} C \xrightarrow{g} C_2$ is exact, by Lemma 3.57, we get $(C_1, f) = \mathrm{Ker}\,(g)$ and $(C_2, g) = \mathrm{Coker}\,(f)$. Conversely, assume that 1) and 2) hold. Then, by Proposition 3.18, f is a monomorphism and, by Proposition 3.22, g is an epimorphism. Moreover, in view of 2) $\mathrm{Im}\,(f) = \mathrm{KerCoker}\,(f) = \mathrm{Ker}\,(g)$. By Proposition 3.61, we conclude. □

Theorem 3.63. *Let $0_{\mathcal{C}} \to C_1 \xrightarrow{f} C \xrightarrow{g} C_2 \to 0_{\mathcal{C}}$ be an exact sequence in an abelian category*



Then the following statements are equivalent:

(a) *there exists $\lambda : C \longrightarrow C_1$ such that $\lambda \circ f = \mathrm{Id}_{C_1}$, i.e. f splits;*

(b) *there exists $\gamma : C_2 \longrightarrow C$ such that $g \circ \gamma = \mathrm{Id}_{C_2}$, i.e. g cosplits;*

(c) *there exists an isomorphism $\alpha : C \longrightarrow \bigtimes_{i\in\{1,2\}} C_i$ such that*

$$\alpha \circ f = \varepsilon_1 \qquad \text{and} \qquad \pi_2 \circ \alpha = g.$$

If (a) holds, we can consider $\alpha = \varepsilon_1 \circ \lambda + \varepsilon_2 \circ g$. If (b) holds, we can consider $\alpha^{-1} = f \circ \pi_1 + \gamma \circ \pi_2$.

Proof. $(a) \Rightarrow (c)$. We set $I = \{1, 2\}$.

Construction of α. Assume that $\lambda : C \longrightarrow C_1$ and $\lambda \circ f = \mathrm{Id}_{C_1}$. Let

$$\alpha = \varepsilon_1 \circ \lambda + \varepsilon_2 \circ g.$$

We have

$$\alpha \circ f = \varepsilon_1 \circ \lambda \circ f + \varepsilon_2 \circ g \circ f = \varepsilon_1 \circ \mathrm{Id}_{C_1} + \varepsilon_2 \circ 0 = \varepsilon_1$$

i.e.

$$\alpha \circ f = \varepsilon_1 \tag{3.3}$$

and

$$\pi_2 \circ \alpha = \pi_2 \circ \varepsilon_1 \circ \lambda + \pi_2 \circ \varepsilon_2 \circ g = g$$

i.e.

$$\pi_2 \circ \alpha = g. \tag{3.4}$$

We also have

$$\pi_1 \circ \alpha = \pi_1 \circ \varepsilon_1 \circ \lambda + \pi_2 \circ \varepsilon_2 \circ g = \lambda$$

i.e.

$$\pi_1 \circ \alpha = \lambda. \tag{3.5}$$

α **is an epimorphism.** Let $\xi : \bigtimes_{i\in I} C_i \longrightarrow X$ be a morphism such that $\xi \circ \alpha = 0$ then

$$\xi \circ \varepsilon_1 \overset{(3.3)}{=} \xi \circ \alpha \circ f = 0$$

so that

$$0 = \xi \circ \alpha = \xi \circ \varepsilon_1 \circ \lambda + \xi \circ \varepsilon_2 \circ g = \xi \circ \varepsilon_2 \circ g$$

and since g is an epimorphism, we deduce that $\xi \circ \varepsilon_2 = 0$. Then

$$\begin{aligned} \xi &= \xi \circ \mathrm{Id}_{\bigtimes_{i\in I} C_i} = \xi \circ (\varepsilon_1 \circ \pi_1 + \varepsilon_2 \circ \pi_2) \\ &= \xi \circ \varepsilon_1 \circ \pi_1 + \xi \circ \varepsilon_2 \circ \pi_2 = 0_X^{\bigtimes_{i\in I} C_i} \end{aligned}$$

i.e. α is an epimorphism.

α **is a monomorphism.** Let $\zeta : X \longrightarrow C$ be a morphism such that $\alpha \circ \zeta = 0$. Then, composing with π_2, we have

$$0 = \pi_2 \circ \alpha \circ \zeta \overset{(3.4)}{=} g \circ \zeta.$$

Since the given sequence is exact, by Proposition 3.61, we have that $(C_1, f) = \mathrm{Ker}\,(g)$. By the universal property of the kernel, there exists a unique morphism $\eta : X \longrightarrow C_1$ such that

$$f \circ \eta = \zeta$$

so that

$$0 = \alpha \circ \zeta = \alpha \circ f \circ \eta \overset{(3.3)}{=} \varepsilon_1 \circ \eta.$$

Since ε_1 is a monomorphism, we get that $\eta = 0$ and thus

$$\zeta = f \circ \eta = f \circ 0 = 0,$$

i.e. α is a monomorphism. By Proposition 3.34, we deduce that α is an isomorphism.

$(b) \Rightarrow (c)$.

Construction of β**.** Assume there exists $\gamma : C_2 \longrightarrow C$ such that

$$g \circ \gamma = \mathrm{Id}_{C_2}.$$

Let

$$\beta = f \circ \pi_1 + \gamma \circ \pi_2.$$

Then we have

$$\beta \circ \varepsilon_1 = (f \circ \pi_1 + \gamma \circ \pi_2) \circ \varepsilon_1 = f$$

and

$$\beta \circ \varepsilon_2 = (f \circ \pi_1 + \gamma \circ \pi_2) \circ \varepsilon_2 = \gamma.$$

Moreover we have

$$g \circ \beta = g \circ f \circ \pi_1 + g \circ \gamma \circ \pi_2 = 0 \circ \pi_1 + \mathrm{Id}_{C_2} \circ \pi_2 = \pi_2$$

i.e.

$$g \circ \beta = \pi_2 \tag{3.6}$$

β **is an epimorphism.** Let $\xi : C \longrightarrow X$ be a morphism such that $\xi \circ \beta = 0$. We have to prove that $\xi = 0$. We have

$$\begin{aligned} 0 = \xi \circ \beta = \xi \circ \beta \circ \mathrm{Id}_{\times_{i \in I} C_i} &= \xi \circ \beta \circ (\varepsilon_1 \circ \pi_1 + \varepsilon_2 \circ \pi_2) \\ &= \xi \circ \beta \circ \varepsilon_1 \circ \pi_1 + \xi \circ \beta \circ \varepsilon_2 \circ \pi_2 \\ &= 0 \circ \varepsilon_1 \circ \pi_1 + \xi \circ \gamma \circ \pi_2 = \xi \circ \gamma \circ \pi_2 \end{aligned}$$

and since π_2 is an epimorphism we get that

$$\xi \circ \gamma = 0.$$

Since g is an epimorphism by Lemma 3.57 we have $(C_2, g) = \mathrm{Coker}\,(f)$ so that from

$$0 = \xi \circ \beta \circ \varepsilon_1 = \xi \circ f$$

we infer there exists a unique $\eta : C_2 \longrightarrow X$ such that $\xi = \eta \circ g$. Then we have

$$\eta = \eta \circ \mathrm{Id}_{C_2} = \eta \circ g \circ \gamma = \xi \circ \gamma = 0.$$

Thus

$$\xi = \eta \circ g = 0 \circ g = 0.$$

β **is a monomorphism.** Let $\zeta : X \longrightarrow \times_{i \in I} C_i$ be a morphism such that $\beta \circ \zeta = 0$, we have to prove that $\zeta = 0^X_{\times_{i \in I} C_i}$. We compute

$$\pi_2 \circ \zeta = g \circ \beta \circ \zeta = g \circ 0 = 0.$$

Then we have

$$\zeta = \mathrm{Id}_{\times_{i \in I} C_i} \circ \zeta = (\varepsilon_1 \circ \pi_1 + \varepsilon_2 \circ \pi_2) \circ \zeta = \varepsilon_1 \circ \pi_1 \circ \zeta + \varepsilon_2 \circ \pi_2 \circ \zeta = \varepsilon_1 \circ \pi_1 \circ \zeta.$$

Moreover $0 = \beta \circ \zeta = \beta \circ \varepsilon_1 \circ \pi_1 \circ \zeta = f \circ \pi_1 \circ \zeta$ and since f is a monomorphism we deduce that

$$\pi_1 \circ \zeta = 0.$$

Thus

$$\zeta = \varepsilon_1 \circ \pi_1 \circ \zeta = \varepsilon_1 \circ 0 = 0.$$

By Proposition 3.34, we deduce that β is an isomorphism. Set

$$\alpha = \beta^{-1}$$

From

$$\beta \circ \varepsilon_1 = f \text{ and } g \circ \beta \overset{3.6}{=} \pi_2$$

we deduce that

$$\alpha \circ f = \varepsilon_1 \text{ and } \pi_2 \circ \alpha = g.$$

$(c) \Rightarrow (a)$. Assume that there exists an isomorphism $\alpha : C \longrightarrow \bigtimes_{i\in\{1,2\}} C_i$ such that

$$\alpha \circ f = \varepsilon_1 \qquad \text{and} \qquad \pi_2 \circ \alpha = g.$$

We set $\lambda = \pi_1 \circ \alpha$. Then

$$\lambda \circ f = \pi_1 \circ \alpha \circ f = \pi_1 \circ \varepsilon_1 = \mathrm{Id}_{C_1}.$$

$(c) \Rightarrow (b)$. Assume that there exists an isomorphism $\alpha : C \longrightarrow \bigtimes_{i\in\{1,2\}} C_i$ such that

$$\alpha \circ f = \varepsilon_1 \qquad \text{and} \qquad \pi_2 \circ \alpha = g.$$

We set $\gamma = \alpha^{-1} \circ \varepsilon_2$. Then we get

$$g \circ \gamma = g \circ \alpha^{-1} \circ \varepsilon_2 = \pi_2 \circ \varepsilon_2 = \mathrm{Id}_{C_2}.$$

□

Definition 3.64. *If one of the conditions in Theorem 3.63 holds, we say that the exact sequence*

$$0_{\mathcal{C}} \to C_1 \xrightarrow{f} C \xrightarrow{g} C_2 \to 0_{\mathcal{C}}$$

splits.

Corollary 3.65. *The sequence*

$$0_{\mathcal{C}} \to C_1 \xrightarrow{\varepsilon_1} \bigtimes_{i\in\{1,2\}} C_i \xrightarrow{\pi_2} C_2 \to 0_{\mathcal{C}}$$

is exact and splits.

Proof. First we prove that the sequence is exact. By Proposition 3.48 and Proposition 3.41 ε_1 is a monomorphism and π_2 is an epimorphism. Thus, $\mathrm{Im}(\varepsilon_1) = \mathrm{KerCoker}(\varepsilon_1) = \varepsilon_1$. We prove that $(C_1, \varepsilon_1) = \mathrm{Ker}(\pi_2)$. We already have that $\pi_2 \circ \varepsilon_1 = 0$. Let $\xi : X \longrightarrow \bigtimes_{i\in\{1,2\}} C_i$ such that $\pi_2 \circ \xi = 0$. We have to prove that there exists $\overline{\xi} : X \longrightarrow C_1$ such that $\xi = \varepsilon_1 \circ \overline{\xi}$. We have

$$\xi = \mathrm{Id}_{\bigtimes_{i\in\{1,2\}} C_i} \circ \xi = \varepsilon_1 \circ \pi_1 \circ \xi + \varepsilon_2 \circ \pi_2 \circ \xi = \varepsilon_1 \circ \pi_1 \circ \xi.$$

Thus we set $\overline{\xi} = \pi_1 \circ \xi$. Assume now that there exists another morphism $\overline{\overline{\xi}}$ such that $\xi = \varepsilon_1 \circ \overline{\overline{\xi}}$. Since also $\xi = \varepsilon_1 \circ \overline{\xi}$ and ε_1 is a monomorphism, we deduce that $\overline{\overline{\xi}} = \overline{\xi}$. In order to prove that it splits let us consider $\lambda = \pi_1$ or $\gamma = \varepsilon_2$, from that we deduce $\alpha = \mathrm{Id}_{\bigtimes_{i\in\{1,2\}} C_i}$. □

Chapter 4

Limits and Colimits

4.1 Limits

Definition 4.1. *A category is called* small *if the class of objects is actually a set.*

Definition 4.2. *Let $F : \mathcal{I} \to \mathcal{C}$ be a covariant functor where $\mathcal{I}$ is a small category. A* cone *on F is an ordered pair*

$$\left(X, (\alpha_I)_{I\in\mathcal{I}}\right)$$

where

- *X is an object of $\mathcal{C}$*
- *$(\alpha_I)_{I\in\mathcal{I}}$ is a family of morphisms of $\mathcal{C}$*
- *$\alpha_I : X \longrightarrow F(I)$ for every $I \in \mathcal{I}$*

such that for every morphism $I \stackrel{\lambda}{\longrightarrow} J$ in $\mathcal{I}$, the following diagram is commutative

$$\begin{array}{ccc} & X & \\ {}^{\alpha_I}\swarrow & & \searrow^{\alpha_J} \\ F(I) & \xrightarrow[F(\lambda)]{} & F(J)\,. \end{array}$$

In this case the family of morphisms $(\alpha_I)_{I\in\mathcal{I}}$ is called compatible with F.

Definition 4.3. *Let $F : \mathcal{I} \to \mathcal{C}$ be a covariant functor where $\mathcal{I}$ is a small category. A* limit *(also called* projective limit*) of the functor F is a cone $\left(X, (\alpha_I)_{I\in\mathcal{I}}\right)$ on F satisfying the following universal property: for any cone $\left(Y, (\xi_I)_{I\in\mathcal{I}}\right)$ on F, there exists a morphism $\xi : Y \longrightarrow X$ such that, for every I, the following diagram commutes*

$$\begin{array}{ccc} Y & \xrightarrow{\xi} & X\,. \\ {}_{\xi_I}\searrow & & \swarrow_{\alpha_I} \\ & F(I)\,. & \end{array}$$

Moreover such ξ is unique with respect to this property.

Proposition 4.4. *Let $\left(X,(\alpha_I)_{I\in\mathcal{I}}\right)$ and $\left(X',(\alpha'_I)_{I\in\mathcal{I}}\right)$ be limits of F. Then there exists a unique morphism $\alpha : X' \longrightarrow X$ such that $\alpha_I \circ \alpha = \alpha'_I$ for every I. Moreover α is an isomorphism.*

Proof. Exercise. □

Notation 4.5. *In the following we denote by $\varprojlim F$ the limit of F whenever it exists.*

Example 4.6. *Let $\mathcal{I}$ be a small and discrete category (i.e. $\mathrm{Hom}\,(I,I) = \{\mathrm{Id}_I\}$ and $\mathrm{Hom}\,(I,J) = \varnothing$ if $I \neq J$). Then a functor $F : \mathcal{I} \to \mathcal{C}$ identifies with a family $(C_I)_{I\in\mathcal{I}}$ of objects of $\mathcal{C}$. In this case a cone on F is an ordered pair $\left(X,(\alpha_I)_{I\in\mathcal{I}}\right)$ where*

$$\alpha_I : X \to C_I \text{ is a morphism in } \mathcal{C} \text{ for every } I \in \mathcal{I}.$$

Therefore, in this case,

$$\varprojlim F = \prod_{I\in\mathcal{I}} F(I)$$

Example 4.7. *Let $\mathcal{I} = \{I,J,K\}$ with morphisms $u_K^I : I \longrightarrow K$ and $u_K^J : J \longrightarrow K$ and the identity maps. Then a functor $F : \mathcal{I} \to \mathcal{C}$ identifies with a couple of morphisms*

$$\gamma_1 = F\left(u_K^I\right) : C_1 = F(I) \to C_3 = F(K), \gamma_2 = F\left(u_K^J\right) : C_2 = F(J) \to C_3 = F(K).$$

A cone on F identifies with a 4-tuple $(X, \xi_1 : X \to C_1, \xi_2 : X \to C_2, \xi_3 : X \to C_3)$ such that

$$\gamma_1 \circ \xi_1 = \xi_3 = \gamma_2 \circ \xi_2.$$

Thus a cone on F further identifies with a triple $(X, \xi_1 : X \to C_1, \xi_2 : X \to C_2)$ such that

$$\gamma_1 \circ \xi_1 = \gamma_2 \circ \xi_2.$$

In this case the limit of F is a triple $(P, \pi_1 : P \to C_1, \pi_2 : P \to C_2)$ such that

$$\gamma_1 \circ \pi_1 = \gamma_2 \circ \pi_2$$

with the property that, given any triple $(X, \xi_1 : X \to C_1, \xi_2 : X \to C_2)$ such that

$$\gamma_1 \circ \xi_1 = \gamma_2 \circ \xi_2,$$

there exists a unique $\xi : X \to P$ such that

$$\pi_1 \circ \xi = \xi_1 \text{ and } \pi_2 \circ \xi = \xi_2.$$

In this case $\varprojlim F$ is called the pullback *of γ_1 and γ_2.*

If the arrival category is preadditive and $\gamma_1 = 0_{F(K)}^{F(I)}$, then a cone on F further identifies with a pair $(X, \xi_2 : P \to C_2)$ such that

$$\gamma_2 \circ \xi_2 = 0.$$

Consequently the pullback in this case is just $\mathrm{Ker}\,(\gamma_2)$.

Proposition 4.8. *Let $\mathcal{C}$ be a preadditive category with $0_{\mathcal{C}}$ and let $F : \mathcal{I} \to \mathcal{C}$ be a covariant functor where $\mathcal{I}$ is a small category. Assume that $\mathcal{C}$ has kernels and products of families of objects labeled by $\mathcal{I}$ or by* $\mathrm{Hom}(\mathcal{I})$*, the set of morphisms between objects of $\mathcal{I}$. Then $\varprojlim F$ exists in $\mathcal{C}$.*

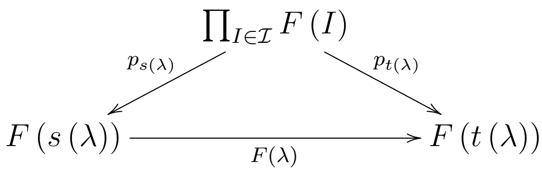
Proof. For every $\lambda \in \mathrm{Hom}(\mathcal{I})$, $\lambda : I \to J$ we set

$$s(\lambda) = I \text{ and } t(\lambda) = J.$$

Let us consider the products

$$\left(\prod_{I\in\mathcal{I}} F(I), (p_I)_{I\in\mathcal{I}}\right) \text{ and } \left(\prod_{\lambda\in\mathrm{Hom}(\mathcal{I})} F(t(\lambda)), \left(q_{t(\lambda)}\right)_{\lambda\in\mathrm{Hom}(\mathcal{I})}\right).$$

Note that, if $\lambda \in \mathrm{Hom}(\mathcal{I})$, the diagram



is, in general, non commutative. For every $\lambda \in \mathrm{Hom}(\mathcal{I})$, we set

$$\pi_\lambda = F(\lambda) \circ p_{s(\lambda)} - p_{t(\lambda)} : \prod_{I\in\mathcal{I}} F(I) \longrightarrow F(t(\lambda)).$$

By the universal property of $\prod_{\lambda\in\mathrm{Hom}(\mathcal{I})} F(t(\lambda))$, there exists a unique morphism

$$\pi = \Delta(\pi_\lambda)_{\lambda\in\mathrm{Hom}(\mathcal{I})} : \prod_{I\in\mathcal{I}} F(I) \longrightarrow \prod_{\lambda\in\mathrm{Hom}(\mathcal{I})} F(t(\lambda))$$

such that

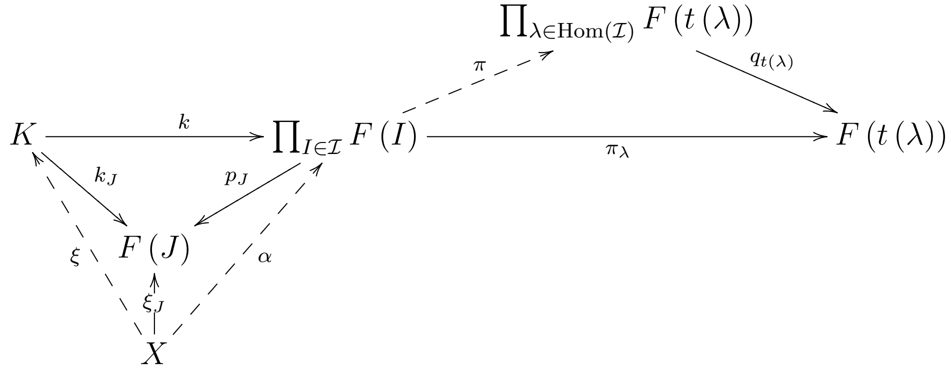
$$q_{t(\lambda)} \circ \pi = \pi_\lambda \text{ for every } \lambda \in \mathrm{Hom}(\mathcal{I}). \tag{4.1}$$

Let

$$(K, k) = \mathrm{Ker}(\pi)$$

and, for every $I \in \mathcal{I}$, set

$$k_I = p_I \circ k : K \longrightarrow F(I).$$



We want to prove that

$$\left(K,(k_I)_{I\in\mathcal{I}}\right)=\varprojlim F.$$

$\left(K,(k_I)_{I\in\mathcal{I}}\right)$ **is a cone.** For every $\lambda\in\mathrm{Hom}\,(\mathcal{I})$, we compute

$$F\,(\lambda)\circ k_{s(\lambda)}=F\,(\lambda)\circ p_{s(\lambda)}\circ k$$

Since $(K,k)=\mathrm{Ker}\,(\pi)$ we have

$$\left(F\,(\lambda)\circ p_{s(\lambda)}-p_{t(\lambda)}\right)\circ k=\pi_\lambda\circ k\overset{(4.1)}{=}q_{t(\lambda)}\circ\pi\circ k=q_{t(\lambda)}\circ 0=0$$

so that we get

$$F\,(\lambda)\circ p_{s(\lambda)}\circ k=p_{t(\lambda)}\circ k=k_{t(\lambda)}$$

which infers

$$F\,(\lambda)\circ k_{s(\lambda)}=k_{t(\lambda)}.$$

We prove that the universal property holds. Let $\left(X,(\xi_I)_{I\in\mathcal{I}}\right)$ be a cone on F i.e.

$$\xi_{t(\lambda)}=F\,(\lambda)\circ\xi_{s(\lambda)}\text{ for every }\lambda\in\mathrm{Hom}\,(\mathcal{I})\,.$$

Construction of $\xi:X\to K$. By the universal property of $\prod_{I\in\mathcal{I}}F\,(I)$, there exists a unique morphism

$$\eta=\Delta\,(\xi_I)_{I\in\mathcal{I}}:X\longrightarrow\prod_{I\in\mathcal{I}}F\,(I)\text{ such that }p_I\circ\eta=\xi_I\text{ for every }I\in\mathcal{I}.$$

We want to prove that $\pi\circ\eta=0$ which is equivalent to $q_{t(\mu)}\circ\pi\circ\eta=0$ for every $\mu\in\mathrm{Hom}\,(\mathcal{I})$. For every $\mu\in\mathrm{Hom}\,(\mathcal{I})$ we have

$$\begin{aligned}q_{t(\mu)}\circ\pi\circ\eta&=\left(F\,(\mu)\circ p_{s(\mu)}-p_{t(\mu)}\right)\circ\eta=F\,(\mu)\circ p_{s(\mu)}\circ\eta-p_{t(\mu)}\circ\eta\\&=F\,(\mu)\circ\xi_{s(\mu)}-\xi_{t(\mu)}=0\end{aligned}$$

where the last equality follows because $\left(X,(\xi_J)_{J\in\mathcal{I}}\right)$ is a cone on F. Since $(K,k)=\mathrm{Ker}\,(\pi)$, by the universal property of the kernel, there exists a unique morphism $\xi:X\longrightarrow K$ such that $k\circ\xi=\eta$.

$k_J\circ\xi=\xi_J$ **and ξ is unique.** For every $J\in\mathcal{I}$, we have:

$$k_J\circ\xi=p_J\circ k\circ\xi=p_J\circ\eta=\xi_J.$$

Now, let ξ' be another morphism such that

$$k_J\circ\xi'=\xi_J\text{ for every }J\in\mathcal{I}.$$

Then, for every $J\in\mathcal{I}$, we have:

$$p_J\circ k\circ\xi'=k_J\circ\xi'=\xi_J=p_J\circ\eta$$

which yields, in view of Exercise 3.42, that

$$k\circ\xi'=\eta.$$

Then the universal property of the kernel infers that $\xi'=\xi$. □

Corollary 4.9. *Let $\mathcal{C}$ be an abelian category. Then $\mathcal{C}$ has limits labeled by small categories $\mathcal{I}$ such that* $\mathrm{Hom}(\mathcal{I})$ *is a finite set. In particular $\mathcal{C}$ has pullbacks.*

Proof. Since $\mathrm{Hom}(\mathcal{I})$ is a finite set, also $\mathrm{Ob}(\mathcal{C})$ is finite. It follows by Proposition 4.8. The last assertion follows in view of Example 4.7. □

Definition 4.10. *A category $\mathcal{C}$ is called* complete *if for every small category $\mathcal{I}$ and for every covariant functor $F : \mathcal{I} \to \mathcal{C}$, there exists $\varprojlim F$.*

Theorem 4.11. *Let $\mathcal{C}$ be a preadditive category with $0_{\mathcal{C}}$. Then $\mathcal{C}$ is complete if and only if $\mathcal{C}$ has products and kernels.*

Proof. In view of Example 4.6 and Example 4.7, if $\mathcal{C}$ is complete it has products and kernels.

Conversely, let us assume that $\mathcal{C}$ has arbitrary products and kernels.

Let $F : \mathcal{I} \to \mathcal{C}$ be a covariant functor. Then, by Proposition 4.8, $\varprojlim F$exists in $\mathcal{C}$. □

Definition 4.12. *Let $(I, \leq)$ be a partially ordered set. We consider the small category $\mathcal{I} = \mathcal{I}(I, \leq)$ having I as the set of objects and whose homomorphism are defined by setting*

$$\mathrm{Hom}_{\mathcal{I}}(i, j) = \left\{u_j^i\right\} \text{ if and only if } i \leq j.$$

A functor $F : \mathcal{I}^{\circ} \to \mathcal{C}$ is called inverse system *in $\mathcal{C}$* labeled *by $\mathcal{I} = \mathcal{I}(I, \leq)$.*

Definition 4.13. *The limit of an inverse system $F : \mathcal{I}^{\circ} \to \mathcal{C}$ is called an* inverse limit.

4.14. *Let $(I, \leq)$ be a partially ordered set and let $F : \mathcal{I}^{\mathrm{op}} \to \mathcal{C}$ be an inverse system in $\mathcal{C}$ labeled by $\mathcal{I} = \mathcal{I}(I, \leq)$. For every $i \in I$ set*

$$C_i = F(i)$$

and for every $i, j \in I, i \leq j$, set

$$\beta_i^j = F\left(u_j^i\right) : C_j \to C_i \text{ for every } i, j \in I, i \leq j.$$

Then we have

$$\beta_i^j \circ \beta_j^k = F\left(u_j^i\right) \circ F\left(u_k^j\right) = F\left(u_k^j \circ u_j^i\right) = F\left(u_k^i\right) = \beta_i^k \text{ for every } i, j, k \in I, i \leq j \leq k \text{ and}$$
$$\beta_i^i = \mathrm{Id}_{C_i} \text{ for every } i \in I.$$

Hence an inverse system in $\mathcal{C}$ labeled by $\mathcal{I} = \mathcal{I}(I, \leq)$ identifies with an ordered pair

$$\left((C_i)_{i \in I}, \left(\beta_i^j\right)_{i,j \in I, i \leq j}\right)$$

where

- $(C_i)_{i\in I}$ *is a family of objects of* $\mathcal{C}$,
- $\left(\beta_i^j\right)_{i,j\in I,i\leq j}$ *is a family of morphisms in* $\mathcal{C}$ *such that*

$$\beta_i^j : C_j \to C_i \text{ for every } i,j \in I, i \leq j.$$

$$\beta_i^j \circ \beta_k^j = \beta_i^k \text{ for every } i,j,k \in I, i \leq j \leq k \text{ and}$$
$$\beta_i^i = \mathrm{Id}_{C_i} \text{ for every } i \in I.$$

Then an inverse limit of such an inverse system is an ordered pair $\left(L, (\lambda_i)_{i\in I}\right)$ *where each* $\lambda_i : L \to C_i$ *is a morphism in* $\mathcal{C}$ *such that*

$$\beta_i^j \circ \lambda_j = \lambda_i \text{ for every } i,j \in I, i \leq j$$

and with the property that if X *is a set and* $(\xi_i)_{i\in I}$ *is a family of morphism* $\xi_i : X \to C_i$ *such that*

$$\beta_i^j \circ \xi_j = \xi_i \text{ for every } i,j \in I, i \leq j$$

then there exists a unique morphism $\xi : X \to L$ *such that*

$$\lambda_i \circ \xi = \xi_i \text{ for every } i \in I.$$

In this case we denote this limit also by

$$\varprojlim \left((C_i)_{i\in I}, \left(\beta_i^j\right)_{i,j\in I,i\leq j} \right).$$

Exercise 4.15. *Let* $(I, \leq) = (\mathbb{N}, \leq)$. *Show that an inverse system in* $\mathcal{C}$ *labeled by* $\mathcal{I} = \mathcal{I}(\mathbb{N}, \leq)$ *identifies with* $\left((C_n)_{n\in\mathbb{N}}, (\beta_n^{n+1})_{n\in\mathbb{N}}\right)$ *were* $(C_n)_{n\in\mathbb{N}}$*is a sequence of objects and* $(\beta_n^{n+1})_{n\in\mathbb{N}}$ *is a sequence of morphisms of* $\mathcal{C}$ *, where*

$$\beta_n^{n+1} : C_{n+1} \to C_n \text{ for every } n \in \mathbb{N}.$$

Therefore an inverse limit for such an inverse system is a couple $\left(L, (\lambda_n)_{n\in\mathbb{N}}\right)$ *where each* $\lambda_n : L \to C_n$ *is a morphism in* $\mathcal{C}$ *such that*

$$\beta_n^{n+1} \circ \lambda_{n+1} = \lambda_n$$

and with the property that if X *is a set and* $(\xi_n)_{n\in\mathbb{N}}$ *is a family of morphism* $\xi_n : X \to C_n$ *such that*

$$\beta_n^{n+1} \circ \xi_{n+1} = \xi_n$$

then there exists a unique morphism $\xi : X \to L$ *such that*

$$\lambda_n \circ \xi = \xi_n \text{ for every } n \in \mathbb{N}.$$

In this case we denote this limit also by

$$\varprojlim \left(C_n, \beta_n^{n+1}\right)_{n\in\mathbb{N}}$$

or even by

$$\varprojlim C_n.$$

Exercise 4.16. *Let $\left((C_n)_{n\in\mathbb{N}}, (\beta_n^{n+1})_{n\in\mathbb{N}}\right)$ be an inverse system in a category C with arbitrary kernels and products. Let us consider the product*

$$\left(\prod_{n\in\mathbb{N}} C_n, (p_n)_{n\in\mathbb{N}}\right)$$

of the family $(C_n)_{n\in\mathbb{N}}$. For every $m \in \mathbb{N}$ we set

$$\pi_m = \beta_m^{m+1} \circ p_{m+1} - p_m : \prod_{n\in\mathbb{N}} C_n \longrightarrow C_m.$$

Let $\pi : \prod_{n\in\mathbb{N}} C_n \longrightarrow \prod_{m\in\mathbb{N}} C_m$ be the diagonal morphism of the $(\pi_m)_{m\in\mathbb{N}}$. Let

$$(K, k) = \operatorname{Ker}(\pi).$$

Show that the limit of the inverse system $\left((C_n)_{n\in\mathbb{N}}, (\beta_n^{n+1})_{n\in\mathbb{N}}\right)$ is

$$\left(K, (p_m \circ k)_{m\in\mathbb{N}}\right).$$

Example 4.17. *Let A be a ring and let $\mathfrak{I}$ be a left ideal of a ring A. For every $n \in \mathbb{N}$, let*

$$\beta_n^{n+1} : A/\mathfrak{I}^{n+1} \longrightarrow A/\mathfrak{I}^n$$

be the left A-module homomorphism defined by

$$\beta_n^{n+1}\left(a + \mathfrak{I}^{n+1}\right) = a + \mathfrak{I}^n \text{ for every } a \in A.$$

Then

$$\left((A/\mathfrak{I}^n)_{n\in\mathbb{N}}, (\beta_n^{n+1})_{n\in\mathbb{N}}\right)$$

is an inverse system in A-Mod. We have

$$\begin{aligned}\varprojlim A/\mathfrak{I}^n &= \left\{(a_n + \mathfrak{I}^n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} A/\mathfrak{I}^n \mid \beta_n^{n+1}\left(a_{n+1} + \mathfrak{I}^{n+1}\right) = a_n + \mathfrak{I}^n \text{ for every } n \in \mathbb{N}\right\} \\ &= \left\{(a_n + \mathfrak{I}^n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} A/\mathfrak{I}^n \mid a_{n+1} - a_n \in \mathfrak{I}^n \text{ for every } n \in \mathbb{N}\right\}.\end{aligned}$$

If A is a commutative local ring and $\mathfrak{I}$ is its maximal ideal, then $\varprojlim A/\mathfrak{I}^n$ is called completion of A in the $\mathfrak{I}$-adic topology.

Exercise 4.18. *Show that if $A = k[X]$ and $\mathfrak{I} = (X)$, then*

$$\varprojlim A/\mathfrak{I}^n \cong k[[X]].$$

4.19. *If $\mathcal{I}$ is a small category and $\mathcal{C}$ is an arbitrary category, one can define the functor category Fun $(\mathcal{I},\mathcal{C})$, whose objects are the functors $F:\mathcal{I}\to\mathcal{C}$ and the morphisms are the functorial morphisms between such functors (Exercise: check that Fun $(\mathcal{I},\mathcal{C})$ is a category). The set of all functorial morphisms $F\to G$ will be written* $\mathrm{Hom}_{Fun}(F,G)$. *Note that* $\mathrm{Hom}_{Fun}(F,G)$ *is indeed a set since there is an obvious identification with a subset of*

$$\prod_{I\in\mathcal{I}}\mathrm{Hom}_{\mathcal{C}}(F(I),G(I)).$$

Definition 4.20. *Let $\mathcal{C}$ and $\mathcal{D}$ be preadditive categories. A functor $F:\mathcal{C}\to\mathcal{D}$ is called* additive *if, for all morphisms $f,g:C\to C'$ in $\mathcal{C}$, we have*

$$F(f+g)=F(f)+F(g)$$

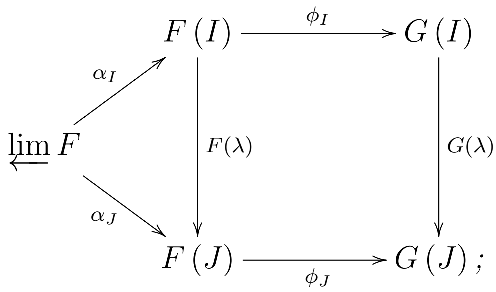
4.21. *If $\mathcal{I}$ and $\mathcal{C}$ are preadditive categories and $\mathcal{I}$ is small, we will denote by* $\mathrm{Hom}(\mathcal{I},\mathcal{C})$ *the full subcategory of Fun $(\mathcal{I},\mathcal{C})$ consisting of all additive functors.*

Definition 4.22. *Let $F:\mathcal{C}\to\mathcal{D}$ be an additive functor between abelian categories $\mathcal{C}$ and $\mathcal{D}$. We say that F is e* right exact *if, for every exact sequence $C'\xrightarrow{\alpha'}C\xrightarrow{\alpha''}C''\longrightarrow 0$ in $\mathcal{C}$, the sequence $F(C')\xrightarrow{F(\alpha')}F(C)\xrightarrow{F(\alpha'')}F(C'')\longrightarrow 0$ is exact in $\mathcal{D}$.*

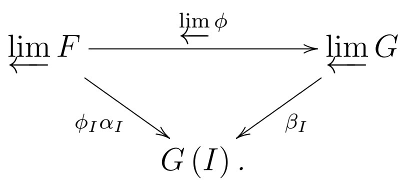
Definition 4.23. *Let $F:\mathcal{C}\to\mathcal{D}$ be an additive functor between abelian categories $\mathcal{C}$ and $\mathcal{D}$. We say that F is e* left exact *if, for every exact sequence $0\longrightarrow C'\xrightarrow{\alpha'}C\xrightarrow{\alpha''}C''$ in $\mathcal{C}$, the sequence $0\longrightarrow F(C')\xrightarrow{F(\alpha')}F(C)\xrightarrow{F(\alpha'')}F(C'')$ is exact in $\mathcal{D}$.*

Exercise 4.24. *Let $F:\mathcal{C}\to\mathcal{D}$ be an additive functor and let $0\longrightarrow C'\xrightarrow{\alpha'}C\xrightarrow{\alpha''}C''\longrightarrow 0$ be a split short exact sequence in $\mathcal{C}$. Prove that the sequence $0\longrightarrow F(C')\xrightarrow{F(\alpha')}F(C)\xrightarrow{F(\alpha'')}F(C'')\longrightarrow 0$ is a split short exact sequence in $\mathcal{D}$.*

Remark 4.25. *If $\phi:F\to G$ is a functorial morphism between covariant functors from $\mathcal{I}$ to $\mathcal{C}$ which admit limits $\left(\varprojlim F,(\alpha_I)_{I\in\mathcal{I}}\right)$ and $\left(\varprojlim G,(\beta_I)_{I\in\mathcal{I}}\right)$ respectively, then the diagram,*



is commutative i.e. $\varprojlim F$ is a cone on G with morphisms $\phi_I\circ\alpha_I$. Then there exists a unique morphism $\varprojlim\phi:\varprojlim F\longrightarrow\varprojlim G$ such that



If $\mathcal{C}$ is complete we can consider the functor $\varprojlim : \mathrm{Fun}(\mathcal{I}, \mathcal{C}) \to \mathcal{C}$.

Theorem 4.26. *Let $\mathcal{C}$ be a complete preabelian category and let $F, G, H : \mathcal{I} \to \mathcal{C}$ be functors, where $\mathcal{I}$ is a small category. Assume that $F \xrightarrow{\phi} G \xrightarrow{\psi} H$ are functorial morphisms such that, for every $I \in \mathcal{I}$, the sequence*

$$0_{\mathcal{C}} \to F(I) \xrightarrow{\phi_I} G(I) \xrightarrow{\psi_I} H(I)$$

is exact. Then the sequence

$$0_{\mathcal{C}} \to \varprojlim F \xrightarrow{\varprojlim \phi} \varprojlim G \xrightarrow{\varprojlim \psi} \varprojlim H$$

is also exact.

Proof. **$\varprojlim \phi$ is a monomorphim.** Let $\xi : X \longrightarrow \varprojlim F$ be a morphism such that $\varprojlim \phi \circ \xi = 0$. Then, for every $I \in \mathcal{I}$, we have

$$0 = \beta_I \circ \left(\varprojlim \phi\right) \circ \xi = \phi_I \circ \alpha_I \circ \xi.$$

Since ϕ_I is a monomorphism, we deduce that,for every $I \in \mathcal{I}$,

$$\alpha_I \circ \xi = 0$$

so that

$$\xi = 0.$$

$\mathrm{Im}\left(\varprojlim \phi\right) =$ **Ker** $\left(\varprojlim \psi\right)$**.** Since $\varprojlim \phi$ is a monomorphism and $\mathcal{C}$ is preabelian by Proposition 3.34, we have

$$\left(\varprojlim F, \varprojlim \phi\right) = \mathrm{KerCoker}\left(\varprojlim \phi\right) = \mathrm{Im}\left(\varprojlim \phi\right).$$

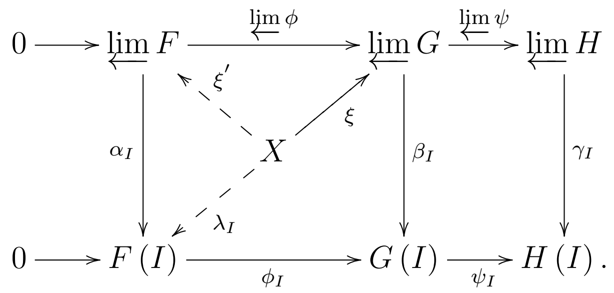
Thus we have to prove that

$$\left(\varprojlim F, \varprojlim \phi\right) = \mathrm{Ker}\left(\varprojlim \psi\right).$$

We prove that $\varprojlim \psi \circ \varprojlim \phi = 0$. In fact, for every $I \in \mathcal{I}$,we have

$$\gamma_I \circ \left(\varprojlim \psi \circ \varprojlim \phi\right) = \psi_I \circ \beta_I \circ \varprojlim \phi = \psi_I \circ \phi_I \circ \alpha_I = 0$$

since by assumption the sequence $0_{\mathcal{C}} \to F(I) \xrightarrow{\phi_I} G(I) \xrightarrow{\psi_I} H(I)$ is exact. Now, let $\xi : X \longrightarrow \varprojlim G$ be a morphism such that $\varprojlim \psi \circ \xi = 0$. Then, for every $I \in \mathcal{I}$, $\gamma_I \circ \varprojlim \psi \circ \xi = 0$ and thus $0 = \gamma_I \circ \varprojlim \psi \circ \xi = \psi_I \circ \beta_I \circ \xi$. We have to prove that there exists $\xi' : X \longrightarrow \varprojlim F$ such that $\xi = \varprojlim \phi \circ \xi'$. Since ϕ_I is a monomorphism, we have $(F(I), \phi_I) = \mathrm{KerCoker}(\phi_I) = \mathrm{Im}(\phi_I) = \mathrm{Ker}(\psi_I)$, thus, for every $I \in \mathcal{I}$, there exists a unique morphism $X \xrightarrow{\lambda_I} F(I)$ such that $\phi_I \circ \lambda_I = \beta_I \circ \xi$:



Then we have a family of morphisms (λ_I). We prove that $\left(X, (\lambda_I)_{I\in\mathcal{I}}\right)$ is a cone on F. Given a morphism $I \xrightarrow{\mu} J$, we have to prove that $\lambda_J = F(\mu) \circ \lambda_I$ or equivalently $\phi_J \circ \lambda_J = \phi_J \circ F(\mu) \circ \lambda_I$, since ϕ_J is a monomorphism. Since $\left(\varprojlim G, (\beta_I)_{I\in\mathcal{I}}\right)$ is a cone on G, we have

$$\phi_J \circ \lambda_J = \beta_J \circ \xi = G(\mu) \circ \beta_I \circ \xi = G(\mu) \circ \phi_I \circ \lambda_I = \phi_J \circ F(\mu) \circ \lambda_I.$$

By the universal property of $\varprojlim F$, there exists a unique morphism $\xi' : X \longrightarrow \varprojlim F$ such that $\alpha_I \circ \xi' = \lambda_I$, for every $I \in \mathcal{I}$. We now have to prove that $\varprojlim \phi \circ \xi' = \xi$. For every $I \in \mathcal{I}$, we have

$$\beta_I \circ \varprojlim \phi \circ \xi' = \phi_I \circ \alpha_I \circ \xi' = \phi_I \circ \lambda_I = \beta_I \circ \xi$$

from which we deduce that $\varprojlim \phi \circ \xi' = \xi$. Assume now that there exists another morphism ξ'' such that $\varprojlim \phi \circ \xi'' = \xi$. Since we also have $\varprojlim \phi \circ \xi' = \xi$ and $\varprojlim \phi$ is a monomorphism, we deduce that $\xi'' = \xi'$. □

4.2 Colimits

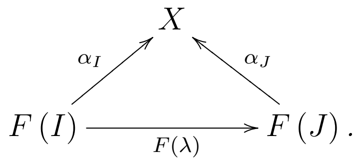
Definition 4.27. *Let $F : \mathcal{I} \to \mathcal{C}$ be a covariant functor where $\mathcal{I}$ is a small category. A* cocone *on F is an ordered pair*

$$\left(X, (\alpha_I)_{I\in\mathcal{I}}\right)$$

where

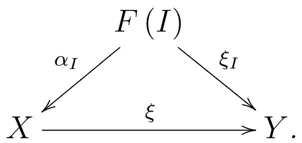
- *X is an object of $\mathcal{C}$*
- *$(\alpha_I)_{I\in\mathcal{I}}$ is a family of morphisms of $\mathcal{C}$*
- *$\alpha_I : F(I) \longrightarrow X$ for every $I \in \mathcal{I}$*

such that for every morphism $I \xrightarrow{\lambda} J$ *in* $\mathcal{I}$*, the following diagram is commutative*



In this case the family of morphisms $(\alpha_I)_{I\in\mathcal{I}}$ *is called* compatible with F.

Definition 4.28. *Let* $F : \mathcal{I} \to \mathcal{C}$ *be a covariant functor where* $\mathcal{I}$ *is a small category. A* colimit *(also called* inductive limit*) of the functor* F *is a cocone* $\left(X, (\alpha_I)_{I\in\mathcal{I}}\right)$ *on* F *satisfying the following universal property: for any cocone* $\left(Y, (\xi_I)_{I\in\mathcal{I}}\right)$ *on* F*, there exists a morphism* $\xi : X \longrightarrow Y$ *such that, for every* I*, the following diagram commutes*



Moreover such ξ *is unique with respect to this property.*

Proposition 4.29. *Let* $\left(X, (\alpha_I)_{I\in\mathcal{I}}\right)$ *and* $\left(X', (\alpha'_I)_{I\in\mathcal{I}}\right)$ *be limits of* F*. Then there exists a unique isomorphism* $\alpha : X \longrightarrow X'$ *such that* $\alpha \circ \alpha_I = \alpha'_I$ *for every* I*. Moreover* α *is an isomorphism.*

Proof. Exercise. □

Notation 4.30. *In the following we denote by* $\varinjlim F$ *the colimit of* F *whenever it exists.*

Example 4.31. *Let* $\mathcal{I}$ *be a small and discrete category (i.e.* $\mathrm{Hom}(I, I) = \{\mathrm{Id}_I\}$ *and* $\mathrm{Hom}(I, J) = \varnothing$ *if* $I \neq J$*). Then a functor* $F : \mathcal{I} \to \mathcal{C}$ *identifies with a family* $(C_I)_{I\in\mathcal{I}}$ *of objects of* $\mathcal{C}$*. In this case a cocone on* F *is an ordered pair* $\left(X, (\alpha_I)_{I\in\mathcal{I}}\right)$ *where*

$$\alpha_I : C_I \to X \text{ is a morphism in } \mathcal{C} \text{ for every } I \in \mathcal{I}.$$

Therefore, in this case,

$$\varinjlim F = \coprod_{I\in\mathcal{I}} F(I).$$

Example 4.32. *Let* $\mathcal{I} = \{I, J, K\}$ *with morphisms* $v_K^I : K \longrightarrow I$ *and* $v_K^J : K \longrightarrow J$ *and the identity maps. Then a functor* $F : \mathcal{I} \to \mathcal{C}$ *identifies with a couple of morphisms*

$$\vartheta_1 = F\left(v_K^I\right) : C_3 = F(K) \to C_1 = F(I), \vartheta_2 = F\left(v_K^J\right) : C_3 = F(K) \to C_2 = F(J).$$

A cocone on F *identifies with a 4-tuple* $(X, \lambda_1 : C_1 \to X, \lambda_2 : C_2 \to X, \lambda_3 : C_3 \to X)$ *such that*

$$\lambda_1 \circ \vartheta_1 = \lambda_3 = \lambda_2 \circ \vartheta_2.$$

Thus a cocone on F further identifies with a triple $(X, \lambda_1 : X \to C_1, \lambda_2 : X \to C_2)$ such that

$$\lambda_1 \circ \vartheta_1 = \lambda_2 \circ \vartheta_2.$$

In this case the colimit of F is a triple $(E, \eta_1 : C_1 \to E, \eta_2 : C_2 \to E)$ such that

$$\eta_1 \circ \vartheta_1 = \eta_2 \circ \vartheta_2$$

with the property that, given any triple $(X, \lambda_1 : C_1 \to X, \lambda_2 : C_2 \to X)$ such that

$$\lambda_1 \circ \vartheta_1 = \lambda_2 \circ \vartheta_2,$$

there exists a unique $\lambda : E \to X$ such that

$$\lambda \circ \eta_1 = \lambda_1 \text{ and } \lambda \circ \eta_2 = \lambda_2.$$

In this case $\varinjlim F$ is called the pushout *of ϑ_1 and ϑ_2.*

If the arrival category is preadditive and $\vartheta_2 = 0$, then a cone on F further identifies with a pair $(X, \lambda_1 : C_2 \to X)$ such that

$$\lambda_1 \circ \vartheta_1 = 0.$$

Consequently the pullback in this case is just $\mathrm{Coker}(\vartheta_1)$.

