Module Theory

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Chapter 1

Modules

1.1 Homomorphisms and Quotients

Definition 1.1. *Let R be a ring. A* ***left*** *R-**module** is a pair $(M, {}^{R}\mu_M)$ where $(M+, 0)$ is an abelian group and*

$$\mu = {}^{R}\mu_M : R \times M \to M$$

is a map such that, setting

$$a \cdot x = \mu\left((a, x)\right),$$

the following properties are satisfied :

M1 $a \cdot (x + y) = a \cdot x + a \cdot y$;

M2 $(a + b) \cdot x = a \cdot x + b \cdot x$;

M3 $(a \cdot_R b)\, x = a \cdot (b \cdot x)$;

M4 $1_R \cdot x = x$

for every $a, b \in R$ and every $x, y \in M$.
In this case we will say that M is a ***left*** *R-**module**. The notation ${}_RM$ will be used to mean that M is a left R-module.*

Definition 1.2. *Let R be a ring and let R^{op} denote the opposite ring of R. A* ***right*** *R-**module** is a left R^{op}-module i.e. it is a pair (M, μ') where $(M+, 0)$ is an abelian group and*

$$\mu' = \mu_M^R : R \times M \to M$$

is a map such that, setting

$$a \cdot x = \mu'\left((a, x)\right),$$

M1′ $a \cdot (x + y) = a \cdot x + a \cdot y$;

M2' $(a+b)\cdot x = a\cdot x + b\cdot x$;

M3' $(a\cdot_{R^{op}} b)\cdot x = a\cdot(b\cdot x)$;

M4' $1_R\cdot x = x$

for every $a, b \in R$ *and every* $x, y \in M$.
In this case we will say that M *is a* ***right*** R-***module***. *The notation* M_R *will be used to mean that* M *is a right* R-*module.*
Note that

$$a\cdot_{R^{op}} b = b\cdot_R a$$

so that M3' rewrites as

$$(a\cdot_R b)\cdot x = (b\cdot_{R^{op}} a)\cdot x = b\cdot(a\cdot x)$$

For this reason, if M *is a right* R-*module, one usually writes* $x\cdot a$ *instead of* $a\cdot x$, *for every* $a\in R$, $x\in M$. *With this notation the conditions M1'), M2'), M3'), M4') may be rephrased as follows:*

M1" $(x+y)\cdot a = x\cdot a + y\cdot a$;

M2" $x\cdot(a+b) = x\cdot a + x\cdot b$;

M3" $x\cdot(a\cdot_R b) = (x\cdot a)\cdot b$;

M4" $x\cdot 1_R = x$.

The abelian group M *is called the* ***underlying additive group*** *of the left (resp. right)* R-*module* M.
Given $x, y \in M$ *we will write* $x-y$ *instead of* $x+(-y)$.

Remark 1.3. *If* R *is a commutative ring, then every left* R-*module is, in a natural way, a right* R-*module, and conversely.*
In fact, let M *be a left* R-*module, given* $a, b\in R$, $x\in M$, *we have*

$$a\cdot(b\cdot x) = (a\cdot_R b)\,x = (b\cdot_R a)\,x = b\cdot(a\cdot x)\,.$$

In the same way, if M *is a right* R-*module, given* $a, b\in R$, $x\in M$, *we have:*

$$(x\cdot a)\cdot b = x\,(a\cdot_R b) = x\cdot(b\cdot_R a) = (x\cdot b)\cdot a.$$

Therefore, when R *is a commutative ring, we will, in general, simply say that* M *is an* R-*module.*

Examples 1.4.

1. *Let G be an abelian group with additive notation. G becomes, in a natural way, a $\mathbb{Z}$- module by defining, for every $n \in \mathbb{Z}$ and $x \in G$,*

$$n \cdot x = nx$$

where nx denotes the nth power of x in the additive notation.

2. *Let A be a ring, R be a subring of A. A becomes a left (resp. right) R-module by setting, for every $r \in R$, $a \in A$, ra (resp. ar) to be the product of the element $r \in R \subseteq A$ with the element $a \in A$ (resp. of the element $a \in A$ with the element $r \in R \subseteq A$) in the ring A.*
In particular the rings $R, R[X], R[[X]]$ may be considered as left (resp. right) R-modules.
If D is a commutative domain, $Q(D)$ is a D-module.

3. *More generally, let $f : R \to A$ be a ring homomorphism. Any left A-module $\left(M, {}^{A}\mu_M\right)$ inherits the structure of a left R-module by setting*

$${}^{R}\mu_M\left((r, x)\right) = {}^{A}\mu_M\left((f(r), x)\right) \text{ for every } r \in R \text{ and } x \in M$$

i.e.

$$r \cdot x = f(r) \cdot x \text{ for every } r \in A \text{ and } x \in M.$$

This module is often denoted by $f_*(M)$ and called the R-module obtained by restriction *of the ring of scalars from A to R.*

1.5. If R is a division ring and M is a left (resp. right) R-module we say that M is a **left (resp. right) vector space over** R. If R is a field, we simply say that M is a **vector space over** R.

Proposition 1.6. *Let R be a ring, M a left R-module.*
Then, for every $a, b \in R$ and for every $x, y \in M$ we have :

1. $a \cdot 0_M = 0_M$;

2. $0_R \cdot x = 0_M$;

3. $(-a) \cdot x = -a \cdot x = a \cdot (-x)$; $(-a) \cdot (-x) = a \cdot x$;

4. $a \cdot (x - y) = a \cdot x - a \cdot y$;

5. $(a - b) \cdot x = a \cdot x - b \cdot x$.

6. $n(a \cdot x) = (na) \cdot x = a \cdot (nx)$ *for every* $n \in \mathbb{Z}, a \in R, x \in M$.

Proof. 1) Let us start from : $a\cdot 0_M = a\,(0_M + 0_M) = a\cdot 0_M + a\cdot 0_M$. Adding $-\,(a\cdot 0_M)$ to both sides we find : $0_M = a\cdot 0_M$.

2) First we look at the obvious : $0_R\cdot x = (0_R + 0_R)\,x = 0_R\cdot x + 0_R\cdot x$. Adding $-\,(x\cdot 0_R)$ to both sides we find : $0_M = 0_R\cdot x$.

3) From $(-a)\,x + ax = ((-a)+a)x = 0_R\cdot x = 0_M$ we obtain that $(-a)\,x = -ax$. In a similar way it follows from

$$ax + a\,(-x) = a(x + (-x)) = a\cdot 0_M = 0_M$$

that $a\,(-x) = -ax$. Moreover : $(-a)\,(-x) = -(a(-x)) = -(-(ax)) = ax$.

4) We calculate:

$$a(x-y) = a(x+(-y)) = ax + a(-y) = ax + (-(ay)) = ax - ay.$$

5) We calculate:

$$(a-b)\,x = (a+(-b))x = ax + (-b)\,x = ax + (-bx) = ax - bx.$$

6) It is easily proved by Induction. □

1.7. *Let M be an abelian group and let $A = \mathrm{End}(M)$ denote the ring of endomorphisms of M. Then M becomes a left A-module by setting*

$$f\cdot x = f\,(x) \text{ every } f\in A \text{ and } x\in M.$$

In fact,note that

$$(f\cdot_A g)\,x = (f\circ g)\cdot x = (f\circ g)\,(x) = f\,(g\,(x)) = f\cdot(g\cdot x) \text{ for every } f,g\in A \text{ and } x\in M.$$

Now let $\varphi : R \to \mathrm{End}(M)$ be a ring morphism. Then, in view of Example 3 in 1.4, we can consider the left R-module $\varphi_*\,(M)$ i.e. M becomes a left R-module by setting

$$r\cdot m = \varphi(r)(m) \text{ for all } r\in R \text{ and for all } m\in M.$$

Conversely let M be a left R-module and let $\mathrm{End}(M)$ denote the ring of endomorphisms of the abelian group underlying the R-module structure of M. For every $r\in R$ consider the map

$$\begin{array}{rccc} t_r : & M & \to & M \\ & m & \mapsto & r\cdot m \end{array}.$$

Clearly $t_r \in \mathrm{End}(M)$ and the map

$$\begin{array}{rccc} \psi : & R & \to & \mathrm{End}(M) \\ & r & \mapsto & t_r \end{array}$$

is a ring morphism. In this way we get:

Theorem 1.8. *Let R be a ring and let M be an abelian group. The ring morphisms $\varphi : R \to \mathrm{End}(M)$ correspond bijectively to the left R-module structures on M.*

Proof. Using notation as above, given a ring morphisms $\varphi : R \to \mathrm{End}(M)$ we have:

$$\psi(r)(m) = r \cdot m = \varphi(r)(m).$$

Conversely, if M is a left R-module we have: $r \cdot m = \psi(r)(m)$.□ □

To get an analogous result for right R-modules we have to consider the ring $\mathrm{End}(M)^{op}$ which has the same addition as $\mathrm{End}(M)$ but where multiplication is defined by

$$f \cdot g = g \circ f.$$

Rephrasing the foregoing theorem we obtain :

Theorem 1.9. *Let R be a ring and let M be an abelian group. The ring morphisms $\varphi : R \to \mathrm{End}(M)^{op}$ correspond bijectively to the right R-module structures on M.*

Definitions 1.10. *Let R be a ring and let M be a left R-module. A subset N of M is said to be an R-**submodule** (or simply **submodule**) of M if :*

1. *N is a subgroup of M;*
2. *$a \in R$ and $x \in N$ implies that $a \cdot x \in N$, for every $a \in R$ and $x \in N$.*

We write $N \leq_R M$ to mean that N is a submodule of M.
We denote by $\mathcal{L}(_R M)$ the set of all the submodules of $_R M$. Given a subset X of M we set $\mathcal{L}(_R M, X) = \mathcal{L}(_R M) \cap \mathcal{L}(M, X)$.

Remark 1.11. *If N is a submodule of a left module M, then N is itself an R-module with respect to*

$$\begin{array}{rccc} f: & R \times N & \to & N \\ & (a, x) & \mapsto & a \cdot x \end{array}$$

where $a \cdot x$ is the product of a and x in M.

Examples 1.12.

1. *Let R be a ring. Then the submodules of $_R R$ are exactly the left ideals of R.*
2. *Let R be a ring. For every $n \in \mathbb{N}$ we let*

   $$I_n = \{f \in R[X] \mid \deg(f) \leq n\}$$

   I_n is a subgroup of $R[X]$ as, given $f, g \in R[X]$

   $$\deg(f) + \deg(-g) \leq \max(\deg(f), \deg(g))$$

   I_n is not an ideal of $R[X]$ (why?), but it is a submodule of $R[X]$ considered as a left module on R. In fact, for every $r \in R$, $f \in R[X]$ we have $\deg(rf) \leq \deg(f)$.

Proposition 1.13. *Let R be a ring and let M be a R-left module. A subset N of M is a submodule of M if and only if :*

1. $N \neq \varnothing$;
2. *for every $x, y \in N$ we have that $x + y \in N$;*
3. *for every $a \in R$, $x \in N$ we have that $a \cdot x \in N$.*

Proof. Let N be a subset of M such that 1), 2) and 3) are verified. For every $x, y \in N$ we have that

$$x - y = x + (-1)y$$

and hence $x - y \in N$. Therefore N is a subgroup and, by 3), also a submodule of M
The converse is trivial. □ □

Definitions 1.14. *Let M, M' be left modules over the ring R. A map $f : M \to M'$ is called a (left) R-**module homomorphism** if :*

1. *f is a group homomorphism, that is if, for every $x, y \in M$ we have*

$$f(x + y) = f(x) + f(y);$$

2. *for every $r \in R$ and for every $x \in M$ we have*

$$f(r \cdot x) = r \cdot f(x)$$

If $f : M \to M'$ is an R-module homomorphism we say that:

- *f is an **injective homomorphism** if the map f is injective ;*
- *f is a **surjective homomorphism** if the map f is surjective ;*
- *f is an **isomorphism** if the map f is bijective.*

*We will say that M and M' are **isomorphic** and we will write $M \cong M'$ if there exists an isomorphism $f : M \to M'$. Observe that, in this case, the inverse map of f, $f^{-1} : M' \to M$ is also a module isomorphism (the proof is left as an exercise).*

1.15. The definitions of submodule of a right R-module and of right R-module homomorphism are similar to those given in 1.10 and 1.14.
If R is a division ring, the submodules of a left (resp. right) R-module are called **subspaces** of M and the R-module homomorphisms are also called **vector spaces homomorphisms** or **linear maps**.

Example 1.16. *Let R be a ring. Given an element $a \in R$ the map*

$$\begin{array}{rccc} \mu_a : & R & \to & R \\ & r & \mapsto & r \cdot_R a \end{array}$$

is a left R-module homomorphism from ${}_R R$ into ${}_R R$. Observe that, if $a \neq 1$, then μ_a is not a ring homomorphism.

1.2 Quotient Module and Isomorphism Theorems

Theorem 1.17 (Correspondence Theorem for Submodules)**.**
Let R be a ring and let $f : M \to M'$ be a left R-module homomorphism. Then

1. *if $L \leq_R M$, $f(L) \leq_R M'$;*
2. *if $L' \leq_R M'$, $f^{\leftarrow}(L') \leq M$.*

Hence, in particular :

$$\mathrm{Im}(f) = f(M) \leq_R M' \text{ and } \mathrm{Ker}(f) = f^{\leftarrow}(\{0_{M'}\}) \leq_R M.$$

The assignment $L \mapsto f(L)$ defines a partially ordered set homomorphism

$$\phi : \mathcal{L}({}_R M, \mathrm{Ker}(f)) \to \mathcal{L}({}_R \mathrm{Im}(f))$$

whose inverse,

$$\phi^{-1} : \mathcal{L}({}_R \mathrm{Im}(f)) \to \mathcal{L}({}_R M, \mathrm{Ker}(f)),$$

is defined by $\phi^{-1}(L') = f^{\leftarrow}(L')$.
In particular the submodules of $\mathrm{Im}(f)$ are exactly those the form $f(L)$ where L is a submodule of M containing $\mathrm{Ker}(f)$.

Proof. Exercise. □ □

Theorem 1.18. *Let R be a ring, let M be a left R-module and let N be a submodule of M. We define a left R-module structure on the abelian group M/N by setting, for every $r \in R$ and for every $x \in M$,*

$$r \cdot (x + N) = (r \cdot x) + N.$$

Moreover, with respect to this structure, the canonical projection $p_N : M \to M/N$ becomes a surjective R-module homomorphism.

Proof. We have first to show that (1) is well defined, that is, given any $r \in R$, $x, x' \in M$ such that $x+N = x'+N$ (i.e. $x - x' \in N$), we have that $(r \cdot x)+N = 9r \cdot x' + N$ (i.e. $r \cdot x - r \cdot x' \in N$).
But $x - x' \in N$ implies that $r \cdot x - r \cdot x' = r \cdot (x - x') \in N$ as N is a submodule of M.
Let now $a, b \in R$, $x, y \in R$. We have:

$$a \cdot [(x+N)+(y+N)] = a \cdot [(x+y)+N] = (a \cdot (x+y)) + N = (a \cdot x + a \cdot y) + N = \\ = (a \cdot x + N) + (a \cdot y + N) = a \cdot (x+N) + a \cdot (y+N);$$

$$(a+b) \cdot (x+N) = ((a+b) \cdot x) + N = (a \cdot x + b \cdot x) + N \\ = (a \cdot x + N) + (b \cdot x + N) = a \cdot (x+N) + b \cdot (x+N);$$

$$(a \cdot b)(x+N) = ((a \cdot_R b)x) + N = (a \cdot (b \cdot x)) + N = a \cdot (b \cdot x + N) = a \cdot (b \cdot (x+N));$$

$$1_R \cdot (x+N) = (1_R \cdot x) + N = x + N.$$

Finally:

$$p_N(a \cdot x) = a \cdot x + N = a \cdot (x+N) = a \cdot p_N(x).$$

□

Definition 1.19. *Let M be a left module over a ring R and let N be a submodule of M. The left R-module (defined in Theorem 1.18) having the quotient group M/N for its underlying abelian group is called the **quotient module** (or a **factor module**) **of** M **modulo** N and is denoted by ${}_R(M/N)$ or simply by M/N.*