Definition 4.33. *A category $\mathcal{C}$ is called* cocomplete *if for every small category $\mathcal{I}$ and for every covariant functor $F : \mathcal{I} \to \mathcal{C}$, there exists $\varinjlim F$.*

Theorem 4.34. *Let $\mathcal{C}$ be a preadditive category with $0_{\mathcal{C}}$. Then $\mathcal{C}$ is cocomplete if and only if $\mathcal{C}$ has coproducts and cokernels.*

Proof. In view of Example 3.49 and Example 4.32, if $\mathcal{C}$ is cocomplete it has coproducts and cokernels.

Conversely, let us assume that $\mathcal{C}$ has arbitrary coproducts and cokernels.

Let $F : \mathcal{I} \to \mathcal{C}$ be a covariant functor.

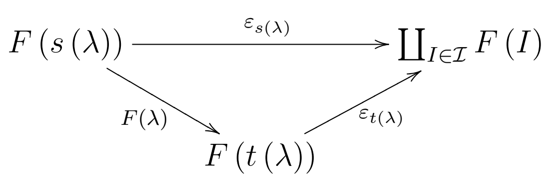
Construction of $\varinjlim F$. Denote by $\mathrm{Hom}(\mathcal{I})$ the set of morphisms between objects of $\mathcal{I}$. For every $\lambda \in \mathrm{Hom}(\mathcal{I})$, $\lambda : I \to J$ we set

$$s(\lambda) = I \text{ and } t(\lambda) = J.$$

Let us consider the coproducts

$$\left(\coprod_{I \in \mathcal{I}} F(I), (\varepsilon_I)_{I \in \mathcal{I}}\right) \text{ and } \left(\coprod_{\lambda \in \mathrm{Hom}(\mathcal{I})} F(s(\lambda)), \left(e_{s(\lambda)}\right)_{\lambda \in \mathrm{Hom}(\mathcal{I})}\right).$$

Note that, if $\lambda \in \mathrm{Hom}(\mathcal{I})$, the diagram



is, in general, non commutative. For every $\lambda \in \mathrm{Hom}(\mathcal{I})$, we set

$$\eta_\lambda = \varepsilon_{t(\lambda)} \circ F(\lambda) - \varepsilon_{s(\lambda)} : F(s(\lambda)) \longrightarrow \coprod_{I \in \mathcal{I}} F(I).$$

By the universal property of $\coprod_{\lambda \in \mathrm{Hom}(\mathcal{I})} F(s(\lambda))$, there exists a unique morphism

$$\eta = \nabla(\eta_\lambda)_{\lambda \in \mathrm{Hom}(\mathcal{I})} : \coprod_{\lambda \in \mathrm{Hom}(\mathcal{I})} F(s(\lambda)) \longrightarrow \coprod_{I \in \mathcal{I}} F(I)$$

such that

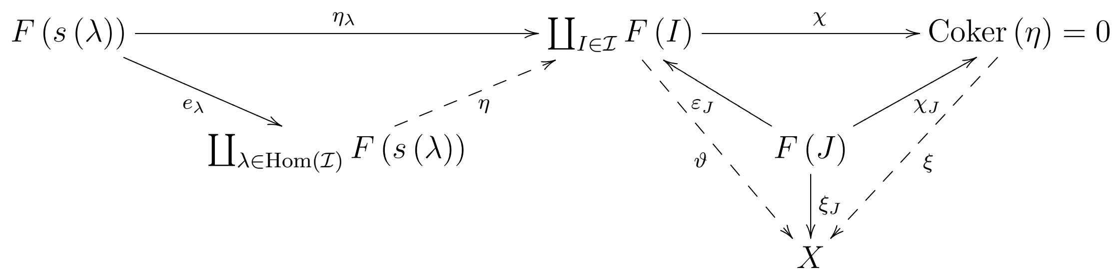
$$\eta \circ e_\lambda = \eta_\lambda \text{ for every } \lambda \in \mathrm{Hom}(\mathcal{I}). \tag{4.2}$$

Let

$$(Q, \chi) = \mathrm{Coker}(\eta)$$

and, for every $I \in \mathcal{I}$, set

$$\chi_I = \chi \circ \varepsilon_I : K \longrightarrow F(I).$$



We want to prove that

$$\left(Q, (\chi_I)_{I \in \mathcal{I}}\right) = \varinjlim F.$$

$\left(Q, (\chi_I)_{I \in \mathcal{I}}\right)$ **is a cocone.** For every $\lambda \in \mathrm{Hom}(\mathcal{I})$, we compute

$$\chi_{t(\lambda)} \circ F(\lambda) = \chi \circ \varepsilon_{t(\lambda)} \circ F(\lambda)$$

Since $(Q, \chi) = \mathrm{Coker}(\eta)$ we have

$$\chi \circ \left(\varepsilon_{t(\lambda)} \circ F(\lambda) - \varepsilon_{s(\lambda)}\right) = \chi \circ \eta_\lambda \overset{(4.2)}{=} \chi \circ \eta \circ \varepsilon_\lambda = 0 \circ \varepsilon_\lambda = 0$$

so that we get

$$\chi \circ \varepsilon_{t(\lambda)} \circ F(\lambda) = \chi \circ \varepsilon_{s(\lambda)} = \chi_{s(\lambda)}$$

which infers

$$\chi_{t(\lambda)} \circ F(\lambda) = \chi_{s(\lambda)}.$$

We prove that the universal property holds. Let $\left(X, (\xi_I)_{I \in \mathcal{I}}\right)$ be a cocone on F i.e.

$$\xi_{t(\lambda)} \circ F(\lambda) = \xi_{s(\lambda)} \text{ for every } \lambda \in \mathrm{Hom}(\mathcal{I}).$$

Construction of ξ :$Q\to X$. By the universal property of , there exists a unique morphism

$$\vartheta = \nabla\left(\xi_I\right)_{I\in\mathcal{I}} : \coprod_{I\in\mathcal{I}} F\left(I\right) \longrightarrow X \text{ such that } \vartheta\circ\varepsilon_I = \xi_I \text{ for every } I\in\mathcal{I}.$$

We want to prove that $\vartheta\circ\eta = 0$ which is equivalent to $\vartheta\circ\eta\circ e_\lambda = 0$ for every $\lambda\in\mathrm{Hom}\left(\mathcal{I}\right)$. For every $\lambda\in\mathrm{Hom}\left(\mathcal{I}\right)$ we have

$$\begin{aligned}\vartheta\circ\eta\circ e_\lambda &= \vartheta\circ\eta_\lambda = \vartheta\circ\varepsilon_{t(\lambda)}\circ F\left(\lambda\right) - \vartheta\circ\varepsilon_{s(\lambda)}\\ &= \xi_{t(\lambda)}F\left(\lambda\right) - \xi_{t(\mu)} = 0\end{aligned}$$

where the last equality follows because $\left(X, \left(\xi_J\right)_{J\in\mathcal{I}}\right)$ is a cocone on F. Since $(Q,\chi) = \mathrm{Coker}\left(\eta\right)$, by the universal property of the cokernel, there exists a unique morphism $\xi : Q \longrightarrow X$ such that $\xi\circ\chi = \vartheta$.

$\xi\circ\chi_J = \xi_J$ and ξ is unique. For every $J\in\mathcal{I}$, we have:

$$\xi\circ\chi_J = \xi\circ\chi\circ\varepsilon_J = \vartheta\circ\varepsilon_J = \xi_J.$$

Now, let ξ' be another morphism such that

$$\xi'\circ\chi_J == \xi_J \text{ for every } J\in\mathcal{I}.$$

Then, for every $J\in\mathcal{I}$, we have:

$$\xi'\circ\chi\circ\varepsilon_J = \xi'\circ\chi_J = \xi_J = \vartheta\circ\varepsilon_J$$

which yields, in view of Exercise 3.49, that

$$\xi'\circ\chi = \vartheta.$$

Then the universal property of the cokernel infers that $\xi' = \xi$. □

Definition 4.35. *Let $(I,\leq)$ be a partially ordered set. We consider the small category $\mathcal{I} = \mathcal{I}\left(I,\leq\right)$ having I as the set of objects and whose homomorphism are defined by setting*

$$\mathrm{Hom}_{\mathcal{I}}\left(i,j\right) = \left\{u_j^i\right\} \textit{ if and only if } i\leq j.$$

A functor $F:\mathcal{I}\to\mathcal{C}$ is called a direct system *in $\mathcal{C}$* labeled *by $\mathcal{I} = \mathcal{I}\left(I,\leq\right)$.*

Definition 4.36. *The colimit of a direct system $F:\mathcal{I}\to\mathcal{C}$ is called a* direct limit*.*

4.37. *Let $(I,\leq)$ be a partially ordered set and let $F:\mathcal{I}\to\mathcal{C}$ be a direct system in $\mathcal{C}$ labeled by $\mathcal{I} = \mathcal{I}\left(I,\leq\right)$. For every $i\in I$ set*

$$C_i = F\left(i\right)$$

and for every $i, j \in I, i \leq j$, set

$$\gamma_j^i = F\left(u_j^i\right) : C_i \to C_j \text{ for every } i, j \in I, i \leq j.$$

Then we have

$$\gamma_k^j \circ \gamma_j^i = F\left(u_k^j\right) \circ F\left(u_j^i\right) = F\left(u_k^j \circ u_j^i\right) = F\left(u_k^i\right) = \gamma_k^i \text{ for every } i, j, k \in I, i \leq j \leq k \text{ and}$$
$$\gamma_i^i = \mathrm{Id}_{C_i} \text{ for every } i \in I.$$

Hence a direct system in $\mathcal{C}$ labeled by $\mathcal{I} = \mathcal{I}(I, \leq)$ identifies with an ordered pair

$$\left((C_i)_{i \in I}, \left(\gamma_j^i\right)_{i,j \in I, i \leq j}\right)$$

where

- $(C_i)_{i \in I}$ *is a family of objects of* $\mathcal{C}$,
- $\left(\gamma_j^i\right)_{i,j \in I, i \leq j}$ *is a family of morphisms in* $\mathcal{C}$ *such that*

$$\gamma_j^i : C_i \to C_j \text{ for every } i, j \in I, i \leq j.$$

$$\gamma_k^j \circ \gamma_j^i = \gamma_k^i \text{ for every } i, j, k \in I, i \leq j \leq k \text{ and}$$
$$\gamma_i^i = \mathrm{Id}_{C_i} \text{ for every } i \in I.$$

Then a direct limit of such a direct system is an ordered pair $\left(L, (\lambda_i)_{i \in I}\right)$ where each $\lambda_i : C_i \to L$ is a morphism in $\mathcal{C}$ such that

$$\lambda_j \circ \gamma_j^i = \lambda_i \text{ for every } i, j \in I, i \leq j$$

and with the property that if X is a set and $(\xi_i)_{i \in I}$ is a family of morphism $\xi_i : C_i \to X$ such that

$$\xi_j \circ \gamma_j^i = \xi_i \text{ for every } i, j \in I, i \leq j$$

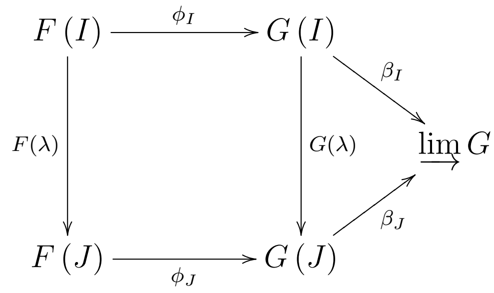
then there exists a unique morphism $\xi : L \to X$ such that

$$\xi \circ \gamma_i = \xi_i \text{ for every } i \in I.$$

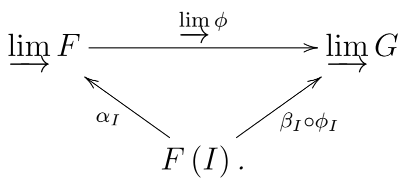
In this case we denote this direct limit also by

$$\varinjlim \left((C_i)_{i \in I}, \left(\gamma_j^i\right)_{i,j \in I, i \leq j}\right).$$

Remark 4.38. *If $\phi : F \to G$ is a functorial morphism between covariant functors from $\mathcal{I}$ to $\mathcal{C}$ which admit colimits $\left(\varinjlim F, (\alpha_I)_{I\in\mathcal{I}}\right)$ and $\left(\varinjlim G, (\beta_I)_{I\in\mathcal{I}}\right)$ respectively, then the diagram*



is commutative i.e. $\varinjlim G$ is a cocone on F with morphisms $\beta_I \circ \phi_I$. Then there exists a unique morphism $\varinjlim \phi : \varinjlim F \longrightarrow \varinjlim G$ such that



If $\mathcal{C}$ is cocomplete we can consider the functor $\varinjlim : \mathrm{Fun}(\mathcal{I}, \mathcal{C}) \to \mathcal{C}$.

Example 4.39. *Let R be a ring and let $\left((M_i)_{i\in I}, \left(f_j^i : M_i \to M_j\right)_{i,j\in I, i\leq j}\right)$ be a direct system in Mod-R. Assume that $(I, \leq)$ is a **direct set** i.e. for every $i, j \in I$ there exists a $k \in I$ such that $i \leq k$ and $j \leq k$. Let*

$$\dot{\bigcup_{i\in I}} M_i = \left\{ (x, i) \in \left(\bigcup_{i\in I} M_i \right) \times I \mid x \in M_i \right\}$$

be the disjoint union of the family $(M_i)_{i\in I}$. We define an equivalence relation $\sim$ on this disjoint union by setting, for every (x, i) and (y, j) in $\dot{\bigcup}_{i\in I} M_i$

$$(x, i) \sim (y, j) \Leftrightarrow \text{there is a } k \in I \text{ such that } f_k^i(x) = f_k^j(y).$$

Let

$$L = \frac{\dot{\bigcup}_{i\in I} M_i}{\sim}.$$

and let

$$\pi : \dot{\bigcup_{i\in I}} M_i \to \frac{\dot{\bigcup}_{i\in I} M_i}{\sim} = L$$

be the canonical projection. For each $(x,i) \in \dot{\bigcup\limits_{i\in I}} M_i$ *we set*

$$[(x,i)] = \pi\left((x,i)\right).$$

We define a right R-module structure on L by setting

$$[(x,i)] + [(y,j)] = \left[\left(f_k^i(x) + f_k^j(y)\right)\right] \text{ where } i \le k, j \le k$$

and

$$[(x,i)] \cdot r = [(xr,i)] \text{ for every } r \in R.$$

It is straightforward to prove that these are good definitions and that L becomes a right R-module. For every $j \in I$ *let* $\varepsilon_j : M_j \to \dot{\bigcup\limits_{i\in I}} M_i$ *be the canonical injection i.e.*

$$\varepsilon_j(x) = (x,j) \text{ for every } x \in M_j.$$

Set

$$\lambda_i = \varepsilon_j \circ \pi.$$

Then it is easy to show that $\left(L, (\lambda_i)_{i\in I}\right)$ *is the direct limit of the direct system* $\left((M_i)_{i\in I}, \left(f_j^i : M_i \to M_j\right)_{i,j\in I, i\le j}\right)$.

Exercise 4.40. *Let R be a ring and let* $\left((M_i)_{i\in I}, \left(f_j^i : M_i \to M_j\right)_{i,j\in I, i\le j}\right)$ *be a direct system in Mod-R. Let* $M = \bigoplus\limits_{i\in I} M_i$ *and, for every* $i \in I$, *let* $\varepsilon_i : M_i \to M$ *be the canonical injection. For every* $i,j \in I$,, *let* $\eta_{i\le j} = \varepsilon_j \circ f_j^i - \varepsilon_i$ *and let* $H = \sum\limits_{\substack{i,j\in I\\ i\le j}} \operatorname{Im}(\eta_{i\le j})$. *Set* $L = \frac{M}{H}$, *let* $\pi : M \to \frac{M}{H}$ *be the canonical projection and, for every* $i \in I$, *let* $\lambda_i = \pi \circ \varepsilon_i : M_i \to L$. *Show that*

$$\left(L, (\lambda_i)_{i\in I}\right) = \varinjlim \left((M_i)_{i\in I}, \left(f_j^i : M_i \to M_j\right)_{i,j\in I, i\le j}\right).$$

Theorem 4.41. *Let* $\mathcal{C}$ *be a cocomplete preabelian category and let* $F, G, H : \mathcal{I} \to \mathcal{C}$ *be functors, where* $\mathcal{I}$ *is a small category. Assume that* $F \xrightarrow{\phi} G \xrightarrow{\psi} H$ *are functorial morphisms such that, for every* $I \in \mathcal{I}$, *the sequence*

$$F(I) \xrightarrow{\phi_I} G(I) \xrightarrow{\psi_I} H(I) \to 0_{\mathcal{C}}$$

is exact. Then the sequence

$$\varinjlim F \xrightarrow{\varinjlim \phi} \varinjlim G \xrightarrow{\varinjlim \psi} \varinjlim H \to 0_{\mathcal{C}}$$

is also exact.

Proof. $\varinjlim \psi$ **is an epimorphism.** Let $\xi : \varinjlim H \longrightarrow X$ be a morphism such that $\xi \circ \varinjlim \psi = 0$. Then, for every $I \in \mathcal{I}$, we have

$$0 = \xi \circ \left(\varinjlim \psi\right) \circ \beta_I = \xi \circ \gamma_I \circ \psi_I.$$

Since ψ_I is an epimorphism, we deduce that, for every $I \in \mathcal{I}$,

$$\xi \circ \gamma_I = 0$$

so that

$$\xi = 0.$$

We prove that

$$\operatorname{Coker}\left(\varinjlim \phi\right) = \left(\varinjlim H, \varinjlim \psi\right)$$

from which it will follow that

$$\operatorname{Im}\left(\varinjlim \phi\right) = \operatorname{KerCoker}\left(\varinjlim \phi\right) = \operatorname{Ker}\left(\varinjlim \psi\right)$$

We prove that $\varinjlim \psi \circ \varinjlim \phi = 0$. Since, by assumption, the sequence $F(I) \xrightarrow{\phi_I} G(I) \xrightarrow{\psi_I} H(I) \to 0_{\mathcal{C}}$ is exact, for every $I \in \mathcal{I}$,we have

$$\left(\varinjlim \psi \circ \varinjlim \phi\right) \circ \alpha_I = \varinjlim \psi \circ \beta_I \circ \phi_I = \gamma_I \circ \psi_I \circ \phi_I = 0.$$

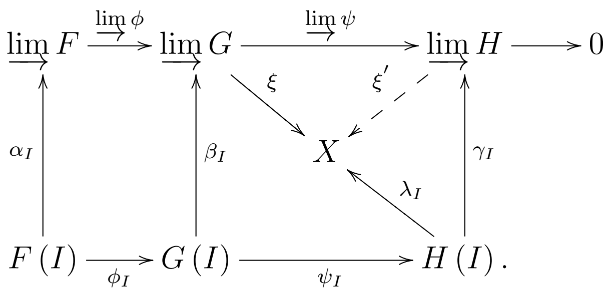
This means that

$$\varinjlim \psi \circ \varinjlim \phi = 0$$

Now, let $\xi : \varinjlim G \longrightarrow X$ be a morphism such that $\xi \circ \left(\varinjlim \phi\right) = 0$. Then, for every $I \in \mathcal{I}$, we have $\xi \circ \left(\varinjlim \phi\right) \circ \alpha_I = 0$ and thus

$$0 = \xi \circ \left(\varinjlim \phi\right) \circ \alpha_I = \xi \circ \beta_I \circ \phi_I.$$

We have to prove that there exists $\xi' : \varinjlim H \longrightarrow X$ such that $\xi = \xi' \circ \varinjlim \psi$. Since ψ_I is an epimorphism, we have that $(H(I), \psi_I) = \operatorname{CokerKer}(\psi_I) = \operatorname{CokerIm}(\phi_I) = \operatorname{CokerKerCoker}(\phi_I) = \operatorname{Coker}(\phi_I)$, thus, for every $I \in \mathcal{I}$, there exists a unique morphism $H(I) \xrightarrow{\lambda_I} X$ such that $\lambda_I \circ \psi_I = \xi \circ \beta_I$:



Then we have a family of morphisms $(\lambda_I)_{I \in \mathcal{I}}$. We prove that $\left(X, (\lambda_I)_{I \in \mathcal{I}}\right)$ is a cocone on H. Given a morphism $I \xrightarrow{\mu} J$, we have to prove that $\lambda_J \circ H(\mu) =$

λ_I or equivalently $\lambda_J \circ H(\mu) \circ \psi_I = \lambda_I \circ \psi_I$, since ψ_I is an epimorphism. Since $\left(\varinjlim G, (\beta_I)_{I\in\mathcal{I}}\right)$ is a cocone on G, we have

$$\lambda_I \circ \psi_I = \xi \circ \beta_I = \xi \circ \beta_J \circ G(\mu) = \lambda_J \circ \psi_J \circ G(\mu) = \lambda_J \circ H(\mu) \circ \psi_I.$$

By the universal property of $\varinjlim H$, there exists a unique morphism $\xi' : \varinjlim H \longrightarrow X$ such that $\xi' \circ \gamma_I = \lambda_I$, for every $I \in \mathcal{I}$. We now have to prove that $\xi' \circ \varinjlim \psi = \xi$. For every $I \in \mathcal{I}$, we have

$$\xi' \circ \varinjlim \psi \circ \beta_I = \xi' \circ \gamma_I \circ \psi_I = \lambda_I \circ \psi_I = \xi \circ \beta_I$$

from which we deduce that $\xi' \circ \varinjlim \psi = \xi$. Assume now that there exists another morphism ξ'' such that $\xi'' \circ \varinjlim \psi = \xi$. Since we also have $\xi' \circ \varinjlim \psi = \xi$ and $\varinjlim \psi$ is an epimorphism, we deduce that $\xi'' = \xi'$. $\square$

Chapter 5

Adjoint functors

Let $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$ be covariant functors.Then we define functors

$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{A}}\left(L\left(\bullet\right), \blacktriangle\right): & \mathcal{B}^{\mathrm{op}} \times \mathcal{A} & \longrightarrow & Sets, \\ \mathrm{Hom}_{\mathcal{B}}\left(\bullet, R\left(\blacktriangle\right)\right): & \mathcal{B}^{\mathrm{op}} \times \mathcal{A} & \longrightarrow & Sets, \end{array}$$

by setting

$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{A}}\left(L\left(\bullet\right), \blacktriangle\right)\left(B, A\right) & = & \mathrm{Hom}_{\mathcal{A}}\left(L\left(B\right), A\right) \\ \mathrm{Hom}_{\mathcal{B}}\left(\bullet, R\left(\blacktriangle\right)\right)\left(B, A\right) & = & \mathrm{Hom}_{\mathcal{B}}\left(B, R\left(A\right)\right) \end{array}$$

for every $(B, A) \in \mathcal{B}^{\mathrm{op}} \times \mathcal{A}$. Given $(f, g) \in \mathrm{Hom}_{\mathcal{B}^{\mathrm{op}} \times \mathcal{A}}\left(\left(B_1, A_1\right), \left(B_2, A_2\right)\right)$ i.e. $f \in \mathrm{Hom}_{\mathcal{B}}\left(B_2, B_1\right)$ and $g \in \mathrm{Hom}_{\mathcal{A}}\left(A_1, A_2\right)$ we set

$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{A}}\left(L\left(\bullet\right), \blacktriangle\right)\left(f, g\right) & = & \mathrm{Hom}_{\mathcal{A}}\left(L\left(f\right), g\right) \\ \mathrm{Hom}_{\mathcal{B}}\left(\bullet, R\left(\blacktriangle\right)\right)\left(f, g\right) & = & \mathrm{Hom}_{\mathcal{B}}\left(f, R\left(g\right)\right) \end{array}$$

where

$$\begin{array}{rccc} \mathrm{Hom}_{\mathcal{A}}\left(L\left(f\right), g\right): & \mathrm{Hom}_{\mathcal{A}}\left(L\left(B_1\right), A_1\right) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}\left(L\left(B_2\right), A_2\right) \\ & \left(L\left(B_1\right) \xrightarrow{\xi} A_1\right) & \longmapsto & \left(L\left(B_2\right) \xrightarrow{L(f)} L\left(B_1\right) \xrightarrow{\xi} A_1 \xrightarrow{g} A_2\right) \\ & & & = g \circ \xi \circ L\left(f\right) \\ \mathrm{Hom}_{\mathcal{B}}\left(f, R\left(g\right)\right): & \mathrm{Hom}_{\mathcal{B}}\left(B_1, R\left(A_1\right)\right) & \longrightarrow & \mathrm{Hom}_{\mathcal{B}}\left(B_2, R\left(A_2\right)\right) \\ & \left(B_1 \xrightarrow{\zeta} R\left(A_1\right)\right) & \longmapsto & \left(B_2 \xrightarrow{f} B_1 \xrightarrow{\zeta} R\left(A_1\right) \xrightarrow{R(g)} R\left(A_2\right)\right) \\ & & & = R\left(g\right) \circ \zeta \circ f \end{array}$$

Definition 5.1. *Let $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$ be covariant functors. The pair of functors (L, R) is called an* adjunction *if there exists a functorial isomorphism $\Lambda : \mathrm{Hom}_{\mathcal{A}}\left(L\left(\bullet\right), \blacktriangle\right) \to \mathrm{Hom}_{\mathcal{B}}\left(\bullet, R\left(\blacktriangle\right)\right)$, i.e. for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$, there exist an isomorphism Λ_A^B :$Hom_{\mathcal{A}}\left(L\left(B\right), A\right) \to Hom_{\mathcal{B}}\left(B, R\left(A\right)\right)$ such that, for every $f \in \mathrm{Hom}_{\mathcal{B}}\left(B_2, B_1\right)$ and $g \in \mathrm{Hom}_{\mathcal{A}}\left(A_1, A_2\right)$ we have*

$$\mathrm{Hom}_{\mathcal{B}}\left(f, R\left(g\right)\right) \circ \Lambda_{A_1}^{B_1} = \Lambda_{A_2}^{B_2} \circ \mathrm{Hom}_{\mathcal{A}}\left(L\left(f\right), g\right), \tag{5.1}$$

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{B}}\left(L\left(B_1\right), A_1\right) & \xrightarrow{\Lambda_{A_1}^{B_1}} & \mathrm{Hom}_{\mathcal{A}}\left(A_1, R\left(B_1\right)\right) \\ \Big\downarrow{\scriptstyle \mathrm{Hom}_{\mathcal{B}}(L(f),g)} & & \Big\downarrow{\scriptstyle \mathrm{Hom}_{\mathcal{A}}(f,R(g))} \\ \mathrm{Hom}_{\mathcal{B}}\left(L\left(A_2\right), B_2\right) & \xrightarrow[\Lambda_{A_2}^{B_2}]{} & \mathrm{Hom}_{\mathcal{A}}\left(A_2, R\left(B_2\right)\right), \end{array}$$

i.e. for every $\xi : L\left(B_1\right) \longrightarrow A_1$

$$\Lambda_{A_2}^{B_2}\left[g \circ \xi \circ \left(L\left(f\right)\right)\right] = R\left(g\right) \circ \Lambda_{A_1}^{B_1}\left(\xi\right) \circ f. \tag{5.2}$$

The equality (5.2) *is equivalent to the following equalities*

$$\Lambda_{A_2}^{B}\left(g \circ \xi\right) = R\left(g\right) \circ \Lambda_{A_1}^{B}\left(\xi\right) \tag{5.3}$$

$$\Lambda_{A}^{B_2}\left[\xi \circ L\left(f\right)\right] = \Lambda_{A}^{B_1}\left(\xi\right) \circ f \tag{5.4}$$

Definition 5.2. *Let* $L : \mathcal{B} \to \mathcal{A}$ *and* $R : \mathcal{A} \to \mathcal{B}$ *be covariant functors. We say that* L is a left adjoint of R, *or equivalently, that* R is a right adjoint to L *if the pair* (L, R) *is an adjunction.*

Example 5.3. *Let* ${}_R M_S$ *be a bimodule,*

$$R = \mathrm{Hom}_S\left({}_R M_S, \bullet\right) : \text{Mod-}S \to \text{Mod-}R$$

and

$$L = \bullet \otimes_R {}_R M_S \colon \text{Mod-}R \to \text{Mod-}S.$$

We set

$$\begin{array}{rccc} \Lambda_A^B : & \mathrm{Hom}_S\left(B \otimes_R M, A\right) & \longrightarrow & \mathrm{Hom}_R\left(B, \mathrm{Hom}_S\left(M, A\right)\right) \\ & \left(B \otimes_R M \xrightarrow{\xi} A\right) & \longmapsto & \left(\begin{array}{ccc} B & \longrightarrow & \mathrm{Hom}_S\left(M, A\right) \\ a & \longmapsto & \left(\begin{array}{ccc} M & \longrightarrow & A \\ m & \longmapsto & \xi\left(a \otimes m\right) \end{array}\right) \end{array}\right) \end{array}$$

and

$$\begin{array}{rccc} \Gamma_A^B : & \mathrm{Hom}_R\left(B, \mathrm{Hom}_S\left(M, A\right)\right) & \longrightarrow & \mathrm{Hom}_S\left(B \otimes_R M, A\right) \\ & \left(B \xrightarrow{\zeta} \mathrm{Hom}_S\left(M, A\right)\right) & \longmapsto & \left(\begin{array}{ccc} B \otimes_R M & \longrightarrow & A \\ a \otimes m & \longmapsto & \zeta\left(a\right)\left(m\right) \end{array}\right). \end{array}$$

We will prove that Γ_A^B *is the inverse of* Λ_A^B.

$\Lambda_A^B\left(\xi\right)\left(a\right)$ ***is a morphism in*** Mod-S***.*** *Let* $\alpha = \Lambda_A^B\left(\xi\right)\left(a\right)$*. We have*

$$\begin{aligned} \alpha\left(m_1 s_1 + m_2 s_2\right) &= \xi\left(a \otimes \left(m_1 s_1 + m_2 s_2\right)\right) \\ &= \xi\left(a \otimes \left(m_1 s_1\right) + a \otimes \left(m_2 s_2\right)\right) \\ &= \xi\left(\left(a \otimes m_1\right) s_1 + \left(a \otimes m_2\right) s_2\right) \\ &= \xi\left(a \otimes m_1\right) s_1 + \xi\left(a \otimes m_2\right) s_2 \\ &= \alpha\left(m_1\right) s_1 + \alpha\left(m_2\right) s_2 \end{aligned}$$

for every $m_1, m_2 \in M$ *and* $s_1, s_2 \in S$.

$\Lambda_A^B(\xi)$ ***is a morphism in*** Mod-R***.*** *Let* $a_1, a_2 \in B$ *and* $r_1, r_2 \in R$. *We have*

$$\begin{aligned}
\Lambda_A^B(\xi)(a_1r_1 + a_2r_2)(m) &= \xi((a_1r_1 + a_2r_2) \otimes m) \\
&= \xi(a_1r_1 \otimes m + a_2r_2 \otimes m) \\
&= \xi(a_1r_1 \otimes m) + \xi(a_2r_2 \otimes m) \\
&= \xi(a_1 \otimes r_1m) + \xi(a_2 \otimes r_2m) \\
&= \Lambda_A^B(\xi)(a_1)(r_1m) + \Lambda_A^B(\xi)(a_2)(r_2m) \\
&= \Lambda_A^B(\xi)(a_1)\, r_1(m) + \Lambda_A^B(\xi)(a_2)\, r_2(m).
\end{aligned}$$

$\Gamma_A^B(\zeta)$ ***is well-defined.*** *We have to prove that the assignment* $(a, m) \mapsto \zeta(a)(m)$ *is balanced. Additivity is trivial, moreover*

$$\begin{aligned}
\zeta(ar)(m) &= (\zeta(a)\, r)(m) && \text{since } \zeta \text{ is a morphism in } \mathrm{Mod}\text{-}R \\
&= \zeta(a)(rm) && \text{by definition of } \cdot \text{ in } (\mathrm{Hom}_S(M, A))_R\,.
\end{aligned}$$

$\Gamma_A^B(\zeta)$ ***is a morphism in*** Mod-S***.*** *We have*

$$\begin{aligned}
\Gamma_A^B(\zeta)((a_1 \otimes m_1)\, s_1 + (a_2 \otimes m_2)\, s_2) &= \Gamma_A^B(\zeta)(a_1 \otimes m_1s_1 + a_2 \otimes m_2s_2) \\
&= \Gamma_A^B(\zeta)(a_1 \otimes m_1s_1) + \Gamma_A^B(\zeta)(a_2 \otimes m_2s_2) \\
&= \zeta(a_1)(m_1s_1) + \zeta(a_2)(m_2s_2) \\
&= \zeta(a_1)(m_1)\, s_1 + \zeta(a_2)(m_2)\, s_2 \\
&= \Gamma_A^B(\zeta)(a_1 \otimes m_1)\, s_1 + \Gamma_A^B(\zeta)(a_2 \otimes m_2)\, s_2.
\end{aligned}$$

$\Gamma_A^B = \left(\Lambda_A^B\right)^{-1}$***.*** *Given* $\xi \in \mathrm{Hom}_S(B \otimes_R M, A)\,, \xi : B \otimes_R M \longrightarrow A$, *we have*

$$\begin{aligned}
\Gamma_A^B\left(\Lambda_A^B(\xi)\right)(\overline{a} \otimes \overline{m}) &= \Gamma_A^B(a \mapsto (m \mapsto \xi(a \otimes m)))(\overline{a} \otimes \overline{m}) \\
&= (a \otimes m \mapsto \xi(a \otimes m))(\overline{a} \otimes \overline{m}) \\
&= \xi(\overline{a} \otimes \overline{m})\,.
\end{aligned}$$

Given $\zeta \in \mathrm{Hom}_S(B \otimes_R M, A)\,, \zeta : B \longrightarrow \mathrm{Hom}_S(M, A)$, *we have*

$$\begin{aligned}
\Lambda_A^B\left(\Gamma_A^B(\zeta)\right)(\bar{a})(\bar{m}) &= \Lambda_A^B(a \otimes m \mapsto \zeta(a)(m))(\overline{a})(\overline{m}) \\
&= (a \mapsto (m \mapsto \zeta(a)(m)))(\overline{a})(\overline{m}) \\
&= (m \mapsto \zeta(\overline{a})(m))(\overline{m}) \\
&= \zeta(\overline{a})(\overline{m})
\end{aligned}$$

(L, R) ***is an adjunction.*** *We have to prove that the diagram*

$$\begin{array}{ccc}
\mathrm{Hom}_S(A_1 \otimes_R M, B_1) & \xrightarrow{\Lambda_{A_1}^{B_1}} & \mathrm{Hom}_R(A_1, \mathrm{Hom}_S(M, B_1)) \\
{\scriptstyle \mathrm{Hom}_S(f \otimes_R M, g)} \downarrow & & \downarrow {\scriptstyle \mathrm{Hom}_R(f, \mathrm{Hom}_S(M, g))} \\
\mathrm{Hom}_S(A_2 \otimes_R M, B_2) & \xrightarrow[\Lambda_{A_2}^{B_2}]{} & \mathrm{Hom}_R(A_1, \mathrm{Hom}_S(M, B_2))\,.
\end{array}$$

is commutative. Starting from $\xi : B_1 \otimes_R M \longrightarrow A$*:*

$$\begin{aligned} \mathrm{Hom}_R\left(f, \mathrm{Hom}_S\left(M, g\right)\right)\left(\Lambda_{A_1}^{B_1}\left(\xi\right)\right) &= \mathrm{Hom}_R\left(f, \mathrm{Hom}_S\left(M, g\right)\right)\left(a_1 \mapsto \left(m \mapsto \xi\left(a_1 \otimes m\right)\right)\right) \\ &= \mathrm{Hom}_S\left(M, g\right)\left(a_1 \mapsto \left(m \mapsto \xi\left(a_1 \otimes m\right)\right)\right) f \\ &= \mathrm{Hom}_S\left(M, g\right)\left(a_2 \mapsto \left(m \mapsto \xi\left(f\left(a_2\right) \otimes m\right)\right)\right) \\ &= a_2 \mapsto \left(m \mapsto g\xi\left(f\left(a_2\right) \otimes m\right)\right) \end{aligned}$$

and

$$\begin{aligned} \Lambda_{A_2}^{B_2}\left(\mathrm{Hom}_S\left(f \otimes_R M, g\right)\left(\xi\right)\right) &= \Lambda_{A_2}^{B_2}\left(g\xi\left(f \otimes_R M\right)\right) \\ &= \Lambda_{A_2}^{B_2}\left(a_2 \otimes m \mapsto g\xi\left(f\left(a_2\right) \otimes m\right)\right) \\ &= a_2 \mapsto \left(m \mapsto g\xi\left(f\left(a_2\right) \otimes m\right)\right). \end{aligned}$$

is also a functorial isomorphism.

Theorem 5.4. *If* (L, R) *and* (L', R) *are adjunctions, then* $L \cong L'$.

Proof. Let $\Lambda : \mathrm{Hom}_{\mathcal{A}}\left(L\left(\bullet\right), \blacktriangle\right) \to \mathrm{Hom}_{\mathcal{B}}\left(\bullet, R\left(\blacktriangle\right)\right)$ and $\Lambda' : \mathrm{Hom}_{\mathcal{A}}\left(L\left(\bullet\right), \blacktriangle\right) \to \mathrm{Hom}_{\mathcal{B}}\left(\bullet, R\left(\blacktriangle\right)\right)$ be the functor isomorphisms.

Construction of the isomorphism.

$\lambda =: \left(\Lambda'\right)^{-1}\Lambda : \mathrm{Hom}_{\mathcal{A}}\left(L\left(\bullet\right), \blacktriangle\right) \to \mathrm{Hom}_{\mathcal{A}}\left(L'\left(\bullet\right), \blacktriangle\right)$ **is a functorial isomorphism as both** Λ and $\left(\Lambda'\right)^{-1}$ are. Hence, given $f : B_2 \longrightarrow B_1$, $g : A_1 \longrightarrow A_2$ and $\xi : L\left(B_1\right) \longrightarrow A_1$, we have that

$$\begin{aligned} &(5.5) & \lambda_{A_2}^{B_2} \circ \mathrm{Hom}_{\mathcal{A}}\left(L\left(f\right), g\right) &= \mathrm{Hom}_{\mathcal{A}}\left(L'\left(f\right), g\right) \circ \lambda_{A_1}^{B_1} \text{ i.e.} \\ &(5.6) & \lambda_{A_2}^{B_2}\left[g \circ \xi \circ L\left(f\right)\right] &= g \circ \lambda_{A_1}^{B_1}\left(\xi\right) \circ L'\left(f\right) \end{aligned}$$

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{A}}\left(L\left(B_1\right), A_1\right) & \xrightarrow{\lambda_{A_1}^{B_1}} & \mathrm{Hom}_{\mathcal{A}}\left(L'\left(B_1\right), A_1\right) \\ \downarrow {\scriptstyle \mathrm{Hom}_{\mathcal{A}}(L(f),g)} & & \downarrow {\scriptstyle \mathrm{Hom}_{\mathcal{A}}(L'(f),g)} \\ \mathrm{Hom}_{\mathcal{A}}\left(L\left(B_2\right), A_2\right) & \xrightarrow[\lambda_{A_2}^{B_2}]{} & \mathrm{Hom}_{\mathcal{A}}\left(L'\left(B_2\right), A_2\right): \end{array} .$$

The equality (5.6) is equivalent to the following equalities

$$\begin{aligned} &(5.7) & \lambda_{A_2}^{B}\left(g \circ \xi\right) &= g \circ \lambda_{A_1}^{B}\left(\xi\right) \\ &(5.8) & \lambda_{A}^{B_2}\left[\xi \circ L\left(f\right)\right] &= \lambda_{A}^{B_1}\left(\xi\right) \circ L'\left(f\right). \end{aligned}$$

In particular, for $g = L\left(f\right), B = B_2$ and $\xi = \mathrm{Id}_{L(B_2)}$, we get from (5.7) that

$$\lambda_{L(B_1)}^{B_2}\left[L\left(f\right)\right] = L\left(f\right) \circ \lambda_{L(B_2)}^{B_2}\left(\mathrm{Id}_{L(B_2)}\right). \tag{5.9}$$

For $A = L\left(B_1\right)$ and $\xi = \mathrm{Id}_{L(B_1)}$, we get from (5.8) that

$$\lambda_{L(B_1)}^{B_2}\left(L\left(f\right)\right) = \lambda_{L(B_1)}^{B_1}\left(\mathrm{Id}_{L(B_1)}\right) \circ L'\left(f\right), \tag{5.10}$$

and for $A_1 = L(B)$, $A_2 = L'(B)$ and $\xi = \mathrm{Id}_{L(B)}$, we get from (5.7) that

$$\lambda^B_{L'(B)}(g) = g \circ \lambda^B_{L(B)}\left(\mathrm{Id}_{L(B)}\right) \tag{5.11}$$

We define $\chi : L' \longrightarrow L$, by setting

$$\chi_B = \lambda^B_{L(B)}\left(\mathrm{Id}_{L(B)}\right).$$

$\chi\colon L' \to L$ **is a morphism of functors.** We have to prove that

$$L(f) \circ \chi_{B_2} = \chi_{B_1} \circ L'(f)$$

$$\begin{array}{ccc} L'(B_2) & \xrightarrow{\chi_{B_2}} & L(B_2) \\ {\scriptstyle L'(f)}\downarrow & & \downarrow{\scriptstyle L(f)} \\ L'(B_1) & \xrightarrow[\chi_{B_1}]{} & L(B_1) \end{array} .$$

We compute

$$L(f) \circ \chi_{B_2} = L(f) \circ \lambda^{B_2}_{L(B_2)}\left(\mathrm{Id}_{L(B_2)}\right) \overset{(5.9)}{=} \lambda^{B_2}_{L(B_1)}(L(f))$$

$$\chi_{B_1} \circ L'(f) = \lambda^{B_1}_{L(B_1)}\left(\mathrm{Id}_{L(B_1)}\right) \circ L'(f) \overset{(5.10)}{=} \lambda^{B_2}_{L(B_1)}(L(f)).$$

χ **is a functorial isomorphism.** We construct the inverse of χ. We set

$$\zeta_B = \left(\Lambda^B_{L'(B)}\right)^{-1} \circ \Lambda'^B_{L'(B)}\left(\mathrm{Id}_{L'(B)}\right) : L(B) \longrightarrow L'(B)$$

We compute

$$\zeta_B \circ \chi_B = \zeta_B \circ \lambda^B_{L(B)}\left(\mathrm{Id}_{L(B)}\right) \overset{(5.11)}{=} \lambda^B_{L'(B)}(\zeta_B) =$$
$$= \left(\Lambda'^B_{L'(B)}\right)^{-1} \circ \Lambda^B_{L'(B)} \circ \left(\Lambda^B_{L'(B)}\right)^{-1} \circ \Lambda'^B_{L'(B)}\left(\mathrm{Id}_{L'(B)}\right) = \mathrm{Id}_{L'(B)}.$$

By symmetry, we also get $\chi_B \circ \zeta_B = \mathrm{Id}_{L(B)}$. □

In an analogous way, one can prove the following result.

Theorem 5.5. *If (L, R) and (L, R') are adjunctions, then $R \cong R'$.*

Theorem 5.6. *Let (L, R) be an adjunction where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$ and let*

$$\Lambda : \mathrm{Hom}_{\mathcal{A}}(L(\bullet), \blacktriangle) \to \mathrm{Hom}_{\mathcal{B}}(\bullet, R(\blacktriangle)).$$

be a functorial isomorphism. Let

$$\eta_B = \Lambda^B_{L(B)}\left(\mathrm{Id}_{L(B)}\right) : B \to RL(B).$$

Then $\eta : \mathrm{Id}_{\mathcal{B}} \to RL$ is a functorial morphism (called unit *of the adjunction). Let*

$$\epsilon_A = \left(\Lambda^{R(A)}_A\right)^{-1}\left(\mathrm{Id}_{R(A)}\right) : LR(A) \to A$$

Then $\epsilon : LR \to \mathrm{Id}_{\mathcal{A}}$ is a morphism functorial (called counit *of the adjunction).*

Moreover we have

1)

$$\Lambda_A^B(\gamma) = R(\gamma) \circ \eta_B, \ \textit{for every } \gamma \in \mathrm{Hom}_{\mathcal{A}}(L(B), A) \tag{5.12}$$

2)

$$\left(\Lambda_A^B\right)^{-1}(\varphi) = \epsilon_A \circ L(\varphi), \ \textit{for every } \varphi \in \mathrm{Hom}_{\mathcal{B}}(B, R(A)) \tag{5.13}$$

3)

$$\epsilon_{L(B)} \circ L(\eta_B) = \mathrm{Id}_{L(B)} \tag{5.14}$$

4)

$$R(\epsilon_A) \circ \eta_{R(A)} = \mathrm{Id}_{R(A)} \tag{5.15}$$

for every $B \in \mathcal{B}$, $A \in \mathcal{A}$, $f : B \longrightarrow R(A)$ *and* $g : L(B) \longrightarrow A$.