Theorem 1.20 (Fundamental Theorem for Quotient Modules)**.** *Let R be a ring and let $f : M \to M'$ be a left R-module homomorphism. If N is a submodule of M contained in $\mathrm{Ker}(f)$, then there exists an R-module homomorphism $\bar{f} : M/N \to M'$ such that the diagram*
commutes, i.e. $f = \bar{f} \circ p_N$.
Moreover:

1. *$\bar{f}$ is unique with respect to this property;*
2. *$\mathrm{Im}(f) = \mathrm{Im}(\bar{f})$ and $\mathrm{Ker}(\bar{f}) = \mathrm{Ker}(f)/N$;*
3. *$\bar{f}$ is injective $\Leftrightarrow N = \mathrm{Ker}(f)$.*

Proof. In view of the Fundamental Theorem for the Quotient Group there exists a group homomorphism $\bar{f} : M/N \to M'$ such that $f = \bar{f} \circ p_N$. Moreover: 1) such a group homomorphism is unique; 2) $\mathrm{Im}(f) = \mathrm{Im}(\bar{f})$, $\mathrm{Ker}(\bar{f}) = \mathrm{Ker}(f)/N$; 3) $\bar{f}$ is injective $\Leftrightarrow N = \mathrm{Ker}(f)$.
Hence we only have to prove that, for every $x \in M$ and $r \in R$:

$$\bar{f}(r(x+N)) = r \cdot \bar{f}(x+N).$$

It is now an easy calculation to arrive at:

$$\bar{f}(r\cdot(x+N)) = \bar{f}(r\cdot x+N) = \bar{f}(p_N(r\cdot x)) = f(r\cdot x) = r\cdot f(x) = r\cdot\bar{f}(p_N(x)) = r\cdot(x+N).$$

□ □

Corollary 1.21 (First Isomorphism Theorem for Modules)**.**
Let R be a ring and $f : M \to M'$ be a left R-module homomorphism. Then the assignment

$$x + \mathrm{Ker}\,(f) \mapsto f\,(x)$$

defines an isomorphism of left R-modules

$$\hat{f} : M/\mathrm{Ker}\,(f) \to \mathrm{Im}\,(f)$$

In particular, if f is surjective, then $\hat{f}$ is an isomorphism and

$$M/\mathrm{Ker}\,(f) \cong M'.$$

Theorem 1.22 (Second Isomorphism Theorem for Modules)**.** *Let L and N be submodules of a module M over a ring R. Then $L \cap N$ and $L + N$ are submodules of M and the assignment $x + (L \cap N) \mapsto x + N$ defines an R-module isomorphism from $L/(L \cap N)$ into $L + N/N$. Therefore:*

$$L/(L \cap N) \cong L + N/N$$

Proof. We know that $L \cap N$ is a subgroup of M. Let $r \in R$, $z \in L \cap N$. Then $rz \in L$ and $rz \in N$, as L and N are submodules of M. Therefore $r \cdot z \in L \cap N$.
We know that $L + N$ is a subgroup of M. Let $r \in R$, $z \in L + N$. Then there exist $x \in L$ and $y \in N$ such that $z = x + y$. Obviously $rx \in L$ and $ry \in N$, and hence $r \cdot z = r \cdot x + r \cdot y \in L + N$.
In view of the Second Isomorphism Theorem for Groups, the assignment $x + (L \cap N) \mapsto x + N$ defines a group isomorphism

$$\varphi : L/(L \cap N) \to L + N/N.$$

Let $r \in R$, $x \in L$, then we calculate:

$$\varphi(r(x + (L \cap N)) = \varphi(rx + (L \cap N)) = rx + N = r(x + N) = r\varphi(x + (L \cap N)).$$

Therefore φ is a left R-module isomorphism. □ □

Theorem 1.23. *Let R be a ring, $f : M \to M'$ be a left R-module homomorphism. For every submodule N of M containing $\mathrm{Ker}\,(f)$ the assignment $x + N \mapsto f\,(x) + f\,(N)$ defines an isomorphism $\hat{f}_N : M/N \to \mathrm{Im}\,(f)\,/f(N)$. Therefore*

$$M/N \cong \mathrm{Im}\,(f)\,/f\,(N)\,.$$

Proof. We know that the assignment $x + N \mapsto f(x) + f(N)$ defines a group isomorphism $\psi = \hat{f}_N : M/N \to \mathrm{Im}(f)/N$. Let $r \in R$, $x \in N$. We have :

$$\psi(r(x+N)) = \psi(rx+N) = f(rx) + f(N) = (rf(x)) + f(N)$$
$$= r(f(x) + f(N)) = r\psi(x+N)$$

Therefore ψ is a left R-module isomorphism. □ □

Corollary 1.24 (Third Isomorphism Theorem for Modules)**.** *Let L and N be submodules of a module M over a ring R and assume that $L \subseteq N$.*
Then the assignment $x + N \mapsto (x + L) + N/L$. defines a left R-module isomorphism from M/L into $M/L/N/L$. Therefore

$$M/N \cong M/L/N/L.$$

Proof. Apply Theorem 1.23 to $p_L : M \to M/L$, recalling that $p_L(N) = N/L$. □

1.3 Product and Direct Sum of a Family of Modules

1.25. Let I and A be nonempty sets. At places, in mathematical literature, a map $f : I \to A$ is called a **family of elements of** A **indexed by** I and we write

$$f = (a_i)_{i \in I} \text{ or } f = (a_i) \text{ where } a_i = f(i) \text{ for every } i \in I\,.$$

In this context the elements of I are called **indexes** and, for every $i \in I$, a_i is called the **i-th element** of the family.
The use of this terminology and notation is traditionally reserved for particular situations. As we do not think that this is the right place to deal with this argument, we will simply use the above terminology and notation, whenever it will be convenient.

In any case the reader should carefully note the difference between the family $(a_i)_{i \in I}$, which is a map from I to A, and the set $\{a_i \mid i \in I\}$, which is the image of the previous map.
In fact, it may happen that $a_i = a_j$ for two distinct indexes $i, j \in I$. It may even happen that the set $\{a_i \mid i \in I\}$ consists of only one element! In this case the family $(a_i)_{i \in I}$ is also called **constant** (in fact, it is a constant map!).
Let $(a_i)_{i \in I}$ be a family of elements of A indexed by I, $(b_j)_{j \in J}$ a family of elements of B indexed by J. Observe that these families are equal if and only if $I = J$, $A = B$ and $a_i = b_i$ for every $i \in I$.
A family of elements of A indexed by $\mathbb{N}$ is called a **sequence of elements of** A.
A family of elements of A indexed by the set $\{1, 2, \dots, n\}$ is usually called an **n-tuple of elements of** A. In this case we write $(a_1, \dots, a_n)$ instead of $(a_i)_{i \in I}$ and a_i, with $1 \le i \le n$, is called the **i-th element** (or **i-th coordinate**) of the n- tuple.
Note that, by the above considerations, two n- tuples of elements of A, $(a_1, \dots, a_n)$

and $(a'_1, \ldots, a'_n)$ coincides if and only if $a_i = a'_i$ for every $i \in \{1, \ldots, n\}$.
We often consider families of sets, i.e. families $(X_i)_{i\in I}$ such that X_i is a set, for every $i \in I$.
If $(X_i)_{i\in I}$ is a family of sets, we define its **union**, denoted by $\bigcup_{i\in I} X_i$, and we read it "union of the $X'_i s$, i ranging in I", as the union of the set of sets $\{X_i \mid i \in I\}$. Thus:

$$\bigcup_{i\in I} X_i = \{x \mid x \in X_i \text{ for some } i \in I\} = \{x \mid \exists\, i \in I \text{ such that } x \in X_i\} .$$

Analogously we define the **intersection** of this family, denoted by $\bigcap_{i\in I} X_i$, and we read it "intersection of the $X'_i s$, i ranging in I", as the intersection of the set of sets $\{X_i \mid i \in I\}$. Thus:

$$\bigcap_{i\in I} X_i = \{x \mid x \in X_i \text{ for every } i \in I\} .$$

If $I = \{1, 2, \ldots, n\}$ we use the notations $\bigcup_{i=1}^{n} X_i$ or $X_1 \cup \ldots \cup X_n$ instead of $\bigcup_{i\in I} X_i$ and the notations $\bigcap_{i=1}^{n} X_i$ or $X_1 \cap \ldots \cap X_n$ instead of $\bigcap_{i\in I} X_i$.
Let $(X_i)_{i\in I}$ be a family of sets. We say that the sets of this family are **pairwise disjoint** if, given $i, j \in I$, from $i \neq j$ it follows that $X_i \cap X_j = \varnothing$. In this case, obviously we have $\bigcap_{i\in I} X_i = \varnothing$.
We remark here that to give a family of sets usually one just gives the set I of indexes and, for every $i \in I$, a set X_i. In fact, the codomain of the family itself, thought of as being a map, is understood to be clear from the context.

Definition 1.26. *Let $(A_i)_{i\in I}$ be a family of nonempty sets. The* ***Cartesian product*** *of such a family is the set, denoted by $\prod_{i\in I} A_i$, to be read "Cartesian product of the A_i's, i ranging in I" given by*

$$\prod_{i\in I} A_i = \left\{ f : I \to \bigcup_{i\in I} A_i \;\mid f(i) \in A_i \;\textit{for every}\; i \in I \right\}.$$

According to 1.25, with the same notations, we write:

$$\prod_{i\in I} A_i = \left\{ (a_i)_{i\in I} \mid a_i \in A_i \;\textit{for every}\; i \in I \right\}.$$

If for every $i \in I$, $A_i = A$ then the set $\prod_{i\in I} A_i$ is usually denoted by A^I and we have:

$$A^I = \{f : I \to A\} .$$

If $I = \{1, 2, \ldots, n\}$ we write $A_1 \times \ldots \times A_n$ or We have

$$A_1 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \;\textit{for every}\; i = 1, \ldots, n\} .$$

If $A_1 = A_2 = \ldots = A_n = A$ we write A^n instead of $A_1 \times \ldots \times A_n$.

1.27. Let $(G_i)_{i\in I}$ be a family of groups. We can state, without using the Axiom of Choice, that $\prod_{i\in I} G_i \neq \varnothing$. In fact, let 1_i be the identity element of G_i. The map

$$f : I \to \bigcup_{i\in I} G_i$$

defined by letting $f(i) = 1_i$ for all $i \in I$, i.e. $f = (1_{G_i})_{i\in I}$, is an element of $\prod_{i\in I} G_i$.
Now we can define a group structure on $\prod_{i\in I} G_i$, as follows.
We define an inner composition law on $\prod_{i\in I} G_i$ by letting, for all $i \in I$ and for every $x = (x_i)_{i\in I}$, $y = (y_i)_{i\in I} \in \prod_{i\in I} G_i$

$$(xy)_i = x_i y_i.$$

Proposition 1.28. *Let $(G_i)_{i\in I}$ be a family of groups. Then, using the $\cdot$ composition law defined in 1.27, $\prod_{i\in I} G_i$ is a group whose identity element is $(1_i)_{i\in I}$.*

Definition 1.29. *Let $(G_i)_{i\in I}$ be a family of groups. In the notations of Proposition 1.28, the group $(\prod_{i\in I} G_i, \cdot, (1_i)_{i\in I})$ is called the* ***direct product of the family of groups*** *$(G_i)_{i\in I}$ and will be simply denoted by $\prod_{i\in I} G_i$. If $I = \{1, 2, .., n\}$ we write $G_1 \times G_2 \times ... \times G_n$ instead of $\prod_{i\in I} G_i$. If $G_i = G$ for all i, then we also write G^I and G^n if $I = \{1, 2, .., n\}$.*

1.30. Let $(G_i)_{i\in I}$ be a family of groups. Consider, for all $j \in I$, the map $\pi_j : \prod_{i\in I} G_i \to G_j$ defined by setting $\pi_j((x_i)_{i\in I}) = x_j$ for all $(x_i)_{i\in I}$. π_j is called the **j-th canonical projection**.

Lemma 1.31. *Let $(A_i)_{i\in I}$ be a family of nonempty sets and let $x \in \prod_{i\in I} A_i$. Then*

$$x = (\pi_i(x))_{i\in I}$$

Therefore if $x, y \in \prod_{i\in I} A_i$, we have

$$x = y \Leftrightarrow \pi_i(x) = \pi_i(y) \ \text{for every } i \in I.$$

Proof. Let $x = (x_i)_{i\in I} \in \prod_{i\in I} G_i$. For every $j \in I$ we have $x_j = \pi_j(x)$, and hence

$$x = (\pi_i(x))_{i\in I}.$$

□

Theorem 1.32. ***(Universal Property of the Direct Product of a family of Groups)*** *Let $(G_i)_{i\in I}$ be a family of groups. Then, for all $j \in I$, the canonical projection $\pi_j : \prod_{i\in I} G_i \to G_j$ is an epimorphism of groups. Moreover, for any group G and any family $(f_i)_{i\in I}$ of homomorphisms $f_i : G \to G_i$, there exists a unique homomorphism $f : G \to \prod_{i\in I} G_i$ such that $\pi_i \circ f = f_i$ for all $i \in I$. This homomorphism is called the* ***diagonal homomorphism of the family*** *$(f_i)_{i\in I}$ of group homomorphism and will be denoted by $\Delta((f_i)_{i\in I})$.*

Proof. Let $j \in I$.
The map $\pi_j : \prod_{i\in I} G_i \to G_j$ is surjective. In fact let $x_j \in G_j$. Consider the element $g = (g_i)_{i\in I} \in \prod_{i\in I} G_i$ defined by $g_i = 1_{G_i}$ for all $i \in I \setminus \{j\}$ and $g_j = x_j$. Then $\pi_j(g) = g_j = x_j$.
The map π_j is a homomorphism. Let $x = (x_i)_{i\in I}$, $y = (y_i)_{i\in I} \in \prod_{i\in I} G_i$. Then

$$\pi_j(xy) = \pi_j((x_i y_i)_{i\in I}) = x_j y_j = \pi_j(x)\,\pi_j(y).$$

Let now $(f_i)_{i\in I}$ be a family of homomorphisms, $f_i : G \to G_i$. We define a map $f : G \to \prod_{i\in I} G_i$ by setting $f(g) = (f_i(g))_{i\in I}$ for all $g \in G$.
f is a homomorphism. Let $g, h \in G$, then:

$$\begin{aligned} f(gh) &= (f_i(gh))_{i\in I} = (f_i(g)f_i(h))_{i\in I} = \\ &= (f_i(g))_{i\in I}\,(f_i(h))_{i\in I} = f(g)f(h). \end{aligned}$$

Given $j \in I$, $\pi_j \circ f = f_j$. In fact, for all $g \in G$,we have :

$$(\pi_j \circ f)(g) = \pi_j((f(g))_{i\in I}) = f_j(g).$$

Let now let $f' : G \to \prod_{i\in I} G_i$ be another homomorphism such that $\pi_i \circ f' = f_i$ for all $i \in I$.
Then by Lemma 1.31 ,

$$f'(g) = (\pi_i(f'(g)))_{i\in I} = ((\pi_i \circ f')(g))_{i\in I} = (f_i(g))_{i\in I} = f(g)$$

for all $g \in G$. □ □

1.33. Let $(G_i)_{i\in I}$ be a family of groups, in additive notation. Given an element $x = (x_i)_{i\in I} \in \prod_{i\in I} G_i$, we set

$$Supp\,(x) = \{i \in I \mid x_i \neq 0_{G_i}\}\,.$$

$Supp\,(x)$ is called the **support** of x. Let F be the subset of $\prod_{i\in I} G_i$ consisting of all the elements with finite support. Obviously the identity element $0 = (0_{G_i})_{i\in I}$ of

$\prod_{i \in I} G_i$ has finite support (it is the only element with empty support); moreover, if $x = (x_i)_{i \in I}$ and $y = (y_i)_{i \in I}$ have finite support also their difference $x - y = (x_i - y_i)_{i \in I}$. In fact

$$Supp(x - y) \subset Supp\,(x) \cup Supp(y).$$

Therefore F is a subgroup of $\prod_{i \in I} G_i$.

Definitions 1.34. *Let $(G_i)_{i \in I}$ be a family of abelian groups. The subgroup of $\prod_{i \in I} G_i$ consisting of all the elements with finite support is called **direct sum** of the family of groups $(G_i)_{i \in I}$ and is denoted by $\bigoplus_{i \in I} G_i$.*
If, for all $i \in I$, $G_i = G$, then the direct sum of the family of groups $(G_i)_{i \in I}$ is also denoted by $G^{(I)}$.