Proof. Let $f : B_2 \to B_1$. We have to prove that

$$RL(f) \circ \eta_{B_2} = \eta_{B_1} \circ f$$

$$\begin{array}{ccc} B_2 & \xrightarrow{\eta_{B_2}} & RL(B_2) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle RL(f)} \\ B_1 & \xrightarrow[\eta_{B_1}]{} & RL(B_1) \end{array}.$$

By (5.3) applied to the case when $g = L(f) : L(B_2) \to L(B_1)$ and $\xi = \mathrm{Id}_{L(B_2)} : L(B_2) \to L(B_2)$, we get

$$\Lambda_{L(B_1)}^{B_2}\left[L(f) \circ \mathrm{Id}_{L(B_2)}\right] = RL(f) \circ \Lambda_{L(B_2)}^{B_2}\left(\mathrm{Id}_{L(B_2)}\right)$$

so that

$$\Lambda_{L(B_1)}^{B_2}[L(f)] = RL(f) \circ \Lambda_{L(B_2)}^{B_2}\left(\mathrm{Id}_{L(B_2)}\right) \tag{5.16}$$

We have

$$\Lambda_{L(B_1)}^{B_2}(L(f)) \overset{(5.16)}{=} RL(f) \circ \Lambda_{L(B_2)}^{B_2}\left(\mathrm{Id}_{L(B_2)}\right) = RL(f) \circ \eta_{B_2}.$$

By (5.4) applied to the case when $\xi = \mathrm{Id}_{L(B_1)} : L(B_1) \to L(B_1)$ and $f = f : B_2 \to B_1$, we get

$$\Lambda_{L(B_1)}^{B_1}\left(\mathrm{Id}_{L(B_1)}\right) \circ f = \Lambda_{L(B_1)}^{B_2}\left[\mathrm{Id}_{L(B_1)} \circ (L(f))\right]$$

so that

$$\Lambda_{L(B_1)}^{B_1}\left(\mathrm{Id}_{L(B_1)}\right) \circ f = \Lambda_{L(B_1)}^{B_2}[(L(f))] \tag{5.17}$$

$$\eta_{B_1} \circ f = \Lambda^{B_1}_{L(B_1)} \left(\mathrm{Id}_{L(B_1)}\right) \circ f \overset{(5.17)}{=} \Lambda^{B_2}_{L(B_1)} \left(L\left(f\right)\right).$$

Let $g : A_1 \longrightarrow A_2$. We have to prove that

$$g \circ \epsilon_{A_1} = \epsilon_{A_2} \circ LR\left(g\right)$$

$$\begin{array}{ccc} LR\left(B_1\right) & \xrightarrow{\epsilon_{B_1}} & B_1 \\ {\scriptstyle LR(g)}\downarrow & & \downarrow{\scriptstyle g} \\ LR\left(B_2\right) & \xrightarrow[\epsilon_{B_2}]{} & B_2 \end{array}.$$

Since Λ is an isomorphism, we will equivalently prove that

$$\Lambda^{R(A_1)}_{A_2} \left(g \circ \epsilon_{A_1}\right) = \Lambda^{R(A_1)}_{A_2} \left(\epsilon_{A_2} \circ LR\left(g\right)\right).$$

By (5.3) applied to the case when $g = g : A_1 \to A_2, \xi = \epsilon_{A_1} : RL\left(A_1\right) \to A_1 = \mathrm{Id}_{L(B_1)} : L\left(B_1\right) \to L\left(B_1\right), f = \mathrm{Id}_{R(A_1)} : R\left(A_1\right) \to R\left(A_1\right)$, we get

$$R\left(g\right) \circ \Lambda^{R(A_1)}_{A_1} \left(\epsilon_{A_1}\right) = \Lambda^{R(A_1)}_{A_2} \left(g \circ \epsilon_{A_1}\right) \tag{5.18}$$

$$\Lambda^{R(A_1)}_{A_2} \left(g \circ \epsilon_{A_1}\right) \overset{(5.18)}{=} R\left(g\right) \circ \Lambda^{R(A_1)}_{A_1} \left(\epsilon_{A_1}\right) = \left[R\left(g\right) \circ \Lambda^{R(A_1)}_{A_1} \circ \left(\Lambda^{R(A_1)}_{A_1}\right)^{-1}\right] \left(\mathrm{Id}_{R(A_1)}\right) = R\left(g\right).$$

By (5.4) applied to the case when $f = R\left(g\right) : R\left(A_1\right) \to R\left(A_2\right)$ and $\xi = \epsilon_{A_2} : LR\left(A_2\right) \to A_2$, we get

$$\Lambda^{R(A_1)}_{A_2} \left(\epsilon_{A_2} \circ LR\left(g\right)\right) = \Lambda^{R(A_2)}_{A_2} \left(\epsilon_{A_2}\right) \circ R\left(g\right) \tag{5.19}$$

$$\begin{aligned} \Lambda^{R(A_1)}_{A_2} \left(\epsilon_{A_2} \circ LR\left(g\right)\right) &\overset{(5.19)}{=} \Lambda^{R(A_2)}_{A_2} \left(\epsilon_{A_2}\right) \circ R\left(g\right) \\ &= \left[\Lambda^{R(A_2)}_{A_2} \circ \left(\Lambda^{R(A_2)}_{A2}\right)^{-1}\right] \left(\mathrm{Id}_{R(A_2)}\right) \circ R\left(g\right) = R\left(g\right). \end{aligned}$$

1) $R\left(\gamma\right) \circ \eta_B = R\left(\gamma\right) \circ \Lambda^B_{L(B)} \left(\mathrm{Id}_{L(B)}\right) \overset{(5.3)}{=} \Lambda^B_A \left(\gamma\right).$

2) In order to prove 2) we apply to both terms Λ which is an isomorphism:

$$\Lambda^B_A \left(\epsilon_A \circ L\left(\varphi\right)\right) \overset{(5.4)}{=} \Lambda^{R(A)}_A \left(\epsilon_A\right) \circ \varphi = \left[\Lambda^{R(A)}_A \circ \left(\Lambda^{R(A)}_A\right)^{-1}\right] \left(\mathrm{Id}_{R(A)}\right) \circ \varphi = \varphi.$$

3) By applying 2) to the first term of the equality we have

$$\epsilon_{L(B)} \circ L\left(\eta_B\right) \overset{2)}{=} \left(\Lambda^B_{L(B)}\right)^{-1} \left(\eta_B\right) = \left[\left(\Lambda^B_{L(B)}\right)^{-1} \circ \Lambda^B_{L(B)}\right] \left(\mathrm{Id}_{L(B)}\right) = \mathrm{Id}_{L(B)}.$$

4) By applying 1) to the first term of the equality we get

$$R\left(\epsilon_A\right) \circ \eta_{R(A)} \overset{1)}{=} \Lambda^{R(A)}_A \left(\epsilon_A\right) = \Lambda^{R(A)}_A \left(\Lambda^{R(A)}_A\right)^{-1} \left(\mathrm{Id}_{R(A)}\right) = \mathrm{Id}_{R(A)}.$$

□

Theorem 5.7. *Let* $L : \mathcal{B} \to \mathcal{A}$ *and* $R : \mathcal{A} \to \mathcal{B}$ *be covariant functors,* $\eta : \mathrm{Id}_{\mathcal{B}} \to RL$ *and* $\epsilon : LR \to \mathrm{Id}_{\mathcal{A}}$ *functorial morphisms such that, for every* $B \in \mathcal{B}$ *and* $A \in \mathcal{A}$*, we have*

$$\epsilon_{L(B)} \circ L(\eta_B) = \mathrm{Id}_{L(B)}$$

and

$$R(\epsilon_A) \circ \eta_{R(A)} = \mathrm{Id}_{R(A)}$$

Then (L, R) *is an adjunction with unit* η *and counit* ϵ*. Namely, for every* $B \in \mathcal{B}$ *and* $A \in \mathcal{A}$*,*

$$\mathrm{Hom}_{\mathcal{A}}(L(B), A) \xrightarrow{\Lambda_A^B} \mathrm{Hom}_{\mathcal{B}}(B, R(A))$$

defined by setting

$$\Lambda_A^B(\xi) = R(\xi) \circ \eta_B$$

is a natural isomorphism with inverse

$$\mathrm{Hom}_{\mathcal{B}}(B, R(A)) \xrightarrow{\Gamma_A^B} \mathrm{Hom}_{\mathcal{A}}(L(B), A)$$

defined by setting

$$\Gamma_A^B(\zeta) = \epsilon_A \circ L(\zeta).$$

Proof. $\mathbf{\Gamma_A^B = \left(\Lambda_A^B\right)^{-1}}$**.** Given $\xi : L(B) \longrightarrow A$ and $\zeta : B \longrightarrow R(A)$, since ϵ and η are functorial morphisms, we have:

$$\begin{aligned}\Gamma_A^B\left(\Lambda_A^B(\xi)\right) &= \Gamma_A^B(R(\xi) \circ \eta_B) \\ &= \epsilon_A \circ L(R(\xi) \circ \eta_B) \\ &= \epsilon_A \circ LR(\xi) \circ (L(\eta_B)) \\ &\overset{\epsilon}{=} \xi \circ \epsilon_{L(B)} \circ L(\eta_B) \\ &= \xi\end{aligned}$$

and

$$\begin{aligned}\Lambda_A^B\left(\Gamma_A^B(\zeta)\right) &= \Lambda_A^B(\epsilon_A \circ L(\zeta)) \\ &= R(\epsilon_A \circ L(\zeta)) \circ \eta_B \\ &= (R(\epsilon_A)) \circ RL(\zeta) \circ \eta_B \\ &\overset{\eta}{=} R(\epsilon_A) \circ \eta_{R(A)} \circ \zeta \\ &= \zeta\end{aligned}$$

$\mathbf{\Lambda}$ gives rise to an adjunction. Given $f : B_2 \longrightarrow B_1$, $g : A_1 \longrightarrow A_2$ and $\xi : L(B_1) \longrightarrow A_1$, we have:

$$\begin{aligned}\Lambda_{A_2}^{B_2}(g \circ \xi \circ L(f)) &= R(g \circ \xi \circ L(f)) \circ \eta_{B_2} \\ &= R(g) \circ R(\xi) \circ RL(f) \circ \eta_{B_2}\end{aligned}$$

$$R(g)\circ\Lambda_{A_1}^{B_1}(\xi)\circ f = R(g)\circ R(\xi)\circ\eta_{B_1}\circ f$$
$$\overset{\eta}{=} R(g)\circ R(\xi)\circ RL(f)\circ\eta_{B_2}$$

η and ϵ are unit and counit. The unit of the adjunction (L,R) is

$$\Lambda_{L(B)}^{B}\left(\mathrm{Id}_{L(B)}\right) = R\left(\mathrm{Id}_{L(B)}\right)\circ\eta_B = \eta_B,$$

whereas the counit is

$$\left(\Lambda_A^{R(A)}\right)^{-1}\left(\mathrm{Id}_{R(A)}\right) = \Gamma_A^{R(A)}\left(\mathrm{Id}_{R(A)}\right) = \epsilon_A\circ L\left(\mathrm{Id}_{R(A)}\right) = \epsilon_A.$$

□

Theorem 5.8. *Let $L:\mathcal{B}\to\mathcal{A}$ and $R:\mathcal{A}\to\mathcal{B}$ be covariant functors determining an equivalence between $\mathcal{B}$ and $\mathcal{A}$, i.e. there are functorial isomorphisms $\eta:\mathrm{Id}_{\mathcal{B}}\to RL$ and $\rho:LR\to\mathrm{Id}_{\mathcal{A}}$. Then (L,R) is an adjunction with unit η and counit ϵ, where $\epsilon_A = \rho_A\circ L\left(\eta_{R(A)}^{-1}\right)\circ\rho_{LR(A)}^{-1}$, for every $A\in\mathcal{A}$.*

Proof. We will prove that the hypothesis of Theorem 5.7 hold. First we want to prove that

$$\eta_{RL(B)} = RL(\eta_B) \text{ and } \rho_{LR(A)} = LR(\rho_A)\,. \tag{5.20}$$

In fact we have

$$\eta_{RL(B)}\circ\eta_B \overset{\eta}{=} RL(\eta_B)\circ\eta_B$$

and

$$\rho_{LR(A)}\circ\rho_A = LR(\rho_A)\circ\rho_A$$

and since η and ρ are iso we conclude. Then we have:

$$\begin{aligned}
\epsilon_{L(B)}\circ L(\eta_B) &= \rho_{L(B)}\circ L\left(\eta_{RL(B)}^{-1}\right)\circ\rho_{LRL(B)}^{-1}\circ L(\eta_B)\\
&\overset{\rho^{-1}}{=} \rho_{L(B)}\circ L\left(\eta_{RL(B)}^{-1}\right)\circ LRL(\eta_B)\circ\rho_{L(B)}^{-1}\\
&\overset{(5.20)}{=} \rho_{L(B)}\circ L\left(\eta_{RL(B)}^{-1}\right)\circ L\left(\eta_{RL(B)}\right)\circ\rho_{L(B)}^{-1}\\
&= \mathrm{Id}_{L(B)}
\end{aligned}$$

$$\begin{aligned}
R(\epsilon_A)\circ\eta_{R(A)} &= R\left(\rho_A\circ L\left(\eta_{R(A)}^{-1}\right)\circ\rho_{LR(A)}^{-1}\right)\circ\eta_{R(A)}\\
&= R(\rho_A)\circ RL\left(\eta_{R(A)}^{-1}\right)\circ R\left(\rho_{LR(A)}^{-1}\right)\circ\eta_{R(A)}\\
&\overset{(5.20)}{=} R(\rho_A)\circ RL\left(\eta_{R(A)}^{-1}\right)\circ RLR\left(\rho_A^{-1}\right)\circ\eta_{R(A)}\\
&\overset{\eta}{=} R(\rho_A)\circ RL\left(\eta_{R(A)}^{-1}\right)\circ\eta_{RLR(A)}\circ R\left(\rho_A^{-1}\right)\\
&= R(\rho_A)\circ RL\left(\eta_{R(A)}^{-1}\right)\circ RL\left(\eta_{R(A)}\right)\circ R\left(\rho_A^{-1}\right)\\
&= \mathrm{Id}_{R(A)}
\end{aligned}$$

□

Theorem 5.9. *Let (L, R) be an adjunction, $L : \mathcal{B} \longrightarrow \mathcal{A}$, $R : \mathcal{A} \longrightarrow \mathcal{B}$ and let $F : \mathcal{I} \longrightarrow \mathcal{A}$ be a functor where $\mathcal{I}$ is a small category. Assume that there exists $\left(\varprojlim F, (\alpha_I)_{I\in\mathcal{I}}\right)$ in $\mathcal{A}$. Then $\left(R\left(\varprojlim F\right), (R(\alpha_I))_{I\in\mathcal{I}}\right)$ is the limit of $RF : \mathcal{I} \longrightarrow \mathcal{B}$.*

Proof. First of all we prove that $\left(R\left(\varprojlim F\right), (R(\alpha_I))_{I\in\mathcal{I}}\right)$ is a cone. Since $\left(\varprojlim F, (\alpha_I)_{I\in\mathcal{I}}\right)$ is a cone on F we have that

$$\alpha_J = F(\lambda) \circ \alpha_I \text{ for every morphism } \lambda : I \to J.$$

By applying R, we get

$$R(\alpha_J) = RF(\lambda) \circ R(\alpha_I).$$

Let now $\left(X, (\xi_I : X \longrightarrow RF(I))_{I\in\mathcal{I}}\right)$ be a cone on RF.

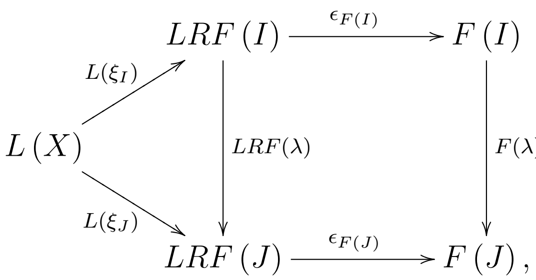
There exists $X \xrightarrow{\xi} R\left(\varprojlim F\right)$. Since $\left(X, (\xi_I)_{I\in\mathcal{I}}\right)$ is a cone we have

$$\xi_J = RF(\lambda) \circ \xi_I$$

so that, by applying L, we get

$$L(\xi_J) = LRF(\lambda) \circ L(\xi_I).$$

We have



where ϵ is the counit of the adjunction. Thus $L(X)$ is a cone on F with morphisms $\epsilon_{F(I)} \circ L(\xi_I)$ and thus there exists a unique morphism

$$\zeta : L(X) \longrightarrow \varprojlim F$$

such that

$$\alpha_I \circ \zeta = \epsilon_{F(I)} \circ L(\xi_I).$$

Let

$$\xi = \Lambda^X_{\varprojlim F}(\zeta) = R(\zeta) \circ \eta_X$$

where Λ is the isomorphism of the adjunction (L, R). Thus $\xi : X \longrightarrow R\left(\varprojlim F\right)$. We will prove that $R(\alpha_I) \circ \xi = \xi_I$. By the properties of the adjunction we have

$$\begin{aligned} R(\alpha_I) \circ \xi &= R(\alpha_I) \circ \Lambda^X_{\varprojlim F}(\zeta) \overset{(5.3)}{=} \Lambda^X_{F(I)}(\alpha_I \circ \zeta) = \Lambda^X_{F(I)}\left(\epsilon_{F(I)} \circ L(\xi_I)\right) = \Lambda^X_{F(I)}\left(\epsilon_{F(I)} \circ L(\xi_I)\right) = \\ &\overset{(5.4)}{=} \Lambda^{RF(I)}_{F(I)}\left(\epsilon_{F(I)}\right) \circ \xi_I = \Lambda^{RF(I)}_{F(I)}\left(\left(\Lambda^{RF(I)}_{F(I)}\right)^{-1}\left(\mathrm{Id}_{RF(I)}\right)\right) \circ \xi_I = \xi_I. \end{aligned}$$

ξ **is unique.** Let $\xi' : X \longrightarrow R\left(\varprojlim F\right)$ be another morphism such that $R(\alpha_I) \circ \xi' = \xi_I$, i.e. we have $R(\alpha_I) \circ \xi' = \xi_I = R(\alpha_I) \circ \xi$. Since $\Lambda^X_{\varprojlim F}$ is an isomorphism, there exists a unique $\zeta' : LX \longrightarrow \varprojlim F$ such that $\xi' = \Lambda^X_{\varprojlim F}(\zeta')$. Then we have

$$R(\alpha_I) \circ \xi' = R(\alpha_I) \circ \Lambda^X_{\varprojlim F}(\zeta') \overset{(5.3)}{=} \Lambda^X_{F(I)}(\alpha_I \circ \zeta')$$

and

$$R(\alpha_I) \circ \xi = R(\alpha_I) \circ \Lambda^X_{\varprojlim F}(\zeta) = \Lambda^X_{F(I)}(\alpha_I \circ \zeta).$$

.Since $\Lambda^X_{F(I)}$ is an isomorphism, we get $\alpha_I \circ \zeta' = \alpha_I \circ \zeta$ for every $I \in \mathcal{I}$, thus, by uniqueness of $\varprojlim F$, $\zeta = \zeta'$ and $\xi = \xi'$. □

Corollary 5.10. *Let (L, R) be an adjunction, $L : \mathcal{B} \longrightarrow \mathcal{A}$, $R : \mathcal{A} \longrightarrow \mathcal{B}$. Assume that both $\mathcal{B}$ and $\mathcal{A}$ are preadditive with zero. If P is a pullback, then $R(P)$ is also a pullback.*

Corollary 5.11. *Let (L, R) be an adjunction, $L : \mathcal{B} \longrightarrow \mathcal{A}$, $R : \mathcal{A} \longrightarrow \mathcal{B}$. Assume that both $\mathcal{B}$ and $\mathcal{A}$ are preadditive with zero and both L and R are additive. If* $\mathrm{Ker}(f)$ *exists in $\mathcal{A}$, then also* $\mathrm{Ker}(R(f))$ *exists and* $R(\mathrm{Ker}(f)) = (\mathrm{Ker}(R(f)))$.

Proof. A kernel is a particular kind of pullback. □

Proposition 5.12. *Let (L, R) be an adjunction, $L : \mathcal{B} \longrightarrow \mathcal{A}$, $R : \mathcal{A} \longrightarrow \mathcal{B}$. Assume that both $\mathcal{A}$ and $\mathcal{B}$ are abelian and both Land R are additive. Then R is a left exact functor.*

Proof. Let $0 \longrightarrow A' \xrightarrow{\alpha'} A \xrightarrow{\alpha''} A''$ be an exact sequence in $\mathcal{A}$. This means that

$$(A', \alpha') = \mathrm{Ker}(\alpha").$$

By Corollary 5.11, we get that

$$(R(A'), R(\alpha')) = \mathrm{Ker}\left(R\left(\alpha^{"}\right)\right)$$

which means that the sequence $0 \longrightarrow R(A') \xrightarrow{R(\alpha')} R(A) \xrightarrow{R(\alpha'')} R(A'')$ is exact in $\mathcal{A}$. □

Theorem 5.13. *Let (L, R) be an adjunction, $L : \mathcal{B} \longrightarrow \mathcal{A}$, $R : \mathcal{A} \longrightarrow \mathcal{B}$ and let $G : \mathcal{I} \longrightarrow \mathcal{B}$ be a functor where $\mathcal{I}$ is a small category. Assume that there exists $\left(\varinjlim G, (\alpha_I)_{I \in \mathcal{I}}\right)$ in $\mathcal{A}$. Then $\left(L\left(\varinjlim F\right), (L(\alpha_I))_{I \in \mathcal{I}}\right)$ is the limit of $LF : \mathcal{I} \longrightarrow \mathcal{A}$.*

Proof. It is analogous to that of Theorem 5.9 and it is left as an exercise to the reader. □

Corollary 5.14. *In the assumption of Theorem 5.9, in particular if X is a pushout, then $L(X)$ is also a pushout.*

Proof. A pullback is a particular kind of colimit. □

Corollary 5.15. *In the assumption of Theorem 5.9 we have* $L(\mathrm{Coker}(f)) = (\mathrm{Coker} L(f))$.

Proof. A cokernel is a particular kind of pushout. □

Proposition 5.16. *Let* (L, H) *be an adjunction,* $L : \mathcal{B} \longrightarrow \mathcal{A}$, $H : \mathcal{A} \longrightarrow \mathcal{B}$. *Assume that both* $\mathcal{B}$ *and* $\mathcal{A}$ *are abelian and both* L *and* H *are additive. Then* L *is a right exact functor.*

Proof. Let $B' \xrightarrow{\alpha'} B \xrightarrow{\alpha''} B'' \longrightarrow 0$ be an exact sequence in $\mathcal{B}$. This means that

$$(B'', \alpha'') = \mathrm{Coker}(\alpha').$$

By Corollary 5.15, we get that

$$(L(B''), L(\alpha'')) = \mathrm{Coker}(L(\alpha'))$$

which means that the sequence $L(B') \xrightarrow{L(\alpha')} L(B) \xrightarrow{L(\alpha'')} L(B'') \longrightarrow 0$ is exact in $\mathcal{A}$. □

Lemma 5.17. *Let* (L, R) *be an adjunction with unit* η *and counit* ϵ, *where* $L : \mathcal{B} \to \mathcal{A}$ *and* $R : \mathcal{A} \to \mathcal{B}$. *For every* $Y \in \mathcal{B}$ *the following conditions are equivalent:*

(1) $\mathcal{L}_{-,Y} = \left(\Lambda_{-}^{Y}\right)^{-1} \circ \mathrm{Hom}_{\mathcal{B}}(-, \eta Y)$ *is a functorial isomorphism*

(2) $\mathrm{Hom}_{\mathcal{B}}(-, \eta Y)$ *is a functorial isomorphism*

(3) ηY *is an isomorphism.*

Proof. Since (L, R) is an adjunction, $\Lambda_X^Z : \mathrm{Hom}_{\mathcal{A}}(LY, X) \to \mathrm{Hom}_{\mathcal{B}}(Y, RX)$ is an isomorphism for every $X \in \mathcal{A}$ and for every $Z \in \mathcal{B}$, so that (1) is equivalent to (2).
(2) ⇒ (3) Since $\mathrm{Hom}_{\mathcal{B}}(-, \eta Y)$ is a functorial isomorphism, in particular $\mathrm{Hom}_{\mathcal{B}}(RLY, \eta Y) : \mathrm{Hom}_{\mathcal{B}}(RLY, Y) \to \mathrm{Hom}_{\mathcal{B}}(RLY, RLY)$ is an isomorphism. Thus, there exists $f \in \mathrm{Hom}_{\mathcal{B}}(RLY, Y)$ such that $(\eta Y) \circ f = \mathrm{Id}_{RLY}$. Moreover we also have $\mathrm{Hom}_{\mathcal{B}}(Y, \eta Y)(\mathrm{Id}_Y) = \eta Y = (\eta Y) \circ f \circ (\eta Y) = \mathrm{Hom}_{\mathcal{B}}(Y, \eta Y)(f \circ (\eta Y))$. Since $\mathrm{Hom}_{\mathcal{B}}(-, \eta Y)$ is a functorial isomorphism, also $\mathrm{Hom}_{\mathcal{B}}(Y, \eta Y)$ is an isomorphism. Thus we deduce that $\mathrm{Id}_Y = f \circ (\eta Y)$. Hence ηY is an isomorphism with two-sided inverse $f : RLY \to Y$.
(3) ⇒ (2) Let h be the two-sided inverse of ηY. Then $\mathrm{Hom}_{\mathcal{B}}(-, h)$ is the inverse of the functor $\mathrm{Hom}_{\mathcal{B}}(-, \eta Y)$. In fact

$$\mathrm{Hom}_{\mathcal{B}}(-, h) \circ \mathrm{Hom}_{\mathcal{B}}(-, \eta Y) = \mathrm{Hom}_{\mathcal{B}}(-, h \circ \eta Y) = \mathrm{Hom}_{\mathcal{B}}(-, \mathrm{Id}_Y)$$
$$\mathrm{Hom}_{\mathcal{B}}(-, \eta Y) \circ \mathrm{Hom}_{\mathcal{B}}(-, h) = \mathrm{Hom}_{\mathcal{B}}(-, \eta Y \circ h) = \mathrm{Hom}_{\mathcal{B}}(-, \mathrm{Id}_{RLY}).$$

□

Proposition 5.18. *Let* (L, R) *be an adjunction with unit* η *and counit* ϵ, *where* $L : \mathcal{B} \to \mathcal{A}$ *and* $R : \mathcal{A} \to \mathcal{B}$. *Then* L *is full and faithful if and only if* η *is a functorial isomorphism.*

Proof. Note that, for every $f \in \mathrm{Hom}_{\mathcal{B}}(Y, Y')$ we have

$$\begin{aligned} \mathcal{L}_{Y,Y'}(f) &= \left[\left(\Lambda^{Y}_{LY'}\right)^{-1} \circ \mathrm{Hom}_{\mathcal{B}}(Y, \eta Y')\right](f) = \left(\Lambda^{Y}_{LY'}\right)^{-1}(\eta Y' \circ f) = \\ &= (\epsilon LY') \circ (L\eta Y') \circ (Lf) \overset{(L,R)\mathrm{adj}}{=} Lf. \end{aligned}$$

To be full and faithful for L means that the map

$$\begin{gathered} \phi : \mathrm{Hom}_{\mathcal{B}}(Y, Y') \longrightarrow \mathrm{Hom}_{\mathcal{A}}(LY, LY') \\ f \mapsto L(f) \end{gathered}$$

is bijective for every $Y, Y' \in \mathcal{B}$. Since this $\phi(f) = L(f) = \mathcal{L}_{Y,Y'}(f)$, ϕ is an isomorphism if and only if $\mathcal{L}_{Y,Y'}$ is an isomorphism for every $Y, Y' \in \mathcal{B}$ and, by Lemma 5.17, if and only if $\eta Y'$ is an isomorphism for every $Y' \in \mathcal{B}$. □

Lemma 5.19. *Let (L, R) be an adjunction with unit η and counit ϵ, where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$. For every $X \in \mathcal{A}$ the following conditions are equivalent:*

(1) $\mathcal{R}_{X,-} = \Lambda^{RX}_{-} \circ \mathrm{Hom}_{\mathcal{A}}(\epsilon X, -)$ *is a functorial isomorphism*

(2) $\mathrm{Hom}_{\mathcal{A}}(\epsilon X, -)$ *is a functorial isomorphism*

(3) ϵX *is an isomorphism.*

Proof. Exercise. □

Proposition 5.20. *Let (L, R) be an adjunction with unit η and counit ϵ, where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$. Then R is full and faithful if and only if ϵ is a functorial isomorphism.*

Proof. Exercise. □

Lemma 5.21. *Let $f : X \to Y$ and $g : Y \to X$ be morphisms in a category $\mathcal{C}$. Assume that $g \circ f = \mathrm{Id}_X$ and that $f \circ g$ is an isomorphism. Then f and g are isomorphisms and $g = f^{-1}$.*

Proof. From $g \circ f = \mathrm{Id}_X$ we infer that $f \circ g \circ f \circ g = f \circ \mathrm{Id}_X \circ g = f \circ g$ i.e.

$$f \circ g \circ f \circ g = f \circ g.$$

Hence

$$f \circ g = (f \circ g)^{-1} \circ f \circ g \circ f \circ g = (f \circ g)^{-1} \circ f \circ g = \mathrm{Id}_Y.$$

□

Proposition 5.22. *Let (L, R) be an adjunction with unit η and counit ϵ, where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$. Then the following assertions are equivalent.*

(a) ϵL *is a functorial isomorphism.*

(b) *$L\eta$ is a functorial isomorphism.*

(c) *$R\epsilon$ is a functorial isomorphism.*

(d) *ηR is a functorial isomorphism.*

(e) *$R\epsilon L$ is a functorial isomorphism.*

(f) *$RL\eta$ is a functorial isomorphism.*

(g) *ηRL is a functorial isomorphism.*

(h) *$LR\epsilon$ is a functorial isomorphism.*

(i) *ϵLR is a functorial isomorphism.*

(l) *$L\eta R$ is a functorial isomorphism.*

(m) *$LR\epsilon L$ is a functorial isomorphism.*

(n) *$LRL\eta$ is a functorial isomorphism.*

(o) *$L\eta RL$ is a functorial isomorphism.*

(p) *ϵLRL is a functorial isomorphism.*

Proof. Since (L, R) is an adjunction, formulas 5.14 and 5.15

$$\begin{aligned} \epsilon L \circ L\eta &= L \\ R\epsilon \circ \eta R &= R \end{aligned}$$

hold. Hence $(a) \Leftrightarrow (b)$ and $(c) \Leftrightarrow (d)$. Moreover we get

$$R\epsilon L \circ RL\eta = RL \tag{5.21}$$
$$R\epsilon L \circ \eta RL = RL \tag{5.22}$$

from which we deduce that $(e) \Leftrightarrow (f) \Leftrightarrow (g)$ and, if any of them holds, we also have

$$RL\eta = \eta RL. \tag{5.23}$$

Always from formulas 5.14 and 5.15, we get

$$\epsilon LR \circ L\eta R = LR \tag{5.24}$$
$$LR\epsilon \circ L\eta R = LR \tag{5.25}$$

from which we deduce that $(h) \Leftrightarrow (i) \Leftrightarrow (l)$ and, if any of them holds, we also have

$$\epsilon LR = LR\epsilon. \tag{5.26}$$

Now, from formulas (5.21), (5.22) and (5.24)we get

$$LR\epsilon L \circ LRL\eta = LRL \tag{5.27}$$

$$LR\epsilon L \circ L\eta RL = LRL \tag{5.28}$$

$$\epsilon LRL \circ L\eta RL = LRL \tag{5.29}$$

from which we deduce that $(m) \Leftrightarrow (n) \Leftrightarrow (o) \Leftrightarrow (p)$. Moreover if one of them holds hold, we obtain

$$LRL\eta = L\eta RL. \tag{5.30}$$

and

$$LR\epsilon L = \epsilon LRL$$

Let $\alpha : F \to G$ be a functorial morphism. Then, by naturality of ϵ, we get the commutative diagram

$$\begin{array}{ccc} LRF & \stackrel{\epsilon F}{\to} & F \\ LR\alpha \downarrow & & \downarrow \alpha \\ LRG & \stackrel{\epsilon G}{\to} & G \end{array}$$

and, by naturality of η, we get the commutative diagram

$$\begin{array}{ccc} F & \stackrel{\eta F}{\to} & RLF \\ \alpha \downarrow & & \downarrow RL\alpha \\ G & \stackrel{\eta G}{\to} & RLG \end{array}$$

so that we have

$$\begin{aligned} \epsilon G \circ LR\alpha &= \alpha \circ \epsilon F \\ \eta G \circ \alpha &= RL\alpha \circ \eta F. \end{aligned}$$

In particular, we get

$$\begin{aligned} \epsilon LR \circ LRL\eta &= L\eta \circ \epsilon L \\ \eta R \circ R\epsilon &= RLR\epsilon \circ \eta RLR \end{aligned}$$

and

$$\begin{aligned} \epsilon G \circ LR\alpha &= \alpha \circ \epsilon F \\ \eta G \circ \alpha &= RL\alpha \circ \eta F. \end{aligned}$$

$(e) \Leftrightarrow (a)$ and $(e) \Leftrightarrow (c)$ Clearly we have only to prove that $(e) \Rightarrow (a)$ and $(e) \Rightarrow (c)$. Since (e) holds, we know that $RL\eta = \eta RL$ are isomorphisms. Hence also $L\eta RL$ is an isomorphism i.e. (o) holds so that $LR\epsilon L = \epsilon LRL$ and $LRL\eta = L\eta RL$ are isomorphisms.

By naturality of ϵ we know that the diagram

$$\begin{array}{ccc} LRL & \overset{\epsilon L}{\to} & L \\ LRL\eta\downarrow & & \downarrow L\eta \\ LRLRL & \overset{\epsilon LRL}{\to} & LRL \end{array}$$

is commutative i.e.

$$L\eta \circ \epsilon L = \epsilon LRL \circ LRL\eta$$

and hence it is an isomorphism. Since $\epsilon L \circ L\eta = L$, by Lemma 5.21 we get that both ϵL and $L\eta$ are isomorphisms.

By $(e) \Leftrightarrow (g)$ we know that $RL\eta$ is iso, so that from $RLR\epsilon \circ RL\eta R = RLR$ we deduce that $RLR\epsilon$ is also an iso. By naturality of η we know that the diagram

$$\begin{array}{ccc} RLR & \overset{\eta RL}{\to} & RLRLR \\ R\epsilon\downarrow & & \downarrow RLR\epsilon \\ R & \overset{\eta R}{\to} & RLR \end{array}$$

is commutative i.e.

$$\eta R \circ R\epsilon = \eta RLR \circ RLR\epsilon$$

From $(e) \Leftrightarrow (f)$ we deduce that also ηRLR is an iso i.e. $\eta R \circ R\epsilon$ is an iso. From Lemma 5.21 we conclude.

Hence we have proved that $(a) = (b) = (c) = (d) = (e) = (f) = (g)$

$(h) \Leftrightarrow (c)$ Clearly we have only to prove that $(h) \Rightarrow (c)$. From

$$\begin{aligned} LR\epsilon \circ L\eta R &= LR \\ \epsilon LR \circ L\eta R &= LR \end{aligned}$$

we deduce that $L\eta R$ is also an iso and $LR\epsilon = \epsilon LR$ is an iso. Hence $R\epsilon LR$ is an iso and from $R\epsilon LR \circ \eta RLR = RLR$ also ηRLR is an iso. We have

$$\begin{array}{ccc} LRLR & \overset{\epsilon LR}{\to} & LR \\ LRL\eta R\downarrow & & \downarrow L\eta R \\ LRLRLR & \overset{\epsilon LRLR}{\to} & LRLR \end{array}$$

$$\epsilon LRLR \circ LRL\eta R = L\eta R \circ \epsilon LR$$

$$\eta R \circ R\epsilon = \eta RLR \circ RLR\epsilon = \eta RLR \circ R\epsilon LR$$

so that $\eta R \circ R\epsilon$ is an iso. From Lemma 5.21 we conclude.

$(p) \Leftrightarrow (a)$ Clearly we have only to prove that $(o) \Rightarrow (a)$. Since $(o) \Leftrightarrow (n)$, $LRL\eta$ is an iso so that, from

$$L\eta \circ \epsilon L = \epsilon LRL \circ LRL\eta$$

we deduce that $L\eta \circ \epsilon L$ is an iso. From Lemma 5.21 we conclude. □

Proposition 5.23. *Let $L : \mathcal{B} \to \mathcal{A}$ be a category equivalence with inverse $H : \mathcal{A} \to \mathcal{B}$. Assume that $\sigma : \mathrm{Id}_{\mathcal{B}} \to HL$ and $\rho : LH \to \mathrm{Id}_{\mathcal{A}}$ be functorial isomorphisms. Then (L, H) is an adjunction with unit $\eta = \sigma$ and counit $\varepsilon = \rho \circ L\eta^{-1}H \circ \rho^{-1}LH$. Alternatively (L, H) is an adjunction with unit $\eta = \sigma^{-1}HL \circ H\varepsilon^{-1}L \circ \sigma$ and counit $\varepsilon = \rho$*

Proof. Let $\eta = \sigma$ and $\varepsilon = \rho \circ L\eta^{-1}H \circ \rho^{-1}LH$. We have

$$\begin{aligned} \varepsilon L \circ L\eta &= \rho L \circ L\eta^{-1}HL \circ \rho^{-1}LHL \circ L\eta \\ &\overset{\rho^{-1}}{=} \rho L \circ L\eta^{-1}HL \circ LHL\eta \circ \rho^{-1}L \\ &\overset{\eta^{-1}}{=} \rho L \circ L\eta \circ L\eta^{-1} \circ \rho^{-1}L = \mathrm{Id}_L. \end{aligned}$$

From

$$\eta HL \circ \eta = HL\eta \circ \eta \tag{5.31}$$

we get

$$\eta HL = HL\eta$$

Similarly from

$$\rho LH \circ \rho = LH\rho \circ \rho$$

$$\rho LH = LH\rho \tag{5.32}$$

$$\begin{aligned} H\varepsilon = H\rho \circ HL\eta^{-1}H \circ H\rho^{-1}LH \circ \eta H &\overset{(5.31)}{=} H\rho \circ \eta^{-1}HLH \circ H\rho^{-1}LH \circ \eta H \\ \overset{(\eta^{-1})}{=} \eta^{-1}H \circ HLH\rho \circ H\rho^{-1}LH \circ \eta H &\overset{(5.32)}{=} \eta^{-1}H \circ H\rho LH \circ H\rho^{-1}LH \circ \eta H = \mathrm{Id}_H \end{aligned}$$

Let $\eta = \sigma^{-1}HL \circ H\varepsilon^{-1}L \circ \sigma$ and $\varepsilon = \rho$. We compute

$$\begin{aligned} H\varepsilon \circ \eta H &= H\varepsilon \circ \sigma^{-1}HLH \circ H\varepsilon^{-1}LH \circ \sigma H \\ &\overset{\sigma^{-1}}{=} \sigma^{-1}H \circ HLH\varepsilon \circ H\varepsilon^{-1}LH \circ \sigma H \\ &\overset{(5.32)}{=} \sigma^{-1}H \circ H\varepsilon LH \circ H\varepsilon^{-1}LH \circ \sigma H = \mathrm{Id}_H \end{aligned}$$

and

$$\begin{aligned} \varepsilon L \circ L\eta &= \varepsilon L \circ L\sigma^{-1}HL \circ LH\varepsilon^{-1}L \circ L\sigma \\ &\overset{(5.31)}{=} \varepsilon L \circ LHL\sigma^{-1} \circ \varepsilon^{-1}LHL \circ L\sigma \overset{\varepsilon^{-1}}{=} \\ &= \varepsilon L \circ \varepsilon^{-1}L \circ L\sigma^{-1} \circ L\sigma = \mathrm{Id}_L \end{aligned}$$

□

Lemma 5.24. *Let (L, R) be an adjunction where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$ such that R is an equivalence of categories. Then L is also an equivalence of categories.*

Proof. By assumption $R : \mathcal{A} \to \mathcal{B}$ is an equivalence of category with inverse $L' : \mathcal{B} \to \mathcal{A}$. By Proposition 5.23 we know that (L', R) is an adjunction. By the uniqueness of the adjoint we have that $L \simeq L'$ which is an equivalence. Thus L is also an equivalence of categories. □

5.1 Some results on equalizers and coequalizers

Definition 5.25. *A functorial morphism $\alpha : C \to D$ is called* functorial monomorphism*, or simply a monomorphism, if for every $\beta, \gamma : B \to C$ such that $\alpha \circ \beta = \alpha \circ \gamma$ we have $\beta = \gamma$.*

Definition 5.26. *A functorial morphism $\alpha : A \to B$ is called* functorial epimorphism*, or simply an epimorphism, if for every $\beta, \gamma : B \to C$ such that $\beta \circ \alpha = \gamma \circ \alpha$ we have $\beta = \gamma$.*

Definition 5.27. *Let $\mathcal{A}$ a category, let $Y, Z \in \mathcal{A}$ and let $f, g : Y \to Z$ be morphisms in $\mathcal{A}$. We say that (E, e) is the* equalizer *in $\mathcal{A}$ of the parallel pair (f, g), and we write $(E, e) = \mathrm{Equ}_{\mathcal{A}}(f, g)$, if*

1) $e : E \to Y$

2)
$$E \xrightarrow{e} Y \underset{g}{\overset{f}{\rightrightarrows}} Z$$
i.e. $f \circ e = g \circ e$

3) *satisfies the universal property, i.e. for every $X \in \mathcal{A}$ and $x : X \to Y$ such that $f \circ x = g \circ x$, there exists a unique morphism in $\mathcal{A}$ $\xi : X \to E$ such that $x = e \circ \xi$.*

Remark 5.28. *In case there exists $(E, e) = \mathrm{Equ}_{\mathcal{A}}(f, g)$, e is a monomorphism. In fact, let $\alpha, \beta : W \to E$ be morphisms in $\mathcal{A}$ such that $e \circ \alpha = e \circ \beta$. Then we have*

$$f \circ e \circ \alpha \overset{e equ}{=} g \circ e \circ \alpha$$

so that $e \circ \alpha$ equalizes (f, g). Since $(E, e) = \mathrm{Equ}_{\mathcal{A}}(f, g)$ there exist a unique morphism $\delta : W \to E$ such that $e \circ \alpha = e \circ \delta$. In particular, we take $\delta = \alpha$. But we also have

$$e \circ \alpha = e \circ \beta$$

so that we can also have $\delta = \beta$. By the uniqueness of the morphism δ we deduce that $\delta = \alpha = \beta$.

Definition 5.29. *Let $\mathcal{A}$ a category, let $Y, Z \in \mathcal{A}$ and let $f, g : Y \to Z$ be morphisms in $\mathcal{A}$. We say that (Q, q) is the* coequalizer *in $\mathcal{A}$ of the parallel pair (f, g), and we write $(Q, q) = \mathrm{Coequ}_{\mathcal{A}}(f, g)$, if*

1) $q : Z \to Q$

2)
$$Y \underset{g}{\overset{f}{\rightrightarrows}} Z \xrightarrow{q} Q$$
i.e. $q \circ f = q \circ g$

3) *satisfies the universal property, i.e. for every* $T \in \mathcal{A}$ *and* $\chi : Z \to T$ *such that* $\chi \circ f = \chi \circ g$*, there exists a unique morphism in* $\mathcal{A}$ $\gamma : Q \to T$ *such that* $\chi = \gamma \circ q$.

Exercise 5.30. *In case there exists* $(Q, q) = \mathrm{Coequ}_{\mathcal{A}}(f, g)$, q *is an epimorphism.*

Remark 5.31. *Let* $\mathcal{A}$ *be a preadditive category, let* $Y, Z \in \mathcal{A}$ *and let* $f, g : Y \to Z$ *be a parallel pair of morphisms in* $\mathcal{A}$. *Then* $\mathrm{Equ}_{\mathcal{A}}(f, g) = \mathrm{Ker}(f - g)$ *and* $\mathrm{Coequ}_{\mathcal{A}}(f, g) = \mathrm{Coker}(f - g)$.

Definition 5.32. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be categories, let* $B, C : \mathcal{A} \to \mathcal{B}$ *be functors and* $\beta, \gamma : B \to C$ *be functorial morphisms. We say that* $(E, i) = \mathrm{Equ}_{Fun}(\beta, \gamma)$ *if*

1) $i : E \to B$

2)

$$E \xrightarrow{i} B \underset{\gamma}{\overset{\beta}{\rightrightarrows}} C$$

i.e. $\beta \circ i = \gamma \circ i$

3) *satisfies the universal property, i.e., for every functorial morphism* $x : X \to B$ *such that* $\beta \circ x = \gamma \circ x$*, there exists a unique functorial morphism* $\xi : X \to E$ *such that* $x = i \circ \xi$.

Definition 5.33. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be categories, let* $B, C : \mathcal{A} \to \mathcal{B}$ *be functors and* $\beta, \gamma : B \to C$ *be functorial morphisms. We say that* $(Q, q) = \mathrm{Coequ}_{Fun}(\beta, \gamma)$ *if*

1) $q : C \to Q$

2)

$$B \underset{\gamma}{\overset{\beta}{\rightrightarrows}} C \xrightarrow{q} Q$$

i.e. $q \circ \beta = q \circ \gamma$

3) *satisfies the universal property, i.e., for every functorial morphism* $\omega : C \to W$ *such that* $\omega \circ \beta = \omega \circ \gamma$*, there exists a unique functorial morphism* $\zeta : Q \to W$ *such that* $\omega = \zeta \circ q$.

Lemma 5.34. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be categories, let* $F, F' : \mathcal{A} \to \mathcal{B}$ *be functors and* $\alpha, \beta : F \to F'$ *be functorial morphisms. If, for every* $X \in \mathcal{A}$*, there exists* $\mathrm{Coequ}_{\mathcal{B}}(\alpha X, \beta X)$*, then there exists the coequalizer* $(C, c) = \mathrm{Coequ}_{\mathrm{Fun}}(\alpha, \beta)$ *in the category of functors. Moreover, for any object* X *in* $\mathcal{A}$*, we have* $(CX, cX) = \mathrm{Coequ}_{\mathcal{B}}(\alpha X, \beta X)$.

Proof. Define a functor $C : \mathcal{A} \to \mathcal{B}$ with object map $(CX, cX) = \mathrm{Coequ}_{\mathcal{B}}(\alpha X, \beta X)$ for every $X \in \mathcal{A}$. For a morphism $f : X \to X'$ in $\mathcal{A}$, naturality of α and β implies that

$$(F'f) \circ (\alpha X) = (\alpha X') \circ (Ff) \ \text{ and } \ (F'f) \circ (\beta X) = (\beta X') \circ (Ff)$$

and hence

$$\begin{array}{rcl} (cX') \circ (F'f) \circ (\alpha X) & = & (cX') \circ (\alpha X') \circ (Ff) \overset{\text{ccoequ}}{=} (cX') \circ (\beta X') \circ (Ff) \\ & = & (cX') \circ (F'f) \circ (\beta X) \end{array}$$

i.e. $(cX') \circ (F'f)$ coequalizes the parallel morphisms βX and αX. In light of this fact, by the universal property of the coequalizer (CX, cX), $Cf : CX \to CX'$ is defined as the unique morphism in $\mathcal{B}$ such that $(Cf) \circ (cX) = (cX') \circ (F'f)$. By construction, c is a functorial morphism $F' \to C$ such that $c \circ \alpha = c \circ \beta$. It remains to prove universality of c. Let $H : \mathcal{A} \to \mathcal{B}$ be a functor and let $\chi : F' \to H$ be a functorial morphism such that $\chi \circ \alpha = \chi \circ \beta$. Then, for any object X in $\mathcal{A}$, $(\chi X) \circ (\alpha X) = (\chi X) \circ (\beta X)$. Since $(CX, cX) = \mathrm{Coequ}_{\mathcal{B}}(\alpha X, \beta X)$, there is a unique morphism $\xi X : CX \to HX$ such that $(\xi X) \circ (cX) = \chi X$. The proof is completed by proving naturality of ξX in X. Take a morphism $f : X \to X'$ in $\mathcal{A}$. Since c and χ functorial morphisms,

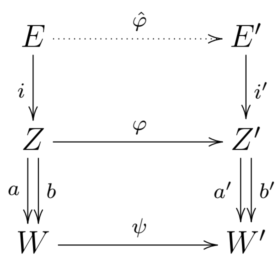
$$\begin{array}{c} (Hf) \circ (\xi X) \circ (cX) = (Hf) \circ (\chi X) \overset{\chi}{=} (\chi X') \circ (F'f) \\ = (\xi X') \circ (cX') \circ (F'f) = (\xi X') \circ (Cf) \circ (cX). \end{array}$$

Since cX is a epimorphism, we get that ξ is a functorial morphism. $\square$

Lemma 5.35. *Let $Z, Z', W, W' : \mathcal{A} \to \mathcal{B}$ be functors, let $a, b : Z \to W$ and $a', b' : Z' \to W'$ be functorial morphisms, let $\varphi : Z \to Z'$ and $\psi : W \to W'$ be functorial isomorphisms such that*

$$\psi \circ a = a' \circ \varphi \qquad \text{and} \qquad \psi \circ b = b' \circ \varphi.$$

Assume that there exist $(E, i) = \mathrm{Equ}_{Fun}(a, b)$ and $(E', i') = \mathrm{Equ}_{Fun}(a', b')$. Then φ induces an isomorphism $\widehat{\varphi} : E \to E'$ such that $\varphi \circ i = i' \circ \widehat{\varphi}$.



$$\begin{array}{ccc} E & \overset{\widehat{\varphi}}{\dashrightarrow} & E' \\ \downarrow i & & \downarrow i' \\ Z & \overset{\varphi}{\longrightarrow} & Z' \\ a \downdownarrows b & & a' \downdownarrows b' \\ W & \overset{\psi}{\longrightarrow} & W' \end{array}$$

Proof. Let us define $\widehat{\varphi}$. Let us compute

$$a' \circ \varphi \circ i = \psi \circ a \circ i \overset{\text{def}i}{=} \psi \circ b \circ i = b' \circ \varphi \circ i$$

and since $(E', i') = \mathrm{Equ}_{Fun}(a', b')$ there exists a unique functorial morphism $\widehat{\varphi} : E \to E'$ such that

$$i' \circ \widehat{\varphi} = \varphi \circ i.$$

Note that $\widehat{\varphi}$ is mono since so are i and i' and φ is an isomorphism. Consider $\varphi^{-1} : Z' \to Z$ and $\psi^{-1} : W' \to W$. Then we have

$$a \circ \varphi^{-1} = \psi^{-1} \circ a' \qquad \text{and} \qquad b \circ \varphi^{-1} = \psi^{-1} \circ b'.$$

Let us compute

$$a \circ \varphi^{-1} \circ i' = \psi^{-1} \circ a' \circ i' \overset{\text{def}i'}{=} \psi^{-1} \circ b' \circ i' = b \circ \varphi^{-1} \circ i'$$

and since $(E, i) = \mathrm{Equ}_{Fun}(a, b)$ there exists a unique functorial morphism $\widehat{\varphi}' : E' \to E$ such that

$$i \circ \widehat{\varphi}' = \varphi^{-1} \circ i'.$$

Then we have

$$i \circ \widehat{\varphi}' \circ \widehat{\varphi} = \varphi^{-1} \circ i' \circ \widehat{\varphi} = \varphi^{-1} \circ \varphi \circ i = i$$

and since i is a monomorphism we deduce that

$$\widehat{\varphi}' \circ \widehat{\varphi} = \mathrm{Id}_E.$$

Similarly

$$i' \circ \widehat{\varphi} \circ \widehat{\varphi}' = \varphi \circ i \circ \widehat{\varphi}' = \varphi \circ \varphi^{-1} \circ i' = i'$$

and since i' is a monomorphism we obtain that

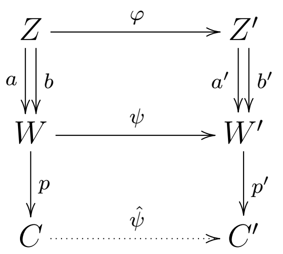
$$\widehat{\varphi} \circ \widehat{\varphi}' = \mathrm{Id}_{E'}.$$

□

Lemma 5.36. *Let $Z, Z', W, W' : \mathcal{A} \to \mathcal{B}$ be functors, let $a, b : Z \to W$ and $a', b' : Z' \to W'$ be functorial morphisms, let $\varphi : Z \to Z'$ and $\psi : W \to W'$ be functorial isomorphisms such that*

$$\psi \circ a = a' \circ \varphi \qquad \textit{and} \qquad \psi \circ b = b' \circ \varphi.$$

Assume that there exist $(C, p) = \mathrm{Coequ}_{Fun}(a, b)$ and $(C', p') = \mathrm{Coequ}_{Fun}(a', b')$. Then ψ induces an isomorphism $\widehat{\psi} : C \to C'$ such that $\widehat{\psi} \circ p = p' \circ \psi$.



$$\begin{array}{ccc} Z & \overset{\varphi}{\longrightarrow} & Z' \\ a \downdownarrows b & & a' \downdownarrows b' \\ W & \overset{\psi}{\longrightarrow} & W' \\ \downarrow p & & \downarrow p' \\ C & \overset{\widehat{\psi}}{\dashrightarrow} & C' \end{array}$$

Proof. Dual to Lemma 5.35. (Exercise). □

Lemma 5.37. *Let $K : \mathcal{B} \to \mathcal{A}$ be a full and faithful functor and let $f, g : X \to Y$ be morphisms in $\mathcal{B}$. If $(KC, Kc) = \mathrm{Coequ}_{\mathcal{A}}(Kf, Kg)$ then $(C, c) = \mathrm{Coequ}_{\mathcal{B}}(f, g)$.*

Proof. Since K is faithful, from $(Kc) \circ (Kf) = (Kc) \circ (Kg)$ we get that $c \circ f = c \circ g$. Let $q : Y \to Q$ be a morphism in $\mathcal{B}$ such that $q \circ f = q \circ g$. Then in $\mathcal{A}$ we get $(Kq) \circ (Kf) = (Kq) \circ (Kg)$ and hence there exists a unique morphism $\xi : KC \to KQ$ such that $\xi \circ (Kc) = Kq$. Since $\xi \in \mathrm{Hom}_{\mathcal{A}}(KC, KQ)$ and K is full, there exists a morphism $\zeta \in \mathrm{Hom}_{\mathcal{B}}(C, Q)$ such that $\xi = K\zeta$. Since K is faithful, from $(K\zeta) \circ (Kc) = Kq$ we get $\zeta \circ c = q$. From the uniqueness of ξ, the one of ζ easily follows. □

Lemma 5.38. *Let $\alpha, \gamma : F \to G$ be functorial morphisms where $F, G : \mathcal{A} \to \mathcal{B}$ are functors. Assume that, for every $X \in \mathcal{A}$ there exists $\mathrm{Coequ}_{\mathcal{B}}(\alpha X, \gamma X)$. Let $(C, c) = \mathrm{Coequ}_{Fun}(\alpha, \gamma)$, where $c : G \to C$. Then, for every $X \in \mathcal{A}$ and $Z \in \mathcal{B}$ we have that*

$$(\mathrm{Hom}_{\mathcal{B}}(CX, Z), \mathrm{Hom}_{\mathcal{B}}(cX, Z)) = \mathrm{Equ}_{\mathrm{Sets}}(\mathrm{Hom}_{\mathcal{B}}(\alpha X, Z), \mathrm{Hom}_{\mathcal{B}}(\gamma X, Z))$$

which means that

$$(\mathrm{Hom}_{\mathcal{B}}(C, -), \mathrm{Hom}_{\mathcal{B}}(c, -)) = \mathrm{Equ}_{\mathrm{Fun}}(\mathrm{Hom}_{\mathcal{B}}(\alpha, -), \mathrm{Hom}_{\mathcal{B}}(\gamma, -))$$

where

$$\mathrm{Hom}_{\mathcal{B}}(C, -) \text{ and } \mathrm{Equ}_{\mathrm{FUn}}(\mathrm{Hom}_{\mathcal{B}}(\alpha, -), \mathrm{Hom}_{\mathcal{B}}(\gamma, -)) : \mathcal{A}^{op} \times \mathcal{B} \to \mathrm{Sets}.$$

Proof. We have that

$$\begin{gathered} \mathrm{Hom}_{\mathcal{B}}(\alpha X, Z) \circ \mathrm{Hom}_{\mathcal{B}}(cX, Z) = \mathrm{Hom}_{\mathcal{B}}((cX) \circ (\alpha X), Z) \\ = \mathrm{Hom}_{\mathcal{B}}((cX) \circ (\gamma X), Z) = \mathrm{Hom}_{\mathcal{B}}(\gamma X, Z) \circ \mathrm{Hom}_{\mathcal{B}}(cX, Z) \end{gathered}$$

i.e. $\mathrm{Hom}_{\mathcal{B}}(cX, Z)$ equalizes $\mathrm{Hom}_{\mathcal{B}}(\alpha X, Z)$ and $\mathrm{Hom}_{\mathcal{B}}(\gamma X, Z)$, for every $X \in \mathcal{A}$ and $Z \in \mathcal{B}$. Let now $\zeta : Q \to \mathrm{Hom}_{\mathcal{B}}(GX, Z)$ be a map such that $\mathrm{Hom}_{\mathcal{B}}(\alpha X, Z) \circ \zeta = \mathrm{Hom}_{\mathcal{B}}(\gamma X, Z) \circ \zeta$. Then, for every $X \in \mathcal{A}$, $Z \in \mathcal{B}$ and for every $q \in Q$ we have

$$\begin{gathered} \zeta(q) \circ (\alpha X) = \mathrm{Hom}_{\mathcal{B}}(\alpha X, Z)(\zeta(q)) = \mathrm{Hom}_{\mathcal{B}}(\gamma X, Z) \circ (\zeta(q)) \\ = \zeta(q) \circ (\gamma X). \end{gathered}$$

Then, for every $X \in \mathcal{A}$ and $Z \in \mathcal{B}$ there exists a unique morphism $\xi_q : CX \to Z$ in $\mathcal{B}$ such that

$$\xi_q \circ (cX) = \zeta(q)$$

i.e.

$$\mathrm{Hom}_{\mathcal{B}}(cX, Z)(\xi_q) = \zeta(q).$$

The assignment $q \mapsto \xi_q$ defines a map $\xi : Q \to \mathrm{Hom}_{\mathcal{B}}(CX, Z)$ such that

$$\mathrm{Hom}_{\mathcal{B}}(cX, Z) \circ \xi = \zeta.$$

$\square$

Chapter 6

MONADS

6.1 Contractible (co)equalizers

Definition 6.1. *Let $\mathcal{A}$ a category, let $Y, Z \in \mathcal{A}$ and let $f, g : Y \to Z$ be morphisms in $\mathcal{A}$. We say that (E, e) is the* equalizer *in $\mathcal{A}$ of the parallel pair (f, g), and we write $(E, e) = \mathrm{Equ}_{\mathcal{A}}(f, g)$, if*

1) $e : E \to Y$

2)

$$E \xrightarrow{e} Y \underset{g}{\overset{f}{\rightrightarrows}} Z$$

i.e. $f \circ e = g \circ e$

3) *satisfies the universal property, i.e. for every $X \in \mathcal{A}$ and $x : X \to Y$ such that $f \circ x = g \circ x$, there exists a unique morphism in $\mathcal{A}$ $\xi : X \to E$ such that $x = e \circ \xi$.*

Remark 6.2. *In case there exists $(E, e) = \mathrm{Equ}_{\mathcal{A}}(f, g)$, e is a monomorphism. In fact, let $\alpha, \beta : W \to E$ be morphisms in $\mathcal{A}$ such that $e \circ \alpha = e \circ \beta$. Then we have*

$$f \circ e \circ \alpha \overset{e equ}{=} g \circ e \circ \alpha$$

so that $e \circ \alpha$ equalizes (f, g). Since $(E, e) = \mathrm{Equ}_{\mathcal{A}}(f, g)$ there exist a unique morphism $\delta : W \to E$ such that $e \circ \alpha = e \circ \delta$. In particular, we take $\delta = \alpha$. But we also have

$$e \circ \alpha = e \circ \beta$$

so that we can also have $\delta = \beta$. By the uniqueness of the morphism δ we deduce that $\delta = \alpha = \beta$.

Definition 6.3. *Let $\mathcal{A}$ a category, let $Y, Z \in \mathcal{A}$ and let $f, g : Y \to Z$ be morphisms in $\mathcal{A}$. We say that (Q, q) is the* coequalizer *in $\mathcal{A}$ of the parallel pair (f, g), and we write $(Q, q) = \mathrm{Coequ}_{\mathcal{A}}(f, g)$, if*

1) $q : Z \to Q$

2)

$$Y \underset{g}{\overset{f}{\rightrightarrows}} Z \xrightarrow{q} Q$$

i.e. $q \circ f = q \circ g$

3) *satisfies the universal property, i.e. for every* $T \in \mathcal{A}$ *and* $\chi : Z \to T$ *such that* $\chi \circ f = \chi \circ g$, *there exists a unique morphism in* $\mathcal{A}$ $\gamma : Q \to T$ *such that* $\chi = \gamma \circ q$.

Exercise 6.4. *In case there exists* $(Q, q) = \mathrm{Coequ}_{\mathcal{A}}(f, g)$, q *is an epimorphism.*

Remark 6.5. *Let* $\mathcal{A}$ *be a preadditive category, let* $Y, Z \in \mathcal{A}$ *and let* $f, g : Y \to Z$ *be a parallel pair of morphisms in* $\mathcal{A}$. *Then* $\mathrm{Equ}_{\mathcal{A}}(f, g) = \mathrm{Ker}(f - g)$ *and* $\mathrm{Coequ}_{\mathcal{A}}(f, g) = \mathrm{Coker}(f - g)$.

Definition 6.6. *1) [McL, page 151] Recall that a functor* $R : \mathcal{A} \to \mathcal{B}$ creates coequalizer *for a pair* $f, g : A \to A'$ *in* $\mathcal{A}$ *whenever to each coequalizer* $(Z, \zeta : RA' \to Z)$ *of* (Rf, Rg) *in* $\mathcal{B}$ *there is a unique object* A'' *in* $\mathcal{A}$ *and a unique morphism* $\gamma : A' \to A''$ *such that*

- $RA'' = Z$,
- $R\gamma = \zeta$ *and*
- (A'', γ) *is a coequalizer of* (f, g) *in* $\mathcal{A}$.

2) [BW, page 94] Let $\mathcal{C}$ *be a category. A* contractible coequalizer *is a* 8*-tuple* $(C, X, Y, c, d_0, d_1, u, v)$ *where*

$$X \; \overset{d_0}{\underset{d_1}{\rightrightarrows}} \!\!\! \overset{v}{\leftarrow} \; Y \; \underset{u}{\overset{c}{\rightleftarrows}} \; C \qquad\qquad X \quad \begin{matrix} \xrightarrow{d_0} \\ \overset{v}{\leftarrow} \\ \xrightarrow{d_1} \end{matrix} \quad Y \quad \underset{u}{\overset{c}{\rightleftarrows}} \quad C$$

such that

$$\begin{aligned} d_0 \circ v &= \mathrm{Id}_Y \\ d_1 \circ v &= u \circ c \\ c \circ u &= \mathrm{Id}_C \\ c \circ d_0 &= c \circ d_1. \end{aligned}$$

3) [BW, page 95] (cf. [Man1, Definitions 1.8 page 167]). An R-contractible coequalizer pair *is a pair of morphisms* (d_0, d_1) *from* X *to* Y *for which there is a contractible coequalizer*

$$RX \quad \begin{matrix} \xrightarrow{Rd_0} \\ \overset{v}{\leftarrow} \\ \xrightarrow{Rd_1} \end{matrix} \quad RY \quad \underset{u}{\overset{c}{\rightleftarrows}} \quad C \; .$$

Note that here the definition differs from [BW, page 95] as we have C and not RC as coequalizer.

4) [BW, 3.6 page 98] A reflexive pair *is a pair of morphisms (d_0, d_1) from X to Y such that if d_0 and d_1 have a common right inverse i.e. there is $e : Y \to X$ such that $d_0 \circ e = d_1 \circ e = \mathrm{Id}_Y$.*

Proposition 6.7. *[BW, Proposition 3.4, page 94]Let $\mathcal{C}$ be a category and let $(C, X, Y, c, d_0, d_1, u, v)$ be a contractible coequalizer. Then $(C, c) = \mathrm{Coequ}_{\mathcal{C}}(d_0, d_1)$.*

Proof. Let $\chi : Y \to Q$ such that

$$\chi \circ d_0 = \chi \circ d_1.$$

We have

$$\chi = \chi \circ \mathrm{Id}_Y = \chi \circ d_0 \circ v = \chi \circ d_1 \circ v = (\chi \circ u) \circ c.$$

Then, let us set

$$\chi' = \chi \circ u : C \to Q$$

so that

$$\chi = \chi' \circ c.$$

Let now $\chi'' : C \to Q$ such that $\chi'' \circ c = \chi$. Then

$$\chi'' = \chi'' \circ \mathrm{Id}_C = \chi'' \circ c \circ u = \chi \circ u = \chi'.$$

□

Proposition 6.8. *[BW, Proposition 3.4, page 94]Let $\mathcal{C}$ be a category, let $(C, X, Y, c, d_0, d_1, u, v)$ be a contractible coequalizer and let $F : \mathcal{C} \to \mathcal{D}$ be a functor, then*

$$FX \underset{Fd_1}{\overset{Fd_0}{\underset{\xleftarrow{Fv}}{\rightrightarrows}}} FY \underset{Fu}{\overset{Fc}{\rightleftarrows}} FC$$

$$FX \quad \begin{matrix} \overset{Fd_0}{\to} \\ \overset{Fv}{\leftarrow} \\ \overset{Fd_1}{\to} \end{matrix} \quad FY \quad \underset{Fu}{\overset{Fc}{\rightleftarrows}} \quad FC$$

is a contractible coequalizer in $\mathcal{D}$.

Proof. Since $(C, X, Y, c, d_0, d_1, u, v)$ is a contractible coequalizer we have

$$\begin{aligned} d_0 \circ v &= \mathrm{Id}_Y \\ d_1 \circ v &= u \circ c \\ c \circ u &= \mathrm{Id}_C \\ c \circ d_0 &= c \circ d_1. \end{aligned}$$

By applying the functor F to them, the equalities still hold. □

Lemma 6.9. *Let* $(L : \mathcal{B} \to \mathcal{A}, R : \mathcal{A} \to \mathcal{B})$ *be an adjunction with unit* η *and counit* ϵ. *Then* $(LB, LRLRLB, LRLB, \epsilon LB, LR\epsilon LB, \epsilon LRLB, L\eta B, LRL\eta B)$ *is a contractible coequalizer.*

$$LRLRLB \;\underset{\overset{\epsilon LRLB}{\longrightarrow}}{\overset{\overset{LR\epsilon LB}{\longrightarrow}}{\overset{LRL\eta B}{\longleftarrow}}}\; LRLB \;\underset{L\eta B}{\overset{\epsilon LB}{\rightleftarrows}}\; LB\;.$$

Proof. We have

$$\begin{aligned} LR\epsilon LB \circ LRL\eta B &= \mathrm{Id}_{LRLB} \\ \epsilon LRLB \circ LRL\eta B &= L\eta B \circ \epsilon LB \\ \epsilon LB \circ L\eta B &= \mathrm{Id}_{LB} \\ \epsilon LB \circ LR\epsilon LB &= \epsilon LB \circ \epsilon LRLB. \end{aligned}$$

□

Lemma 6.10. *Let* $(L : \mathcal{B} \to \mathcal{A}, R : \mathcal{A} \to \mathcal{B})$ *be an adjunction. Let* η *and* ϵ *be the unit and counit of* (L, R) *respectively. Let* $(B, \mu : RLB \to B) \in {}_{RL}\mathcal{B}$. *Then* $(\epsilon LB, L\mu)$ *is a reflexive* R*-contractible coequalizer pair. In particular*

$$RLRLB \;\underset{\overset{RL\mu}{\longrightarrow}}{\overset{\overset{R\epsilon LB}{\longrightarrow}}{\overset{\eta RLB}{\longleftarrow}}}\; RLB \;\underset{\eta B}{\overset{\mu}{\rightleftarrows}}\; B\;.$$

is a contractible coequalizer whence preserved by any functor.

Proof. Let us check it is a reflexive R-contractible coequalizer pair. We have $L\mu \circ L\eta B = \mathrm{Id}_{LB} = \epsilon LB \circ L\eta B$ so that $(\epsilon LB, L\mu)$ is a reflexive pair. Let us check it is an R-contractible coequalizer pair. Since $(B, \mu) \in {}_{RL}\mathcal{B}$ we have $\mu \circ RL\mu = \mu \circ R\epsilon LB$ and $\mu \circ \eta B = \mathrm{Id}_B$. Moreover we have $R\epsilon LB \circ \eta RLB = \mathrm{Id}_{RLB}$, $RL\mu \circ \eta RLB = \eta B \circ \mu$. Thus $(\epsilon LB, L\mu)$ is a reflexive R-contractible coequalizer pair. □

Corollary 6.11. *Let* $(L : \mathcal{B} \to \mathcal{A}, R : \mathcal{A} \to \mathcal{B})$ *be an adjunction. Let* η *and* ϵ *be the unit and counit of* (L, R) *respectively. Then*

$$RLRLRA \;\underset{\overset{RLR\epsilon A}{\longrightarrow}}{\overset{\overset{R\epsilon LRA}{\longrightarrow}}{\overset{\eta RLRA}{\longleftarrow}}}\; RLRA \;\underset{\eta RA}{\overset{R\epsilon A}{\rightleftarrows}}\; RA\;.$$

is a contractible coequalizer whence preserved by any functor.

Proof. Since $(RA, R\epsilon A) \in {}_{RL}\mathcal{B}$, we can apply Lemma 6.10. □

6.2 Monads

Definition 6.12. *A* monad *on a category* $\mathcal{A}$ *is a triple* $\mathbb{A} = (A, m_A, u_A)$, *where* $A : \mathcal{A} \to \mathcal{A}$ *is a functor,* $m_A : AA \to A$ *and* $u_A : \mathcal{A} \to A$ *are functorial morphisms satisfying the associativity and the unitality conditions:*

$$m_A \circ (m_A A) = m_A \circ (A m_A) \quad \text{and} \quad m_A \circ (A u_A) = A = m_A \circ (u_A A).$$

Definition 6.13. *A* morphism between two monads $\mathbb{A} = (A, m_A, u_A)$ *and* $\mathbb{B} = (B, m_B, u_B)$ *on a category* $\mathcal{A}$ *is a functorial morphism* $\varphi : A \to B$ *such that*

$$\varphi \circ m_A = m_B \circ (\varphi\varphi) \quad \text{and} \quad \varphi \circ u_A = u_B.$$

Here $\varphi\varphi = \varphi B \circ A\varphi = B\varphi \circ \varphi A$.

Example 6.14. *Let* $(\mathrm{A}, m_{\mathrm{A}}, u_{\mathrm{A}})$ *an* R*-ring where* R *is an algebra over a commutative ring* k. *This means that*

- A *is an* R-R*-bimodule*
- $m_{\mathrm{A}} : \mathrm{A} \otimes_R \mathrm{A} \to \mathrm{A}$ *is a morphism of* R-R*-bimodules*
- $u_{\mathrm{A}} : R \to \mathrm{A}$ *is a morphism of* R-R*-bimodules satisfying the following*

  $$m_{\mathrm{A}} \circ (m_{\mathrm{A}} \otimes_R \mathrm{A}) = m_{\mathrm{A}} \circ (\mathrm{A} \otimes_R m_{\mathrm{A}}), m_{\mathrm{A}} \circ (\mathrm{A} \otimes_R u_{\mathrm{A}}) = r_{\mathrm{A}} \quad \text{and} \quad m_{\mathrm{A}} \circ (u_{\mathrm{A}} \otimes_R \mathrm{A}) = l_{\mathrm{A}}$$

  where $r_{\mathrm{A}} : \mathrm{A} \otimes_R R \to \mathrm{A}$ *and* $l_{\mathrm{A}} : R \otimes_R \mathrm{A} \to \mathrm{A}$ *are the right and left constraints.* *Let*

  $$\begin{aligned} A &= - \otimes_R \mathrm{A} : Mod\text{-}R \to Mod\text{-}R \\ m_A &= - \otimes_R m_{\mathrm{A}} : - \otimes_R \mathrm{A} \otimes_R \mathrm{A} \to - \otimes_R \mathrm{A} \\ u_A &= (- \otimes_R u_{\mathrm{A}}) \circ r_{-}^{-1} : - \to - \otimes_R R \to - \otimes_R \mathrm{A} \end{aligned}$$

  We prove that $\mathbb{A} = (A, m_A, u_A)$ *is a monad on the category* $Mod\text{-}R$. *For every* $M \in Mod\text{-}R$ *we compute*

  $$\begin{aligned} [m_A \circ (m_A A)](M) &= (M \otimes_R m_{\mathrm{A}}) \circ (M \otimes_R \mathrm{A} \otimes_R m_{\mathrm{A}}) = M \otimes_R [m_{\mathrm{A}} \circ (\mathrm{A} \otimes_R m_{\mathrm{A}})] \\ &= M \otimes_R [m_{\mathrm{A}} \circ (m_{\mathrm{A}} \otimes_R \mathrm{A})] = (M \otimes_R m_{\mathrm{A}}) \circ (M \otimes_R m_{\mathrm{A}} \otimes_R \mathrm{A}) \\ &= [m_A \circ (A m_A)](M) \end{aligned}$$

  $$\begin{aligned} [m_A \circ (A u_A)](M) &= (M \otimes_R m_{\mathrm{A}}) \circ \left[(M \otimes_R u_{\mathrm{A}}) \circ r_M^{-1}\right] \otimes_R \mathrm{A} \\ &= (M \otimes_R m_{\mathrm{A}}) \circ (M \otimes_R u_{\mathrm{A}} \otimes_R \mathrm{A}) \circ \left(r_M^{-1} \otimes_R \mathrm{A}\right) \\ &= (M \otimes_R [m_{\mathrm{A}} \circ (u_{\mathrm{A}} \otimes_R \mathrm{A})]) \circ \left(r_M^{-1} \otimes_R \mathrm{A}\right) \\ &= (M \otimes_R l_{\mathrm{A}}) \circ \left(r_M^{-1} \otimes_R \mathrm{A}\right) = M \otimes_R \mathrm{A} = AM \end{aligned}$$

  and

  $$\begin{aligned} [m_A \circ (u_A A)](M) &= (M \otimes_R m_{\mathrm{A}}) \circ (M \otimes_R \mathrm{A} \otimes_R u_{\mathrm{A}}) \circ r_{M \otimes_R \mathrm{A}}^{-1} \\ &= (M \otimes_R [m_{\mathrm{A}} \circ (\mathrm{A} \otimes_R u_{\mathrm{A}})]) \circ r_{M \otimes_R \mathrm{A}}^{-1} \\ &= (M \otimes_R r_{\mathrm{A}}) \circ r_{M \otimes_R \mathrm{A}}^{-1} = M \otimes_R \mathrm{A} = AM. \end{aligned}$$

Exercise 6.15. *Let R, A be rings.Let $u_A : R \to A$ be a ring homomorphism. Let us denote by m the multiplication of A and by $m_A : A \otimes_R A \to A$ the well-defined induced map. Prove that (A, m_A, u_A) is an R-ring.*

Exercise 6.16. *Prove that every ring is a $\mathbb{Z}$-ring.*

Proposition 6.17 ([H]). *Let (L, R) be an adjunction with unit η and counit ϵ where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$. Then $\mathbb{RL} = (RL, R\epsilon L, \eta)$ is a monad on the category $\mathcal{B}$.*

Proof. We have to prove that

$$(R\epsilon L) \circ (RLR\epsilon L) = (R\epsilon L) \circ (R\epsilon LRL) \quad \text{and} \quad (R\epsilon L) \circ RL\eta = RL = (R\epsilon L) \circ (\eta RL).$$

In fact we have

$$(R\epsilon L) \circ (RLR\epsilon L) \overset{\epsilon}{=} (R\epsilon L) \circ (R\epsilon LRL)$$

and

$$(R\epsilon L) \circ RL\eta \overset{(L,R)}{=} RL \overset{(L,R)}{=} (R\epsilon L) \circ (\eta RL).$$

□

Exercise 6.18. *Let A, B rings and let M be an A-B-bimodule. Consider the functors*

$$\begin{aligned} L &= - \otimes_A M : Mod\text{-}A \to Mod\text{-}B \\ R &= \mathrm{Hom}_B(M, -) : Mod\text{-}B \to Mod\text{-}A. \end{aligned}$$

Then $(L, R) = (- \otimes_A M, \mathrm{Hom}_B(M, -))$ is an adjunction. Compute the monad $\mathbb{RL}$ associated to this adjunction. Moreover, compute the monad $\mathbb{RL}$ in the particular case $A = B = R$ and M è un R-ring.

Definition 6.19. *A* left module functor *for a monad $\mathbb{A} = (A, m_A, u_A)$ on a category $\mathcal{A}$ is a pair $\left(Q, {}^A\mu_Q\right)$ where $Q : \mathcal{B} \to \mathcal{A}$ is a functor and ${}^A\mu_Q : AQ \to Q$ is a functorial morphism satisfying:*

$${}^A\mu_Q \circ \left(A\,{}^A\mu_Q\right) = {}^A\mu_Q \circ (m_A Q) \quad \text{and} \quad Q = {}^A\mu_Q \circ (u_A Q).$$

Example 6.20. *Let* A *be an R-ring. Let $A = \mathrm{A} \otimes_R -$ be a monad associated to the R-ring and let $Q = M \otimes_R -$ where M is a left* A*-module. Then Q is a left A-module functor via the map*

$$\begin{gathered} AQ = (\mathrm{A} \otimes_R M \otimes_R -) \longrightarrow Q = (M \otimes_R -) \\ a \otimes_R m \otimes_R - \mapsto am \otimes_R - \end{gathered}$$

where we denote by am the multiplication of an element $a \in A$ with an element $m \in M$.

Definition 6.21. *A* right module functor *for a monad* $\mathbb{A} = (A, m_A, u_A)$ *on a category* $\mathcal{A}$ *is a pair* $\left(P, \mu_P^A\right)$ *where* $P : \mathcal{A} \to \mathcal{B}$, *is a functor and* $\mu_P^A : PA \to P$ *is a functorial morphism such that*

$$\mu_P^A \circ \left(\mu_P^A A\right) = \mu_P^A \circ (Pm_A) \quad \text{and} \quad P = \mu_P^A \circ (Pu_A).$$

Remark 6.22. *Let* $\mathbb{A} = (A, m_A, u_A)$ *be a monad on a category* $\mathcal{A}$ *and let* $\left(Q, {}^A\mu_Q\right)$ *be a left* $\mathbb{A}$*-module functor and* $\left(P, \mu_P^A\right)$ *be a right* $\mathbb{A}$*-module functor. By the unitality property of* ${}^A\mu_Q$ *and* μ_P^A *we deduce that they are both epimorphism.*

Definition 6.23. *For two monads* $\mathbb{A} = (A, m_A, u_A)$ *on a category* $\mathcal{A}$ *and* $\mathbb{B} = (B, m_B, u_B)$ *on a category* $\mathcal{B}$, *a* $\mathbb{A}$-$\mathbb{B}$-bimodule functor *is a triple* $\left(Q, {}^A\mu_Q, \mu_Q^B\right)$, *where* $Q : \mathcal{B} \to \mathcal{A}$ *is a functor and* $\left(Q, {}^A\mu_Q\right)$ *is a left* $\mathbb{A}$*-module,* $\left(Q, \mu_Q^B\right)$ *is a right* $\mathbb{B}$*-module such that in addition*

$${}^A\mu_Q \circ \left(A\mu_Q^B\right) = \mu_Q^B \circ \left({}^A\mu_Q B\right).$$

Definition 6.24. *A* module *for a monad* $\mathbb{A} = (A, m_A, u_A)$ *on a category* $\mathcal{A}$ *is a pair* $\left(X, {}^A\mu_X\right)$ *where* $X \in \mathcal{A}$ *and* ${}^A\mu_X : AX \to X$ *is a morphism in* $\mathcal{A}$ *such that*

$${}^A\mu_X \circ \left(A^A\mu_X\right) = {}^A\mu_X \circ (m_A X) \quad \text{and} \quad X = {}^A\mu_X \circ (u_A X).$$

A morphism between two $\mathbb{A}$-modules $\left(X, {}^A\mu_X\right)$ *and* $\left(X', {}^A\mu_{X'}\right)$ *is a morphism* $f : X \to X'$ *in* $\mathcal{A}$ *such that*

$${}^A\mu_{X'} \circ (Af) = f \circ {}^A\mu_X.$$

We will denote by ${}_{\mathbb{A}}\mathcal{A}$ *the category of* $\mathbb{A}$*-modules and their morphisms. This is the so-called* Eilenberg-Moore category *which is sometimes also denoted by* $\mathcal{A}^{\mathbb{A}}$.

Remark 6.25. *Let* $\mathbb{A} = (A, m_A, u_A)$ *be a monad on a category* $\mathcal{A}$ *and let* $\left(X, {}^A\mu_X\right) \in {}_{\mathbb{A}}\mathcal{A}$. *From the unitality property of* ${}^A\mu_X$ *we deduce that* ${}^A\mu_X$ *is epi for every* $\left(X, {}^A\mu_X\right) \in {}_{\mathbb{A}}\mathcal{A}$ *and that* $u_A X$ *is mono for every* $\left(X, {}^A\mu_X\right) \in {}_{\mathbb{A}}\mathcal{A}$, *i.e.* u_A *is a monomorphism.*

Example 6.26. *Let* A *be an R-ring. and Let* $A = - \otimes_R \mathrm{A} : Mod\text{-}R \to Mod\text{-}R$ *be the monad associated. We want to understand the category of modules with respect to this monad. The underlying category is* $\mathcal{A} = Mod\text{-}R$. *Let* $X \in Mod\text{-}R$. *We need a map*

$$\begin{aligned} {}^A\mu_X : AX = X \otimes_R \mathrm{A} &\to X \\ x \otimes_R a &\mapsto xa. \end{aligned}$$

This means that X *is endowed with a right* A*-module structure so that* ${}_{\mathbb{A}}\mathcal{A} = Mod\text{-}A$.

Example 6.27. *Let* A *be an R-ring. and Let* $A = \mathrm{A} \otimes_R - : R\text{-}Mod \to R\text{-}Mod$ *be the monad associated. We want to understand the category of modules with respect to this monad. The underlying category is* $\mathcal{A} = R\text{-}Mod$. *Let* $X \in R\text{-}Mod$. *We need a map*

$$\begin{aligned} {}^A\mu_X : AX = \mathrm{A} \otimes_R X &\to X \\ a \otimes_R x &\mapsto ax. \end{aligned}$$

This means that X *is endowed with a left* A*-module structure so that* ${}_{\mathbb{A}}\mathcal{A} = A\text{-}Mod$.

Definition 6.28. *Corresponding to a monad $\mathbb{A} = (A, m_A, u_A)$ on $\mathcal{A}$, there is an adjunction $({}_{\mathbb{A}}F, {}_{\mathbb{A}}U)$ where ${}_{\mathbb{A}}U$ is the forgetful functor and ${}_{\mathbb{A}}F$ is the free functor*

$$\begin{array}{rccc} {}_{\mathbb{A}}U : & {}_{\mathbb{A}}\mathcal{A} & \to & \mathcal{A} \\ & \left(X, {}^{A}\mu_X\right) & \to & X \\ & f & \to & f \end{array} \qquad \begin{array}{rccc} {}_{\mathbb{A}}F : & \mathcal{A} & \to & {}_{\mathbb{A}}\mathcal{A} \\ & X & \to & (AX, m_A X) \\ & f & \to & Af. \end{array}$$

Note that ${}_{\mathbb{A}}U_{\mathbb{A}}F = A$. The unit of this adjunction is given by the unit u_A of the monad $\mathbb{A}$:

$$u_A : \mathcal{A} \to {}_{\mathbb{A}}U_{\mathbb{A}}F = A.$$

The counit $\lambda_A : {}_{\mathbb{A}}F_{\mathbb{A}}U \to {}_{\mathbb{A}}\mathcal{A}$ of this adjunction is defined by setting

$${}_{\mathbb{A}}U\left(\lambda_A\left(X, {}^{A}\mu_X\right)\right) = {}^{A}\mu_X \text{ for every } \left(X, {}^{A}\mu_X\right) \in {}_{\mathbb{A}}\mathcal{A}.$$

In fact, for every $\left(X, {}^{A}\mu_X\right) \in {}_{\mathbb{A}}\mathcal{A}$ we need to define a morphism in ${}_{\mathbb{A}}\mathcal{A}$ between

$${}_{\mathbb{A}}F_{\mathbb{A}}U\left(X, {}^{A}\mu_X\right) \to \left(X, {}^{A}\mu_X\right)$$

i.e. between

$$(AX, m_A X) \to \left(X, {}^{A}\mu_X\right).$$

This needs to be a morphism of $\mathbb{A}$-modules between the underlying objects AX and X. Therefore, we define $\lambda_A\left(X, {}^{A}\mu_X\right)$ as morphism on the underlying objects to be

$${}_{\mathbb{A}}U\left(\lambda_A\left(X, {}^{A}\mu_X\right)\right) = {}^{A}\mu_X \text{ for every } \left(X, {}^{A}\mu_X\right) \in {}_{\mathbb{A}}\mathcal{A}.$$

Then, the adjunction relations are the following

$$(\lambda_{A\mathbb{A}}F) \circ ({}_{\mathbb{A}}Fu_A) = {}_{\mathbb{A}}F \qquad \text{and} \qquad ({}_{\mathbb{A}}U\lambda_A) \circ (u_{A\mathbb{A}}U) = {}_{\mathbb{A}}U.$$

Exercise 6.29. *Prove that ${}_{\mathbb{A}}FX = (AX, m_A X) \in {}_{\mathbb{A}}\mathcal{A}$.*

Exercise 6.30. *Let (L, R) be an adjunction and let $\mathbb{A} = (RL, R\epsilon L, \eta)$ be the monad associated to the adjunction. Prove that $(R, R\epsilon)$ is a left $\mathbb{A}$-module functor and that $(L, \epsilon L)$ is a right $\mathbb{A}$-module functor.*

Proposition 6.31. *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad on a category $\mathcal{A}$. Then ${}_{\mathbb{A}}U$ is a faithful functor. Moreover, given $Z, W \in {}_{\mathbb{A}}\mathcal{A}$ we have that*

$$Z = W \text{ if and only if} {}_{\mathbb{A}}U(Z) = {}_{\mathbb{A}}U(W) \text{ and } {}_{\mathbb{A}}U(\lambda_A Z) = {}_{\mathbb{A}}U(\lambda_A W).$$

In particular, if $F, G : \mathcal{X} \to {}_{\mathbb{A}}\mathcal{A}$ are functors, we have

$$F = G \text{ if and only if } {}_{\mathbb{A}}UF = {}_{\mathbb{A}}UG \text{ and } {}_{\mathbb{A}}U(\lambda_A F) = {}_{\mathbb{A}}U(\lambda_A G)$$

Proposition 6.32. *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad on a category $\mathcal{A}$ and let $\left(X, {}^{A}\mu_X\right)$ be a module for $\mathbb{A}$. Then we have*

$$\left(X, {}^{A}\mu_X\right) = \mathrm{Coequ}_{\mathcal{A}}\left(A^{A}\mu_X, m_A X\right).$$

In particular if $\left(Q, {}^{A}\mu_Q\right)$ is a left $\mathbb{A}$-module functor, then we have

$$\left(Q, {}^{A}\mu_Q\right) = \mathrm{Coequ}_{\mathrm{Fun}}\left(A^{A}\mu_Q, m_A Q\right).$$

Proof. Note that

$$AAX \underset{{}^{A}\mu_X}{\overset{m_A X}{\underset{u_A AX}{\overset{\longrightarrow}{\longleftarrow}}\longrightarrow}} AX \underset{u_A X}{\overset{{}^{A}\mu_X}{\rightleftarrows}} X$$

$$AAX \quad \overset{m_A X}{\rightarrow} \quad \overset{u_A AX}{\leftarrow} \quad \overset{A^A\mu_X}{\rightarrow} \quad AX \quad \underset{u_A X}{\overset{{}^{A}\mu_X}{\rightleftarrows}} \quad X$$

is a contractible coequalizer and thus, by Proposition 6.7, $\left(X, {}^{A}\mu_X\right) = \mathrm{Coequ}_{\mathcal{A}}\left(A^A\mu_X, m_A X\right)$. Let now $\left(Q, {}^{A}\mu_Q\right)$ be a left $\mathbb{A}$-module functor where $Q : \mathcal{B} \to \mathcal{A}$. Then, by the foregoing, for every $Y \in \mathcal{B}$ we have that

$$\left(QY, {}^{A}\mu_Q Y\right) = \left(QY, {}^{A}\mu_{QY}\right) = \mathrm{Coequ}_{\mathcal{A}}\left(A^A\mu_{QY}, m_A QY\right) = \mathrm{Coequ}_{\mathcal{A}}\left(A^A\mu_Q Y, m_A QY\right).$$

Then, by Lemma 5.34, we have that $\left(Q, {}^{A}\mu_Q\right) = \mathrm{Coequ}_{\mathrm{Fun}}\left(A^A\mu_Q, m_A Q\right)$. □

Proposition 6.33. *Let* $\mathbb{A} = (A, m_A, u_A)$ *be a monad on a category* $\mathcal{A}$ *and let* $\left(P, \mu_P^A\right)$ *be a right* $\mathbb{A}$*-module functor, then we have*

(6.1) $$\left(P, \mu_P^A\right) = \mathrm{Coequ}_{\mathrm{Fun}}\left(\mu_P^A A, P m_A\right).$$

Proof. Note that

$$PAA \underset{\mu_P^A A}{\overset{P m_A}{\underset{PAu_A}{\overset{\longrightarrow}{\longleftarrow}}\longrightarrow}} PA \underset{P u_A}{\overset{\mu_P^A}{\rightleftarrows}} P$$

$$PAA \quad \overset{P m_A}{\rightarrow} \quad \overset{PAu_A}{\leftarrow} \quad \overset{\mu_P^A A}{\rightarrow} \quad PA \quad \underset{P u_A}{\overset{\mu_P^A}{\rightleftarrows}} \quad P$$

is a contractible coequalizer and thus, by Proposition 6.7, $\left(P, \mu_P^A\right) = \mathrm{Coequ}_{\mathrm{Fun}}\left(\mu_P^A A, P m_A\right)$. □

Proposition 6.34. *Let* $\mathbb{A} = (A, m_A, u_A)$ *be a monad on a category* $\mathcal{A}$ *and let* $({}_{\mathbb{A}}F, {}_{\mathbb{A}}U)$ *be the adjunction associated. Then* ${}_{\mathbb{A}}U$ *reflects isomorphisms.*

Proof. Let $f : \left(X, {}^{A}\mu_X\right) \to \left(Y, {}^{A}\mu_Y\right)$ be a morphism in ${}_{\mathbb{A}}\mathcal{A}$ such that ${}_{\mathbb{A}}Uf$ has a two-sided inverse f^{-1} in $\mathcal{A}$. Since

$${}^{A}\mu_{X'} \circ (Af) = f \circ {}^{A}\mu_X$$

we get that

$$f^{-1} \circ {}^{A}\mu_{X'} = {}^{A}\mu_X \circ \left(Af^{-1}\right).$$

□

6.3 On Beck's Theorem

Lemma 6.35. *[Bo2, Corollary 4.1.4] Let* $(L : \mathcal{B} \to \mathcal{A}, R : \mathcal{A} \to \mathcal{B})$ *be an adjunction. Then the forgetful functor* $U : {}_{RL}\mathcal{B} \to \mathcal{B}$ *reflects the isomorphisms.*

Proof. Let $f : (B, \mu) \to (B', \mu')$ be a morphism in ${}_{RL}\mathcal{B}$ such that Uf is an isomorphism. We have that

$$\mu' \circ RLUf = Uf \circ \mu$$

so that

$$\left[(Uf)^{-1}\right] \circ \mu' = \mu \circ RL\left[(Uf)^{-1}\right]$$

which entails that $(Uf)^{-1}$ gives rise to a morphism $g : (B', \mu') \to (B, \mu)$ such that $Ug = (Uf)^{-1}$. Hence

$$U(f \circ g) = \mathrm{Id}_{B'} \qquad \text{and} \qquad U(g \circ f) = \mathrm{Id}_B$$

so that

$$f \circ g = \mathrm{Id}_{(B',\mu')} \qquad \text{and} \qquad g \circ f = \mathrm{Id}_{(B,\mu)}.$$

□

Definition 6.36. *Let* (L, R) *be an adjunction where* $L : \mathcal{B} \to \mathcal{A}$ *and* $R : \mathcal{A} \to \mathcal{B}$ *and let* $\mathbb{A} = (A = RL, m_A = R\epsilon L, u_A = \eta)$ *be the associated monad on the category* $\mathcal{B}$*. We can consider the functor*

$$K = {}_RK : \mathcal{A} \to {}_{\mathbb{A}}\mathcal{B}$$

defined by setting

$$K(X) = (RX, R\epsilon X) \qquad \textit{and} \qquad K(f) = R(f).$$

This is called the comparison functor *of the adjunction* (L, R) *. Note that* ${}_{\mathbb{A}}U \circ K = R$

Proposition 6.37 (Beck)**.** *[BW, Theorem 3.13, page 100] Let* (L, R) *be an adjunction where* $L : \mathcal{B} \to \mathcal{A}$ *and* $R : \mathcal{A} \to \mathcal{B}$*. Consider the comparison functor* $K : \mathcal{A} \to {}_{RL}\mathcal{B}$*. Then* K *is full and faithful if and only if for every* $A \in \mathcal{A}$ *we have that* $(A, \epsilon A) = \mathrm{Coequ}_{\mathcal{A}}(LR\epsilon A, \epsilon LRA)$*.*

Proof. Let $U : {}_{RL}\mathcal{B} \to \mathcal{B}$ be the forgetful functor. Let $A \in \mathcal{A}$. By Corollary 6.11,

$$RLRLRA \underset{R\epsilon LRA}{\overset{RLR\epsilon A}{\rightrightarrows}} RLRA \xrightarrow{R\epsilon A} RA.$$

is a contractible coequalizer. In particular it is preserved by L so that $LR\epsilon A$ is an epimorphism.