Remark 1.35. *If I is finite, then*

$$\prod_{i \in I} G_i = \bigoplus_{i \in I} G_i$$

1.36. Let $(G_i)_{i \in I}$ be a family of abelian groups. Fix a $j \in I$ and let

$$\varepsilon_j : G_j \to \bigoplus_{i \in I} G_i$$

be the map defined by setting for all $a \in G_j$

$$\begin{aligned} (\varepsilon_j(a))_i &= a \quad \text{if } i = j \\ (\varepsilon_j(a))_i &= 0_{G_i} \quad \text{if } i \neq j \end{aligned}$$

In other words, $\varepsilon_j\,(a)$ has all its components zero but the j-th, which is a.
The map ε_j is easily verified to be a monomorphism: it is called the **j-th canonical injection**.

Notations 1.37. *For all $i, j \in I$ we denote by $\mathbf{0}_{i,j} : G_j \to G_i$ the costant map equal to 0_{G_i}. Moreover we denote by $\delta_{i,j} : G_j \to G_i$ the map defined by setting*

$$\begin{aligned} \delta_{i,j} &= \mathrm{Id}_{G_i} \quad \textit{if } i = j \\ \delta_{i,j} &= \mathbf{0}_{i,j} \quad \textit{if } i \neq j \end{aligned}$$

Lemma 1.38. *Let $(G_i)_{i \in I}$ be a family of abelian groups. Then, for every $i, j \in I$ we have*

$$\pi_i\,(\varepsilon_j(a)) = \delta_{i,j}\,(a)\,.$$

Proof. Let $i = j$. Then, for all $a \in G_j$, we have $\pi_j(\varepsilon_j(a)) = (\varepsilon_j(a))_j = a = \mathrm{Id}_{G_j}\,(a)\,.$
Let $i \neq j$. Then, for all $a \in G_j$, we have $\pi_i(\varepsilon_j(a)) = (\varepsilon_j(a))_i = 0_{G_i} = \mathbf{0}_{i,j}\,(a)\,.$ □

Exercise 1.39. *Let $(G_i)_{i\in I}$ be a family of abelian groups. Prove that $\varepsilon_j = \Delta\left(\delta_{i,j}\right)_{i\in I}$.*

Lemma 1.40. *Let $(G_i)_{i\in I}$ be a family of abelian groups and let $x \in \bigoplus_{i\in I} G_i$. Then*

$$x = \sum_{i\in Supp(x)} \varepsilon_i \pi_i(x) = \sum_{i\in I} \varepsilon_i \pi_i(x).$$

Proof. Let $j \in I$. Then, in view of Lemma 1.38, we have

$$\pi_j\left(\sum_{i\in Supp(x)} \varepsilon_i \pi_i(x)\right) = \sum_{i\in Supp(x)} \pi_j\left(\varepsilon_i\left(\pi_i(x)\right)\right) = \sum_{i\in Supp(x)} \delta_{i,j}\pi_i(x) = \pi_j(x) \text{ for every } j \in I$$

and hence, by Lemma 1.31, we conclude. □

Theorem 1.41. ***(Universal Property of the Direct Sum of a family of Groups)*** *Let $(G_i)_{i\in I}$ be a family of abelian groups. For all abelian groups G and family of homomorphisms $(f_i)_{i\in I}$, $f_i : G_i \to G$ there exists a unique homomorphism*

$$f : \bigoplus_{i\in I} G_i \to G$$

such that $f \circ \varepsilon_i = f_i$ for all $i \in I$. Such a homomorphism will be called the ***codiagonal homomorphism*** *of the homomorphisms family $(f_i)_{i\in I}$, and will be denoted by $\nabla\left(f_i\right)_{i\in I}$.*

Proof. Define

$$f : \bigoplus_{i\in I} G_i \to G$$

by setting

$$f(x) = \sum_{i\in I} f_i(x_i) \qquad \text{for all } x = (x_i)_{i\in I} \in \bigoplus_{i\in I} G_i.$$

Observe that this makes sense, in fact $x_i \neq 0$ only for finitely many i's.
Let $x = (x_i)_{i\in I}$, $y = (y_i)_{i\in I} \in \bigoplus_{i\in I} G_i$. Then

$$f(x+y) = \sum_{i\in I} f_i((x+y)_i) = \sum_{i\in I} f_i(x_i+y_i) = \sum_{i\in I}(f_i(x_i) + f_i(y_i)).$$

Since G is commutative, we have that

$$f(x+y) = \sum_{i\in I} f_i(x_i) + \sum_{i\in I} f_i(y_i) = f(x) + f(y)$$

so that f is a homomorphism. Let $j \in I$, $a \in G_j$. Then

$$(f \circ \varepsilon_j)(a) = \sum_{i\in I}(f_i(\varepsilon_j(a))_i) = f_j(a)$$

hence $f \circ \varepsilon_j = f_j$ for all $j \in I$.
Let now $f' : \bigoplus_{i \in I} G_i \to G$ be another homomorphism such that $f' \circ \varepsilon_i = f_i$ for all $i \in I$. If $x \in \bigoplus_{i \in I} G_i$ then

$$f(x) \overset{\text{Lem1.40}}{=} f\left(\sum_{i \in I} \varepsilon_i \pi_i(x)\right) = \sum_{i \in I} f \varepsilon_i \pi_i(x) = \sum_{i \in I} f_i \pi_i(x) = \sum_{i \in I} f' \varepsilon_i \pi_i(x) =$$
$$= f'\left(\sum_{i \in I} \varepsilon_i \pi_i(x)\right) \overset{\text{Lem1.40}}{=} f(x)$$

Therefore $f = f'$. □ □

1.42. Let R be a ring and let $(M_i)_{i \in I}$ be a family of left R-modules. We define on the abelian group $\prod_{i \in I} M_i$ a multiplication by the elements of R by setting, for every $r \in R$, $x = (x_i)_{i \in I} \in \prod_{i \in I} M_i$,

$$rx = (rx_i)_{i \in I}.$$

Theorem 1.43. *Let $(M_i)_{i \in I}$ be a family of left modules over a ring R. The abelian group $\prod_{i \in I} M_i$ becomes a left R-module with the multiplication by the elements of R defined as in 1.42. Moreover $\bigoplus_{i \in I} M_i$ is a submodule of this R- module.*

Proof. Exercise. □

Definition 1.44. *Let R be a ring, $(M_i)_{i \in I}$ be a family of left R-modules. The abelian group $\prod_{i \in I} M_i$ with the left R-module structure defined in 1.42 is called the* ***direct product of the family of left R-modules*** *$(M_i)_{i \in I}$ and is denoted by $\prod_{i \in I} M_i$. If $I = \{1, 2, ..., n\}$ we write $M_1 \times ... \times M_n$ instead of $\prod_{i \in I} M_i$. If $M = M_i$ for all $i \in I$, then we also write M^I and M^n if $I = \{1, ..., n\}$. The left R-module $\bigoplus_{i \in I} M_i$ will be called the* ***direct sum of the family of left R-modules*** *$(M_i)_{i \in I}$.*
If, for every $i \in I$, M_i is a fixed left R-module M, we will denote the direct sum considered before by $M^{(I)}$.