Suppose that K is full and faithful and let us prove that

$$LRLRA \underset{\epsilon LRA}{\overset{LR\epsilon A}{\rightrightarrows}} LRA \xrightarrow{\epsilon A} A$$

is a coequalizer too. Clearly ϵA coequalizes $(LR\epsilon A, \epsilon LRA)$. Let $\omega : LRA \to W$ be a morphism in $\mathcal{A}$ which coequalizes $(LR\epsilon A, \epsilon LRA)$. Then $R\omega$ coequalizes $(RLR\epsilon A, R\epsilon LRA)$ so that there is a unique morphism $\widehat{\omega} : RA \to RW$ such that $\widehat{\omega} \circ R\epsilon A = R\omega$. Let us check that $\widehat{\omega}$ is a morphism in ${}_{RL}\mathcal{B}$. We have

$$\epsilon W \circ L\widehat{\omega} \circ LR\epsilon A = \epsilon W \circ LR\omega = \omega \circ \epsilon LRA = \omega \circ LR\epsilon A.$$

Since $LR\epsilon A$ is an epimorphism, we get $\epsilon W \circ L\widehat{\omega} = \omega$. Thus

$$R\epsilon W \circ RL\widehat{\omega} = R\omega = \widehat{\omega} \circ R\epsilon A$$

so that $\widehat{\omega}$ is a morphism in ${}_{RL}\mathcal{B}$ i.e. it defines a morphism $\omega^1 : KA \to KW$ in ${}_{RL}\mathcal{B}$ such that $U\omega^1 = \widehat{\omega}$. Since K is full there is a morphism $h : A \to W$ such that $\omega^1 = Kh$. Then, from $\widehat{\omega} \circ R\epsilon A = R\omega$, we have

$$UKh \circ UK\epsilon A = UK\omega$$

so that, since U and K are both faithful, we get

$$h \circ \epsilon A = \omega.$$

Let us check that h is the unique morphism with this property. Let $h' : A \to W$ be such that $h' \circ \epsilon A = \omega$. By applying R we get $Rh' \circ R\epsilon A = R\omega$. Since $\widehat{\omega} \circ R\epsilon A = R\omega$ and $R\epsilon A$ is an epimorphism, we get $Rh' = \widehat{\omega}$. Thus

$$UKh\prime = Rh' = \widehat{\omega} = UKh$$

whence $h\prime = h$.

Conversely, suppose that

$$LRLRA \overset{LR\epsilon A}{\underset{\epsilon LRA}{\rightrightarrows}} LRA \xrightarrow{\epsilon A} A$$

is a coequalizer and let us prove that $K : \mathcal{A} \to {}_{RL}\mathcal{B}$ is full and faithful. Let $f : KA \to KA'$ be a morphism in ${}_{RL}\mathcal{B}$. Then $Uf : RA \to RA'$ is such that

$$R\epsilon A' \circ RLUf = Uf \circ R\epsilon A. \tag{6.2}$$

Then

$$\begin{aligned} \epsilon A' \circ LUf \circ LR\epsilon A &= \epsilon A' \circ L\left[Uf \circ R\epsilon A\right] = \epsilon A' \circ L\left[R\epsilon A' \circ RLUf\right] \\ &= \epsilon A' \circ LR\epsilon A' \circ LRLUf \\ &= \epsilon A' \circ \epsilon LRA' \circ LRLUf \\ &= \epsilon A' \circ LUf \circ \epsilon LRA \end{aligned}$$

so that there is a unique morphism $\widehat{f} : A \to A'$ such that $\widehat{f} \circ \epsilon A = \epsilon A' \circ LUf$. Thus

$$UK\widehat{f} = R\widehat{f} = R\widehat{f} \circ R\epsilon A \circ \eta RA = R\epsilon A' \circ RLUf \circ \eta RA \overset{(6.2)}{=} Uf \circ R\epsilon A \circ \eta RA = Uf$$

so that $K\widehat{f} = f$ i.e. K is full. Let $g, g\prime : A \to A'$ be morphisms in $\mathcal{A}$ such that $Kg = Kg'$. Then $Rg = Rg'$. Thus $LRg = LRg'$ and hence

$$g \circ \epsilon A = \epsilon A' \circ LRg = \epsilon A' \circ LRg' = g' \circ \epsilon A.$$

Since ϵA is an epimorphism, we get that $g = g'$ i.e. K is faithful. □

Remark 6.38. *A functor $R : \mathcal{A} \to \mathcal{B}$ which has a left adjoint $L : \mathcal{B} \to \mathcal{A}$ for which the corresponding comparison functor $K : \mathcal{A} \to {}_{RL}\mathcal{B}$*
is full and faithful is called of descent type.

Theorem 6.39. *Let $(L : \mathcal{B} \to \mathcal{A}, R : \mathcal{A} \to \mathcal{B})$ be an adjunction. Let η and ϵ be the unit and counit of (L, R) respectively. Consider the comparison functor $K : \mathcal{A} \to {}_{RL}\mathcal{B}$. Set $S := \{(L\mu, \epsilon LB) \mid (B, \mu : RLB \to B) \in {}_{RL}\mathcal{B}\}$. Then the following assertions are equivalent.*

(a) K has a left adjoint, say Λ,

(b) For each element in S we can choose a specific coequalizer in $\mathcal{A}$.

Assume that (b) holds.

Then, for every $(B, \mu) \in {}_{RL}\mathcal{B}$, $\Lambda(B, \mu)$ is defined to be the coequalizer

$$LRLB \underset{\epsilon LB}{\overset{L\mu}{\rightrightarrows}} LB \overset{\pi(B,\mu)}{\longrightarrow} \Lambda(B, \mu)$$

and for every morphism $f : (B, \mu) \to (B', \mu')$ the morphism $\Lambda(f) : \Lambda(B, \mu) \to \Lambda(B', \mu')$ is uniquely defined by

$$\Lambda(f) \circ \pi(B, \mu) = \pi(B', \mu') \circ LU(f).$$

Moreover the unit η^1 and the counit ϵ^1 of the adjunction (Λ, K) are uniquely defined by

$$U\eta^1(B, \mu) \circ \mu = R\pi(B, \mu), \tag{6.3}$$

$$\epsilon^1 A \circ \pi KA = \epsilon A, \tag{6.4}$$

and we have

$$\pi(B, \mu) = \epsilon\Lambda(B, \mu) \circ LU\eta^1(B, \mu). \tag{6.5}$$

Furthermore, Λ is full and faithful if and only if R preserves coequalizers of elements in S.

Proof. Let $U : {}_{RL}\mathcal{B} \to \mathcal{B}$ be the forgetful functor. Then $U \circ K = R$. Let $(B, \mu : RLB \to B) \in {}_{RL}\mathcal{B}$ and consider the pair

$$LRLB \underset{\epsilon LB}{\overset{L\mu}{\rightrightarrows}} LB.$$

Assume that $(L\mu, \epsilon LB)$ has a specific coequalizer that will be denoted by $(\Lambda(B,\mu), \pi(B,\mu) : LB \to \Lambda(B$
Let $f : (B,\mu) \to (B',\mu')$ be a morphism in ${}_{RL}\mathcal{B}$. Then $U(f) : B \to B'$ is such that

$$U(f) \circ \mu = \mu' \circ RLU(f)$$

so that

$$LU(f) \circ L\mu = L\mu' \circ LRLU(f).$$

Moreover, by naturality of the counit we have

$$LU(f) \circ \epsilon LB = \epsilon LB' \circ LRLU(f).$$

Thus

$$\begin{aligned}
\pi(B',\mu') \circ LU(f) \circ L\mu &= \pi(B',\mu') \circ L\mu' \circ LRLU(f) \\
&= \pi(B',\mu') \circ \epsilon LB' \circ LRLU(f) \\
&= \pi(B',\mu') \circ LU(f) \circ \epsilon LB
\end{aligned}$$

so that there is a unique morphism $\Lambda(f) : \Lambda(B,\mu) \to \Lambda(B',\mu')$ such that

$$\Lambda(f) \circ \pi(B,\mu) = \pi(B',\mu') \circ LU(f).$$

Let $f' : (B',\mu') \to (B'',\mu'')$ be a morphism in ${}_{RL}\mathcal{B}$. Then

$$\begin{aligned}
\Lambda(f') \circ \Lambda(f) \circ \pi(B,\mu) &= \Lambda(f') \circ \pi(B',\mu') \circ LU(f) \\
&= \pi(B'',\mu'') \circ LU(f') \circ LU(f) \\
&= \pi(B'',\mu'') \circ LU(f' \circ f) \\
&= \Lambda(f' \circ f) \circ \pi(B,\mu).
\end{aligned}$$

Since $\pi(B,\mu)$ is an epimorphism, we obtain $\Lambda(f') \circ \Lambda(f) = \Lambda(f' \circ f)$. Moreover

$$\Lambda\left(\mathrm{Id}_{(B,\mu)}\right) \circ \pi(B,\mu) = \pi(B,\mu) \circ LU\left(\mathrm{Id}_{(B,\mu)}\right) = \pi(B,\mu)$$

so that $\Lambda\left(\mathrm{Id}_{(B,\mu)}\right) = \mathrm{Id}_{\Lambda(B,\mu)}$. Let us check that (Λ, K) is an adjunction. We produce the unit and counit of this adjunction.

By Lemma 6.10, we have the following coequalizer in $\mathcal{B}$

$$RLRLB \underset{R\epsilon LB}{\overset{RL\mu}{\rightrightarrows}} RLB \xrightarrow{\mu} B.$$

Since $\pi(B,\mu)$ coequalizes $(L\mu, \epsilon LB)$, we have that $R\pi(B,\mu)$ coequalizes $(RL\mu, R\epsilon LB)$. Then there is a unique map $\alpha(B,\mu) : B \to R\Lambda(B,\mu)$ such that

$$\alpha(B,\mu) \circ \mu = R\pi(B,\mu). \tag{6.6}$$

Let us check that $\alpha(B,\mu)$ is a morphism in ${}_{RL}\mathcal{B}$. We have

$$\begin{aligned}
\epsilon\Lambda(B,\mu) \circ L\alpha(B,\mu) &= \epsilon\Lambda(B,\mu) \circ L\alpha(B,\mu) \circ L\mu \circ L\eta B = \epsilon\Lambda(B,\mu) \circ LR\pi(B,\mu) \circ L\eta B \\
&= \pi(B,\mu) \circ \epsilon LB \circ L\eta B = \pi(B,\mu)
\end{aligned}$$

so that

$$\epsilon\Lambda(B,\mu)\circ L\alpha(B,\mu)=\pi(B,\mu) \tag{6.7}$$

and hence

$$R\epsilon\Lambda(B,\mu)\circ RL\alpha(B,\mu)=R\pi(B,\mu)=\alpha(B,\mu)\circ\mu$$

i.e. $\alpha(B,\mu)$ is a morphism in ${}_{RL}\mathcal{B}$. Thus $\alpha(B,\mu)$ defines a morphism $\eta^1(B,\mu):(B,\mu)\to K\Lambda(B,\mu)$ such that $U(\eta^1(B,\mu))=\alpha(B,\mu)$. Note that from (6.6) one gets (6.3). Let us check that $\eta^1(B,\mu)$ is natural. Let $f:(B,\mu)\to(B',\mu')$ be a morphism in ${}_{RL}\mathcal{B}$. Then

$$\begin{aligned}R\Lambda f\circ\alpha(B,\mu)\circ\mu &= R\Lambda f\circ R\pi(B,\mu)=R\pi(B',\mu')\circ RLUf\\ &= \alpha(B',\mu')\circ\mu'\circ RLUf\\ &= \alpha(B',\mu')\circ U(f)\circ\mu\end{aligned}$$

so that

$$R\Lambda f\circ\alpha(B,\mu)=\alpha(B',\mu')\circ Uf$$

whence

$$K\Lambda f\circ\eta^1(B,\mu)=\eta^1(B',\mu')\circ f.$$

Now since $U(\eta^1(B,\mu))=\alpha(B,\mu)$, from (6.7) we deduce (6.5).

We have seen that for all $B\in\mathcal{B}$ we have an equalizer

$$LRLB\underset{\epsilon LB}{\overset{L\mu}{\rightrightarrows}}LB\overset{\pi(B,\mu)}{\longrightarrow}\Lambda(B,\mu).$$

Apply this to $B=KA$ for all $A\in\mathcal{A}$ to get the coequalizer

$$LRLRA\underset{\epsilon LRA}{\overset{LR\epsilon A}{\rightrightarrows}}LRA\overset{\pi KA}{\longrightarrow}\Lambda KA.$$

By naturality of ϵ, we have that ϵA coequalizes $(LR\epsilon A,\epsilon LRA)$ so that there is a unique morphism $\epsilon^1A:\Lambda KA\to A$ such that (6.4) holds. Let us check that ϵ^1A is natural in A. Let $g:A\to A'$ be a morphism in $\mathcal{A}$. Then

$$\begin{aligned}g\circ\epsilon^1A\circ\pi KA &= g\circ\epsilon A=\epsilon A'\circ LRg\\ &= \epsilon^1A'\circ\pi KA'\circ LRg\\ &= \epsilon^1A'\circ\pi KA'\circ LUKg\\ &= \epsilon^1A'\circ\Lambda Kg\circ\pi KA.\end{aligned}$$

Since πKA is an epimorphism, we get

$$g\circ\epsilon^1A=\epsilon^1A'\circ\Lambda Kg.$$

For $(B, \mu : RLB \to B) \in {}_{RL}\mathcal{B}$,

$$\begin{aligned} & \epsilon^1 \Lambda (B, \mu) \circ \Lambda \eta^1 (B, \mu) \circ \pi (B, \mu) \\ = \; & \epsilon^1 \Lambda (B, \mu) \circ \pi K \Lambda (B, \mu) \circ LU \eta^1 (B, \mu) \\ = \; & \epsilon^1 \Lambda (B, \mu) \circ \pi K \Lambda (B, \mu) \circ L\alpha (B, \mu) \\ = \; & \epsilon \Lambda (B, \mu) \circ L\alpha (B, \mu) \overset{(6.7)}{=} \pi (B, \mu) \end{aligned}$$

so that

$$\epsilon^1 \Lambda (B, \mu) \circ \Lambda \eta^1 (B, \mu) = \mathrm{Id}_{\Lambda(B,\mu)}.$$

For all $A \in \mathcal{A}$,

$$\begin{aligned} & UK\epsilon^1 A \circ U\eta^1 KA \\ = \; & UK\epsilon^1 A \circ U\eta^1 KA \circ R\epsilon A \circ \eta RA \\ = \; & R\epsilon^1 A \circ \alpha KA \circ R\epsilon A \circ \eta RA \\ \overset{(6.6)}{=} \; & R\epsilon^1 A \circ R\pi KA \circ \eta RA \\ \overset{(6.4)}{=} \; & R\epsilon A \circ \eta RA \\ = \; & \mathrm{Id}_{RA}. \end{aligned}$$

Thus

$$K\epsilon^1 A \circ \eta^1 KA = \mathrm{Id}_{KA}.$$

We have so proved that (Λ, K) is an adjuntion.

Conversely, assume that K has a left adjoint Λ. For (B, μ) in ${}_{RL}\mathcal{B}$, we set

$$\pi (B, \mu) := \epsilon \Lambda (B, \mu) \circ LU \eta^1 (B, \mu).$$

Let us check that

$$LRLB \underset{\epsilon LB}{\overset{L\mu}{\rightrightarrows}} LB \overset{\pi(B,\mu)}{\longrightarrow} \Lambda (B, \mu)$$

is a coequalizer. Note that $\mu : RLB \to B$ is a morphism in ${}_{RL}\mathcal{B}$

$$\begin{aligned} \pi (B, \mu) \circ L\mu \; &= \; \epsilon \Lambda (B, \mu) \circ LU\eta^1 (B, \mu) \circ L\mu \\ &= \; \epsilon \Lambda (B, \mu) \circ L \left[U\eta^1 (B, \mu) \circ \mu \right] \\ &\overset{\eta^1(B,\mu) \in {}_{RL}\mathcal{B}}{=} \; \epsilon \Lambda (B, \mu) \circ L \left[R\epsilon \Lambda (B, \mu) \circ RLU\eta^1 (B, \mu) \right] \\ &= \; \epsilon \Lambda (B, \mu) \circ LR\epsilon \Lambda (B, \mu) \circ LRLU\eta^1 (B, \mu) \\ &= \; \epsilon \Lambda (B, \mu) \circ \epsilon LR\Lambda (B, \mu) \circ LRLU\eta^1 (B, \mu) \\ &= \; \epsilon \Lambda (B, \mu) \circ LU\eta^1 (B, \mu) \circ \epsilon LB \\ &= \; \pi (B, \mu) \circ \epsilon LB \end{aligned}$$

Let $\zeta : LB \to Z$ be a morphism in $\mathcal{A}$ which equalizes $(L\mu, \epsilon LB)$. Set

$$\widehat{\zeta} := R\zeta \circ \eta B : B \to RZ.$$

Let us check that $\widehat{\zeta}$ is a morphism in ${}_{RL}\mathcal{B}$. We have

$$\epsilon Z\circ L\widehat{\zeta}=\epsilon Z\circ LR\zeta\circ L\eta B=\zeta\circ\epsilon LB\circ L\eta B=\zeta \tag{6.8}$$

so that

$$\begin{aligned} R\epsilon Z\circ RL\widehat{\zeta}\overset{(6.8)}{=}R\zeta &= R\zeta\circ R\epsilon LB\circ\eta RLB=R\left(\zeta\circ\epsilon LB\right)\circ\eta RLB\\ &= R\left(\zeta\circ L\mu\right)\circ\eta RLB=R\zeta\circ RL\mu\circ\eta RLB=R\zeta\circ\eta B\circ\mu=\widehat{\zeta}\circ\mu\end{aligned}$$

Hence $\widehat{\zeta}:B\to RZ$ defines a morphism $\vartheta:(B,\mu)\to KZ$ such that $U\vartheta=\widehat{\zeta}$. Then

$$\begin{aligned}\left(\epsilon^1 Z\circ\Lambda\vartheta\right)\circ\pi\left(B,\mu\right) &= \epsilon^1 Z\circ\Lambda\vartheta\circ\epsilon\Lambda\left(B,\mu\right)\circ LU\eta^1\left(B,\mu\right)\\ &= \epsilon Z\circ LR\epsilon^1 Z\circ LR\Lambda\vartheta\circ LU\eta^1\left(B,\mu\right)\\ &= \epsilon Z\circ LR\epsilon^1 Z\circ LUK\Lambda\vartheta\circ LU\eta^1\left(B,\mu\right)\\ &= \epsilon Z\circ LUK\epsilon^1 Z\circ LU\eta^1 KZ\circ LU\vartheta\\ &= \epsilon Z\circ L\widehat{\zeta}\\ &\overset{(6.8)}{=}\zeta\end{aligned}$$

so that $\left(\epsilon^1 Z\circ\Lambda\vartheta\right)\circ\pi\left(B,\mu\right)=\zeta$. Let us check that the unique morphism $\phi:\Lambda\left(B,\mu\right)\to Z$ such that $\phi\circ\pi\left(B,\mu\right)=\zeta$ is exactly $\epsilon^1 Z\circ\Lambda\vartheta$. Consider the canonical isomorphism $\Phi:\mathcal{A}\left(\Lambda\left(B,\mu\right),Z\right)\to{}_{RL}\mathcal{B}\left(\left(B,\mu\right),KZ\right),\Phi\left(x\right)=Kx\circ\eta^1\left(B,\mu\right)$. Thus, in order to prove that $\phi=\epsilon^1 Z\circ\Lambda\vartheta$ it suffices to check that $\Phi\left(\phi\right)=\Phi\left(\epsilon^1 Z\circ\Lambda\vartheta\right)$ i.e.

$$K\phi\circ\eta^1\left(B,\mu\right)=K\epsilon^1 Z\circ K\Lambda\vartheta\circ\eta^1\left(B,\mu\right).$$

Note that the latter term is $K\epsilon^1 Z\circ K\Lambda\vartheta\circ\eta^1\left(B,\mu\right)=K\epsilon^1 Z\circ\eta^1 KZ\circ\vartheta=\vartheta$ so that we have to prove that

$$K\phi\circ\eta^1\left(B,\mu\right)=\vartheta.$$

or equivalently

$$UK\phi\circ U\eta^1\left(B,\mu\right)=\widehat{\zeta}.$$

Consider the canonical isomorphism $\Theta:\mathcal{A}\left(LB,Z\right)\to\mathcal{B}\left(B,RZ\right),\Theta\left(y\right)=Ry\circ\eta B$. Since $\widehat{\zeta}:=R\zeta\circ\eta B=\Theta\left(\zeta\right)$, in order to prove the last displayed equality it suffices to check that

$$\Theta^{-1}\left[UK\phi\circ U\eta^1\left(B,\mu\right)\right]=\zeta$$

i.e. that $\epsilon Z\circ L\left[UK\phi\circ U\eta^1\left(B,\mu\right)\right]=\zeta$. We have

$$\begin{aligned}\epsilon Z\circ L\left[UK\phi\circ U\eta^1\left(B,\mu\right)\right] &= \epsilon Z\circ LR\phi\circ LU\eta^1\left(B,\mu\right)\\ &= \phi\circ\epsilon\Lambda\left(B,\mu\right)\circ LU\eta^1\left(B,\mu\right)=\phi\circ\pi\left(B,\mu\right)=\zeta.\end{aligned}$$

We have so proved that $\left(\Lambda\left(B,\mu\right),\pi\left(B,\mu\right)\right)$ is a coequalizer for $\left(L\mu,\epsilon LB\right)$.

Let us prove the last part of the statement.

Assume that R preserves coequalizers of elements in S and let us prove that Λ is full and faithful i.e. that $\eta^1(B,\mu)$ is an isomorphism for every $(B,\mu) \in {}_{RL}\mathcal{B}$. Consider the following coequalizer

$$LRLB \underset{\epsilon LB}{\overset{L\mu}{\rightrightarrows}} LB \overset{\pi(B,\mu)}{\longrightarrow} \Lambda(B,\mu)$$

By assumption we have a coequalizer

$$RLRLB \underset{R\epsilon LB}{\overset{RL\mu}{\rightrightarrows}} RLB \overset{R\pi(B,\mu)}{\longrightarrow} R\Lambda(B,\mu).$$

Since μ coequalizes $(RL\mu, R\epsilon LB)$, there is a unique morphism $\xi : R\Lambda(B,\mu) \to B$ such that $\xi \circ R\pi(B,\mu) = \mu$. Let $\alpha(B,\mu) = U\eta^1(B,\mu)$. Then

$$\mathrm{Id}_B = \mu \circ \eta B = \xi \circ R\pi(B,\mu) \circ \eta B \overset{(6.3)}{=} \xi \circ \alpha(B,\mu) \circ \mu \circ \eta B = \xi \circ .$$

Moroever

$$\alpha(B,\mu) \circ \xi \circ R\pi(B,\mu) = \alpha(B,\mu) \circ \mu \overset{(6.3)}{=} R\pi(B,\mu).$$

Since $R\pi(B,\mu)$ is an epimorphism, we get $\alpha(B,\mu) \circ \xi = \mathrm{Id}_{R\Lambda(B,\mu)}$. Therefore $\alpha(B,\mu) = U\eta^1(B,\mu)$ is an isomorphism. By Lemma 6.35 we deduce that $\eta^1(B,\mu)$ is an isomorphism.

Conversely, assume that Λ is full and faithful. Let $(B, \mu : RLB \to B) \in {}_{RL}\mathcal{B}$ and consider the coequalizer

$$LRLB \underset{\epsilon LB}{\overset{L\mu}{\rightrightarrows}} LB \overset{\pi(B,\mu)}{\longrightarrow} \Lambda(B,\mu).$$

Let us check it is preserved by R. Clearly $R\pi(B,\mu)$ coequalizes $(RL\mu, R\epsilon LB)$. Let $\delta : RLB \to D$ be a morphism in $\mathcal{B}$ that coequalizes $(RL\mu, R\epsilon LB)$. Set $\xi := \eta^1(B,\mu)^{-1} : K\Lambda(B,\mu) \to (B,\mu)$ and let $\alpha(B,\mu) = U\eta^1(B,\mu)$. Then

$$\begin{aligned} & [\delta \circ \eta B \circ U\xi] \circ R\pi(B,\mu) \\ & \overset{(6.3)}{=} [\delta \circ \eta B \circ U\xi] \circ \alpha(B,\mu) \circ \mu \\ = \; & \delta \circ \eta B \circ \mu = \delta \circ RL\mu \circ \eta RLB = \delta \circ R\epsilon LB \circ \eta RLB = \delta. \end{aligned}$$

Let now $\omega : R\Lambda(B,\mu) \to D$ be a morphism such that $\omega \circ R\pi(B,\mu) = \delta$. Then

$$\begin{aligned} & \delta \circ \eta B \circ U\xi \\ = \; & \omega \circ R\pi(B,\mu) \circ \eta B \circ U\xi \overset{(6.3)}{=} \omega \circ \alpha(B,\mu) \circ \mu \circ \eta B \circ U\xi \\ = \; & \omega \circ \alpha(B,\mu) \circ U\xi = \omega \circ U\eta^1(B,\mu) \circ U\xi = \omega. \end{aligned}$$

Therefore $(R\Lambda(B,\mu), R\pi(B,\mu))$ is the coequalizer of $(RL\mu, R\epsilon LB)$. □

Corollary 6.40. *Let $(L : \mathcal{B} \to \mathcal{A}, R : \mathcal{A} \to \mathcal{B})$ be an adjunction. Let η and ϵ be the unit and counit of (L, R) respectively. Consider the comparison functor $K : \mathcal{A} \to {}_{RL}\mathcal{B}$ and assume that (2) in Theorem 6.39 holds and denote by $\Lambda : {}_{RL}\mathcal{B} \to \mathcal{A}$ the left adjoint of K constructed therein. Let $U : {}_{RL}\mathcal{B} \to \mathcal{B}$ be the forgetful functor and let $F : \mathcal{B} \to_{RL} \mathcal{B}$ be the free functor. Then we have*

$$UK = R, \qquad KL = F \qquad \text{and} \qquad \Lambda F = L.$$

Moreover, for all $A \in \mathcal{A}$,

$$(\Lambda KA, \pi KA) = \mathrm{Coequ}_{\mathcal{A}}(LR\epsilon A, \epsilon LRA).$$

Proof. For every $A \in \mathcal{A}$, we have $KA = (RA, R\epsilon A)$ and for every $B \in \mathcal{B}$, we have $FB = (RLB, R\epsilon LB)$. Hence the first two equalities are trivial. Now, by Lemma 6.9, $(LB, LRLRLB, LRLB, \epsilon LB, LR\epsilon LB, \epsilon LRLB, L\eta B, LRL\eta B)$ is a contractible coequalizer. In diagram:

$$LRLRLB \ \underset{\overset{\longrightarrow}{\epsilon LRLB}}{\overset{\overset{LR\epsilon LB}{\longrightarrow}}{\overset{LRL\eta B}{\longleftarrow}}}\ LRLB \ \underset{L\eta B}{\overset{\epsilon LB}{\rightleftarrows}}\ LB\ .$$

In particular $(LB, \epsilon LB)$ is the coequalizer of

$$LRLRLB \underset{\epsilon LRLB}{\overset{LR\epsilon LB}{\rightrightarrows}} LRLB.$$

By the construction of Λ given in Theorem 6.39, we deduce that $\Lambda FB = LB$, for every $B \in \mathcal{B}$, and that
$(\Lambda KA, \pi KA) = \mathrm{Coequ}_{\mathcal{A}}(LR\epsilon A, \epsilon LRA)$, for every $A \in \mathcal{A}$. □

Theorem 6.41 (Beck)**.** *[BLV, Theorem 2.1] Let $(L : \mathcal{B} \to \mathcal{A}, R : \mathcal{A} \to \mathcal{B})$ be an adjunction. Let η and ϵ be the unit and counit of (L, R) respectively. Consider the comparison functor $K : \mathcal{A} \to {}_{RL}\mathcal{B}$. The following assertions are equivalent:*

(1) K is a category isomorphism.

(2) K is an equivalence and for any isomorphism $f : RX \to B$ in the category $\mathcal{B}$ there exists a unique pair $(A, g : X \to A)$, where A is an object in $\mathcal{A}$ and g a morphism in $\mathcal{A}$, such that $RA = B$ and $Rg = f$.

Proof. Let $U : {}_{RL}\mathcal{B} \to \mathcal{B}$ be the forgetful functor. Note that both in (1) and (2) the functor K is, in particular, an equivalence so that, in view of Proposition 6.37 and Theorem 6.39 we have that

- for every $A \in \mathcal{A}$ we have that $(A, \epsilon A) = \mathrm{Coequ}_{\mathcal{A}}(LR\epsilon A, \epsilon LRA)$,
- each element in $S := \{(L\mu, \epsilon LB) \mid (B, \mu : RLB \to B) \in {}_{RL}\mathcal{B}\}$ has a coequalizer in $\mathcal{A}$,

- R preserves coequalizers of elements in S.

(1) $\Rightarrow$ (2). Let Λ be a left adjoint of K such that $\Lambda K = \mathrm{Id}_{\mathcal{A}}$ and $K\Lambda = \mathrm{Id}_{{}_{RL}\mathcal{B}}$. Note that the unit and counit of the adjunction (Λ, K) are the identity funtorial morphism $\epsilon^1 : \Lambda K \to \mathrm{Id}_{\mathcal{A}}$ and $\eta^1 : \mathrm{Id}_{{}_{RL}\mathcal{B}} \to K\Lambda$. Let $f : RX \to B$ be an isomorphism in the category $\mathcal{B}$. It is clear that B can be regarded as an object in ${}_{RL}\mathcal{B}$ via $\mu := f \circ R\varepsilon X \circ RLf^{-1} : RLB \to B$. Moreover f defines a morphism $\widehat{f} : KX \to (B, \mu)$ such that $U\widehat{f} = f$. Clearly $\widehat{f}$ is an isomorphism. Now $(B, \mu) = K\Lambda (B, \mu) = KA$ where $A := \Lambda (B, \mu)$. Thus $B = UKA = RA$. Set $g := \Lambda \widehat{f} : X \to A$. Then $Rg = UK\Lambda\widehat{f} = U\widehat{f} = f$. Let now $(A', g' : X \to A')$ be another pair such that $RA' = B$ and $Rg' = f$. Since $\widehat{f}$ is an isomorphism, we have that $g = \Lambda\widehat{f}$ is an isomorphism. Consider

$$\tau := g' \circ g^{-1} : A \to A'$$

Then

$$UK\tau = R\tau = Rg' \circ R\left(g^{-1}\right) = f \circ (Rg)^{-1} = \mathrm{Id}_{RA}.$$

We have $f = U\widehat{f} = UK\Lambda\widehat{f} = R\Lambda\widehat{f}$ so that

$$R\epsilon A' \circ RLf = R\left(\epsilon A' \circ Lf\right) = R\left(\epsilon A' \circ LR\Lambda\widehat{f}\right) = R\left(\Lambda\widehat{f} \circ \epsilon X\right) = R\Lambda\widehat{f} \circ R\epsilon X = f \circ R\epsilon X = \mu \circ RLf.$$

Since RLf is an isomorphism we get $R\epsilon A' = \mu$ so that

$$KA' = (RA', R\epsilon A') = (B, \mu) = K\Lambda (B, \mu) = KA$$

and hence $A' = A$. Since $UK\tau = \mathrm{Id}_{RA} = UK\mathrm{Id}_A$, we get $\tau = \mathrm{Id}_A$ so that $g' = g$.

(2) $\Rightarrow$ (1). Since K has a left adjoint, by Theorem 6.39 the class S has a specific coequalizer. Thus we can consider the left adjoint Λ of K as constructed in Theorem 6.39. Let η^1 and ϵ^1 be the unit and counit of (Λ, K) respectively. Let $(B, \mu : RLB \to B) \in {}_{RL}\mathcal{B}$. Let $f(B, \mu) : R\Lambda (B, \mu) \to B$ denote the inverse of $U\eta^1 (B, \mu)$. By hypothesis there exists a unique pair $(\Lambda' (B, \mu), g(B, \mu) : \Lambda (B, \mu) \to \Lambda' (B, \mu))$, where $\Lambda' (B, \mu)$ is an object in $\mathcal{A}$ and $g(B, \mu)$ a morphism in $\mathcal{A}$, such that $R\Lambda' (B, \mu) = B$ and $Rg(B, \mu) = f(B, \mu)$. Since $f(B, \mu)$ is an isomorphism and $R = UK$, we have that $g(B, \mu)$ is an isomorphism too.

By (6.3) and (6.5), we have

$$R\epsilon\Lambda (B, \mu) \circ RLU\eta^1 (B, \mu) = U\eta^1 (B, \mu) \circ \mu$$

so that

$$f(B, \mu) \circ R\epsilon\Lambda (B, \mu) = \mu \circ RLf(B, \mu).$$

Using this equality we get

$$\begin{aligned} R\epsilon\Lambda' (B, \mu) \circ RLf(B, \mu) &= R\left[\epsilon\Lambda' (B, \mu) \circ Lf(B, \mu)\right] = R\left[\epsilon\Lambda' (B, \mu) \circ LRg(B, \mu)\right] = R\left[g(B, \mu) \circ \epsilon\Lambda\right. \\ &= Rg(B, \mu) \circ R\epsilon\Lambda (B, \mu) = f(B, \mu) \circ R\epsilon\Lambda (B, \mu) = \mu \circ RLf(B, \mu). \end{aligned}$$

Since $f(B,\mu)$ is an isomorphism, we obtain $R\epsilon\Lambda'(B,\mu)=\mu$ so that

$$K\Lambda'(B,\mu)=(R\Lambda'(B,\mu),R\epsilon\Lambda'(B,\mu))=(B,\mu).$$

Let $A\in\mathcal{A}$ and set $\alpha:=\epsilon^1A\circ(gKA)^{-1}:\Lambda'KA\to A$. We have that $R\Lambda'KA=UK\Lambda'KA=UKA=RA$ and

$$R\alpha=R\epsilon^1A\circ R(gKA)^{-1}=R\epsilon^1A\circ fKA^{-1}=R\epsilon^1A\circ U\eta^1KA=U\left[K\epsilon^1A\circ\eta^1KA\right]=\mathrm{Id}_{RA}.$$

By uniqueness in the assumption, we get $(\Lambda'KA,\alpha)=(A,\mathrm{Id}_A)$.

For all $h:(B,\mu)\to(B',\mu')$, set

$$\Lambda'h:=g(B',\mu')\circ\Lambda h\circ g(B,\mu)^{-1}.$$

Then we get a functor $\Lambda':{}_{RL}\mathcal{B}\to\mathcal{A}$ which is an inverse of K. □

Proposition 6.42. *[Li, Proposition 3, page 83] Let $(A,m:AA\to A,u:\mathrm{Id}_{\mathcal{C}}\to A)$ be a monad on a category $\mathcal{C}$ and let $f,g:(M,\mu)\to(N,\nu)$ be a pair of morphisms in ${}_A\mathcal{C}$. Let $U:{}_A\mathcal{C}\to\mathcal{C}$ be the forgetful functor and assume that*

1) (Uf,Ug) has a coequalizer $(C,c:N\to C)$ in $\mathcal{C}$.

2) $(AC,Ac)=\mathrm{Coequ}_{\mathcal{C}}(AUf,AUg)$.

3) AAc is an epimorphism in $\mathcal{C}$.

Then there is a unique morphism $\tau:AC\to C$ such that $c\circ\nu=\tau\circ Ac$. Moroever $(C,\tau)\in{}_A\mathcal{C}$, c defines a morphism $\widehat{c}:(N,\nu)\to(C,\tau)$ in ${}_A\mathcal{C}$ such that $U\widehat{c}=c$ and $((C,\tau),\widehat{c})=\mathrm{Coequ}_{{}_A\mathcal{C}}(f,g)$.

Proof. Let us consider

$$\begin{array}{ccccc} AM & \overset{AUf}{\underset{AUg}{\rightrightarrows}} & AN & \overset{Ac}{\to} & AC\\ \mu\downarrow & & \nu\downarrow & & \\ M & \overset{Uf}{\underset{Ug}{\rightrightarrows}} & N & \overset{c}{\to} & C \end{array}$$

We have

$$c\circ\nu\circ AUf=c\circ Uf\circ\mu=c\circ Ug\circ\mu=c\circ\nu\circ AUg.$$

Since $(AC,Ac)=\mathrm{Coequ}_{\mathcal{C}}(AUf,AUg)$ there exists a unique morphism $\tau:AC\to C$ such that

$$\tau\circ Ac=c\circ\nu.$$

Let us prove that $(C,\tau)\in{}_A\mathcal{C}$. We have

$$\begin{aligned} bg\tau\circ A\tau\circ AAc&=\tau\circ Ac\circ A\nu=c\circ\nu\circ A\nu\\ &=c\circ\nu\circ mN=\tau\circ Ac\circ mN=\tau\circ mC\circ AAc.\end{aligned}$$

Since AAc is an epimorphism in $\mathcal{C}$, we get $\tau\circ A\tau=\tau\circ mC$. Moreover we have

$$\tau\circ uC\circ c=\tau\circ Ac\circ uN=c\circ\nu\circ uN=c$$

and since c is an epimorphism, we get $\tau \circ uC = \mathrm{Id}_C$ so that $(C,\tau) \in {}_A\mathcal{C}$.

Since $\tau \circ Ac = c \circ \nu$, c defines a morphism $\widehat{c} : (N,\nu) \to (C,\tau)$ in ${}_A\mathcal{C}$ such that $U\widehat{c} = c$. Let us check that $((C,\tau), \widehat{c}) = \mathrm{Coequ}_{{}_A\mathcal{C}}(f,g)$. We have

$$U(\widehat{c} \circ f) = U\widehat{c} \circ Uf = c \circ Uf = c \circ Ug = U\widehat{c} \circ Ug = U(\widehat{c} \circ g)$$

so that $\widehat{c} \circ f = \widehat{c} \circ g$. Let $\omega : (N,\nu) \to (Z,\zeta)$ be a morphism in ${}_A\mathcal{C}$ such that $\omega \circ f = \omega \circ g$. Then $U\omega$ coequalizes (Uf, Ug) so that there is a unique morphism $w : C \to Z$ such that $w \circ c = U\omega$. We have

$$w \circ \tau \circ Ac = w \circ c \circ \nu = U\omega \circ \nu = \zeta \circ AU\omega = \zeta \circ Aw \circ Ac.$$

Since Ac is an epimorphism, we obtain $w \circ \tau = \zeta \circ Aw$ so that w defines a morphism $\widehat{w} : (C,\tau) \to (Z,\zeta)$ such that $U\widehat{w} = w$. We have

$$U(\widehat{w} \circ \widehat{c}) = U\widehat{w} \circ U\widehat{c} = w \circ c = U\omega$$

so that $\widehat{w} \circ \widehat{c} = \omega$. Let us check that $\widehat{w}$ is unique. Let $\alpha : (C,\tau) \to (Z,\zeta)$ be a morphism in ${}_A\mathcal{C}$ such that $\alpha \circ \widehat{c} = \omega$. Then

$$U\alpha \circ c = U(\alpha \circ \widehat{c}) = U\omega = w \circ c = U\widehat{w} \circ c.$$

Since c is an epimorphism we get $U\alpha = U\widehat{w}$ and hence $\alpha = \widehat{w}$. □

Theorem 6.43 (Beck)**.** *[BLV, Theorem 2.1 page 5] Let $(L : \mathcal{B} \to \mathcal{A}, R : \mathcal{A} \to \mathcal{B})$ be an adjunction. Let η and ϵ be the unit and counit of (L,R) respectively. Consider the comparison functor $K : \mathcal{A} \to {}_{RL}\mathcal{B}$. The following assertions are equivalent:*

(1) K is an equivalence.

(2) R reflects isomorphisms and for any reflexive R-contractible coequalizer pair we can choose a specific coequalizer in $\mathcal{A}$, which is preserved by R.

(3) R reflects isomorphisms and for every element in $S := \{(L\mu, \epsilon LB) \mid (B, \mu : RLB \to B) \in {}_{RL}\mathcal{B}\}$ we can choose a specific coequalizer in $\mathcal{A}$ which is preserved by R.

(4) For every $A \in \mathcal{A}$ we have that $(A, \epsilon A) = \mathrm{Coequ}_{\mathcal{A}}(LR\epsilon A, \epsilon LRA)$. For every element in $S := \{(L\mu, \epsilon LB) \mid (B, \mu : RLB \to B) \in {}_{RL}\mathcal{B}\}$ we can choose a specific coequalizer in $\mathcal{A}$ which is preserved by R.

Proof. (1) $\Leftrightarrow$ (4). It follows by Proposition 6.37 and Theorem 6.39.

(1) $\Rightarrow$ (2). Let $\Lambda : {}_{RL}\mathcal{B} \to \mathcal{A}$ be a left adjoint of K. Let η^1 and ϵ^1 be the unit and counit of (Λ, K) respectively. Assume that $f : A \to A'$ is a morphism in $\mathcal{A}$ such that Rf is an isomorphism. Since $Rf = UKf$ is an isomorphism, so is $Kf : KA \to KA'$. Since $\epsilon^1 A' \circ \Lambda K f = f \circ \epsilon^1 A$ and the counit is an isomorphism, we get that f is an isomorphism. Let (d_0, d_1) from A to A' be a reflexive R-contractible coequalizer pair. Since the pair is reflexive there is a morphism $e : A' \to A$ such that $d_0 \circ e = d_1 \circ e = \mathrm{Id}_{A'}$. Since it is an R-contractible coequalizer pair, there exists $C \in \mathcal{C}$ and morphism $v : RA' \to RA$, $c : RA' \to C$ and $u : C \to RA'$

$$RA \quad \overset{Rd_0}{\longrightarrow} \quad \overset{v}{\leftarrow} \quad \overset{Rd_1}{\longrightarrow} \quad RA' \underset{u}{\overset{c}{\rightleftarrows}} C\ .$$

such that

$$\begin{aligned} Rd_0 \circ v &= \mathrm{Id}_{RA'}, \\ Rd_1 \circ v &= u \circ c, \\ c \circ u &= \mathrm{Id}_C, \\ c \circ Rd_0 &= c \circ Rd_1. \end{aligned}$$

In particular $(C, c) = \mathrm{Coequ}_{\mathcal{B}}(Rd_0, Rd_1) = \mathrm{Coequ}_{\mathcal{B}}(UKd_0, UKd_1)$. Since

$$RA \underset{Rd_1}{\overset{Rd_0}{\rightrightarrows}} RA' \overset{c}{\to} C$$

is an R-contractible coequalizer pair, in view of Proposition 6.8 and Proposition6.7, it is preserved by any functor.

$$RLRA \underset{RLRd_1}{\overset{RLRd_0}{\rightrightarrows}} RLRA' \overset{RLc}{\to} RLC$$

In particular $(RLC, RLc) = \mathrm{Coequ}_{\mathcal{B}}(RLRd_0, RLRd_1) = \mathrm{Coequ}_{\mathcal{B}}(RLUKd_0, RLUKd_1)$ and also $RLRLc$ is an epimorphism.

Apply Proposition 6.42 to the monad $(RL, R\epsilon L, \eta)$ on the category $\mathcal{B}$ and to the pair $Kd_0, Kd_1 : KA \to KA'$. Thus

there is a unique morphism $m : RLC \to C$ such that

$$m \circ RLc = c \circ R\epsilon A'.$$

Moreover $(C, m) \in {}_{RL}\mathcal{B}$, c defines a morphism $\widehat{c} : KA' \to (C, m)$ in ${}_{RL}\mathcal{B}$ such that $U\widehat{c} = c$ and $((C, m), \widehat{c}) = \mathrm{Coequ}_{{}_{RL}\mathcal{B}}(Kd_0, Kd_1)$. Since Λ is an equivalence we have that $(\Lambda(C, m), \Lambda\widehat{c}) = \mathrm{Coequ}_{\mathcal{A}}(\Lambda Kd_0, \Lambda Kd_1)$. Set $A''' := \Lambda(C, m)$ and $\gamma := \Lambda\widehat{c} \circ (\epsilon^1 A')^{-1} : A' \to A'''$. Since $\epsilon^1 A' \circ \Lambda K d_i = d_i \circ \epsilon^1 A$ and $\epsilon^1 A$ is an isomorphism, it is clear that $(A''', \gamma) = \mathrm{Coequ}_{\mathcal{A}}(d_0, d_1)$.

We have

$$\begin{aligned} U\eta^1(C, m)^{-1} \circ R\gamma &= U\eta^1(C, m)^{-1} \circ R\Lambda\widehat{c} \circ R\left(\epsilon^1 A'\right)^{-1} \\ &= U\eta^1(C, m)^{-1} \circ UK\Lambda\widehat{c} \circ \left(R\epsilon^1 A'\right)^{-1} \\ &= U\widehat{c} \circ \left(U\eta^1 KA'\right)^{-1} \circ \left(UK\epsilon^1 A'\right)^{-1} \\ &= U\widehat{c} = c. \end{aligned}$$

so that

$$R\gamma = U\eta^1(C, m) \circ c. \tag{6.9}$$

Since $(C, m) \in {}_{RL}\mathcal{B}$, we have an isomorphism

$$\eta^1(C, m) : (C, m) \to K\Lambda 8(C, m) = (R\Lambda(C, m), R\epsilon\Lambda(C, m)) = (RA''', R\epsilon A''').$$

Since (C, c) is a coequalizator of (Rd_0, Rd_1) in $\mathcal{B}$, by (6.9) we deduce that $(RA''', R\gamma)$ is a coequalizator of (Rd_0, Rd_1) in $\mathcal{B}$.

(2) $\Rightarrow$ (3). Let $(B, \mu : RLB \to B) \in {}_{RL}\mathcal{B}$. By Lemma 6.10, $(\epsilon LB, L\mu)$ is a reflexive R-contractible coequalizer pair. By assumption, $(\epsilon LB, L\mu)$ has a specific coequalizer in $\mathcal{A}$, which is preserved by R.

(3) $\Rightarrow$ (4) .Let $A \in \mathcal{A}$. Then $(B, \mu) := (RA, R\epsilon A) \in {}_{RL}\mathcal{B}$. By assumption $(\epsilon LRA, LR\epsilon A)$ has a specific coequalizer (C, c) in $\mathcal{A}$, which is preserved by R. Since ϵA coequalizes $(\epsilon LRA, LR\epsilon A)$, there is a unique morphism $h : C \to A$ such that $h \circ c = \epsilon A$. Then $Rh \circ Rc = R\epsilon A$. By Corollary 6.11 and Proposition 6.7, we know that $(RA, R\epsilon A)$ is the coequalizer of $(R\epsilon LRA, RLR\epsilon A)$ in $\mathcal{B}$. Since also (RC, Rc) is the coequalizer of $(R\epsilon LRA, RLR\epsilon A)$ in $\mathcal{B}$, we have that Rh is an isomorphism. Since R reflects isomorphisms we obtain that h is an isomorphism too so that $(A, \epsilon A) = \mathrm{Coequ}_{\mathcal{A}}(LR\epsilon A, \epsilon LRA)$. □

Remark 6.44. *A functor $R : \mathcal{A} \to \mathcal{B}$ which has a left adjoint $L : \mathcal{B} \to \mathcal{A}$ for which the corresponding comparison functor $K : \mathcal{A} \to {}_{RL}\mathcal{B}$*

is an equivalence of categories is called monadic *(*tripleable *in Beck's terminology [[Be, Definition 3, page 8]]). For this reason Theorem 6.43 is also called "Beck's Precise Tripleability Theorem" (cfr.[BW, Theorem 3.14, page 101]).*

6.4 Johnstone for Monads

Proposition 6.45 ([Appel] and [J])**.** *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad on a category $\mathcal{A}$ and let $\mathbb{B} = (B, m_B, u_B)$ be a monad on a category $\mathcal{B}$ and let $Q : \mathcal{A} \to \mathcal{B}$ be a functor. Then there is a bijection between the following collections of data*

$\mathcal{F}$ *functors $\widetilde{Q} : {}_{\mathbb{A}}\mathcal{A} \to {}_{\mathbb{B}}\mathcal{B}$ that are liftings of Q (i.e. ${}_{\mathbb{B}}U\widetilde{Q} = Q_{\mathbb{A}}U$)*

$\mathcal{M}$ *functorial morphisms $\Phi : BQ \to QA$ such that*

$$\Phi \circ (m_B Q) = (Qm_A) \circ (\Phi A) \circ (B\Phi) \qquad \text{and} \qquad \Phi \circ (u_B Q) = Qu_A$$

given by

$$\begin{aligned}
a &: \mathcal{F} \to \mathcal{M} \text{ where } a\left(\widetilde{Q}\right) = \left({}_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F\right) \circ \left({}_{\mathbb{B}}U_{\mathbb{B}}FQu_A\right) \\
b &: \mathcal{M} \to \mathcal{F} \text{ where } {}_{\mathbb{B}}Ub(\Phi) = Q_{\mathbb{A}}U \text{ and } {}_{\mathbb{B}}U\lambda_B b(\Phi) = (Q_{\mathbb{A}}U\lambda_A) \circ \Phi \\
b &: \mathcal{M} \to \mathcal{F} \text{ where } b(\Phi)\left(\left(X, {}^A\mu_X\right)\right) = \left(QX, \left(Q^A\mu_X\right) \circ (\Phi X)\right) \text{ and } b(\Phi)(f) = Q(f).
\end{aligned}$$

Proof. Let $\widetilde{Q} : {}_{\mathbb{A}}\mathcal{A} \to {}_{\mathbb{B}}\mathcal{B}$ be a lifting of the functor $Q : \mathcal{A} \to \mathcal{B}$ (i.e. ${}_{\mathbb{B}}U\widetilde{Q} = Q_{\mathbb{A}}U$). Define a functorial morphism $\phi : {}_{\mathbb{B}}FQ \to \widetilde{Q}_{\mathbb{A}}F$ as the composite

$$\phi := \left(\lambda_B\widetilde{Q}_{\mathbb{A}}F\right) \circ \left({}_{\mathbb{B}}FQu_A\right)$$

where $u_A : \mathcal{A} \to {}_{\mathbb{A}}U_{\mathbb{A}}F = A$ is also the unit of the adjunction $({}_{\mathbb{A}}F, {}_{\mathbb{A}}U)$ and $\lambda_B : {}_{\mathbb{B}}F_{\mathbb{B}}U \to {}_{\mathbb{B}}\mathcal{B}$ is the counit of the adjunction. Let now define

$$\Phi \overset{def}{=} {}_{\mathbb{B}}U\phi : {}_{\mathbb{B}}U_{\mathbb{B}}FQ = BQ \to {}_{\mathbb{B}}U\widetilde{Q}_{\mathbb{A}}F = Q_{\mathbb{A}}U_{\mathbb{A}}F = QA.$$

We have to prove that such a Φ satisfies $\Phi \circ (m_B Q) = (Qm_A) \circ (\Phi A) \circ (B\Phi)$ and $\Phi \circ (u_B Q) = Qu_A$. First, let us compute

$$\begin{aligned}
(Qm_A) \circ (\Phi A) \circ (B\Phi) &= (Qm_A) \circ \left({}_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}FA\right) \circ ({}_{\mathbb{B}}U_{\mathbb{B}}FQu_AA) \\
&\quad \circ \left(B_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F\right) \circ (B_{\mathbb{B}}U_{\mathbb{B}}FQu_A) \\
&\overset{{}_{\mathbb{A}}U\lambda_{A\mathbb{A}}F}{=} (Q_{\mathbb{A}}U\lambda_{A\mathbb{A}}F) \circ \left({}_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}FA\right) \\
&\quad \circ ({}_{\mathbb{B}}U_{\mathbb{B}}FQu_AA) \circ \left(B_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F\right) \circ (B_{\mathbb{B}}U_{\mathbb{B}}FQu_A) \\
&\overset{\widetilde{Q}\text{lifting}}{=} \left({}_{\mathbb{B}}U\widetilde{Q}\lambda_{A\mathbb{A}}F\right) \circ \left({}_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}FA\right) \\
&\quad \circ ({}_{\mathbb{B}}U_{\mathbb{B}}FQu_AA) \circ \left(B_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F\right) \circ (B_{\mathbb{B}}U_{\mathbb{B}}FQu_A) \\
&= {}_{\mathbb{B}}U\left[\left(\widetilde{Q}\lambda_{A\mathbb{A}}F\right) \circ \left(\lambda_B\widetilde{Q}_{\mathbb{A}}FA\right) \circ ({}_{\mathbb{B}}FQu_AA)\right] \\
&\quad \circ \left(B_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F\right) \circ (BBQu_A) \\
&\overset{\lambda_B}{=} {}_{\mathbb{B}}U\left[\left(\lambda_B\widetilde{Q}_{\mathbb{A}}F\right) \circ \left({}_{\mathbb{B}}F_{\mathbb{B}}U\widetilde{Q}\lambda_{A\mathbb{A}}F\right) \circ ({}_{\mathbb{B}}FQu_AA)\right] \\
&\quad \circ \left(B_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F\right) \circ (BBQu_A) \\
&\overset{\widetilde{Q}\text{lifting}}{=} {}_{\mathbb{B}}U\left[\left(\lambda_B\widetilde{Q}_{\mathbb{A}}F\right) \circ ({}_{\mathbb{B}}FQ_{\mathbb{A}}U\lambda_{A\mathbb{A}}F) \circ ({}_{\mathbb{B}}FQu_AA)\right] \\
&\quad \circ \left(B_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F\right) \circ (BBQu_A) \\
&\overset{{}_{\mathbb{A}}U\lambda_{A\mathbb{A}}F}{=} {}_{\mathbb{B}}U\left[\left(\lambda_B\widetilde{Q}_{\mathbb{A}}F\right) \circ ({}_{\mathbb{B}}FQm_A) \circ ({}_{\mathbb{B}}FQu_AA)\right] \\
&\quad \circ \left(B_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F\right) \circ (BBQu_A) \\
&\overset{A\text{monad}}{=} \left({}_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F\right) \circ \left(B_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F\right) \circ (BBQu_A) \\
&\overset{\widetilde{Q}\text{is}B\text{mod}}{=} \left({}_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F\right) \circ \left(m_{B\mathbb{B}}U\widetilde{Q}_{\mathbb{A}}F\right) \circ (BBQu_A) \\
&\overset{m_B}{=} \left({}_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F\right) \circ (BQu_A) \circ (m_BQ) \\
&= \left({}_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F\right) \circ ({}_{\mathbb{B}}U_{\mathbb{B}}FQu_A) \circ (m_BQ) \\
&= ({}_{\mathbb{B}}U\phi) \circ (m_BQ) \\
&= \Phi \circ (m_BQ)\,.
\end{aligned}$$

Moreover we have

$$\begin{aligned}\Phi\circ(u_BQ) &= ({}_{\mathbb{B}}U\phi)\circ(u_BQ)\\ &= \left({}_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F\right)\circ({}_{\mathbb{B}}U_{\mathbb{B}}FQu_A)\circ(u_BQ)\\ &\overset{u_B}{=} \left({}_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F\right)\circ(u_BQ_{\mathbb{A}}U_{\mathbb{A}}F)\circ(Qu_A)\\ &\overset{\widetilde{Q}\text{lifting}}{=} \left({}_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F\right)\circ\left(u_{B\mathbb{B}}U\widetilde{Q}_{\mathbb{A}}F\right)\circ(Qu_A)\\ &\overset{({}_{\mathbb{B}}F,{}_{\mathbb{B}}U)\text{adj}}{=} Qu_A.\end{aligned}$$

Conversely, let Φ be a functorial morphism satisfying $\Phi\circ(m_BQ)=(Qm_A)\circ(\Phi A)\circ(B\Phi)$ and $\Phi\circ(u_BQ)=Qu_A$. We define $\widetilde{Q}:{}_{\mathbb{A}}\mathcal{A}\to{}_{\mathbb{B}}\mathcal{B}$ by setting, for every $\left(X,{}^{A}\mu_X\right)\in{}_{\mathbb{A}}\mathcal{A}$,

$$\widetilde{Q}\left(\left(X,{}^{A}\mu_X\right)\right)=\left(QX,\left(Q^{A}\mu_X\right)\circ(\Phi X)\right).$$

We have to check that $\left(Q\left(X\right),\left(Q^{A}\mu_X\right)\circ(\Phi X)\right)\in{}_{\mathbb{B}}\mathcal{B}$, that is

$${}^{B}\mu_{\widetilde{Q}X}\circ\left(B^{B}\mu_{\widetilde{Q}X}\right)={}^{B}\mu_{\widetilde{Q}X}\circ(m_BQX)\quad\text{and}\quad{}^{B}\mu_{\widetilde{Q}X}\circ(u_BQX)=QX.$$

We compute

$$\begin{aligned}{}^{B}\mu_{\widetilde{Q}X}\circ\left(B^{B}\mu_{\widetilde{Q}X}\right) &= \left(Q^{A}\mu_X\right)\circ(\Phi X)\circ\left(BQ^{A}\mu_X\right)\circ(B\Phi X)\\ &\overset{\Phi}{=} \left(Q^{A}\mu_X\right)\circ\left(QA^{A}\mu_X\right)\circ(\Phi AX)\circ(B\Phi X)\\ &\overset{X\text{module}}{=} \left(Q^{A}\mu_X\right)\circ(Qm_AX)\circ(\Phi AX)\circ(B\Phi X)\\ &\overset{\text{propertyof}\Phi}{=} \left(Q^{A}\mu_X\right)\circ(\Phi X)\circ(m_BQX)\\ &= {}^{B}\mu_{\widetilde{Q}X}\circ(m_BQX).\end{aligned}$$

Moreover we have

$$\begin{aligned}{}^{B}\mu_{\widetilde{Q}X}\circ(u_BQX) &= \left(Q^{A}\mu_X\right)\circ(\Phi X)\circ(u_BQX)\\ &\overset{\text{propertyof}\Phi}{=} \left(Q^{A}\mu_X\right)\circ(Qu_AX)\\ &\overset{X\text{module}}{=} QX.\end{aligned}$$

Now, let $f:\left(X,{}^{A}\mu_X\right)\to\left(Y,{}^{A}\mu_Y\right)$ a morphism of left $\mathbb{A}$-modules, that is a morphism $f:X\to Y$ in $\mathcal{A}$ such that

$${}^{A}\mu_Y\circ(Af)=f\circ{}^{A}\mu_X.$$

We have to prove that $\widetilde{Q}\left(f\right):\widetilde{Q}X=\left(QX,{}^{B}\mu_{QX}\right)\to\widetilde{Q}Y=\left(QX,{}^{B}\mu_{QY}\right)$ is a morphism of left $\mathbb{B}$-modules. We set $\widetilde{Q}\left(f\right)=Q\left(f\right)$ and we compute

$${}^{B}\mu_{\widetilde{Q}Y}\circ\left(B\widetilde{Q}f\right)\overset{?}{=}\left(\widetilde{Q}f\right)\circ{}^{B}\mu_{\widetilde{Q}X}$$

i.e. by definition of the functor $\widetilde{Q}$

$$^{B}\mu_{QY} \circ (BQf) \overset{?}{=} (Qf) \circ {}^{B}\mu_{QX}$$

$$\begin{aligned} ^{B}\mu_{QY} \circ (BQf) &= \left(Q^{A}\mu_{Y}\right) \circ (\Phi Y) \circ (BQf) \\ &\overset{\Phi}{=} \left(Q^{A}\mu_{Y}\right) \circ (QAf) \circ (\Phi X) \\ &\overset{f \text{morph} A\text{-mod}}{=} (Qf) \circ \left(Q^{A}\mu_{X}\right) \circ (\Phi X) \\ &= (Qf) \circ {}^{B}\mu_{QX}. \end{aligned}$$

Let now check that $\widetilde{Q}$ is a lifting of Q. Let $\left(X,^{A}\mu_{X}\right) \in {}_{\mathbb{A}}\mathcal{A}$ and compute

$${}_{\mathbb{B}}U\widetilde{Q}\left(\left(X,^{A}\mu_{X}\right)\right) = {}_{\mathbb{B}}U\left(QX,^{B}\mu_{QX}\right) = QX = Q_{\mathbb{A}}U\left(\left(X,^{A}\mu_{X}\right)\right)$$

and thus on the objects

$${}_{\mathbb{B}}U\widetilde{Q} = Q_{\mathbb{A}}U.$$

Let $f : \left(X,^{A}\mu_{X}\right) \to \left(Y,^{A}\mu_{Y}\right) \in {}_{\mathbb{A}}\mathcal{A}$ be a morphism, we have

$${}_{\mathbb{B}}U\widetilde{Q}\left(f\right) : QX \to QY = Q_{\mathbb{A}}U\left(f\right) : QX \to QY.$$

Therefore $\widetilde{Q}$ is a lifting of the functor Q.

We have to prove that it is a bijection. Let us start with $\widetilde{Q} : {}_{\mathbb{A}}\mathcal{A} \to {}_{\mathbb{B}}\mathcal{B}$ a lifting of the functor $Q : \mathcal{A} \to \mathcal{B}$. Then we construct $\Phi : BQ \to QA$ given by

$$\Phi = \left({}_{\mathbb{B}}U\lambda_{B}\widetilde{Q}_{\mathbb{A}}F\right) \circ \left({}_{\mathbb{B}}U_{\mathbb{B}}FQu_{A}\right)$$

and using this functorial morphism we define a functor $\overline{Q} : {}_{\mathbb{A}}\mathcal{A} \to {}_{\mathbb{B}}\mathcal{B}$ as follows: for every $\left(X,^{A}\mu_{X}\right) \in {}_{\mathbb{A}}\mathcal{A}$

$$\overline{Q}\left(\left(X,^{A}\mu_{X}\right)\right) = \left(QX, \left(Q^{A}\mu_{X}\right) \circ (\Phi X)\right).$$

Since both $\widetilde{Q}$ and $\overline{Q}$ are lifting of Q, we have that ${}_{\mathbb{B}}U\widetilde{Q} = Q_{\mathbb{A}}U = {}_{\mathbb{B}}U\overline{Q}$. We have to prove that ${}_{\mathbb{B}}U\left(\lambda_{B}\overline{Q}\right) = {}_{\mathbb{B}}U\left(\lambda_{B}\widetilde{Q}\right)$. Let $Z \in {}_{\mathbb{A}}\mathcal{A}$. We compute

$$\begin{aligned} {}_{\mathbb{B}}U\left(\lambda_{B}\overline{Q}Z\right) &= \left(Q_{\mathbb{A}}U\lambda_{A}Z\right) \circ \left({}_{\mathbb{B}}U\lambda_{B}\widetilde{Q}_{\mathbb{A}}F_{\mathbb{A}}UZ\right) \circ \left({}_{\mathbb{B}}U_{\mathbb{B}}FQu_{A\mathbb{A}}UZ\right) \\ &\overset{\widetilde{Q}\text{lifting}Q}{=} \left({}_{\mathbb{B}}U\widetilde{Q}\lambda_{A}Z\right) \circ \left({}_{\mathbb{B}}U\lambda_{B}\widetilde{Q}_{\mathbb{A}}F_{\mathbb{A}}UZ\right) \circ \left({}_{\mathbb{B}}U_{\mathbb{B}}FQu_{A\mathbb{A}}UZ\right) \\ &\overset{\lambda_{B}}{=} \left({}_{\mathbb{B}}U\lambda_{B}\widetilde{Q}Z\right) \circ \left({}_{\mathbb{B}}U_{\mathbb{B}}F_{\mathbb{B}}U\widetilde{Q}\lambda_{A}Z\right) \circ \left({}_{\mathbb{B}}U_{\mathbb{B}}FQu_{A\mathbb{A}}UZ\right) \\ &= \left({}_{\mathbb{B}}U\lambda_{B}\widetilde{Q}Z\right) \circ \left({}_{\mathbb{B}}U_{\mathbb{B}}F_{\mathbb{B}}\left[Q_{\mathbb{A}}U\lambda_{A}Z \circ Qu_{A\mathbb{A}}UZ\right]\right) \\ &\overset{(u_{A},\lambda_{A})\text{adj}}{=} {}_{\mathbb{B}}U\lambda_{B}\widetilde{Q}Z. \end{aligned}$$

Conversely, let us start with a functorial morphism $\Phi : BQ \to QA$ satisfying $\Phi \circ (m_B Q) = (Qm_A) \circ (\Phi A) \circ (B\Phi)$ and $\Phi \circ (u_B Q) = Qu_A$. Then we construct a functor $\widetilde{Q} : {}_{\mathbb{A}}\mathcal{A} \to {}_{\mathbb{B}}\mathcal{B}$ by setting, for every $\left(X, {}^A\mu_X\right) \in {}_{\mathbb{A}}\mathcal{A}$,

$$\widetilde{Q}\left(\left(X, {}^A\mu_X\right)\right) = \left(QX, \left(Q^A\mu_X\right) \circ (\Phi X)\right)$$

which lifts $Q : \mathcal{A} \to \mathcal{B}$. Now, we define a functorial morphism $\Psi : BQ \to QA$ given by

$$\Psi = \left({}_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F\right) \circ \left({}_{\mathbb{B}}U_{\mathbb{B}}FQu_A\right).$$

Then we have

$$\begin{aligned} \Psi &= \left({}_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F\right) \circ \left({}_{\mathbb{B}}U_{\mathbb{B}}FQu_A\right) \\ &\overset{\mathrm{def}\widetilde{Q}}{=} \left(Q_{\mathbb{A}}U\lambda_{A\mathbb{A}}F\right) \circ \left(\Phi_{\mathbb{A}}F\right) \circ \left({}_{\mathbb{B}}U_{\mathbb{B}}FQu_A\right) \\ &= (Qm_A) \circ (\Phi A) \circ (BQu_A) \\ &\overset{\Phi}{=} (Qm_A) \circ (QAu_A) \circ \Phi \\ &\overset{A\mathrm{monad}}{=} \Phi. \end{aligned}$$

□

Corollary 6.46. *Let $\mathcal{X}, \mathcal{A}$ be categories, let $\mathbb{A} = (A, m_A, u_A)$ be a monad on a category $\mathcal{A}$ and let $F : \mathcal{X} \to \mathcal{A}$ be a functor. Then there exists a bijective correspondence between the following collections of data:*

- $\mathcal{H}$ *Left $\mathbb{A}$-module actions ${}^A\mu_F : AF \to F$*
- $\mathcal{G}$ *Functors ${}_AF : \mathcal{X} \to {}_{\mathbb{A}}\mathcal{A}$ such that ${}_{\mathbb{A}}U_AF = F$,*

given by

$$\begin{gathered} \widetilde{a} : \mathcal{H} \to \mathcal{G} \text{ where } {}_{\mathbb{A}}U\widetilde{a}\left({}^A\mu_F\right) = F \text{ and } {}_{\mathbb{A}}U\lambda_A\widetilde{a}\left({}^A\mu_F\right) = {}^A\mu_F \text{ i.e.} \\ \widetilde{a}\left({}^A\mu_F\right)(X) = \left(FX, {}^A\mu_F X\right) \text{ and } \widetilde{a}\left({}^A\mu_F\right)(f) = F(f) \\ \widetilde{b} : \mathcal{G} \to \mathcal{H} \text{ where } \widetilde{b}\left({}_AF\right) = {}_{\mathbb{A}}U\lambda_{AA}F : AF \to F. \end{gathered}$$

Proof. Apply Proposition 6.45 to the case $\mathcal{A} = \mathcal{X}, \mathcal{B} = \mathcal{A}$, $\mathbb{A} = \mathrm{Id}_{\mathcal{X}}$ and $\mathbb{B} = \mathbb{A}$. Then $\widetilde{Q} = {}_AF$ is the lifting of F and $\Phi = {}^A\mu_F$ satisfies ${}^A\mu_F \circ (m_AF) = {}^A\mu_F \circ \left(A^A\mu_F\right)$ and ${}^A\mu_F \circ (u_AF) = F$ that is $\left(F, {}^A\mu_F\right)$ is a left $\mathbb{A}$-module functor. □

Corollary 6.47. *Let (L, R) be an adjunction with $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$ and let $\mathbb{A} = (A, m_A, u_A)$ be a monad on $\mathcal{B}$. Then there is a bijective correspondence between the following collections of data*

- $\mathfrak{K}$ *Functors $K : \mathcal{A} \to {}_{\mathbb{A}}\mathcal{B}$ such that ${}_{\mathbb{A}}U \circ K = R$,*

$\mathfrak{L}$ *functorial morphism* $\alpha : AR \to R$ *such that* (R, α) *is a left module functor for the monad* $\mathbb{A}$

given by

$$\begin{aligned} \Phi &: \mathfrak{K} \to \mathfrak{L} \text{ where } \Phi(K) = {}_{\mathbb{A}}U\lambda_A K : AR \to R \\ \Omega &: \mathfrak{L} \to \mathfrak{K} \text{ where } \Omega(\alpha)(X) = (RX, \alpha X) \text{ and } {}_{\mathbb{A}}U\Omega(\alpha)(f) = R(f). \end{aligned}$$

Proof. Apply Corollary 6.46 to the case "F" $= R : \mathcal{A} \to \mathcal{B}$ where (L, R) is an adjunction with $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$ and $\mathbb{A} = (A, m_A, u_A)$ a monad on $\mathcal{B}$. $\square$

6.5 The comparison functor for monads

The dual version, for comonads, of this subsection can be found in [GT].

Proposition 6.48. *Let* (L, R) *be an adjunction where* $L : \mathcal{B} \to \mathcal{A}$ *and* $R : \mathcal{A} \to \mathcal{B}$ *with unit* η *and counit* ϵ *and let* $\mathbb{A} = (A, m_A, u_A)$ *be a monad on the category* $\mathcal{B}$. *There exists a bijective correspondence between the following collections of data:*

$\mathfrak{M}$ *monad morphisms* $\psi : \mathbb{A} = (A, m_A, u_A) \to \mathbb{RL} = (RL, R\epsilon L, \eta)$

$\mathfrak{R}$ *functorial morphism* $r : LA \to L$ *such that* (L, r) *is a right module functor for the monad* $\mathbb{A}$

$\mathfrak{L}$ *functorial morphism* $l : AR \to R$ *such that* (R, l) *is a left module functor for the monad* $\mathbb{A}$

given by

$$\begin{aligned} \Theta &: \mathfrak{M} \to \mathfrak{R} \text{ where } \Theta(\psi) = (\epsilon L) \circ (L\psi) \\ \Xi &: \mathfrak{R} \to \mathfrak{M} \text{ where } \Xi(r) = (Rr) \circ (\eta A) \\ \Gamma &: \mathfrak{M} \to \mathfrak{L} \text{ where } \Gamma(\psi) = (R\epsilon) \circ (\psi R) \\ \Lambda &: \mathfrak{L} \to \mathfrak{M} \text{ where } \Lambda(l) = (lL) \circ (A\eta). \end{aligned}$$

Proof. For a given $\psi \in \mathfrak{M}$, we compute

$$\begin{aligned} \Theta(\psi) \circ (\Theta(\psi) A) &= (\epsilon L) \circ (L\psi) \circ (\epsilon LA) \circ (L\psi A) \\ &\overset{\epsilon}{=} (\epsilon L) \circ (\epsilon LRL) \circ (LRL\psi) \circ (L\psi A) \\ &\overset{\epsilon,\psi}{=} (\epsilon L) \circ (LR\epsilon L) \circ (L\psi\psi) \overset{\psi\text{morphmon}}{=} (\epsilon L) \circ (L\psi) \circ (Lm_A) = \Theta(\psi) \circ (Lm_A) \end{aligned}$$

and

$$\Theta(\psi) \circ (Lu_A) = (\epsilon L) \circ (L\psi) \circ (Lu_A) \overset{\psi\text{morphmon}}{=} (\epsilon L) \circ (L\eta) = L.$$

Therefore we deduce that $\Theta(\psi) \in \mathfrak{R}$. For a given $r \in \mathfrak{R}$, we compute

$$\begin{aligned}
(R\epsilon L) \circ (\Xi(r)\Xi(r)) &\overset{\Xi(r)}{=} (R\epsilon L) \circ (RL\Xi(r)) \circ (\Xi(r)A) \\
&= (R\epsilon L) \circ (RLRr) \circ (RL\eta A) \circ (RrA) \circ (\eta AA) \\
&\overset{\epsilon}{=} (Rr) \circ (R\epsilon LA) \circ (RL\eta A) \circ (RrA) \circ (\eta AA) \\
&\overset{(L,R)}{=} (Rr) \circ (RrA) \circ (\eta AA) \overset{(L,r)}{=} (Rr) \circ (RLm_A) \circ (\eta AA) \\
&\overset{\eta}{=} (Rr) \circ (\eta A) \circ m_A = \Xi(r) \circ m_A
\end{aligned}$$

and

$$\Xi(r) \circ u_A = (Rr) \circ (\eta A) \circ u_A \overset{\eta}{=} (Rr) \circ (RLu_A) \circ \eta \overset{(L,r)}{=} \eta.$$

Therefore we deduce that $\Xi(r) \in \mathfrak{M}$. For a given $\psi \in \mathfrak{M}$, we compute

$$\begin{aligned}
\Gamma(\psi) \circ [A\Gamma(\psi)] &= (R\epsilon) \circ (\psi R) \circ (AR\epsilon) \circ (A\psi R) \\
\overset{\psi}{=} (R\epsilon) \circ (RLR\epsilon) \circ (\psi RLR) \circ (A\psi R) &\overset{\epsilon,\psi}{=} (R\epsilon) \circ (R\epsilon LR) \circ (\psi\psi R) \\
&\overset{\psi\text{morphmon}}{=} (R\epsilon) \circ (\psi R) \circ (m_A R) = \Gamma(\psi) \circ (m_A R)
\end{aligned}$$

and

$$\Gamma(\psi) \circ (u_A R) = (R\epsilon) \circ (\psi R) \circ (u_A R) \overset{\psi\text{morphmon}}{=} (R\epsilon) \circ (\eta R) = R.$$

Therefore we deduce that $\Gamma(\psi) \in \mathfrak{L}$. For a given $l \in \mathfrak{L}$, we compute

$$\begin{aligned}
(R\epsilon L) \circ (\Lambda(l)\Lambda(l)) &\overset{\Lambda(l)}{=} (R\epsilon L) \circ (\Lambda(l)RL) \circ (A\Lambda(l)) \\
&= (R\epsilon L) \circ (lLRL) \circ (A\eta RL) \circ (AlL) \circ (AA\eta) \\
&\overset{l}{=} (lL) \circ (AR\epsilon L) \circ (A\eta RL) \circ (AlL) \circ (AA\eta) \\
&\overset{(L,R)}{=} (lL) \circ (AlL) \circ (AA\eta) \overset{(R,l)}{=} (lL) \circ (m_A RL) \circ (AA\eta) \\
&\overset{m_A}{=} (lL) \circ (A\eta) \circ m_A = \Lambda(l) \circ m_A
\end{aligned}$$

and

$$\Lambda(l) \circ u_A = (lL) \circ (A\eta) \circ u_A \overset{u_A}{=} (lL) \circ (u_A RL) \circ \eta \overset{(R,l)}{=} \eta.$$

Therefore we deduce that $\Lambda(l) \in \mathfrak{M}$. Let now $\psi \in \mathfrak{M}$ and let us calculate

$$\Xi\Theta(\psi) = (R\epsilon L) \circ (RL\psi) \circ (\eta A) \overset{\eta}{=} (R\epsilon L) \circ (\eta RL) \circ \psi = \psi.$$

Let now $r \in \mathfrak{R}$ and let us calculate

$$\Theta\Xi(r) = (\epsilon L) \circ (LRr) \circ (L\eta A) \overset{\epsilon}{=} r \circ (\epsilon LA) \circ (L\eta A) = r.$$

Let now $\psi \in \mathfrak{M}$ and let us calculate

$$\Lambda\Gamma(\psi) = (R\epsilon L) \circ (\psi RL) \circ (A\eta) \overset{\psi}{=} (R\epsilon L) \circ (RL\eta) \circ \psi = \psi.$$

Let now $l \in \mathfrak{L}$ and let us calculate

$$\Gamma\Lambda(l) = (R\epsilon) \circ (lLR) \circ (A\eta R) \overset{l}{=} l \circ (AR\epsilon) \circ (A\eta R) = l.$$

□

Theorem 6.49. *Let (L, R) be an adjunction where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$ and let $\mathbb{A} = (A, m_A, u_A)$ be a monad on the category $\mathcal{B}$. There exists a bijective correspondence between the following collections of data:*

$\mathfrak{K}$ *Functors $K : \mathcal{A} \to {}_{\mathbb{A}}\mathcal{B}$ such that ${}_{\mathbb{A}}U \circ K = R$*

$\mathfrak{M}$ *monad morphisms $\psi : \mathbb{A} = (A, m_A, u_A) \to \mathbb{RL} = (RL, R\epsilon L, \eta)$*

given by

$$\begin{aligned} \Psi &: \mathfrak{K} \to \mathfrak{M} \text{ where } \Psi(K) = ([{}_{\mathbb{A}}U\lambda_A K]\, L) \circ (A\eta) \\ \Upsilon &: \mathfrak{M} \to \mathfrak{K} \text{ where } \Upsilon(\psi)(X) = (RX, (R\epsilon X) \circ (\psi RX)) \text{ and } \Upsilon(\psi)(f) = Rf. \end{aligned}$$

Proof. By Corollary 6.47, there exists a bijective correspondence between $\mathfrak{K}$ and the collection $\mathfrak{L}$ of functorial morphisms $\alpha : AR \to R$ such that (R, α) is a left module functor for the monad $\mathbb{A}$ given by

$$\begin{aligned} \Phi &: \mathfrak{K} \to \mathfrak{L} \text{ where } \Phi(K) = {}_{\mathbb{A}}U\lambda_A K : AR \to R \\ \Omega &: \mathfrak{L} \to \mathfrak{K} \text{ where } \Omega(\alpha)(X) = (RX, \alpha X) \text{ and } {}_{\mathbb{A}}U\Omega(\alpha)(f) = Rf. \end{aligned}$$

By Proposition 6.48, there exists a bijective correspondence between $\mathfrak{L}$ and the collection $\mathfrak{M}$ of monad morphisms $\psi : \mathbb{A} = (A, m_A, u_A) \to \mathbb{RL} = (RL, R\epsilon L, \eta)$ given by

$$\begin{aligned} \Lambda &: \mathfrak{L} \to \mathfrak{M} \text{ where } \Lambda(l) = (lL) \circ (A\eta) \\ \Gamma &: \mathfrak{M} \to \mathfrak{L} \text{ where } \Gamma(\psi) = (R\epsilon) \circ (\psi R). \end{aligned}$$

We compute

$$(\Lambda \circ \Phi)(K) = ({}_{\mathbb{A}}U\lambda_A KL) \circ (A\eta) = \Psi(K)$$

and

$$\begin{aligned} [(\Omega \circ \Gamma)(\psi)](X) &= (RX, (R\epsilon X) \circ (\psi RX)) = \Upsilon(\psi)(Y) \\ [(\Omega \circ \Gamma)(\psi)](f) &= Rf = \Upsilon(\psi)(f). \end{aligned}$$

□

Remark 6.50. *When $\mathbb{A} = \mathbb{RL} = (RL, R\epsilon L, \eta)$ and $\psi = \mathrm{Id}_{\mathbb{RL}}$ the functor $K = \Upsilon(\psi) : \mathcal{A} \to {}_{\mathbb{RL}}\mathcal{B}$ such that ${}_{\mathbb{RL}}U \circ K = R$ is called the* Eilenberg-Moore comparison functor.

Corollary 6.51. *Let $\mathbb{A} = (A, m_A, u_A)$ and $\mathbb{B} = (B, m_B, u_B)$ be monads on a category $\mathcal{B}$. There exists a bijective correspondence between the following collections of data:*

$\mathcal{K}$ *Functors $K : {}_{\mathbb{A}}\mathcal{B} \to {}_{\mathbb{B}}\mathcal{B}$ such that ${}_{\mathbb{B}}U \circ K = {}_{\mathbb{A}}U$,*

$\mathcal{M}$ *monad morphisms $\psi : \mathbb{A} \to \mathbb{B}$*

given by

$$\begin{aligned}\Psi &: \mathcal{K} \to \mathcal{M} \text{ where } \Psi(K) = ([{}_{\mathbb{A}}U\lambda_A K]\,{}_{\mathbb{A}}F) \circ (Au_A)\\ \Upsilon &: \mathcal{M} \to \mathcal{K} \text{ where } \Upsilon(\psi)(X) = ({}_{\mathbb{A}}UX, ({}_{\mathbb{A}}U\lambda_A X) \circ (\psi\,{}_{\mathbb{A}}UX)) \text{ and } \Upsilon(\psi)(f) = {}_{\mathbb{A}}U(f).\end{aligned}$$

Proof. Apply Theorem 6.49 to the case "R" $= {}_{\mathbb{A}}U : {}_{\mathbb{A}}\mathcal{B} \to \mathcal{B}$ and "L" $= {}_{\mathbb{A}}F : \mathcal{B} \to {}_{\mathbb{A}}\mathcal{B}$ and note that $(RL, R\epsilon L, \eta) = ({}_{\mathbb{A}}U\,{}_{\mathbb{A}}F, {}_{\mathbb{A}}U\lambda_{A\mathbb{A}}F, u_A) = (A, m_A, u_A)$. □

Proposition 6.52. *Let (L, R) be an adjunction where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$, let $\mathbb{A} = (A, m_A, u_A)$ be a monad on the category $\mathcal{B}$ and let $\psi : \mathbb{A} = (A, m_A, u_A) \to \mathbb{RL} = (RL, R\epsilon L, \eta)$ be a monad morphism. Let $r = \Theta(\psi) = (\epsilon L) \circ (L\psi)$ and $l = \Gamma(\psi) = (R\epsilon) \circ (\psi R)$. Then the isomorphism $a_{X,Y} : \mathrm{Hom}_{\mathcal{A}}(LY, X) \to \mathrm{Hom}_{\mathcal{B}}(Y, RX)$ of the adjunction (L, R) induces an isomorphism*

$$\widetilde{a}_{-,Y} : \mathrm{Equ}_{\mathrm{Hom}_{\mathcal{A}}(LY,-)}\left(\mathrm{Hom}_{\mathcal{A}}(rY, -), \mathrm{Hom}_{\mathcal{A}}\left(L^A\mu_Y, -\right)\right) \to \mathrm{Hom}_{{}_{\mathbb{A}}\mathcal{B}}\left(\left(Y, {}^A\mu_Y\right), K_\psi -\right)$$

for every $\left(Y, {}^A\mu_Y\right) \in {}_{\mathbb{A}}\mathcal{B}$.

Proof. Let

$$a_{X,Y} : \mathrm{Hom}_{\mathcal{A}}(LY, X) \to \mathrm{Hom}_{\mathcal{B}}(Y, RX)$$

be the isomorphism of the adjunction (L, R), for every $Y \in \mathcal{B}$ and for every $X \in \mathcal{A}$. Recall that $a_{X,Y}(\xi) = (R\xi) \circ (\eta Y)$ and $a^{-1}_{X,Y}(\zeta) = (\epsilon X) \circ (L\zeta)$.

Let us check that we can apply Lemma 5.35 to the case $Z = \mathrm{Hom}_{\mathcal{A}}(LY, -)$, $Z' = \mathrm{Hom}_{\mathcal{B}}(Y, R-)$, $W = \mathrm{Hom}_{\mathcal{A}}(LAY, -)$, $W' = \mathrm{Hom}_{\mathcal{B}}(AY, R-)$, $a = \mathrm{Hom}_{\mathcal{A}}(rY, -)$, $b = \mathrm{Hom}_{\mathcal{A}}\left(L^A\mu_Y, -\right)$, $a' = (\Gamma(\psi) -) \circ (A-)$, $b' = \mathrm{Hom}_{\mathcal{B}}\left({}^A\mu_Y, R-\right)$, $E = \mathrm{Equ}_{\mathrm{Fun}}\left(\mathrm{Hom}_{\mathcal{A}}(rY, -), \mathrm{Hom}_{\mathcal{A}}\right.$ and
$E' = \mathrm{Equ}_{\mathrm{Fun}}\left((\Gamma(\psi) -) \circ (A-), - \circ {}^A\mu_Y\right)$ and $\varphi = a_{-,Y}$, $\psi = a_{-,AY}$, .

$$\begin{array}{ccc}
E = \mathrm{Equ}_{\mathrm{Fun}}\left(\mathrm{Hom}_{\mathcal{A}}(rY, -), \mathrm{Hom}_{\mathcal{A}}\left(L^A\mu_Y, -\right)\right) & \overset{\tilde{a}_{-,Y}}{\dashrightarrow} & E' = \mathrm{Equ}_{\mathrm{Fun}}\left((\Gamma(\psi) -) \circ (A-), - \circ {}^A\mu_Y\right)\\
\downarrow i & & \downarrow i'\\
Z = \mathrm{Hom}_{\mathcal{A}}(LY, -) & \xrightarrow{a_{-,Y}} & Z' = \mathrm{Hom}_{\mathcal{B}}(Y, R-)\\
a = \mathrm{Hom}_{\mathcal{A}}(rY, -) \downdownarrows b = \mathrm{Hom}_{\mathcal{A}}\left(L^A\mu_Y, -\right) & & a' = (\Gamma(\psi) -) \circ (A-) \downdownarrows b' = \mathrm{Hom}_{\mathcal{B}}\left({}^A\mu_Y, R-\right)\\
W = \mathrm{Hom}_{\mathcal{A}}(LAY, -) & \xrightarrow{a_{-,AY}} & W' = \mathrm{Hom}_{\mathcal{B}}(AY, R-)
\end{array}$$

$$\begin{array}{ccc}
E = \mathrm{Equ}_{\mathrm{Fun}}\left(\mathrm{Hom}_{\mathcal{A}}(rY, -), \mathrm{Hom}_{\mathcal{A}}\left(L^A\mu_Y, -\right)\right) & \overset{\widetilde{a}_{-,Y}}{\dashrightarrow} & E' = \mathrm{Equ}_{\mathrm{Fun}}\left((\Gamma(\psi) -) \circ (A-), - \circ {}^A\mu_Y\right)\\
\downarrow i & & \downarrow i'\\
Z = \mathrm{Hom}_{\mathcal{A}}(LY, -) & \overset{a_{-,Y}}{\longrightarrow} & Z' = \mathrm{Hom}_{\mathcal{B}}(Y, R-)\\
a = \mathrm{Hom}_{\mathcal{A}}(rY, -) \downdownarrows b = \mathrm{Hom}_{\mathcal{A}}\left(L^A\mu_Y, -\right) & & a' = (\Gamma(\psi) -) \circ (A-) \downdownarrows b' = \mathrm{Hom}_{\mathcal{B}}\left({}^A\mu_Y, R-\right)\\
W = \mathrm{Hom}_{\mathcal{A}}(LAY, -) & \overset{a_{-,AY}}{\longrightarrow} & W' = \mathrm{Hom}_{\mathcal{B}}(AY, R-)
\end{array}$$

For every $\left(Y, {}^A\mu_Y\right) \in {}_{\mathbb{A}}\mathcal{B}$, $X \in \mathcal{A}$ and for every $\xi \in \mathrm{Hom}_{\mathcal{A}}(LY, X)$, since $l = \Gamma(\psi) = (R\epsilon) \circ (\psi R)$ and $r = \Theta(\psi) = (\epsilon L) \circ (L\psi)$, we have

$$\begin{aligned}
&([(\Gamma(\psi)X) \circ (A-)] \circ a_{X,Y})(\xi) \overset{\text{defa}}{=} [(\Gamma(\psi)X) \circ (A-)]((R\xi) \circ (\eta Y)) \\
&= (\Gamma(\psi)X) \circ (AR\xi) \circ (A\eta Y) = (R\epsilon X) \circ (\psi RX) \circ (AR\xi) \circ (A\eta Y) \overset{\psi}{=} \\
&= (R\epsilon X) \circ (RLR\xi) \circ (RL\eta Y) \circ (\psi Y) \overset{\epsilon}{=} (R\xi) \circ (R\epsilon LY) \circ (RL\eta Y) \circ (\psi Y) \\
&= (R\xi) \circ (\psi Y) = (R\xi) \circ (R\epsilon LY) \circ (\eta RLY) \circ (\psi Y) \overset{\eta}{=} (R\xi) \circ (R\epsilon LY) \circ (RL\psi Y) \circ (\eta AY) \\
&\overset{\text{defr}}{=} (R\xi) \circ (RrY) \circ (\eta AY) = a_{X,AY}(\xi \circ rY) = [a_{X,AY} \circ \mathrm{Hom}_{\mathcal{A}}(rY, X)](\xi)
\end{aligned}$$

so that we obtain

$$[(\Gamma(\psi)X) \circ (A-)] \circ a_{X,Y} = a_{X,AY} \circ \mathrm{Hom}_{\mathcal{A}}(rY, X).$$

Now, let us compute

$$\begin{aligned}
\left[\mathrm{Hom}_{\mathcal{B}}\left({}^A\mu_Y, RX\right) \circ a_{X,Y}\right](\xi) &\overset{\text{defa}}{=} \mathrm{Hom}_{\mathcal{B}}\left({}^A\mu_Y, RX\right)((R\xi) \circ (\eta Y)) = \\
&= (R\xi) \circ (\eta Y) \circ {}^A\mu_Y
\end{aligned}$$

and on the other hand

$$\begin{aligned}
\left(a_{X,AY} \circ \mathrm{Hom}_{\mathcal{A}}\left(L^A\mu_Y, X\right)\right)(\xi) = a_{X,AY}\left(\xi \circ \left(L^A\mu_Y\right)\right) &\overset{\text{defa}}{=} (R\xi) \circ \left(RL^A\mu_Y\right) \circ (\eta AY) \\
&\overset{\eta}{=} (R\xi) \circ (\eta Y) \circ {}^A\mu_Y
\end{aligned}$$

so that we get

$$\mathrm{Hom}_{\mathcal{B}}\left({}^A\mu_Y, RX\right) \circ a_{X,Y} = a_{X,AY} \circ \mathrm{Hom}_{\mathcal{A}}\left(L^A\mu_Y, X\right).$$

Since $K_\psi(X) = \Upsilon(\psi)(X) = (RX, (R\epsilon X) \circ (\psi RX))$, for every $\zeta \in \mathrm{Hom}_{\mathcal{B}}(Y, RX)$ we have

$$[(\Gamma(\psi)X) \circ (A-)](\zeta) = (\Gamma(\psi)X) \circ (A\zeta) = (R\epsilon X) \circ (\psi RX) \circ (A\zeta) = {}^A\mu_{RX} \circ (A\zeta)$$

and

$$\mathrm{Hom}_{\mathcal{B}}\left({}^A\mu_Y, RX\right)(\zeta) = \zeta \circ {}^A\mu_Y$$

so that

$$\begin{aligned}
[(\Gamma(\psi)X) \circ (A-)](\zeta) = \mathrm{Hom}_{\mathcal{B}}\left({}^A\mu_Y, RX\right)(\zeta) \text{ if and only if} \\
\zeta \in \mathrm{Hom}_{{}_{\mathbb{A}}\mathcal{B}}\left(\left(Y, {}^A\mu_Y\right), (RX, (R\epsilon X) \circ (\psi RX))\right).
\end{aligned}$$

Thus we get

$$\begin{aligned}
&\mathrm{Equ}_{\mathrm{Hom}_{\mathcal{B}}(Y,RX)}\left((\Gamma(\psi)X) \circ (A-), - \circ {}^A\mu_Y\right) \\
&= \left\{f \in \mathrm{Hom}_{\mathcal{B}}(Y, RX) \mid (\Gamma(\psi)X) \circ (A\zeta) = \zeta \circ {}^A\mu_Y\right\} \\
&= \left\{f \in \mathrm{Hom}_{\mathcal{B}}(Y, RX) \mid (R\epsilon X) \circ (\psi RX) \circ (A\zeta) = \zeta \circ {}^A\mu_Y\right\} \\
&= \left\{f \in \mathrm{Hom}_{\mathcal{B}}\left({}_{\mathbb{A}}U\left(Y, {}^A\mu_Y\right), {}_{\mathbb{A}}U(K_\psi X)\right) \mid {}^A\mu_{{}_{\mathbb{A}}U(K_\psi X)} \circ (A\zeta) = \zeta \circ {}^A\mu_Y\right\} \\
&\mathrm{Hom}_{{}_{\mathbb{A}}\mathcal{B}}\left(\left(Y, {}^A\mu_Y\right), K_\psi X\right)
\end{aligned}$$

so that $\mathrm{Equ}_{\mathrm{Fun}}\left(\left(\Gamma\left(\psi\right)-\right)\circ\left(A-\right),\mathrm{Hom}_{\mathcal{B}}\left({}^{A}\mu_{Y},R-\right)\right) = \mathrm{Hom}_{{}_{\mathbb{A}}\mathcal{B}}\left(\left(Y,{}^{A}\mu_{Y}\right),K_{\psi}-\right)$. □

Proposition 6.53. *Let (L,R) be an adjunction where $L:\mathcal{B}\to\mathcal{A}$ and $R:\mathcal{A}\to\mathcal{B}$, let $\mathbb{A}=(A,m_A,u_A)$ be a monad on the category $\mathcal{B}$ and let $\psi:\mathbb{A}=(A,m_A,u_A)\to\mathbb{RL}=(RL,R\epsilon L,\eta)$ be a monad morphism. Let $r=\Theta(\psi)=(\epsilon L)\circ(L\psi)$. Then the functor $K_{\psi}=\Upsilon(\psi):\mathcal{A}\to{}_{\mathbb{A}}\mathcal{B}$ has a left adjoint $D_{\psi}:{}_{\mathbb{A}}\mathcal{B}\to\mathcal{A}$ if and only, for every $\left(Y,{}^{A}\mu_{Y}\right)\in{}_{\mathbb{A}}\mathcal{B}$, there exists* $\mathrm{Coequ}_{\mathcal{A}}\left(rY,L^{A}\mu_{Y}\right)$. *In this case, there exists a functorial morphism $d_{\psi}:L_{\mathbb{A}}U\to D_{\psi}$ such that*

$$(D_{\psi},d_{\psi})=\mathrm{Coequ}_{Fun}\left(r_{\mathbb{A}}U,L_{\mathbb{A}}U\lambda_{A}\right)$$

and thus

$$\left[D_{\psi}\left(\left(Y,{}^{A}\mu_{Y}\right)\right),d_{\psi}\left(Y,{}^{A}\mu_{Y}\right)\right]=\mathrm{Coequ}_{\mathcal{A}}\left(rY,L^{A}\mu_{Y}\right).$$

Proof. Assume first that, for every $\left(Y,{}^{A}\mu_{Y}\right)\in{}_{\mathbb{A}}\mathcal{B}$, there exists $\mathrm{Coequ}_{\mathcal{A}}\left(rY,L^{A}\mu_{Y}\right)$. By Proposition 6.52, the isomorphism $a_{X,Y}:\mathrm{Hom}_{\mathcal{A}}(LY,X)\to\mathrm{Hom}_{\mathcal{B}}(Y,RX)$ of the adjunction (L,R) induces an isomorphism

$$\widetilde{a}_{X,Y}:\mathrm{Equ}_{\mathrm{Sets}}\left(\mathrm{Hom}_{\mathcal{A}}(rY,X),\mathrm{Hom}_{\mathcal{A}}\left(L^{A}\mu_{Y},X\right)\right)\to\mathrm{Hom}_{{}_{\mathbb{A}}\mathcal{B}}\left(\left(Y,{}^{A}\mu_{Y}\right),K_{\psi}X\right).$$