Theorem 1.45. ***(Universal Property of the Direct Product of a family of Modules)*** *Let R be a ring, $(M_i)_{i \in I}$ be a family of left R-modules. Then, for every $j \in I$, the canonical projection $\pi_j : \prod_{i \in I} M_i \to M_j$ is a surjective module homomorphism..*
Moreover, for every left R-module M and for every family $(f_i)_{i \in I}$ of homomorphisms

$f_i : M \to M_i$, there exists a unique R-module homomorphism $f : M \to \prod_{i\in I} M_i$ such that $\pi_i \circ f = f_i$ for every $i \in I$.
This homomorphism is called the ***diagonal homomorphism*** *of the family $(f_i)_{i\in I}$ and will be denoted by $\Delta((f_i)_{i\in I})$.*

Proof. Exercise (see Theorem 1.32). □

Exercise 1.46. *Let $\Delta = \Delta((f_i)_{i\in I})$ where, for each $i \in I$, $f_i : M \to M_i$ is a left R-module homomorphism. Then*

$$\mathrm{Ker}\,(\Delta) = \bigcap_{i\in I} \mathrm{Ker}\,(f_i)$$

Corollary 1.47. *Let $(M_i)_{i\in I}$ be a family of left R-modules and let $f : M \to \prod_{i\in I} M_i$ be a left R-module homomorphism. Then $f = \Delta((\pi_i \circ f)_{i\in I})$. Therefore if $f, g : M \to \prod_{i\in I} M_i$ are left R-module homomorphisms we have*

$$f = g \Leftrightarrow \pi_i \circ f = \pi_i \circ g \text{ for every } i \in I.$$

Proof. For each $i \in I$ we have $\pi_i \circ \Delta((\pi_i \circ f)_{i\in I}) = \pi_i \circ f$. Hence, by the uniqueness of the diagonal homomorphism, we get $f = \Delta((\pi_i \circ f)_{i\in I})$. □

Theorem 1.48. *(**Universal Property of the Direct Sum of a family of Modules**) Let R be a ring, $(M_i)_{i\in I}$ be a family of left R-modules. Then, for every $j \in I$, the canonical injection $\varepsilon_j : M_j \to \bigoplus_{i\in I} M_i$ is an injective R-module homomorphism. Moreover, for every left R-module M and for every family $(f_i)_{i\in I}$ of R-module homomorphisms $f_i : M_i \to M$, there exists a unique R-module homomorphism $f : \bigoplus_{i\in I} M_i \to M$ such that $f \circ \varepsilon_i = f_i$ for every $i \in I$.*
This homomorphism is called the ***codiagonal homomorphism*** *of the family $(f_i)_{i\in I}$ of homomorphism and will be denoted by $\nabla((f_i)_{i\in I})$.*

Proof. Exercise (see Theorem 1.41). □

Corollary 1.49. *Let $(M_i)_{i\in I}$ be a family of left R-modules and let $f : \bigoplus_{i\in I} M_i \to M$ be a left R-module homomorphism. Then $f = \nabla((f\circ\varepsilon_i)_{i\in I})$. Therefore if $f, g : \bigoplus_{i\in I} M_i \to M$ are left R-module homomorphisms we have*

$$f = g \Leftrightarrow f \circ \varepsilon_i = f \circ \varepsilon_i \text{ for every } i \in I.$$

Proof. For each $i \in I$ we have $\nabla((f \circ \varepsilon_i)_{i\in I}) \circ \varepsilon_i = f \circ \varepsilon_i$. Hence, by the uniqueness of the codiagonal homomorphism, we get $f = \nabla((f \circ \varepsilon_i)_{i\in I})$. □

Lemma 1.50. *Let R be a ring, M be a left R-module and let $(N_i)_{i\in I}$ be a family of submodules of M. Then $\bigcap_{i\in I} N_i$ is a submodule of M.*

Proof. Exercise. □

1.4 Sum and Direct Sum of Submodules. Cyclic Modules

Definitions 1.51. *Let M be a left module over a ring R. Given $n \in \mathbb{N}$, $n \geq 1$, $r_1, ..., r_n \in R$, $x_1, ..., x_n \in M$, the element $\sum_{i=1}^{n} r_i x_i$ of M, is called a **linear combination with coefficients in** R **of the elements** $x_1, .., x_n$; $r_1, ..., r_n$ are called **coefficients** of the linear combination.*
Let S be a subset of M. In view of Lemma 1.50, the intersection

$$\bigcap_{L \in \mathcal{L}(_R M, S)} L$$

*of all submodules of M that contain S is a submodule of M that contains S. Clearly it is the smallest submodule of M containing S. This submodule is called the **submodule of** M **generated by** S **and is denoted by** RS. If $S = \{s\}$ we write Rs instead of $R\{s\}$.*
*If $(M_i)_{i \in I}$ is a family of submodules of M then the submodule of M generated by $\bigcup_{i \in I} M_i$ is called the **sum of the family of submodules** $(M_i)_{i \in I}$ and is denoted by $\sum_{i \in I} M_i$.*
If $I = \{1, ..., n\}$ we write $M_1 + ... + M_n$ or $\sum_{i=1}^{n} M_i$ instead of $\sum_{i \in I} M_i$.

Theorem 1.52. *Let R be a ring, M a left R-submodule and let S be a subset of M. Then, if $S = \varnothing$, $RS = \{0\}$. If $S \neq \varnothing$ then*

$$RS = \left\{ \sum_{i=1}^{n} r_i s_i \mid n \in \mathbb{N}, n \geq 1, r_i \in R, s_i \in S \textit{ for every } i = 1, ..., n \right\}$$

In other words RS is the set of all the linear combinations with coefficients in R of the elements of S.
If $S = \{s_1, ..., s_k\}$ then $RS = \left\{ \sum_{i=1}^{k} r_i s_i \mid r_i \in R \right\}$.
In particular

$$Rs = \{rs \mid r \in R\}.$$

Proof. If $S = \varnothing$, $\{0\} \supseteq S$ and then $RS = \{0\}$.
Assume then that $S \neq \varnothing$ and let

$$N = \left\{ \sum_{i=1}^{n} r_i s_i \mid n \in \mathbb{N}, n \geq 1, r_i \in R, s_i \in S \text{ for every } i = 1, ..., n \right\}$$

$N \supseteq S$: in fact, for every $s \in S$ we have $s = 1 \cdot s$.
N is a <u>submodule of M</u>. N is clearly a subgroup of M. Let now $r \in R$, $y \in N$.

Then there exist $n \in \mathbb{N}$, $n \geq 1$ and $r_1, ..., r_n \in R$, $s_1, ..., s_n \in S$ such that $y = \sum_{i=1}^{n} r_i s_i$. Then we have

$$ry = r \sum_{i=1}^{n} r_i s_i = \sum_{i=1}^{n} (rr_i) s_i.$$

Therefore $N \supseteq RS$.
Conversely, let L be a submodule of M containing S. Then, for every $r \in R$ and $s \in S$, L contains rs. It follows that L contains any linear combination with coefficients in R of elements of S and so L contains N and hence $RS \supseteq N$. □ □

Corollary 1.53. *Let R be a ring, M a left R-module and let $(M_i)_{i \in I}$ be a family of submodules of M. Then*

$$\sum_{i \in I} M_i = \left\{ \sum_{j=1}^{n} x_{i_j} \mid n \in \mathbb{N},\ n \geq 1, i_j \in I \text{ and } x_{i_j} \in M_{i_j} \text{ for every } j = 1, ..., n \right\}$$

In particular, if $I = \{1, ..., k\}$,

$$M_1 + ... + M_k = \left\{ \sum_{i=1}^{k} x_i \mid x_i \in M_i \right\}.$$

Proof. If $x_i \in M_i$, then, for every $r \in R$, $rx_i \in M_i$. □

Corollary 1.54. *Let R be a ring, M a left R-module and let S be a nonempty subset of M. Then*

$$RS = \sum_{s \in S} Rs.$$

In particular, if $S = \{s_1, ..., s_k\}$, $RS = Rs_1 + ... + Rs_k$.

Definitions 1.55. *Let M be a left module over a ring R. We say that :*

- *a subset S of M is a* ***set of generators*** *of M if $RS = M$;*
- *M is* ***finitely generated*** *if M admits a set of generators which is a finite set;*
- *M is* ***cyclic*** *if there exists an $m \in M$ such that $\{m\}$ is a set of generators of M, i.e.$M = Rm$;*
- *an element $(s_1, ..., s_n) \in M^n$ is said to be* ***linearly independent*** *(**over** R) if, given any $r_1, ..., r_n \in R$, $\sum_{i=1}^{k} r_i s_i = 0$ implies $r_i = 0$ for every i, i.e. if the only zero linear combination with coefficients in R of the elements $s_1, ..., s_n$ is that one with all coefficients equal to 0 ;*

- *an element* $(s_1, ..., s_n) \in M^n$ *is said to be* ***linearly dependent*** *(**over*** R*) if it is not linearly independent, i.e. if there exists a zero linear combination with coefficients* R *of* $s_1, ..., s_n$ *where the coefficients are not all zero;*

- *an element* $(s_1, ..., s_n) \in M^n$ *is called a* ***basis*** *of* M *if* $(s_1, ..., s_n)$ *is a linearly independent element and* $\{s_1, ..., s_n\}$ *is a set of generators of* M.