Let $\left(D_{\psi}\left(\left(Y,{}^{A}\mu_{Y}\right)\right),d_{\psi}\left(Y,{}^{A}\mu_{Y}\right)\right)$ denote the coequalizer

$$LAY\underset{L^{A}\mu_{Y}}{\overset{rY}{\rightrightarrows}}LY\xrightarrow{d_{\psi}\left(Y,{}^{A}\mu_{Y}\right)}D_{\psi}\left(Y,{}^{A}\mu_{Y}\right)$$

where $d_{\psi}\left(Y,{}^{A}\mu_{Y}\right):LY\to D_{\psi}\left(\left(Y,{}^{A}\mu_{Y}\right)\right)$ is the canonical projection. Then, by Lemma 5.38, we have

$$\begin{aligned}&\left(\mathrm{Hom}_{\mathcal{A}}\left(D_{\psi}\left(\left(Y,{}^{A}\mu_{Y}\right)\right),X\right),\mathrm{Hom}_{\mathcal{A}}\left(d_{\psi}\left(\left(Y,{}^{A}\mu_{Y}\right)\right),X\right)\right)\\&=\mathrm{Equ}_{\mathrm{Sets}}\left(\mathrm{Hom}_{\mathcal{A}}(rY,X),\mathrm{Hom}_{\mathcal{A}}\left(L^{A}\mu_{Y},X\right)\right).\end{aligned}$$

Thus, for every $\left(Y,{}^{A}\mu_{Y}\right)\in{}_{\mathbb{A}}\mathcal{B}$ and for every $X\in\mathcal{A}$, $a_{X,Y}$ induces an isomorphism $\widetilde{a}_{X,Y}:\mathrm{Hom}_{\mathcal{A}}\left(D_{\psi}\left(\left(Y,{}^{A}\mu_{Y}\right)\right),X\right)\to\mathrm{Hom}_{{}_{\mathbb{A}}\mathcal{B}}\left(\left(Y,{}^{A}\mu_{Y}\right),K_{\psi}X\right)$ such that the following diagram is commutative

(6.10)
$$\begin{array}{ccc}\mathrm{Hom}_{\mathcal{A}}\left(D_{\psi}\left(\left(Y,{}^{A}\mu_{Y}\right)\right),X\right) & \overset{\tilde{a}_{X,Y}}{\dashrightarrow} & \mathrm{Hom}_{{}_{\mathbb{A}}\mathcal{B}}\left(\left(Y,{}^{A}\mu_{Y}\right),K_{\psi}X\right)\\ {\scriptstyle \mathrm{Hom}_{\mathcal{A}}\left(d_{\psi}\left(\left(Y,{}^{A}\mu_{Y}\right)\right),X\right)}\downarrow & & \downarrow\\ \mathrm{Hom}_{\mathcal{A}}(LY,X) & \xrightarrow{a_{X,Y}} & \mathrm{Hom}_{\mathcal{B}}(Y,RX)\\ {\scriptstyle \mathrm{Hom}_{\mathcal{A}}(rY,X)}\downarrow\downarrow{\scriptstyle \mathrm{Hom}_{\mathcal{A}}\left(L^{A}\mu_{Y},X\right)} & & {\scriptstyle (\Gamma(\psi)X)\circ(A-)}\downarrow\downarrow{\scriptstyle \mathrm{Hom}_{\mathcal{B}}\left({}^{A}\mu_{Y},RX\right)}\\ \mathrm{Hom}_{\mathcal{A}}(LAY,X) & \xrightarrow{a_{X,AY}} & \mathrm{Hom}_{\mathcal{B}}(AY,RX)\end{array}$$

$$\begin{array}{ccc}\mathrm{Hom}_{\mathcal{A}}\left(D_{\psi}\left(\left(Y,{}^{A}\mu_{Y}\right)\right),X\right) & \overset{\widetilde{a}_{X,Y}}{\dashrightarrow} & \mathrm{Hom}_{{}_{\mathbb{A}}\mathcal{B}}\left(\left(Y,{}^{A}\mu_{Y}\right),K_{\psi}X\right)\\ \mathrm{Hom}_{\mathcal{A}}\left(d_{\psi}\left(\left(Y,{}^{A}\mu_{Y}\right)\right),X\right)\downarrow & & \downarrow\\ \mathrm{Hom}_{\mathcal{A}}(LY,X) & \overset{a_{X,Y}}{\longrightarrow} & \mathrm{Hom}_{\mathcal{B}}(Y,RX)\\ \downarrow\downarrow & & \downarrow\downarrow\\ \mathrm{Hom}_{\mathcal{A}}(LAY,X) & \overset{a_{X,AY}}{\longrightarrow} & \mathrm{Hom}_{\mathcal{B}}(AY,RX)\end{array}$$

i.e. (D_ψ, K_ψ) is an adjunction.
Conversely, assume now that the functor $K_\psi = \Upsilon(\psi) : \mathcal{A} \to {}_{\mathbb{A}}\mathcal{B}$ has a left adjoint $D_\psi : {}_{\mathbb{A}}\mathcal{B} \to \mathcal{A}$. Let $\widetilde{\eta} : \mathrm{Id}_{{}_{\mathbb{A}}\mathcal{B}} \to K_\psi D_\psi$ be the unit of the adjunction (D_ψ, K_ψ) and let

$$d_\psi = a^{-1}_{D_\psi, {}_{\mathbb{A}}U}\left({}_{\mathbb{A}}U\widetilde{\eta}\right) = (\epsilon D_\psi) \circ (L_{\mathbb{A}}U\widetilde{\eta}) : L_{\mathbb{A}}U \to D_\psi.$$

We will prove that

$$(D_\psi, d_\psi) = \mathrm{Coequ}_{Fun}\left(r_{\mathbb{A}}U, L_{\mathbb{A}}U\lambda_A\right).$$

First of all let us compute

$$\begin{aligned} d_\psi \circ (r_{\mathbb{A}}U) &= d_\psi \circ (\epsilon L_{\mathbb{A}}U) \circ (L\psi_{\mathbb{A}}U) = (\epsilon D_\psi) \circ (L_{\mathbb{A}}U\widetilde{\eta}) \circ (\epsilon L_{\mathbb{A}}U) \circ (L\psi_{\mathbb{A}}U) \\ &\overset{\epsilon}{=} (\epsilon D_\psi) \circ (LR\epsilon D_\psi) \circ (LRL_{\mathbb{A}}U\widetilde{\eta}) \circ (L\psi_{\mathbb{A}}U) \\ &\overset{\psi}{=} (\epsilon D_\psi) \circ (LR\epsilon D_\psi) \circ (L\psi_{\mathbb{A}}UK_\psi D_\psi) \circ (LA_{\mathbb{A}}U\widetilde{\eta}) \\ &= (\epsilon D_\psi) \circ (LR\epsilon D_\psi) \circ (L\psi RD_\psi) \circ (LA_{\mathbb{A}}U\widetilde{\eta}) \end{aligned}$$

and also

$$\begin{aligned} d_\psi \circ (L_{\mathbb{A}}U\lambda_A) &= (\epsilon D_\psi) \circ (L_{\mathbb{A}}U\widetilde{\eta}) \circ (L_{\mathbb{A}}U\lambda_A) \overset{\widetilde{\eta}\mathrm{morph}_{\mathbb{A}}\mathcal{B}}{=} (\epsilon D_\psi) \circ (L_{\mathbb{A}}U\lambda_A K_\psi D_\psi) \circ (LA_{\mathbb{A}}U\widetilde{\eta}) \\ &\overset{\mathrm{def}K_\psi}{=} (\epsilon D_\psi) \circ (LR\epsilon D_\psi) \circ (L\psi RD_\psi) \circ (LA_{\mathbb{A}}U\widetilde{\eta}) \end{aligned}$$

so that

$$d_\psi \circ (r_{\mathbb{A}}U) = d_\psi \circ (L_{\mathbb{A}}U\lambda_A).$$

Now, we will prove that the following diagram is commutative

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{A}}\left(D_\psi\left(\left(Y, {}^A\mu_Y\right)\right), X\right) & \overset{\widetilde{a}_{X,(Y,{}^A\mu_Y)}}{\dashrightarrow} & \mathrm{Hom}_{{}_{\mathbb{A}}\mathcal{B}}\left(\left(Y, {}^A\mu_Y\right), K_\psi X\right) \\ {\scriptstyle \mathrm{Hom}_{\mathcal{A}}\left(d_\psi\left(\left(Y,{}^A\mu_Y\right)\right),X\right)}\downarrow & & \downarrow{\scriptstyle {}_{\mathbb{A}}U} \\ \mathrm{Hom}_{\mathcal{A}}(LY, X) & \xrightarrow{a_{X,Y}} & \mathrm{Hom}_{\mathcal{B}}(Y, RX). \end{array}$$

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{A}}\left(D_\psi\left(\left(Y, {}^A\mu_Y\right)\right), X\right) & \overset{\widetilde{a}_{X,(Y,{}^A\mu_Y)}}{\dashrightarrow} & \mathrm{Hom}_{{}_{\mathbb{A}}\mathcal{B}}\left(\left(Y, {}^A\mu_Y\right), K_\psi X\right) \\ \mathrm{Hom}_{\mathcal{A}}\left(d_\psi\left(\left(Y, {}^A\mu_Y\right)\right), X\right)\downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{A}}(LY, X) & \overset{a_{X,Y}}{\longrightarrow} & \mathrm{Hom}_{\mathcal{B}}(Y, RX). \end{array}$$

In fact, for every $\zeta \in \mathrm{Hom}_{\mathcal{A}}\left(D_\psi\left(\left(Y, {}^A\mu_Y\right)\right), X\right)$, we have

$$\begin{aligned} {}_{\mathbb{A}}U\widetilde{a}_{X,(Y,{}^A\mu_Y)}(\zeta) &\overset{\mathrm{def}\widetilde{a}}{=} {}_{\mathbb{A}}U\left[(K_\psi\zeta) \circ \left(\widetilde{\eta}\left(Y, {}^A\mu_Y\right)\right)\right] = \left({}_{\mathbb{A}}UK_\psi\zeta\right) \circ \left({}_{\mathbb{A}}U\widetilde{\eta}\left(Y, {}^A\mu_Y\right)\right) \\ &\overset{\mathrm{def}K_\psi}{=} (R\zeta) \circ \left({}_{\mathbb{A}}U\widetilde{\eta}\left(Y, {}^A\mu_Y\right)\right) \end{aligned}$$

and on the other hand

$$\begin{aligned}
\left[a_{X,Y} \circ \mathrm{Hom}_{\mathcal{A}}\left(d_{\psi}\left(\left(Y, {}^{A}\mu_{Y}\right)\right), X\right)\right](\zeta) &= a_{X,Y}\left(\zeta \circ d_{\psi}\left(Y, {}^{A}\mu_{Y}\right)\right) \\
&\overset{\mathrm{def}d_{\psi}}{=} a_{X,Y}\left(\zeta \circ \left(\epsilon D_{\psi}\left(Y, {}^{A}\mu_{Y}\right)\right) \circ \left(L_{\mathbb{A}} U \widetilde{\eta}\left(Y, {}^{A}\mu_{Y}\right)\right)\right) \\
&\overset{\mathrm{def}a}{=} (R\zeta) \circ \left(R\epsilon D_{\psi}\left(Y, {}^{A}\mu_{Y}\right)\right) \circ \left(R L_{\mathbb{A}} U \widetilde{\eta}\left(Y, {}^{A}\mu_{Y}\right)\right) \circ (\eta Y) \\
&\overset{\eta}{=} (R\zeta) \circ \left(R\epsilon D_{\psi}\left(Y, {}^{A}\mu_{Y}\right)\right) \circ \left(\eta_{\mathbb{A}} U K_{\psi} D_{\psi}\left(Y, {}^{A}\mu_{Y}\right)\right) \circ \left({}_{\mathbb{A}} U \widetilde{\eta}\left(Y, {}^{A}\mu_{Y}\right)\right) \\
&\overset{\mathrm{def}K_{\psi}}{=} (R\zeta) \circ \left(R\epsilon D_{\psi}\left(Y, {}^{A}\mu_{Y}\right)\right) \circ \left(\eta R D_{\psi}\left(Y, {}^{A}\mu_{Y}\right)\right) \circ \left({}_{\mathbb{A}} U \widetilde{\eta}\left(Y, {}^{A}\mu_{Y}\right)\right) \\
&\overset{(L,R)}{=} (R\zeta) \circ \left({}_{\mathbb{A}} U \widetilde{\eta}\left(Y, {}^{A}\mu_{Y}\right)\right)
\end{aligned}$$

so that, for every $\left(Y, {}^{A}\mu_{Y}\right) \in {}_{\mathbb{A}}\mathcal{B}$ we have

$$\left[{}_{\mathbb{A}} U \circ \widetilde{a}_{-,\left(Y, {}^{A}\mu_{Y}\right)}\right] = \left[a_{-,Y} \circ \mathrm{Hom}_{\mathcal{A}}\left(d_{\psi}\left(\left(Y, {}^{A}\mu_{Y}\right)\right), -\right)\right].$$

Since $a_{-,Y}$ and $\widetilde{a}_{-,\left(Y, {}^{A}\mu_{Y}\right)}$ are isomorphisms, we deduce that $\mathrm{Hom}_{\mathcal{A}}\left(d_{\psi}\left(\left(Y, {}^{A}\mu_{Y}\right)\right), -\right)$ is mono. Applying the commutativity of this diagram in the particular case of $\left(Y, {}^{A}\mu_{Y}\right) = K_{\psi} X$, we get that

$$\begin{aligned}
(\widetilde{\epsilon} X) \circ (d_{\psi} K_{\psi} X) &= \mathrm{Hom}_{\mathcal{A}}(d_{\psi} K_{\psi} X, X)((\widetilde{\epsilon} X)) \\
&= \mathrm{Hom}_{\mathcal{A}}(d_{\psi} K_{\psi} X, X)\left(\widetilde{a}_{X, K_{\psi} X}^{-1}\left(\mathrm{Id}_{K_{\psi} X}\right)\right) \\
&= \left[\mathrm{Hom}_{\mathcal{A}}(d_{\psi} K_{\psi} X, X) \circ \widetilde{a}_{X, K_{\psi} X}^{-1}\right]\left(\mathrm{Id}_{K_{\psi} X}\right) \\
&= a_{X, {}_{\mathbb{A}} U K_{\psi} X}^{-1} {}_{\mathbb{A}} U\left(\mathrm{Id}_{K_{\psi} X}\right) = a_{X,RX}^{-1}\left(\mathrm{Id}_{{}_{\mathbb{A}} U K_{\psi} X}\right) \\
&= a_{X,RX}^{-1}\left(\mathrm{Id}_{RX}\right) = \epsilon X
\end{aligned}$$

i.e.

$$(\widetilde{\epsilon} X) \circ (d_{\psi} K_{\psi} X) = \epsilon X. \tag{6.11}$$

Now, we have to prove the universal property of the coequalizer. Let $X \in \mathcal{A}$ and let $\xi : LY \to X$ be a morphism in $\mathcal{A}$ such that $\xi \circ (rY) = \xi \circ \left(L^{A}\mu_{Y}\right)$ that is

$$\xi \circ (\epsilon LY) \circ (L\psi Y) = \xi \circ \left(L^{A}\mu_{Y}\right).$$

This means that $\xi \in \mathrm{Equ}_{Sets}\left(\mathrm{Hom}_{\mathcal{A}}(rY, X), \mathrm{Hom}_{\mathcal{A}}\left(L^{A}\mu_{Y}, X\right)\right) \simeq \mathrm{Hom}_{{}_{\mathbb{A}}\mathcal{B}}\left(\left(Y, {}^{A}\mu_{Y}\right), K_{\psi} X\right)$ by Proposition 6.52. Then, $a_{X,Y}(\xi) \in \mathrm{Hom}_{{}_{\mathbb{A}}\mathcal{B}}\left(\left(Y, {}^{A}\mu_{Y}\right), (RX, (R\epsilon X) \circ (\psi RX))\right) = \mathrm{Hom}_{{}_{\mathbb{A}}\mathcal{B}}\left(\left(Y, {}^{A}\mu_{Y}\right), K_{\psi} X\right)$. We want to prove that there exists a unique morphism $\xi' : D_{\psi}\left(Y, {}^{A}\mu_{Y}\right) \to X$ such that $\xi' \circ \left(d_{\psi}\left(Y, {}^{A}\mu_{Y}\right)\right) = \xi$. By hypothesis we have that the map

$$\mathrm{Hom}_{\mathcal{A}}\left(D_{\psi}\left(\left(Y, {}^{A}\mu_{Y}\right)\right), X\right) \xrightarrow{\widetilde{a}_{X,\left(Y, {}^{A}\mu_{Y}\right)}} \mathrm{Hom}_{{}_{\mathbb{A}}\mathcal{B}}\left(\left(Y, {}^{A}\mu_{Y}\right), K_{\psi} X\right)$$

$$\mathrm{Hom}_{\mathcal{A}}\left(D_{\psi}\left(\left(Y, {}^{A}\mu_{Y}\right)\right), X\right) \overset{\widetilde{a}_{X,\left(Y, {}^{A}\mu_{Y}\right)}}{\longrightarrow} \mathrm{Hom}_{{}_{\mathbb{A}}\mathcal{B}}\left(\left(Y, {}^{A}\mu_{Y}\right), K_{\psi} X\right)$$

is bijective. Hence, given $(R\xi)\circ(\eta Y) \in \mathrm{Hom}_{\mathbb{A}\mathcal{B}}\left(\left(Y, {}^{A}\mu_{Y}\right), K_{\psi}X\right)$, $\widetilde{a}^{-1}_{X,(Y,{}^{A}\mu_{Y})}\left((R\xi)\circ(\eta Y)\right) = (\widetilde{\epsilon}X)\circ(D_{\psi}R\xi)\circ(D_{\psi}\eta Y) \in \mathrm{Hom}_{\mathcal{A}}\left(D_{\psi}\left(\left(Y, {}^{A}\mu_{Y}\right)\right), X\right)$. We want to prove that

$$(\widetilde{\epsilon}X)\circ(D_{\psi}R\xi)\circ(D_{\psi}\eta Y)\circ\left(d_{\psi}\left(\left(Y, {}^{A}\mu_{Y}\right)\right)\right) = \xi.$$

In fact we have

$$\begin{aligned}
&(\widetilde{\epsilon}X)\circ(D_{\psi}R\xi)\circ(D_{\psi}\eta Y)\circ\left(d_{\psi}\left(Y, {}^{A}\mu_{Y}\right)\right)\\
&\overset{d_{\psi}}{=} (\widetilde{\epsilon}X)\circ\left(d_{\psi}\left(RX, {}^{A}\mu_{RX}\right)\right)\circ(LR\xi)\circ(L\eta Y)\\
&= (\widetilde{\epsilon}X)\circ(d_{\psi}K_{\psi}X)\circ(LR\xi)\circ(L\eta Y)\\
&\overset{(6.11)}{=} (\epsilon X)\circ(LR\xi)\circ(L\eta Y)\\
&\overset{\epsilon}{=} \xi\circ(\epsilon LY)\circ(L\eta Y) \overset{(L,R)}{=} \xi.
\end{aligned}$$

Let us denote by $\xi' = (\widetilde{\epsilon}X)\circ(D_{\psi}R\xi)\circ(D_{\psi}\eta Y)$ the morphism such that $\xi'\circ\left(d_{\psi}\left(Y, {}^{A}\mu_{Y}\right)\right) = \xi$. We have to prove that ξ' is unique with respect to this property. Let $\xi'' : D_{\psi}\left(Y, {}^{A}\mu_{Y}\right) \to X$ be another morphism in $\mathcal{A}$ such that $\xi''\circ d_{\psi}\left(Y, {}^{A}\mu_{Y}\right) = \xi$. Then we have

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{A}}\left(d_{\psi}\left(Y, {}^{A}\mu_{Y}\right), X\right)(\xi'') &= \xi''\circ d_{\psi}\left(Y, {}^{A}\mu_{Y}\right) = \xi = \xi'\circ d_{\psi}\left(Y, {}^{A}\mu_{Y}\right)\\
&= \mathrm{Hom}_{\mathcal{A}}\left(d_{\psi}\left(Y, {}^{A}\mu_{Y}\right), X\right)(\xi')
\end{aligned}$$

and since $\mathrm{Hom}_{\mathcal{A}}\left(d_{\psi}\left(Y, {}^{A}\mu_{Y}\right), X\right)$ is mono, we deduce that

$$\xi'' = \xi'.$$

□

Corollary 6.54. *Let (L, R) be an adjunction where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$. Let $r = \Theta(\mathrm{Id}_{\mathbb{RL}}) = \epsilon L$. Then the functor $K = \Upsilon(\mathrm{Id}_{\mathbb{RL}}) : \mathcal{A} \to {}_{\mathbb{RL}}\mathcal{B}$ has a left adjoint $D : {}_{\mathbb{RL}}\mathcal{B} \to \mathcal{A}$ if and only, for every $\left(Y, {}^{RL}\mu_{Y}\right) \in {}_{\mathbb{RL}}\mathcal{B}$, there exists $\mathrm{Coequ}_{\mathcal{A}}\left(\epsilon LY, L^{RL}\mu_{Y}\right)$. In this case, there exists a functorial morphism $d : L_{\mathbb{RL}}U \to D$ such that*

$$(D, d) = \mathrm{Coequ}_{Fun}\left(\epsilon L_{\mathbb{RL}}U, L_{\mathbb{RL}}U\lambda_{RL}\right)$$

and thus

$$\left[D\left(\left(Y, {}^{RL}\mu_{Y}\right)\right), d\left(Y, {}^{RL}\mu_{Y}\right)\right] = \mathrm{Coequ}_{\mathcal{A}}\left(\epsilon LY, L^{RL}\mu_{Y}\right).$$

Proof. We can apply Proposition 6.53 where "ψ" $= \mathrm{Id}_{\mathbb{RL}}$. □

Remark 6.55. *In the setting of Proposition 6.53, for every $X \in \mathcal{A}$, we note that the counit of the adjunction (D_{ψ}, K_{ψ}) is given by*

$$\widetilde{\epsilon}X = \widetilde{a}^{-1}_{X,K_{\psi}X}\left(\mathrm{Id}_{K_{\psi}X}\right) : D_{\psi}K_{\psi}(X) \to X.$$

We will consider diagram (6.10) *in the particular case of* $\left(Y, {}^{A}\mu_{Y}\right) = K_{\psi}X$. *Note that, since* $K_{\psi}X = (RX, (R\epsilon X)\circ(\psi RX)) = (RX, lX)$, *we have*

$$\begin{aligned}\left(D_{\psi}K_{\psi}\left(X\right), d_{\psi}K_{\psi}\left(X\right)\right) &= \left(D_{\psi}\left(RX, lX\right), d_{\psi}K_{\psi}\left(X\right)\right) = \mathrm{Coequ}_{\mathcal{B}}\left(rRX, LlX\right)\\ &= \mathrm{Coequ}_{\mathcal{B}}\left(\left(\epsilon LRX\right)\circ\left(L\psi RX\right), \left(LR\epsilon X\right)\circ\left(L\psi RX\right)\right)\end{aligned}$$

i.e.

$$\left(D_{\psi}K_{\psi}\left(X\right), d_{\psi}K_{\psi}\left(X\right)\right) = \mathrm{Coequ}_{\mathcal{B}}\left(rRX, LlX\right) \tag{6.12}$$

where $l = \Gamma(\psi) = (R\epsilon)\circ(\psi R)$. *We compute*

$$\begin{aligned}&\left(\widetilde{\epsilon}X\right)\circ\left(d_{\psi}K_{\psi}X\right) = \mathrm{Hom}_{\mathcal{A}}\left(d_{\psi}K_{\psi}X, X\right)\left(\left(\widetilde{\epsilon}X\right)\right)\\ &= \mathrm{Hom}_{\mathcal{A}}\left(d_{\psi}K_{\psi}X, X\right)\left(\widetilde{a}_{X,K_{\psi}X}^{-1}\left(\mathrm{Id}_{K_{\psi}X}\right)\right)\\ &= \left[\mathrm{Hom}_{\mathcal{A}}\left(d_{\psi}K_{\psi}X, X\right)\circ\widetilde{a}_{X,K_{\psi}X}^{-1}\right]\left(\mathrm{Id}_{K_{\psi}X}\right)\\ &\overset{(6.10)}{=} a_{X,K_{\psi}X}^{-1}{}_{\mathbb{A}}U\left(\mathrm{Id}_{K_{\psi}X}\right) = a_{X,K_{\psi}X}^{-1}\left(\mathrm{Id}_{{}_{\mathbb{A}}UK_{\psi}X}\right)\\ &= a_{X,K_{\psi}X}^{-1}\left(\mathrm{Id}_{RX}\right) = \epsilon X\end{aligned}$$

so that

$$\left(\widetilde{\epsilon}X\right)\circ\left(d_{\psi}K_{\psi}X\right) = \epsilon X.$$

Since $\widetilde{\epsilon}X = \widetilde{a}_{X,K_{\psi}X}^{-1}\left(\mathrm{Id}_{K_{\psi}X}\right)$ *and* $\widetilde{a}_{X,K_{\psi}X}^{-1}$ *is an isomorphism, we deduce that* $\widetilde{\epsilon}X : D_{\psi}K_{\psi}(X) \to X$ *is defined as the unique morphism such that*

$$\left(\widetilde{\epsilon}X\right)\circ\left(d_{\psi}K_{\psi}X\right) = \epsilon X. \tag{6.13}$$

On the other hand, for every $\left(Y, {}^{A}\mu_{Y}\right) \in {}_{\mathbb{A}}\mathcal{B}$, *the unit of the adjunction* (D_{ψ}, K_{ψ}), $\widetilde{\eta} : {}_{\mathbb{A}}\mathcal{B} \to K_{\psi}D_{\psi}$, *is given by*

$$\widetilde{\eta}\left(Y, {}^{A}\mu_{Y}\right) = \widetilde{a}_{D_{\psi}(Y,{}^{A}\mu_{Y}),Y}\left(\mathrm{Id}_{D_{\psi}(Y,{}^{A}\mu_{Y})}\right) : \left(Y, {}^{A}\mu_{Y}\right) \to K_{\psi}D_{\psi}\left(\left(Y, {}^{A}\mu_{Y}\right)\right).$$

Then by commutativity of the diagram (6.10), *we deduce that*

$$\begin{aligned}&{}_{\mathbb{A}}U\widetilde{\eta}\left(Y, {}^{A}\mu_{Y}\right) = {}_{\mathbb{A}}U\widetilde{a}_{D_{\psi}(Y,{}^{A}\mu_{Y}),Y}\left(\mathrm{Id}_{D_{\psi}(Y,{}^{A}\mu_{Y})}\right)\\ &= a_{D_{\psi}(Y,{}^{A}\mu_{Y}),Y}\circ\mathrm{Hom}_{\mathcal{A}}\left(d_{\psi}\left(\left(Y, {}^{A}\mu_{Y}\right)\right), D_{\psi}\left(Y, {}^{A}\mu_{Y}\right)\right)\left(\mathrm{Id}_{D_{\psi}(Y,{}^{A}\mu_{Y})}\right)\\ &= a_{D_{\psi}(Y,{}^{A}\mu_{Y}),Y}\left(d_{\psi}\left(\left(Y, {}^{A}\mu_{Y}\right)\right)\right) = \left(Rd_{\psi}\left(Y, {}^{A}\mu_{Y}\right)\right)\circ\left(\eta Y\right).\end{aligned}$$

Thus we obtain that

$${}_{\mathbb{A}}U\widetilde{\eta}\left(Y, {}^{A}\mu_{Y}\right) = \left(Rd_{\psi}\left(Y, {}^{A}\mu_{Y}\right)\right)\circ\left(\eta Y\right). \tag{6.14}$$

Observe that, for every $Y \in \mathcal{B}$ *we have that* ${}_{\mathbb{A}}F(Y) = (AY, m_{A}Y)$. *Moreover*

$$\begin{aligned}&\left(D_{\psi\mathbb{A}}F\left(Y\right), d_{\psi\mathbb{A}}F\left(Y\right)\right) = \left(D_{\psi}\left(AY, m_{A}Y\right), d_{\psi}\left(AY, m_{A}Y\right)\right)\\ &= \mathrm{Coequ}_{\mathcal{A}}\left(rAY, Lm_{A}Y\right) \overset{(6.1)}{=} \left(LY, rY\right)\end{aligned}$$

so that we get

$$(D_{\psi\mathbb{A}}F, d_{\psi\mathbb{A}}F) = (L, r)\,. \tag{6.15}$$

In particular

$$d_{\psi}\left(AY, m_A Y\right) = rY. \tag{6.16}$$

Corollary 6.56. *In the setting of Proposition 6.53, assume that, for every* $\left(Y, {}^{A}\mu_Y\right) \in {}_{\mathbb{A}}\mathcal{B}$*, there exists* $\mathrm{Coequ}_{\mathcal{A}}\left(rY, L^{A}\mu_Y\right)$*. Then, for every* $Y \in \mathcal{B}$ *we have*

$${}_{\mathbb{A}}U\widetilde{\eta}\left(AY, m_A Y\right) = \psi Y$$

and hence

$${}_{\mathbb{A}}U\widetilde{\eta}_{\mathbb{A}}F = \psi$$

where $\widetilde{\eta}$ *denotes the unit of the adjunction* (D_ψ, K_ψ)*.*

Proof. Let us calculate

$$\begin{aligned} {}_{\mathbb{A}}U\widetilde{\eta}\left(AY, m_A Y\right) &\overset{(6.14)}{=} \left(Rd_{\psi}\left(AY, m_A Y\right)\right) \circ (\eta AY) \\ &\overset{(6.16)}{=} (RrY) \circ (\eta AY) = \Xi\left(r\right)(Y) = \psi Y. \end{aligned}$$

□

Corollary 6.57. *In the setting of Proposition 6.53, assume that, for every* $\left(Y, {}^{A}\mu_Y\right) \in {}_{\mathbb{A}}\mathcal{B}$*, there exists* $\mathrm{Coequ}_{\mathcal{A}}\left(rY, L^{A}\mu_Y\right)$*. Then* D_ψ *is full and faithful if and only if* $\widetilde{\eta}$ *is a functorial isomorphism.*

Proof. By Proposition 6.53, (D_ψ, K_ψ) is an adjunction with unit $\widetilde{\eta} : {}_{\mathbb{A}}\mathcal{B} \to K_\psi D_\psi$. Then we can apply Proposition 5.18. □

Lemma 6.58. *In the setting of Proposition 6.53, assume that, for every* $\left(Y, {}^{A}\mu_Y\right) \in {}_{\mathbb{A}}\mathcal{B}$*, there exists* $\mathrm{Coequ}_{\mathcal{A}}\left(rY, L^{A}\mu_Y\right)$*.Then, for every* $\left(Y, {}^{A}\mu_Y\right) \in {}_{\mathbb{A}}\mathcal{B}$*, the following diagram*

$$\begin{array}{ccccc} AA_{\mathbb{A}}U\left(Y, {}^{A}\mu_Y\right) & \overset{m_{A\mathbb{A}}U\left(Y,{}^{A}\mu_Y\right)}{\underset{A_{\mathbb{A}}U\lambda_A\left(Y,{}^{A}\mu_Y\right)}{\rightrightarrows}} & A_{\mathbb{A}}U\left(Y, {}^{A}\mu_Y\right) & \xrightarrow{{}_{\mathbb{A}}U\lambda_A\left(Y,{}^{A}\mu_Y\right)} & {}_{\mathbb{A}}U\left(Y, {}^{A}\mu_Y\right) \\ {\scriptstyle \psi A_{\mathbb{A}}U\left(Y,{}^{A}\mu_Y\right)}\downarrow & & \downarrow{\scriptstyle \psi_{\mathbb{A}}U\left(Y,{}^{A}\mu_Y\right)} & & \downarrow{\scriptstyle {}_{\mathbb{A}}U\widetilde{\eta}\left(Y,{}^{A}\mu_Y\right)} \\ RLA_{\mathbb{A}}U\left(Y, {}^{A}\mu_Y\right) & \overset{Rr_{\mathbb{A}}U\left(Y,{}^{A}\mu_Y\right)}{\underset{RL_{\mathbb{A}}U\lambda_A\left(Y,{}^{A}\mu_Y\right)}{\rightrightarrows}} & RL_{\mathbb{A}}U\left(Y, {}^{A}\mu_Y\right) & \xrightarrow{Rd_{\psi}\left(Y,{}^{A}\mu_Y\right)} & RD_{\psi}\left(Y, {}^{A}\mu_Y\right) \end{array}$$

$$\begin{array}{ccccc} AA_{\mathbb{A}}U\left(Y, {}^{A}\mu_Y\right) & \overset{m_{A\mathbb{A}}U\left(Y,{}^{A}\mu_Y\right)}{\underset{A_{\mathbb{A}}U\lambda_A\left(Y,{}^{A}\mu_Y\right)}{\rightrightarrows}} & A_{\mathbb{A}}U\left(Y, {}^{A}\mu_Y\right) & \overset{{}_{\mathbb{A}}U\lambda_A\left(Y,{}^{A}\mu_Y\right)}{\rightarrow} & {}_{\mathbb{A}}U\left(Y, {}^{A}\mu_Y\right) \\ \psi A_{\mathbb{A}}U\left(Y, {}^{A}\mu_Y\right)\downarrow & & \downarrow \psi_{\mathbb{A}}U\left(Y, {}^{A}\mu_Y\right) & & \downarrow {}_{\mathbb{A}}U\widetilde{\eta}\left(Y, {}^{A}\mu_Y\right) \\ RLA_{\mathbb{A}}U\left(Y, {}^{A}\mu_Y\right) & \overset{Rr_{\mathbb{A}}U\left(Y,{}^{A}\mu_Y\right)}{\underset{RL_{\mathbb{A}}U\lambda_A\left(Y,{}^{A}\mu_Y\right)}{\rightrightarrows}} & RL_{\mathbb{A}}U\left(Y, {}^{A}\mu_Y\right) & \overset{Rd_{\psi}\left(Y,{}^{A}\mu_Y\right)}{\rightarrow} & RD_{\psi}\left(Y, {}^{A}\mu_Y\right) \end{array}$$

serially commutes. Therefore we get

$$({}_{\mathbb{A}}U\widetilde{\eta}) \circ ({}_{\mathbb{A}}U\lambda_A) = (Rd_\psi) \circ (\psi_{\mathbb{A}}U) \quad \textit{and} \quad (\psi_{\mathbb{A}}U) \circ (m_{A\mathbb{A}}U) = (Rr_{\mathbb{A}}U) \circ (\psi A_{\mathbb{A}}U)$$

Proof. Let us compute

$$\begin{aligned}
{}_{\mathbb{A}}U\widetilde{\eta}\left(Y,{}^{A}\mu_{Y}\right)\circ{}^{A}\mu_{Y} &\overset{(6.14)}{=} \left(Rd_{\psi}\left(Y,{}^{A}\mu_{Y}\right)\right)\circ(\eta Y)\circ{}^{A}\mu_{Y}\\
&\overset{\eta}{=} \left(Rd_{\psi}\left(Y,{}^{A}\mu_{Y}\right)\right)\circ\left(RL^{A}\mu_{Y}\right)\circ(\eta AY)\\
&\overset{\psi\text{morphmonads}}{=} \left(Rd_{\psi}\left(Y,{}^{A}\mu_{Y}\right)\right)\circ\left(RL^{A}\mu_{Y}\right)\circ(\psi AY)\circ(u_{A}AY)\\
&\overset{\text{def}d_{\psi}}{=} \left(Rd_{\psi}\left(Y,{}^{A}\mu_{Y}\right)\right)\circ(RrY)\circ(\psi AY)\circ(u_{A}AY)\\
&\overset{\text{def}r}{=} \left(Rd_{\psi}\left(Y,{}^{A}\mu_{Y}\right)\right)\circ(R\epsilon LY)\circ(RL\psi Y)\circ(\psi AY)\circ(u_{A}AY)\\
&\overset{\psi\text{morphmonads}}{=} \left(Rd_{\psi}\left(Y,{}^{A}\mu_{Y}\right)\right)\circ(\psi Y)\circ(m_{A}Y)\circ(u_{A}AY)\\
&\overset{A\text{monad}}{=} \left(Rd_{\psi}\left(Y,{}^{A}\mu_{Y}\right)\right)\circ(\psi Y)
\end{aligned}$$

so that we deduced

$${}_{\mathbb{A}}U\widetilde{\eta}\left(Y,{}^{A}\mu_{Y}\right)\circ{}^{A}\mu_{Y} = \left(Rd_{\psi}\left(Y,{}^{A}\mu_{Y}\right)\right)\circ(\psi Y)$$

and thus

$$\left({}_{\mathbb{A}}U\widetilde{\eta}\right)\circ\left({}_{\mathbb{A}}U\lambda_{A}\right) = (Rd_{\psi})\circ(\psi_{\mathbb{A}}U)\,.$$

Let us calculate

$$\begin{aligned}
(\psi_{\mathbb{A}}U)\circ(m_{A\mathbb{A}}U) &\overset{\psi\text{monadsmorph}}{=} (R\epsilon L_{\mathbb{A}}U)\circ(RL\psi_{\mathbb{A}}U)\circ(\psi A_{\mathbb{A}}U)\\
&\overset{\text{def}r}{=} (Rr_{\mathbb{A}}U)\circ(\psi A_{\mathbb{A}}U)\,.
\end{aligned}$$

□

Theorem 6.59. *Let (L,R) be an adjunction where $L:\mathcal{B}\to\mathcal{A}$ and $R:\mathcal{A}\to\mathcal{B}$, let $\mathbb{A}=(A,m_A,u_A)$ be a monad on the category $\mathcal{B}$ and let $\psi:\mathbb{A}=(A,m_A,u_A)\to\mathbb{RL}=(RL,R\epsilon L,\eta)$ be a monad morphism. Let $r=\Theta(\psi)=(\epsilon L)\circ(L\psi)$. Assume that, for every $\left(Y,{}^{A}\mu_{Y}\right)\in{}_{\mathbb{A}}\mathcal{B}$, there exists $\mathrm{Coequ}_{\mathcal{A}}\left(rY,L^{A}\mu_{Y}\right)$. Then we can consider the functor $K_{\psi}=\Upsilon(\psi):\mathcal{A}\to{}_{\mathbb{A}}\mathcal{B}$. Its left adjoint $D_{\psi}:{}_{\mathbb{A}}\mathcal{B}\to\mathcal{A}$ is full and faithful if and only if*

1) *R preserves the coequalizer*

$$(D_{\psi},d_{\psi}) = \mathrm{Coequ}_{Fun}\left(r_{\mathbb{A}}U,L_{\mathbb{A}}U\lambda_{A}\right)$$

2) *$\psi:\mathbb{A}\to\mathbb{RL}$ is a monad isomorphism.*

Proof. Recall that, by Corollary 6.56,

$$ {}_{\mathbb{A}}U\widetilde{\eta}_{\mathbb{A}}F=\psi. \tag{6.17}$$

By Corollary 6.57, D_{ψ} is full and faithful if and only if $\widetilde{\eta}$ is a functorial isomorphism.

Let us assume that $\widetilde{\eta}$ is a functorial isomorphism, hence ψ is an isomorphism too. Recall that, by Lemma 6.58, we have

$$({}_{\mathbb{A}}U\widetilde{\eta}) \circ ({}_{\mathbb{A}}U\lambda_A) = (Rd_\psi) \circ (\psi_{\mathbb{A}}U) \tag{6.18}$$

so that

$$({}_{\mathbb{A}}U\lambda_A) = \left({}_{\mathbb{A}}U\widetilde{\eta}^{-1}\right) \circ (Rd_\psi) \circ (\psi_{\mathbb{A}}U) \tag{6.19}$$

Let us consider the diagram

$$RLA_{\mathbb{A}}U \underset{RL_{\mathbb{A}}U\lambda_A}{\overset{Rr_{\mathbb{A}}U}{\rightrightarrows}} RL_{\mathbb{A}}U \xrightarrow{Rd_\psi} RD_\psi$$

$$RLA_{\mathbb{A}}U \quad \underset{RL_{\mathbb{A}}U\lambda_A}{\overset{Rr_{\mathbb{A}}U}{\rightrightarrows}} \quad RL_{\mathbb{A}}U \quad \overset{Rd_\psi}{\longrightarrow} \quad RD_\psi$$

We have to prove that $(RD_\psi, Rd_\psi) = \mathrm{Coequ}_{Fun}(Rr_{\mathbb{A}}U, RL_{\mathbb{A}}U\lambda_A)$. Since R is a functor, we clearly have $(Rd_\psi) \circ (Rr_{\mathbb{A}}U) = (Rd_\psi) \circ (RL_{\mathbb{A}}U\lambda_A)$. Let $Q : {}_{\mathbb{A}}\mathcal{B} \to \mathcal{X}$ be a functor and let $\chi : RL_{\mathbb{A}}U \to Q$ be a functorial morphism such that

$$\chi \circ (Rr_{\mathbb{A}}U) = \chi \circ (RL_{\mathbb{A}}U\lambda_A).$$

We compute

$$\chi \circ (\psi_{\mathbb{A}}U) = \chi \circ (\psi_{\mathbb{A}}U) \circ (m_{A\mathbb{A}}U) \circ (u_A A_{\mathbb{A}}U) \overset{\psi\text{morpmonads}}{=} \chi \circ (R\epsilon L_{\mathbb{A}}U) \circ (\psi\psi_{\mathbb{A}}U) \circ (u_A A_{\mathbb{A}}U)$$
$$= \chi \circ (R\epsilon L_{\mathbb{A}}U) \circ (RL\psi_{\mathbb{A}}U) \circ (\psi A_{\mathbb{A}}U) \circ (u_A A_{\mathbb{A}}U) \overset{\text{defr}}{=} \chi \circ (Rr_{\mathbb{A}}U) \circ (\psi A_{\mathbb{A}}U) \circ (u_A A_{\mathbb{A}}U)$$
$$\overset{\chi}{=} \chi \circ (RL_{\mathbb{A}}U\lambda_A) \circ (\psi A_{\mathbb{A}}U) \circ (u_A A_{\mathbb{A}}U) \overset{\psi}{=} \chi \circ (\psi_{\mathbb{A}}U) \circ (A_{\mathbb{A}}U\lambda_A) \circ (u_A A_{\mathbb{A}}U)$$
$$\overset{u_A}{=} \chi \circ (\psi_{\mathbb{A}}U) \circ (u_{A\mathbb{A}}U) \circ ({}_{\mathbb{A}}U\lambda_A)$$
$$\overset{\psi\text{morpmonads}}{=} \chi \circ (\eta_{\mathbb{A}}U) \circ ({}_{\mathbb{A}}U\lambda_A)$$
$$\overset{(6.19)}{=} \chi \circ (\eta_{\mathbb{A}}U) \circ \left({}_{\mathbb{A}}U\widetilde{\eta}^{-1}\right) \circ (Rd_\psi) \circ (\psi_{\mathbb{A}}U).$$

Since ψ is an isomorphism we deduce that

$$\chi = \left[\chi \circ (\eta_{\mathbb{A}}U) \circ \left({}_{\mathbb{A}}U\widetilde{\eta}^{-1}\right)\right] \circ (Rd_\psi).$$

Let now $\omega : RD_\psi \to Q$ be a functorial morphism such that

$$\chi = \omega \circ (Rd_\psi).$$

We compute

$$\left[\chi \circ (\eta_{\mathbb{A}}U) \circ \left({}_{\mathbb{A}}U\widetilde{\eta}^{-1}\right)\right] \circ ({}_{\mathbb{A}}U\widetilde{\eta}) \circ ({}_{\mathbb{A}}U\lambda_A)$$
$$\overset{(6.18)}{=} \left[\chi \circ (\eta_{\mathbb{A}}U) \circ \left({}_{\mathbb{A}}U\widetilde{\eta}^{-1}\right)\right] \circ (Rd_\psi) \circ (\psi_{\mathbb{A}}U)$$
$$= \chi \circ (\psi_{\mathbb{A}}U) = \omega \circ (Rd_\psi) \circ (\psi_{\mathbb{A}}U) \overset{(6.18)}{=} \omega \circ ({}_{\mathbb{A}}U\widetilde{\eta}) \circ ({}_{\mathbb{A}}U\lambda_A)$$

and since ${}_{\mathbb{A}}U\lambda_A$ is an epimorphism (it is a coequalizer) and $\widetilde{\eta}$ is an isomorphism, we deduce that

$$\omega = \chi \circ (\eta_{\mathbb{A}}U) \circ \left({}_{\mathbb{A}}U\widetilde{\eta}^{-1}\right).$$

Conversely, assume that 1) and 2) hold. Then ψ is a functorial isomorphism. Consider the diagram

$$\begin{array}{ccccc} AA_{\mathbb{A}}U\left(Y,{}^{A}\mu_Y\right) & \overset{m_{A\mathbb{A}}U\left(Y,{}^{A}\mu_Y\right)}{\underset{A_{\mathbb{A}}U\lambda_A\left(Y,{}^{A}\mu_Y\right)}{\rightrightarrows}} & A_{\mathbb{A}}U\left(Y,{}^{A}\mu_Y\right) & \xrightarrow{{}_{\mathbb{A}}U\lambda_A\left(Y,{}^{A}\mu_Y\right)} & {}_{\mathbb{A}}U\left(Y,{}^{A}\mu_Y\right) \\ \psi A_{\mathbb{A}}U\left(Y,{}^{A}\mu_Y\right)\downarrow & & \downarrow\psi_{\mathbb{A}}U\left(Y,{}^{A}\mu_Y\right) & & \downarrow{}_{\mathbb{A}}U\widetilde{\eta}\left(Y,{}^{A}\mu_Y\right) \\ RLA_{\mathbb{A}}U\left(Y,{}^{A}\mu_Y\right) & \overset{Rr_{\mathbb{A}}U\left(Y,{}^{A}\mu_Y\right)}{\underset{RL_{\mathbb{A}}U\lambda_A\left(Y,{}^{A}\mu_Y\right)}{\rightrightarrows}} & RL_{\mathbb{A}}U\left(Y,{}^{A}\mu_Y\right) & \xrightarrow{Rd_\psi\left(Y,{}^{A}\mu_Y\right)} & RD_\psi\left(Y,{}^{A}\mu_Y\right) \end{array}$$

$$\begin{array}{ccccc} AA_{\mathbb{A}}U\left(Y,{}^{A}\mu_Y\right) & \overset{m_{A\mathbb{A}}U\left(Y,{}^{A}\mu_Y\right)}{\underset{A_{\mathbb{A}}U\lambda_A(Y,{}^{A}\mu_Y)}{\rightrightarrows}} & A_{\mathbb{A}}U\left(Y,{}^{A}\mu_Y\right) & \overset{{}_{\mathbb{A}}U\lambda_A\left(Y,{}^{A}\mu_Y\right)}{\rightarrow} & {}_{\mathbb{A}}U\left(Y,{}^{A}\mu_Y\right) \\ \psi A_{\mathbb{A}}U\left(Y,{}^{A}\mu_Y\right)\downarrow & & \downarrow\psi_{\mathbb{A}}U\left(Y,{}^{A}\mu_Y\right) & & \downarrow{}_{\mathbb{A}}U\widetilde{\eta}\left(Y,{}^{A}\mu_Y\right) \\ RLA_{\mathbb{A}}U\left(Y,{}^{A}\mu_Y\right) & \overset{Rr_{\mathbb{A}}U\left(Y,{}^{A}\mu_Y\right)}{\underset{RL_{\mathbb{A}}U\lambda_A(Y,{}^{A}\mu_Y)}{\rightrightarrows}} & RL_{\mathbb{A}}U\left(Y,{}^{A}\mu_Y\right) & \overset{Rd_\psi\left(Y,{}^{A}\mu_Y\right)}{\rightarrow} & RD_\psi\left(Y,{}^{A}\mu_Y\right) \end{array}$$

of Lemma 6.58 where the first row is always a coequalizer (see Proposition 6.32) and the last row is also a coequalizer by the assumption 1). Then we can apply Lemma 5.36 and hence we get that ${}_{\mathbb{A}}U\widetilde{\eta}$ is a functorial isomorphism. Since,by Proposition 6.34, ${}_{\mathbb{A}}U$ reflects isomorphism we deduce that $\widetilde{\eta}$ is a functorial isomorphism. □

Corollary 6.60. *Let (L, R) be an adjunction where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$. Let $r = \Theta\left(\mathrm{Id}_{\mathbb{RL}}\right) = \epsilon L$. Assume that, for every $\left(Y, {}^{RL}\mu_Y\right) \in {}_{\mathbb{RL}}\mathcal{B}$, there exists $\mathrm{Coequ}_{\mathcal{A}}\left(\epsilon LY, L^{RL}\mu_Y\right)$. Then we can consider the functor $K = \Upsilon\left(\mathrm{Id}_{\mathbb{RL}}\right) : \mathcal{A} \to {}_{\mathbb{RL}}\mathcal{B}$. Its left adjoint $D : {}_{\mathbb{RL}}\mathcal{B} \to \mathcal{A}$ is full and faithful if and only if R preserves the coequalizer*

$$(D, d) = \mathrm{Coequ}_{Fun}\left(\epsilon L_{\mathbb{RL}}U, L_{\mathbb{RL}}U\lambda_{RL}\right).$$

Proof. We can apply Theorem 6.59 with "ψ" $= \mathrm{Id}_{\mathbb{RL}}$. □

Theorem 6.61. *Let (L, R) be an adjunction where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$, let $\mathbb{A} = (A, m_A, u_A)$ be a monad on the category $\mathcal{B}$ and let $\psi : \mathbb{A} = (A, m_A, u_A) \to \mathbb{RL} = (RL, R\epsilon L, \eta)$ be a monad morphism. Let $r = \Theta(\psi) = (\epsilon L) \circ (L\psi)$ and $l = \Gamma(\psi) = (R\epsilon) \circ (\psi R)$. Assume that, for every $\left(Y, {}^{A}\mu_Y\right) \in {}_{\mathbb{A}}\mathcal{B}$, there exists $\mathrm{Coequ}_{\mathcal{A}}\left(rY, L^{A}\mu_Y\right)$. Then we can consider the functor $K_\psi = \Upsilon(\psi) : \mathcal{A} \to {}_{\mathbb{A}}\mathcal{B}$ and its left adjoint $D_\psi : {}_{\mathbb{A}}\mathcal{B} \to \mathcal{A}$. The functor K_ψ is an equivalence of categories if and only if*

1) *R preserves the coequalizer*

$$(D_\psi, d_\psi) = \mathrm{Coequ}_{Fun}\left(r_{\mathbb{A}}U, L_{\mathbb{A}}U\lambda_A\right)$$

2) *R reflects isomorphisms and*

3) $\psi : \mathbb{A} \to \mathbb{RL}$ *is a monad isomorphism.*

Proof. If K_ψ is an equivalence then, by Lemma 5.24, D_ψ is an equivalence of categories so that, by Theorem 6.59, 1) and 3) hold. By Proposition 6.34, the functor ${}_{\mathbb{A}}U$ reflects isomorphisms. Since $R = {}_{\mathbb{A}}U \circ K_\psi$ we get that 2) holds.

Conversely assume that 1), 2) and 3) hold. By Theorem 6.59 , D_ψ is full and faithful and hence by Corollary 6.57 $\widetilde{\eta}$ is a functorial isomorphism. Let us prove that $\widetilde{\epsilon}$ is an isomorphism as well. Since R reflects isomorphisms, it is enough to prove that $R\widetilde{\epsilon}$ is an isomorphism. As observed in Remark 6.55, $\widetilde{\epsilon}X : D_\psi K_\psi(X) \to X$ is defined as the unique morphism such that

$$(\widetilde{\epsilon}X) \circ (d_\psi K_\psi X) = \epsilon X.$$

Hence we get

$$(R\widetilde{\epsilon}X) \circ (Rd_\psi K_\psi X) = R\epsilon X \tag{6.20}$$

so that

$$(R\widetilde{\epsilon}X) \circ (Rd_\psi K_\psi X) \circ (\eta RX) = (R\epsilon X) \circ (\eta RX) = RX.$$

We will prove that $(Rd_\psi K_\psi X) \circ (\eta RX)$ is also a left inverse of $R\widetilde{\epsilon}X$. We have

$$\begin{aligned} &(Rd_\psi K_\psi X) \circ (\eta RX) \circ (R\widetilde{\epsilon}X) \circ (Rd_\psi K_\psi X) \\ &\overset{(6.20)}{=} (Rd_\psi K_\psi X) \circ (\eta RX) \circ (R\epsilon X) \\ &\overset{(L,R)\mathrm{adj}}{=} (Rd_\psi K_\psi X) \end{aligned}$$

and since R preserves coequalizers $(D_\psi, d_\psi) = \mathrm{Coequ}_{Fun}(r_{\mathbb{A}}U, L_{\mathbb{A}}U\lambda_A)$, $Rd_\psi K_\psi X$ is an epimorphism, so that

$$(Rd_\psi K_\psi X) \circ (\eta RX) \circ (R\widetilde{\epsilon}X) = RD_\psi K_\psi X$$

so that $R\widetilde{\epsilon}$ is a functorial isomorphism. □

Definition 6.62. *Let* $\mathbb{A} = (A, m_A, u_A)$ *be a monad on the category* $\mathcal{B}$ *and let* $\left(R, {}^A\mu_R\right)$ *be a left* $\mathbb{A}$*-module functor. We say that* $\left(R, {}^A\mu_R\right)$ *is an* $\mathbb{A}$-coGalois functor *if* R *has a left adjoint* L *and if the canonical morphism*

$$cocan := \left({}^A\mu_R L\right) \circ (A\eta) : \mathbb{A} \to \mathbb{RL}$$

is a monad isomorphism, where η *denotes the unit of the adjunction* (L, R)*.*

Corollary 6.63. *Let* $\left(R, {}^A\mu_R\right)$ *be a left A-coGalois functor where* $R : \mathcal{A} \to \mathcal{B}$ *preserves coequalizers,* R *reflects isomorphisms and* $\mathbb{A} = (A, m_A, u_A)$ *is a monad on* $\mathcal{B}$*. Assume that, for every* $\left(Y, {}^A\mu_Y\right) \in {}_{\mathbb{A}}\mathcal{B}$*, there exists* $\mathrm{Coequ}_{\mathcal{A}}\left(rY, L^A\mu_Y\right)$ *where* $r = (\epsilon L) \circ (Lcocan)$ *where* L *is the left adjoint of* R *and* ϵ *is the counit of the adjunction* (L, R)*. Then we can consider the functor* $K_{cocan} : \mathcal{A} \to {}_{\mathbb{A}}\mathcal{B}$ *and its left adjoint* $D_{cocan} : {}_{\mathbb{A}}\mathcal{B} \to \mathcal{A}$*. Then the functor* K_{cocan} *is an equivalence of categories.*

Proof. We can apply Theorem 6.61 to the case $\psi = cocan$. □

Theorem 6.64 (Beck's Theorem for monads)**.** *Let* (L, R) *be an adjunction where* $L : \mathcal{B} \to \mathcal{A}$ *and* $R : \mathcal{A} \to \mathcal{B}$*. Let* $r = \Theta(\mathrm{Id}_{\mathbb{RL}}) = \epsilon L$ *and assume that, for every* $\left(Y, {}^{RL}\mu_Y\right) \in {}_{\mathbb{RL}}\mathcal{B}$*, there exists* $\mathrm{Coequ}_{\mathcal{A}}\left(\epsilon LY, L^{RL}\mu_Y\right)$*. Then we can consider the functor* $K = \Upsilon(\mathrm{Id}_{\mathbb{RL}}) : \mathcal{A} \to {}_{\mathbb{RL}}\mathcal{B}$ *and its left adjoint* $D : {}_{\mathbb{RL}}\mathcal{B} \to \mathcal{A}$*. The functor* K *is an equivalence of categories if and only if*

1) *R preserves the coequalizer*

$$(D, d) = \mathrm{Coequ}_{Fun}\left(\epsilon L_{\mathbb{RL}} U, L_{\mathbb{RL}} U \lambda_{RL}\right).$$

2) *R reflects isomorphisms.*

Proof. Apply Theorem 6.61 taking $\psi = \mathrm{Id}_{\mathbb{RL}}$. □

Definition 6.65. *Let* $\mathbb{A} = (A, m_A, u_A)$ *be a monad on the category* $\mathcal{B}$ *and let* $R : \mathcal{A} \to \mathcal{B}$ *be a functor. The functor* R *is called* ψ-monadic *if it has a left adjoint* $L : \mathcal{B} \to \mathcal{A}$ *for which there exists* $\psi : \mathbb{A} \to \mathbb{RL}$ *a monad morphism such that the functor* $K_\psi = \Upsilon(\psi) : \mathcal{A} \to {}_{\mathbb{A}}\mathcal{B}$ *is an equivalence of categories.*

Definition 6.66. *Let* $R : \mathcal{A} \to \mathcal{B}$ *be a functor. The functor* R *is called* monadic *if it has a left adjoint* $L : \mathcal{B} \to \mathcal{A}$ *for which the functor* $K = \Upsilon(\mathrm{Id}_{\mathbb{RL}}) : \mathcal{A} \to {}_{\mathbb{RL}}\mathcal{B}$ *is an equivalence of categories.*

Lemma 6.67. *Let* $\mathbb{A} = (A, m_A, u_A)$ *be a monad on the category* $\mathcal{B}$ *and let* $R : \mathcal{A} \to \mathcal{B}$ *be a* ψ*-monadic functor and let*

$$X \overset{c_0}{\underset{c_1}{\rightrightarrows}} X' \tag{6.21}$$

$$X \overset{c_0}{\underset{c_1}{\rightrightarrows}} X'$$

be an R*-contractible coequalizer pair in* $\mathcal{A}$*. Then* (6.21) *has a coequalizer* $c : X' \to X''$ *in* $\mathcal{A}$ *and*

$$RX \overset{Rc_0}{\underset{Rc_1}{\rightrightarrows}} RX' \xrightarrow{Rc} RX''$$

$$RX \overset{Rc_0}{\underset{Rc_1}{\rightrightarrows}} RX' \overset{Rc}{\to} RX''$$

is a coequalizer in $\mathcal{B}$*.*

Proof. Since R is a ψ-monadic functor, we know that $K_\psi = \Upsilon(\psi) : \mathcal{A} \to {}_{\mathbb{A}}\mathcal{B}$ is an equivalence of categories. Then instead of considering

$$X \overset{c_0}{\underset{c_1}{\rightrightarrows}} X'$$

$$X \underset{c_1}{\overset{c_0}{\rightrightarrows}} X'$$

in the category $\mathcal{A}$, we can consider

$$K_\psi X \underset{K_\psi c_1}{\overset{K_\psi c_0}{\rightrightarrows}} K_\psi X'$$

$$K_\psi X \quad \underset{K_\psi c_1}{\overset{K_\psi c_0}{\rightrightarrows}} \quad K_\psi X'$$

in ${}_{\mathbb{A}}\mathcal{B}$, which is a ${}_{\mathbb{A}}U$-contractible coequalizer pair. Let us denote by $\left(Y, {}^A\mu_Y\right) := K_\psi X$ and $\left(Y', {}^A\mu_{Y'}\right) := K_\psi X'$ so that we can rewrite the ${}_{\mathbb{A}}U$-contractible coequalizer pair as follows

$$\left(Y, {}^A\mu_Y\right) \underset{K_\psi c_1}{\overset{K_\psi c_0}{\rightrightarrows}} \left(Y', {}^A\mu_{Y'}\right).$$

$$\left(Y, {}^A\mu_Y\right) \quad \underset{K_\psi c_1}{\overset{K_\psi c_0}{\rightrightarrows}} \quad \left(Y', {}^A\mu_{Y'}\right).$$

We want to prove that this pair has a coequalizer in ${}_{\mathbb{A}}\mathcal{B}$. Since the pair $(K_\psi c_0, K_\psi c_1)$ is a ${}_{\mathbb{A}}U$-contractible coequalizer pair, we have that

$$RX \;\overset{Rc_0}{\underset{Rc_1}{\rightrightarrows}}\; RX' \;\underset{u}{\overset{q}{\rightleftarrows}}\; Q \qquad (v : RX' \to RX)$$

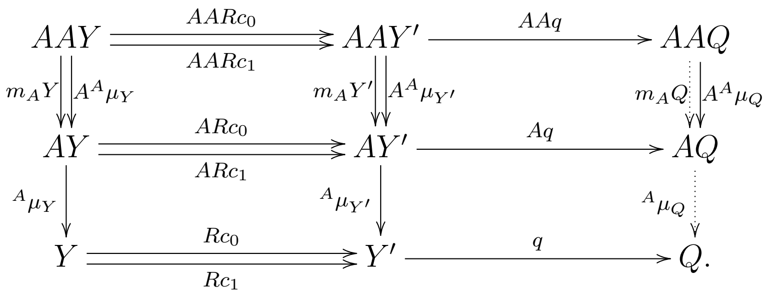
$$RX \quad \begin{matrix} \overset{Rc_0}{\rightarrow} \\ \overset{v}{\leftarrow} \\ \overset{Rc_1}{\rightarrow} \end{matrix} \quad RX' \quad \underset{u}{\overset{q}{\rightleftarrows}} \quad Q$$

is a contractible coequalizer in $\mathcal{B}$, i.e.

$$Y \;\overset{Rc_0}{\underset{Rc_1}{\rightrightarrows}}\; Y' \;\underset{u}{\overset{q}{\rightleftarrows}}\; Q \qquad (v : Y' \to Y)$$

$$Y \quad \begin{matrix} \overset{Rc_0}{\rightarrow} \\ \overset{v}{\leftarrow} \\ \overset{Rc_1}{\rightarrow} \end{matrix} \quad Y' \quad \underset{u}{\overset{q}{\rightleftarrows}} \quad Q$$

is a contractible coequalizer and thus, by Proposition 6.7, a coequalizer in $\mathcal{B}$. Let us consider the following diagram



$$\begin{array}{ccccc} AAY & \overset{AARc_0}{\underset{AARc_1}{\rightrightarrows}} & AAY' & \overset{AAq}{\longrightarrow} & AAQ \\ m_AY \downdownarrows A^A\mu_Y & & m_AY' \downdownarrows A^A\mu_{Y'} & & m_AQ \downdownarrows A^A\mu_Q \\ AY & \overset{ARc_0}{\underset{ARc_1}{\rightrightarrows}} & AY' & \overset{Aq}{\longrightarrow} & AQ \\ \downarrow {}^A\mu_Y & & \downarrow {}^A\mu_{Y'} & & \downarrow {}^A\mu_Q \\ Y & \overset{Rc_0}{\underset{Rc_1}{\rightrightarrows}} & Y' & \overset{q}{\longrightarrow} & Q. \end{array}$$

By Proposition 6.8, all the rows are contractible coequalizers. Since $Rc_0 = {}_{\mathbb{A}}UK_\psi c_0$ and $Rc_1 = {}_{\mathbb{A}}UK_\psi c_1$, we have that the lower left square serially commutes. Moreover, since we also have that m_A is a functorial morphism, the upper left square serially commutes. We also have that $q \circ {}^A\mu_{Y'}$ coequalizes (ARc_0, ARc_1) and, since $(AQ, Aq) = \mathrm{Coequ}_{\mathcal{B}}(ARc_0, ARc_1)$, by the universal property of the coequalizer, there exists a unique morphism ${}^A\mu_Q : AQ \to Q$ such that

$$ {}^A\mu_Q \circ (Aq) = q \circ {}^A\mu_{Y'}. \tag{6.22}$$

Let us prove that $\left(Q, {}^A\mu_Q\right) \in {}_{\mathbb{A}}\mathcal{B}$ and thus formula (6.22) will say that q is a morphism in ${}_{\mathbb{A}}\mathcal{B}$. Since m_A is a functorial morphism and by definition of ${}^A\mu_Q$, the upper right square serially commutes. We have

$$\begin{aligned} {}^A\mu_Q \circ \left(A^A\mu_Q\right) \circ (AAq) &\overset{(6.22)}{=} {}^A\mu_Q \circ (Aq) \circ \left(A^A\mu_{Y'}\right) \\ &\overset{(6.22)}{=} q \circ {}^A\mu_{Y'} \circ \left(A^A\mu_{Y'}\right) \overset{{}^A\mu_{Y'}\mathrm{ass}}{=} q \circ {}^A\mu_{Y'} \circ (m_AY') \\ &\overset{(6.22)}{=} {}^A\mu_Q \circ (Aq) \circ (m_AY') \overset{m_A}{=} {}^A\mu_Q \circ (m_AQ) \circ (AAq) \end{aligned}$$

and since AAq is an epimorphism we get

$$ {}^A\mu_Q \circ \left(A^A\mu_Q\right) = {}^A\mu_Q \circ (m_AQ)$$

that is that ${}^A\mu_Q$ is associative. Moreover we have

$$\begin{aligned} {}^A\mu_Q \circ (u_AQ) \circ q &\overset{u_A}{=} {}^A\mu_Q \circ (Aq) \circ (u_AY') \\ &\overset{(6.22)}{=} q \circ {}^A\mu_{Y'} \circ (u_AY') = q \end{aligned}$$

and since q is an epimorphism we get that

$$^{A}\mu_{Q}\circ(u_{A}Q)=Q$$

so that $^{A}\mu_{Q}$ is also unital. Therefore $\left(Q,{}^{A}\mu_{Q}\right)\in{}_{\mathbb{A}}\mathcal{B}$ and q is a morphism in ${}_{\mathbb{A}}\mathcal{B}$. Now we want to prove that it is a coequalizer in ${}_{\mathbb{A}}\mathcal{B}$. Let $\left(Z,{}^{A}\mu_{Z}\right)\in{}_{\mathbb{A}}\mathcal{B}$ and $\chi:\left(Y',{}^{A}\mu_{Y'}\right)\to\left(Z,{}^{A}\mu_{Z}\right)$ be a morphism in ${}_{\mathbb{A}}\mathcal{B}$ such that $\chi\circ(K_{\psi}c_{0})=\chi\circ(K_{\psi}c_{1})$. Then, by regarding χ as a morphism in $\mathcal{B}$ we also have that

$$\chi\circ(Rc_{0})=\chi\circ(Rc_{1}).$$

Since $(Q,q)=\mathrm{Coequ}_{\mathcal{B}}(Rc_{0},Rc_{1})$, there exists a unique morphism $\xi:Q\to Z$ such that

$$\xi\circ q=\chi.$$

Now we want to prove that ξ is a morphism in ${}_{\mathbb{A}}\mathcal{B}$. In fact, let us consider the following diagram

$$\begin{array}{ccccc} AY' & \xrightarrow{Aq} & AQ & \xrightarrow{A\xi} & AZ \\ {}^{A}\mu_{Y'}\downarrow & & {}^{A}\mu_{Q}\downarrow & & {}^{A}\mu_{Z}\downarrow \\ Y' & \xrightarrow{q} & Q & \xrightarrow{\xi} & Z. \end{array}$$

$$\begin{array}{ccccc} AY' & \overset{Aq}{\longrightarrow} & AQ & \overset{A\xi}{\longrightarrow} & AZ \\ \downarrow{}^{A}\mu_{Y'} & & \downarrow{}^{A}\mu_{Q} & & \downarrow{}^{A}\mu_{Z} \\ Y' & \overset{q}{\longrightarrow} & Q & \overset{\xi}{\longrightarrow} & Z. \end{array}$$

Since $q\in{}_{\mathbb{A}}\mathcal{B}$, the left square commutes. Since $\chi\in{}_{\mathbb{A}}\mathcal{B}$ we have

$$^{A}\mu_{Z}\circ(A\xi)\circ(Aq)={}^{A}\mu_{Z}\circ(A\chi)=\chi\circ{}^{A}\mu_{Y'}=\xi\circ q\circ{}^{A}\mu_{Y'}$$

so that we have

$$\xi\circ{}^{A}\mu_{Q}\circ(Aq)\overset{(6.22)}{=}\xi\circ q\circ{}^{A}\mu_{Y'}={}^{A}\mu_{Z}\circ(A\xi)\circ(Aq)$$

and since Aq is an epimorphism, we deduce that

$$\xi\circ{}^{A}\mu_{Q}={}^{A}\mu_{Z}\circ(A\xi)$$

i.e. $\xi\in{}_{\mathbb{A}}\mathcal{B}$. Therefore $(Q,q)=\mathrm{Coequ}_{{}_{\mathbb{A}}\mathcal{B}}(K_{\psi}c_{0},K_{\psi}c_{1})$. Now, since $K_{\psi}:\mathcal{A}\to{}_{\mathbb{A}}\mathcal{B}$, there exist $X'',c\in\mathcal{A}$ such that

$$K_{\psi}X''=Q\text{ and }K_{\psi}c=q$$

and thus $(X'',c)=\mathrm{Coequ}_{\mathcal{A}}(c_{0},c_{1})$. Moreover, since

$$RX\ \underset{Rc_{1}}{\overset{Rc_{0}}{\underset{\xleftarrow{\ v\ }}{\rightrightarrows}}}\ RX'\ \underset{u}{\overset{q}{\rightleftarrows}}\ Q$$

$$
RX \quad \begin{array}{c} \overset{Rc_0}{\to} \\ \overset{v}{\leftarrow} \\ \overset{Rc_1}{\to} \end{array} \quad RX' \quad \underset{u}{\overset{q}{\rightleftarrows}} \quad Q
$$

is a contractible coequalizer and $(Q, q) = ({}_{\mathbb{A}}UK_{\psi}X'', {}_{\mathbb{A}}UK_{\psi}c)$, we deduce that $({}_{\mathbb{A}}UK_{\psi}X'', {}_{\mathbb{A}}UK_{\psi}c)$ is a contractible coequalizer of (Rc_0, Rc_1). Then (RX'', Rc) is a contractible coequalizer of (Rc_0, Rc_1) so that $(RX'', Rc) = \mathrm{Coequ}_{\mathcal{B}}(Rc_0, Rc_1)$. □

Theorem 6.68 (Generalized Beck's Precise Tripleability Theorem)**.** *Let $R : \mathcal{A} \to \mathcal{B}$ be a functor and let $\mathbb{A} = (A, m_A, u_A)$ be a monad on the category $\mathcal{B}$. Then R is ψ-monadic if and only if*

1) *R has a left adjoint $L : \mathcal{B} \to \mathcal{A}$,*

2) *$\psi : \mathbb{A} \to \mathbb{RL}$ is a monad isomorphism where $\mathbb{RL} = (RL, R\epsilon L, \eta)$ with η and ϵ unit and counit of (L, R),*

3) *for every $\left(Y, {}^{A}\mu_Y\right) \in {}_{\mathbb{A}}\mathcal{B}$, there exist* $\mathrm{Coequ}_{\mathcal{A}}\left(rY, L^{A}\mu_Y\right)$, *where $r = \Theta(\psi) = (\epsilon L) \circ (L\psi)$, and R preserves the coequalizer*

$$\mathrm{Coequ}_{Fun}(r_{\mathbb{A}}U, L_{\mathbb{A}}U\lambda_A),$$

4) *R reflects isomorphisms.*

In this case in $\mathcal{A}$ there exist coequalizers of R-contractible coequalizer pairs and R preserves them.

Proof. Assume first that R is ψ-monadic. Then by definition R has a left adjoint $L : \mathcal{B} \to \mathcal{A}$ and a monad morphism $\psi : \mathbb{A} \to \mathbb{RL}$ such that the functor $K_{\psi} = \Upsilon(\psi) : \mathcal{A} \to {}_{\mathbb{A}}\mathcal{B}$ is an equivalence of categories. Let K'_{ψ} be an inverse of K_{ψ}. Then in particular $K'_{\psi} : {}_{\mathbb{A}}\mathcal{B} \to \mathcal{A}$ is a left adjoint of K_{ψ} so that, by Proposition 6.53, for every $\left(Y, {}^{A}\mu_Y\right) \in {}_{\mathbb{A}}\mathcal{B}$, there exists $\mathrm{Coequ}_{\mathcal{A}}\left(rY, L^{A}\mu_Y\right)$ where $r = \Theta(\psi) = (\epsilon L)\circ(L\psi)$ and thus $\left(K'_{\psi}, k'_{\psi}\right) = \mathrm{Coequ}_{Fun}(r_{\mathbb{A}}U, L_{\mathbb{A}}U\lambda_A)$ where $k'_{\psi}\left(Y, {}^{A}\mu_Y\right) : LY \to \mathrm{Coequ}_{Fun}(r_{\mathbb{A}}U, L_{\mathbb{A}}U\lambda_A)$ is the canonical projection. Then we can apply Theorem 6.61 to get that R preserves the coequalizer $\left(K'_{\psi}, k'_{\psi}\right) = \mathrm{Coequ}_{Fun}(r_{\mathbb{A}}U, L_{\mathbb{A}}U\lambda_A)$, R reflects isomorphisms and $\psi : \mathbb{A} \to \mathbb{RL}$ is a monads isomorphism.

Conversely, by assumption 1) R has a left adjoint $L : \mathcal{B} \to \mathcal{A}$ so that (L, R) is an adjunction and by 2) there exist $\mathrm{Coequ}_{\mathcal{A}}\left(rY, L^{A}\mu_Y\right)$, for every $\left(Y, {}^{A}\mu_Y\right) \in {}_{\mathbb{A}}\mathcal{B}$ so that we can apply Proposition 6.53. Thus the functor $K_{\psi} = \Upsilon(\psi) : \mathcal{A} \to {}_{\mathbb{A}}\mathcal{B}$ has a left adjoint $D_{\psi} : {}_{\mathbb{A}}\mathcal{B} \to \mathcal{A}$. Now, by applying Theorem 6.61 in the converse direction, we deduce that $K_{\psi} = \Upsilon(\psi) : \mathcal{A} \to {}_{\mathbb{A}}\mathcal{B}$ is an equivalence of categories, i.e. R is monadic. If R is ψ-monadic, by Lemma 6.67, in $\mathcal{A}$ there exist coequalizers of reflexive R-contractible coequalizer pairs and R preserves them. □

Corollary 6.69 (Beck's Precise Tripleability Theorem)**.** *Let $R : \mathcal{A} \to \mathcal{B}$ be a functor. Then R is monadic if and only if*

1) *R has a left adjoint* $L:\mathcal{B}\to\mathcal{A}$,

2) *for every* $\left(Y,{}^{RL}\mu_Y\right)\in{}_{\mathbb{RL}}\mathcal{B}$, *there exist* $\mathrm{Coequ}_{\mathcal{A}}\left(\epsilon LY, L^{RL}\mu_Y\right)$ *and R preserves the coequalizer*

$$\mathrm{Coequ}_{Fun}\left(\epsilon L_{\mathbb{RL}}U, L_{\mathbb{RL}}U\lambda_{RL}\right),$$

3) *R reflects isomorphisms.*

In this case in $\mathcal{A}$ *there exist coequalizers of R-contractible coequalizer pairs and R preserves them.*

Proof. Apply Theorem 6.68 to the case that $\psi=\mathrm{Id}_{\mathbb{RL}}$. □

6.6 BECK1 for Monads

Lemma 6.70. *Let* (L,R) *be an adjunction, where* $L:\mathcal{B}\to\mathcal{A}$ *and* $R:\mathcal{A}\to\mathcal{B}$, *with unit* η *and counit* ϵ. *Then for every* $X\in\mathcal{A}$, $(RX, RLRX, RLRLRX, R\epsilon X, R\epsilon LRX, RLR\epsilon X, \eta RX, \eta RLRX)$ *is a contractible coequalizer and in particular, for every* $X\in\mathcal{A}$

$$(RX, R\epsilon X)=\mathrm{Coequ}_{\mathcal{B}}\left(R\epsilon LRX, RLR\epsilon X\right).$$

Proof. Consider the following diagram

$$RLRLRX \underset{RLR\epsilon X}{\overset{R\epsilon LRX}{\underset{\eta RLRX}{\rightleftarrows}}\!\!\!\rightarrow} RLRX \underset{\eta RX}{\overset{R\epsilon X}{\rightleftarrows}} RX$$

and let us compute

$$(R\epsilon LRX)\circ(\eta RLRX)=\mathrm{Id}_{RLRX}$$
$$(RLR\epsilon X)\circ(\eta RLRX)\overset{\eta}{=}(\eta RX)\circ(R\epsilon X)$$
$$(R\epsilon X)\circ(\eta RX)=\mathrm{Id}_{RX}$$
$$(R\epsilon X)\circ(R\epsilon LRX)\overset{\epsilon}{=}(R\epsilon X)\circ(RLR\epsilon X).$$

Thus $(RX, RLRX, RLRLRX, R\epsilon X, R\epsilon LRX, RLR\epsilon X, \eta RX, \eta RLRX)$ is a contractible coequalizer for every $X\in\mathcal{A}$ and by Proposition 6.7 we get that $(RX, R\epsilon X)=\mathrm{Coequ}_{\mathcal{B}}\left(R\epsilon LRX, RLR\epsilon X\right)$. □

Lemma 6.71. *Let* (L,R) *be an adjunction where* $L:\mathcal{B}\to\mathcal{A}$ *and* $R:\mathcal{A}\to\mathcal{B}$, *let* $\mathbb{A}=(A,m_A,u_A)$ *be a monad on the category* $\mathcal{B}$ *and let* $\psi:\mathbb{A}=(A,m_A,u_A)\to\mathbb{RL}=(RL,R\epsilon L,\eta)$ *be a monad morphism. Let* $K_\psi=\Upsilon(\psi)=(R,(R\epsilon)\circ(\psi R))$ *and* ${}_{\mathbb{A}}UK_\psi(f)={}_{\mathbb{A}}U\Upsilon(\psi)(f)=R(f)$ *for every morphism f in* $\mathcal{A}$. *For every* $X\in\mathcal{A}$ *we have*

$$(K_\psi X, K_\psi\epsilon X)=\mathrm{Coequ}_{{}_{\mathbb{A}}\mathcal{B}}\left(K_\psi\epsilon LRX, K_\psi LR\epsilon X\right). \tag{6.23}$$

Proof. By Lemma 6.70 we have that $(RX, R\epsilon X) = \mathrm{Coequ}_{\mathcal{B}}(R\epsilon LRX, RLR\epsilon X)$. Let $\chi : K_\psi LRX = (RLRX, (R\epsilon LRX) \circ (\psi RLRX)) \to Q$ be a morphism in ${}_{\mathbb{A}}\mathcal{B}$ such that

$$\chi \circ (K_\psi \epsilon LRX) = \chi \circ (K_\psi LR\epsilon X).$$

Then

$$({}_{\mathbb{A}}U\chi) \circ (R\epsilon LRX) = ({}_{\mathbb{A}}U\chi) \circ (RLR\epsilon X) \tag{6.24}$$

and hence there exists a unique $\omega : {}_{\mathbb{A}}UK_\psi X = RX \to {}_{\mathbb{A}}UQ$ in $\mathcal{B}$ such that

$${}_{\mathbb{A}}U\chi = \omega \circ (R\epsilon X) = \omega \circ ({}_{\mathbb{A}}UK_\psi \epsilon X) \tag{6.25}$$

Let us prove that ω gives rise to a morphism in ${}_{\mathbb{A}}\mathcal{B}$. Since χ is a morphism in ${}_{\mathbb{A}}\mathcal{B}$ we have that

$$({}_{\mathbb{A}}U\chi) \circ (R\epsilon LRX) \circ (\psi RLRX) = ({}_{\mathbb{A}}U\lambda_A Q) \circ (A_{\mathbb{A}}U\chi) \tag{6.26}$$

Let us compute

$$\begin{aligned} ({}_{\mathbb{A}}U\lambda_A Q) \circ (A\omega) \circ (AR\epsilon X) &\overset{(6.25)}{=} ({}_{\mathbb{A}}U\lambda_A Q) \circ (A_{\mathbb{A}}U\chi) \\ &\overset{(6.26)}{=} ({}_{\mathbb{A}}U\chi) \circ (R\epsilon LRX) \circ (\psi RLRX) \\ &\overset{(6.24)}{=} ({}_{\mathbb{A}}U\chi) \circ (RLR\epsilon X) \circ (\psi RLRX) \\ &\overset{\psi}{=} ({}_{\mathbb{A}}U\chi) \circ (\psi RX) \circ (AR\epsilon X) \\ &\overset{(6.25)}{=} \omega \circ (R\epsilon X) \circ (\psi RX) \circ (AR\epsilon X) \end{aligned}$$

so that

$$({}_{\mathbb{A}}U\lambda_A Q) \circ (A\omega) \circ (AR\epsilon X) = \omega \circ (R\epsilon X) \circ (\psi RX) \circ (AR\epsilon X).$$

Since $(AR\epsilon X) \circ (A\eta RX) = ARX$, we deduce that $AR\epsilon X$ is epi and thus

$$({}_{\mathbb{A}}U\lambda_A Q) \circ (A\omega) = \omega \circ (R\epsilon X) \circ (\psi RX)$$

i.e. $\omega : {}_{\mathbb{A}}UK_\psi X = RX \to {}_{\mathbb{A}}UQ$ is a morphism of left $\mathbb{A}$-modules. □

Proposition 6.72. *Let (L, R) be an adjunction where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$, let $\mathbb{A} = (A, m_A, u_A)$ be a monad on the category $\mathcal{B}$ and let $\psi : \mathbb{A} = (A, m_A, u_A) \to \mathbb{RL} = (RL, R\epsilon L, \eta)$ be a monad morphism. Let $K_\psi = \Upsilon(\psi) = (R, (R\epsilon) \circ (\psi R))$ and ${}_{\mathbb{A}}UK_\psi(f) = {}_{\mathbb{A}}U\Upsilon(\psi)(f) = R(f)$ for every morphism f in $\mathcal{A}$. If ψY is an epimorphism for every $Y \in \mathcal{B}$, the assignment $\widetilde{\mathcal{K}}_{LRX,X'} : \mathrm{Hom}_{\mathcal{A}}(LRX, X') \to \mathrm{Hom}_{{}_{\mathbb{A}}\mathcal{B}}(K_\psi LRX, K_\psi X')$ defined by setting*

$$\widetilde{\mathcal{K}}_{LRX,X'}(f) = K_\psi(f)$$

is an isomorphism whose inverse is defined by

$$\widetilde{\mathcal{K}}^{-1}_{LRX,X'}(h) = (\epsilon X') \circ (L_{\mathbb{A}}Uh) \circ (L\eta RX).$$

Proof. Let $f \in \mathrm{Hom}_{\mathcal{A}}(LRX, X')$. We compute

$$\widetilde{\mathcal{K}}^{-1}_{LRX,X'}\left(\widetilde{\mathcal{K}}_{LRX,X'}(f)\right) = (\epsilon X') \circ (L_{\mathbb{A}} U K_{\psi} f) \circ (L\eta RX)$$
$$= (\epsilon X') \circ (LRf) \circ (L\eta RX) \overset{\epsilon}{=} f \circ (\epsilon LRX) \circ (L\eta RX) = f.$$

Let $h \in \mathrm{Hom}_{\mathbb{A}\mathcal{B}}(K_{\psi} LRX, K_{\psi} X')$. This means that

$$({}_{\mathbb{A}}Uh) \circ (R\epsilon LRX) \circ (\psi RLRX) = (R\epsilon X') \circ (\psi RX') \circ (A_{\mathbb{A}}Uh)$$
$$\overset{\psi}{=} (R\epsilon X') \circ (RL_{\mathbb{A}}Uh) \circ (\psi RLRX)$$

Since ψY is an epimorphism for every $Y \in \mathcal{B}$, we deduce that

$$({}_{\mathbb{A}}Uh) \circ (R\epsilon LRX) = (R\epsilon X') \circ (RL_{\mathbb{A}}Uh) \tag{6.27}$$

We compute

$$(R\epsilon X') \circ (RL_{\mathbb{A}}Uh) \circ (RL\eta RX) \overset{(6.27)}{=} ({}_{\mathbb{A}}Uh) \circ (R\epsilon LRX) \circ (RL\eta RX)$$
$$= {}_{\mathbb{A}}Uh$$

and thus

$$(K_{\psi}\epsilon X') \circ (K_{\psi} L_{\mathbb{A}}Uh) \circ (K_{\psi} L\eta RX) = h$$

i.e.

$$\widetilde{\mathcal{K}}_{LRX,X'}\left(\widetilde{\mathcal{K}}^{-1}_{LRX,X'}(h)\right) = K_{\psi}\left(\widetilde{\mathcal{K}}^{-1}_{LRX,X'}(h)\right)$$
$$= (K_{\psi}\epsilon X') \circ (K_{\psi} L_{\mathbb{A}}Uh) \circ (K_{\psi} L\eta RX) = h.$$

□

Proposition 6.73. *Let (L, R) be an adjunction where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$, let $\mathbb{A} = (A, m_A, u_A)$ be a monad on the category $\mathcal{B}$ and let $\psi : \mathbb{A} = (A, m_A, u_A) \to \mathbb{RL} = (RL, R\epsilon L, \eta)$ be a monad morphism. Let $K_{\psi} = \Upsilon(\psi) = (R, (R\epsilon) \circ (\psi R))$ and ${}_{\mathbb{A}}UK_{\psi}(f) = {}_{\mathbb{A}}U\Upsilon(\psi)(f) = R(f)$ for every morphism f in $\mathcal{A}$. If K_{ψ} is full and faithful then, for every $X \in \mathcal{A}$, we have*

$$(X, \epsilon X) = \mathrm{Coequ}_{\mathcal{A}}(LR\epsilon X, \epsilon LRX).$$

Proof. By Lemma 6.71 we have

$$(K_{\psi}X, K_{\psi}\epsilon X) = \mathrm{Coequ}_{\mathbb{A}\mathcal{B}}(K_{\psi}\epsilon LRX, K_{\psi}LR\epsilon X).$$

Then we can apply Lemma 5.37 and deduce that $(X, \epsilon X) = \mathrm{Coequ}_{\mathcal{B}}(\epsilon LRX, LR\epsilon X)$.
□

Theorem 6.74 (Generalized Beck's Theorem for Monads)**.** *Let (L,R) be an adjunction where $L:\mathcal{B}\to\mathcal{A}$ and $R:\mathcal{A}\to\mathcal{B}$, let $\mathbb{A}=(A,m_A,u_A)$ be a monad on the category $\mathcal{B}$ and let $\psi:\mathbb{A}=(A,m_A,u_A)\to\mathbb{RL}=(RL,R\epsilon L,\eta)$ be a monads morphism such that ψY is an epimorphism for every $Y\in\mathcal{B}$. Let $K_\psi=\Upsilon(\psi)=(R,(R\epsilon)\circ(\psi R))$ and ${}_{\mathbb{A}}UK_\psi(f)={}_{\mathbb{A}}U\Upsilon(\psi)(f)=R(f)$ for every morphism f in $\mathcal{A}$. Then $K_\psi:\mathcal{A}\to{}_{\mathbb{A}}\mathcal{B}$ is full and faithful if and only if for every $X\in\mathcal{A}$ we have that $(X,\epsilon X)=\mathrm{Coequ}_{\mathcal{A}}(LR\epsilon X,\epsilon LRX)$.*

Proof. If K_ψ is full and faithful then we can apply Proposition 6.73 to get that for every $X\in\mathcal{A}$ we have that $(X,\epsilon X)=\mathrm{Coequ}_{\mathcal{A}}(LR\epsilon X,\epsilon LRX)$.

Conversely assume that for every $X\in\mathcal{A}$ we have that $(X,\epsilon X)=\mathrm{Coequ}_{\mathcal{A}}(LR\epsilon X,\epsilon LRX)$. We want to prove that $\widetilde{\mathcal{K}}_{X,X'}$ is bijective for every $X,X'\in\mathcal{A}$. Let us consider the following diagram

$$\begin{array}{ccc}
0 & & 0\\
\downarrow & & \downarrow\\
\mathrm{Hom}_{\mathcal{A}}(X,X') & \overset{\widetilde{\mathcal{K}}_{X,X'}}{\dashrightarrow} & \mathrm{Hom}_{{}_{\mathbb{A}}\mathcal{B}}(K_\psi X,K_\psi X')\\
\mathrm{Hom}_{\mathcal{A}}(\epsilon X,X')\downarrow & & \downarrow \mathrm{Hom}_{{}_{\mathbb{A}}\mathcal{B}}(K_\psi\epsilon X,K_\psi X')\\
\mathrm{Hom}_{\mathcal{A}}(LRX,X') & \overset{\widetilde{\mathcal{K}}_{LRX,X'}}{\longrightarrow} & \mathrm{Hom}_{{}_{\mathbb{A}}\mathcal{B}}(K_\psi LRX,K_\psi X')\\
\mathrm{Hom}_{\mathcal{A}}(LR\epsilon X,X')\downdownarrows \mathrm{Hom}_{{}_{\mathbb{A}}\mathcal{B}}(\epsilon LRX,X') & & \mathrm{Hom}_{{}_{\mathbb{A}}\mathcal{B}}(K_\psi LR\epsilon X,K_\psi X')\downdownarrows \mathrm{Hom}_{{}_{\mathbb{A}}\mathcal{B}}(K_\psi\epsilon LRX,K_\psi X')\\
\mathrm{Hom}_{\mathcal{A}}(LRLRX,X') & \overset{\widetilde{\mathcal{K}}_{LRLRX,X'}}{\longrightarrow} & \mathrm{Hom}_{{}_{\mathbb{A}}\mathcal{B}}(K_\psi LRLRX,K_\psi X')
\end{array}$$

$$\begin{array}{ccc}
0 & & 0\\
\downarrow & & \downarrow\\
\mathrm{Hom}_{\mathcal{A}}(X,X') & \overset{\widetilde{\mathcal{K}}_{X,X'}}{\longrightarrow} & \mathrm{Hom}_{{}_{\mathbb{A}}\mathcal{B}}(K_\psi X,K_\psi X')\\
\mathrm{Hom}_{\mathcal{A}}(\epsilon X,X')\downarrow & & \downarrow \mathrm{Hom}_{{}_{\mathbb{A}}\mathcal{B}}(K_\psi\epsilon X,K_\psi X')\\
\mathrm{Hom}_{\mathcal{A}}(LRX,X') & \overset{\widetilde{\mathcal{K}}_{LRX,X'}}{\longrightarrow} & \mathrm{Hom}_{{}_{\mathbb{A}}\mathcal{B}}(K_\psi LRX,K_\psi X')\\
\mathrm{Hom}_{\mathcal{A}}(LR\epsilon X,X')\downdownarrows \mathrm{Hom}_{\mathcal{A}}(\epsilon LRX,X') & & \mathrm{Hom}_{{}_{\mathbb{A}}\mathcal{B}}(K_\psi LR\epsilon X,K_\psi X')\downdownarrows \mathrm{Hom}_{{}_{\mathbb{A}}\mathcal{B}}(K_\psi\epsilon LRX,K
\\
\mathrm{Hom}_{\mathcal{A}}(LRLRX,X') & \overset{\widetilde{\mathcal{K}}_{LRLRX,X'}}{\longrightarrow} & \mathrm{Hom}_{{}_{\mathbb{A}}\mathcal{B}}(K_\psi LRLRX,K_\psi X')
\end{array}$$

Since $(X,\epsilon X)=\mathrm{Coequ}_{\mathcal{A}}(LR\epsilon X,\epsilon LRX)$ the left column of the diagram is exact by Lemma 5.38. By Lemma 6.71 we have $(K_\psi X,K_\psi\epsilon X)=\mathrm{Coequ}_{{}_{\mathbb{A}}\mathcal{B}}(K_\psi\epsilon LRX,K_\psi LR\epsilon X)$ so that also the right column is exact by Lemma 5.38. Let $f\in\mathrm{Hom}_{\mathcal{A}}(X,X')$ and $g\in\mathrm{Hom}_{\mathcal{A}}(LRX,X')$. Since

$$K_\psi(f\circ(\epsilon X))=(K_\psi f)\circ(K_\psi\epsilon X)$$

$$K_\psi(g\circ(\epsilon LRX))=(K_\psi g)\circ(K_\psi\epsilon LRX)\ \text{ and }\ K_\psi(g\circ(LR\epsilon X))=(K_\psi g)\circ(K_\psi LR\epsilon X)$$

the diagram is serially commutative. By Proposition 6.72, $\widetilde{\mathcal{K}}_{LRX,X'}$ and $\widetilde{\mathcal{K}}_{LRLRX,X'}$ are isomorphisms and so is $\widetilde{\mathcal{K}}_{X,X'}$ by Lemma 5.35. □

Corollary 6.75 (Beck's Theorem for Monads). *Let (L, R) be an adjunction where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$. Then $K = \Upsilon(\mathrm{Id}_{RL}) : \mathcal{A} \to {}_{\mathbb{RL}}\mathcal{B}$ is full and faithful if and only if for every $X \in \mathcal{A}$ we have that $(X, \epsilon X) = \mathrm{Coequ}_{\mathcal{A}}(LR\epsilon X, \epsilon LRX)$.*

6.7 Grothendieck

Let $\mathcal{C}$be an abelian category. Following Grothendieck's terminology we say that

$AB3$= cocomplete$\Rightarrow \mathcal{C}$ has inductive limits
$AB3^*$=complete$\Rightarrow \mathcal{C}$ has projective limits
$AB4$=the direct sum $\bigoplus_{i\in I} f_i$of a family $(f_i)_{i\in I}$ of monomorphisms is a monomorphism=direct sums are left exact
$AB4^*$=the direct product $\prod_{i\in I} f_i$of a family $(f_i)_{i\in I}$ of epimorphisms is an epiomorphism=direct product are left exact.
$AB5$=direct inductive limits are exact.

Theorem 6.76. *(Popescu Proposition 8.5 page 54)Let $\mathcal{C}$be an $AB3$-category and an $AB3^*$-category. TFAE.*

(a) For any family of objects $(X_i)_{i\in I}$ of $\mathcal{C}$,the canonical morphism $t : \bigoplus\limits_{i\in I} X_i \longrightarrow \prod\limits_{i\in I} X_i$ is a monomorphism.

(b) If $(X_i)_{i\in I}$ is a family of objects of $\mathcal{C}$ and $f : Y \to \bigoplus\limits_{i\in I} X_i$ is a morphism such that $p_i f = 0$ for any $i \in I$, then $f = 0$

Following Mitchell, we say that $\mathcal{C}$is a C_2-category if $\mathcal{C}$is both an $AB3$-category and an $AB3^*$-category satisfying the equivalent conditions of the previous Theorem.

Theorem 6.77. *(Popescu Corollary 8.10 page 61) Let $\mathcal{C}$be an $AB5$-category and an $AB3^*$-category. then is a$\mathcal{C}$is a C_2-category.*

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