Theorem 1.56. *Let* R *be a ring,* M *a left* R*-module and let* $(x_1, ..., x_n) \in M^n$*. Then there exists a left* R*-module homomorphism*

$$\upsilon : R^n \to M$$

such that $\upsilon(e_i) = x_i$*, where* $e_i = (0, ..., 1, 0, ..., 0)$ *(all components except the* i*-th are* 0 *and the* i*-th component is* 1*) for every* $i = 1, ..., n$*.*
Moreover :

1. *this homomorphism is unique;*

2. $\mathrm{Im}(\upsilon) = R\{x_1, ..., x_n\}$*;*

3. υ *is injective* $\Leftrightarrow (x_1, ..., x_n)$ *is linearly independent.*

Proof. Define $\upsilon : R^n \to M$ by setting

$$\upsilon((r_1, ..., r_n)) = \sum_{i=1}^{n} r_i x_i \text{ for every } (r_1, ..., r_n) \in R^n$$

Clearly we have that $\upsilon(e_i) = x_i$ for every $i = 1, ..., n$.
<u>υ is a left R-module homomorphism</u>. In fact let $(r_1, ..., r_n)$, $(s_1, ..., s_n) \in R^n$, $r \in R$. We have that

$$\begin{aligned}
&\upsilon((r_1, ..., r_n) + (s_1, ..., s_n)) = \\
&\quad = \upsilon((r_1 + s_1, ..., r_n + s_n)) = \sum_{i=1}^{n}(r_i + s_i)x_i = \\
&\quad = \sum_{i=1}^{n} r_i x_i + \sum_{i=1}^{n} s_i x_i = \upsilon((r_1, ..., r_n)) + \upsilon((s_1, ..., s_n))
\end{aligned}$$

$$\upsilon(r(r_1, ..., r_n)) = \upsilon((rr_1, ..., rr_n)) = \sum_{i=1}^{n} rr_i x_i = r\sum_{i=1}^{n} r_i x_i = r\upsilon((r_1, ..., r_n)).$$

<u>υ is unique</u>. Let $\upsilon' : R^n \to M$ be another left R-module homomorphism such that $\upsilon'(e_i) = x_i$ for every $i = 1, ..., n$. Let $(r_1, ..., r_n) \in R^n$. Then $(r_1, ..., r_n) = \sum_{i=1}^{n} r_i e_i$ and hence

$$\upsilon'((r_1, ..., r_n)) = \upsilon'(\sum_{i=1}^{n} r_i e_i) = \sum_{i=1}^{n} r_i \upsilon'(e_i) = \sum_{i=1}^{n} r_i x_i = \upsilon((r_1, ..., r_n))$$

Clearly $\mathrm{Im}\,(\upsilon) = RS$.
Since $\mathrm{Ker}\,(\upsilon) = \left\{(r_1, ..., r_n) \in R^n \mid \sum_{i=1}^{n} r_i x_i = 0\right\}$, it is clear that $\mathrm{Ker}\,(\upsilon) = 0$ if and only if $(x_1, ..., x_n)$ is linearly independent. □ □

Corollary 1.57. *With notation as in Theorem 1.56, $(x_1, ..., x_n)$ is a basis of M if and only if*

$$\upsilon : R^n \to M$$

is an isomorphism.

Corollary 1.58. *The element $(e_1, ..., e_n)$ is a basis of R^n.*

Corollary 1.59. *Let R be a ring and let $\varphi : M \to M'$ be a left R-module isomorphism. If $(x_1, ..., x_n)$ is a basis of M, then $(\varphi(x_1), ..., \varphi(x_n))$ is a basis of M'.*

Proof. Let $\upsilon : R^n \to M$ and $\upsilon' : R^n \to M'$ be the R-module homomorphisms such that $\upsilon(e_i) = x_i$ and $\upsilon'(e_i) = \varphi(x_i)$ for every $i = 1, ..., n$. Then $(\varphi \circ \upsilon)(e_i) = \varphi(x_i)$ for every $i = 1, ..., n$, and hence $\varphi \circ \upsilon = \upsilon'$. Since φ and υ are isomorphisms, so is υ'. □ □

1.60. Let R be a ring, let M be a left R-module and let $x \in M$. The map

$$\begin{array}{rccc} \mu_x : & R & \to & M \\ & r & \mapsto & rx \end{array}$$

is a left R-module homomorphism and $\mathrm{Im}\,(\mu_x) = Rx$ *by Theorem 1.56.* Thus $\mathrm{Ker}\,(\mu_x) = \{r \in R \mid rx = 0\}$ is a submodule of ${}_RR$, that is a left ideal of R. This ideal is called the **(left) annihilator of** x **in** R and is denoted by $Ann_R(x)$.The First Theorem of Isomorphism for Modules now allows to identity:

$$R/Ann_R(x) \cong Rx.$$

Corollary 1.61. *Let R be a ring. The cyclic left R- modules are exactly those isomorphic to modules of the form R/I where I is a left ideal of R.*

Proof. If $M = Rx$ then, as observed in 1.60, we have that $R/Ann_R(x) \cong M$.
Conversely, let I be a left ideal of R and let $f : {}_R(R/I) \to {}_RM$ be an isomorphism. We let $x = f(1 + I)$. Then, for every $y \in M$, there exists an $r \in R$ such that $y = f(r + I)$ and we have that $y = f(r(1 + I)) = rf(1 + I) = rx$. Therefore $M = Rx$. □ □

Remark 1.62. *Let R be ring. In general it is not true that every non-zero finitely generated left R- module has a basis. Moreover it can be proved that this holds if and only if R is a division ring.*

Example 1.63.

1. Let $n \in \mathbb{N}$, $n > 0$. Then $\mathbb{Z}/n\mathbb{Z}$ is a cyclic $\mathbb{Z}$-module: $\mathbb{Z}/n\mathbb{Z} = \mathbb{Z}(1 + n\mathbb{Z})$. But $\mathbb{Z}/n\mathbb{Z}$ does not admit a basis. In fact, for every $x \in \mathbb{Z}/n\mathbb{Z}$ we have that $n\mathbb{Z} \subseteq Ann_{\mathbb{Z}}(x)$.

Proposition 1.64. *Let $(f_i : M_i \to M)_{i\in I}$ be a family of morphisms of left R-modules and let $f = \nabla (f_i)_{i\in I}$. Then*

$$\mathrm{Im}(f) = \sum_{i\in I} \mathrm{Im}(f_i)$$

Proof. Let $(x_i)_{i\in I} \in \bigoplus_{i\in I} M_i$ and let $F = Supp(x)$. We have

$$f(x) = \sum_{i\in F} f_i(x_i) \in \sum_{i\in I} f_i(M_i).$$

Conversely, let $m \in \sum_{i\in I} f_i(M_i)$. Then there exists a finite subset F of I and, for each $i \in F$ an element $x_i \in M_i$ such that

$$m = \sum_{i\in F} f_i(x_i).$$

Let $z = \sum_{i\in F} \varepsilon_i(x_i)$. Then we have

$$f(z) = \sum_{i\in I} f(\varepsilon_i(x_i)) = \sum_{i\in F} (f \circ \varepsilon_i)(x_i) = \sum_{i\in F} f_i(x_i) = m.$$

□

Notations 1.65. *Let R be a ring, M a left R-module and let $(M_i)_{i\in I}$ be a family of submodules of M. Let $u_i : M_i \to M$ be the canonical inclusion and let*

$$u = \nabla((u_i)_{i\in I}) : \bigoplus_{i\in I} M_i \to M$$

be the codiagonal morphism of the family $(u_i)_{i\in I}$.

Corollary 1.66. *Let R be a ring, M a left R-module and let $(M_i)_{i\in I}$ be a family of submodules of M. Within the notations of 1.65 we have that*

$$\mathrm{Im}(u) = \sum_{i\in I} M_i.$$

Proof. It follows from Proposition (1.64). □

Proposition 1.67. *Let R be a ring, M a left R-module and let $(M_i)_{i\in I}$ be a family of submodules of M. Within the notations of 1.65, the following statements are equivalent:*

(a) u is injective.

(b) For every $i \in I$, we have that

$$M_i \cap \sum_{j \in I \setminus \{i\}} M_j = \{0\}$$

Proof. $(a) \Rightarrow (b)$. Let $i \in I$, and let $x \in M_i \cap \sum_{j \in I \setminus \{i\}} M_j$. Then there exists a finite subset F of $I \setminus \{i\}$ and, for every $j \in J$, an element $x_j \in M_j$ such that

$$x = \sum_{j \in F} x_j.$$

Since $x \in M_i$, we can consider $z = \varepsilon_i(x)$. Let $w = \sum_{j \in F} \varepsilon_j(x_j)$. Then we have

$$\begin{aligned} u(z) &= u(\varepsilon_i(x)) = (u \circ \varepsilon_i)(x) = u_i(x) = x = \sum_{j \in F} x_j = \sum_{j \in F} u_j(x_j) = \\ &= \sum_{j \in F} (u \circ \varepsilon_j)(x_j) = u\left(\sum_{j \in F} \varepsilon_j(x_j)\right) = u(w). \end{aligned}$$

Since u is injective, we deduce that $z = w$ and hence

$$Supp(z) \subseteq Supp(z) \cap Supp(w) \subseteq \{i\} \cap (I \setminus \{i\}) = \varnothing$$

so that $z = 0$.

$(b) \Rightarrow (a)$. Let $0 \neq x \in \bigoplus_{i \in I} M$. Then there is an $i_0 \in F = Supp(x)$. Assume that $x \in \operatorname{Ker}(u)$. Then we have

$$0 = u(x) = \sum_{t \in F} x_t$$

and hence

$$x_{i_0} = -\sum_{t \in F \setminus \{i_0\}} x_t \in M_{i_0} \cap \sum_{j \in I \setminus \{i_0\}} M_j = \{0\}$$

so that we get $x_{i_0} = 0$. Contradiction. □

Definition 1.68. *Let R be a ring, M a left R-module and let $(M_i)_{i \in I}$ be a family of submodules of M. Within the notations of 1.65, we will say that M is an* internal direct sum *of the family $(M_i)_{i \in I}$ if $u : \bigoplus_{i \in I} M_i \to M$ is an isomorphism. In this case we will also write*

$$M = \dot{\bigoplus_{i \in I}} M_i$$

Corollary 1.69. *Let R be a ring, M a left R-module and let $(M_i)_{i \in I}$ be a family of submodules of M. Then M is an internal direct sum of the family $(M_i)_{i \in I}$ if and only if*

1) $M=\sum_{i\in I}M_i$;

2) *For every $i\in I$, we have that $M_i\cap\sum_{j\in I\setminus\{i\}}M_j=\{0\}$.*

Proof. It follows from Corollary 1.66 and Proposition 1.67. $\square$

Exercise 1.70. *Let R be a ring, M a left R-module and let $(M_i)_{i\in I}$ be a family of submodules of M. Show that M is an internal direct sum of the family $(M_i)_{i\in I}$ if and only if every element $x\in M$ can be written as*

$$x=\sum_{i\in I}x_i \text{ where the } x_i=0 \text{ for almost every } i\in I$$

and moreover this representation is unique.

Definition 1.71. *Let R be a ring and let L be a submodule of a left R-module M. We will say that L is a direct summand of M if there exists a left submodule H of M such that*

$$M=L\dot{\oplus}H.$$

Remark 1.72. *In Definition 1.71, the submodule H is, in general, not unique. For example, if $R=k$ is a field, $M=k\times k$ and $L=k(1,0)$, then H can be chosen to be any $k(a,b)$ with $b\neq 0$.*

1.5 Exact sequences and split exact sequences

Notations 1.73. *Let R be a ring and let M be a left R-module. In the following, for every $r\in R$ and $x\in M$, the element $r\cdot x$ will be often denoted simply by rx.*

The left R-module with only one element 0 will be simply denoted by 0 instead of $\{0\}$.

Definition 1.74. *A sequence of left R-module homomorphisms*

$$\cdots\xrightarrow{f_{n-1}}M_{n-1}\xrightarrow{f_n}M_n\xrightarrow{f_{n+1}}M_{n+1}\xrightarrow{f_{n+2}}\cdots$$

is said to be exact *if*

$$\operatorname{Im}(f_n)=\operatorname{Ker}(f_{n+1}) \text{ for every } n\in\mathbb{Z}$$

A sequence of the form

$$0\to L\xrightarrow{f}M\xrightarrow{g}N\to 0$$

is called a short sequence*.*

Exercise 1.75. *Consider a short sequence of left R-module homomorphisms*

$$0\to L\xrightarrow{f}M\xrightarrow{g}N\to 0.$$

Show that this sequence is exact if and only if

1) *f is injective,*

2) *g is surjective,*

3) $\mathrm{Im}\,(f) = \mathrm{Ker}\,(g)$.

Examples 1.76.

1) *Let $g : M \to N$ be a surjective homomorphism. Then*

$$0 \to \mathrm{Ker}\,(g) \xrightarrow{i} M \xrightarrow{g} N \to 0,$$

where $i : \mathrm{Ker}\,(g) \to M$ is the canonical inclusion, is an exact sequence. In particular, for every submodule L of a module M, the sequence

$$0 \to L \xrightarrow{i} M \xrightarrow{p_L} P/L \to 0$$

is exact.

2) *Let $f : L \to M$ be an injective morphism. Then the sequence*

$$0 \to L \xrightarrow{f} M \xrightarrow{p} M/\mathrm{Im}\,(f) \to 0$$

is exact.

Proposition 1.77. *Let $\xi : M \to H$ and $\eta : M \to N$ be left R-module homomorphisms and assume that*

- *η is surjective*
- $\mathrm{Ker}\,(\eta) \subseteq \mathrm{Ker}\,(\xi)$.

Then there exists an homomorphism $\sigma : N \to H$ such that

$$\sigma \circ \eta = \xi.$$

Moreover such an homomorphism is unique with respect to this property.

Proof. Since $\mathrm{Ker}\,(\eta) \subseteq \mathrm{Ker}\,(\xi)$, ny the Fundamental Theorem of the Quotient Module 1.20, there exists an homomorphism $\overline{\xi} : M/\mathrm{Ker}\,(\eta) \to H$ such that $\xi = \overline{\xi} \circ p_{\mathrm{Ker}(\eta)}$. By the First Isomorphism Theorem for Modules 1.21 $\widehat{\eta} : M/\mathrm{Ker}\,(\eta) \to \mathrm{Im}\,(\eta) = N$ is an isomorphism. Let $\gamma : N \to M/\mathrm{Ker}\,(\eta)$ be a two-sided inverse of $\widehat{\eta}$ and set $\sigma = \overline{\xi} \circ \gamma$. We compute

$$\sigma \circ \eta = \sigma \circ \widehat{\eta} \circ p_{\mathrm{Ker}(\eta)} = \overline{\xi} \circ \gamma \circ \widehat{\eta} \circ p_{\mathrm{Ker}(\eta)} = \overline{\xi} \circ p_{\mathrm{Ker}(\eta)} = \xi.$$

The last assertion follows directly from the surjectivity of η. □

Proposition 1.78. *Let $\varphi : L \to M$ and $\vartheta : U \to M$ be left R-module homomorphisms and assume that*

- *φ is injective*
- $\mathrm{Im}\,(\vartheta) \subseteq \mathrm{Im}\,(\varphi)$.

Then there exists an homomorphism $\pi : U \to L$ such that

$$\varphi \circ \pi = \vartheta.$$

Moreover such an homomorphism is unique with respect to this property.

Proof. Since φ is injective, we know that $\varphi^{|\mathrm{Im}(\varphi)}$ is bijective. Let $h : \mathrm{Im}\,(\varphi) \to L$ be the two-sided inverse of $\varphi^{|\mathrm{Im}(\varphi)}$. Since $\mathrm{Im}\,(\vartheta) \subseteq \mathrm{Im}\,(\varphi)$ we can consider $\vartheta^{|\mathrm{Im}(\varphi)}$. Let $i : \mathrm{Im}\,(\varphi) \to M$ be the canonical injection. Set $\pi = h \circ \vartheta^{|\mathrm{Im}(\varphi)}$. We compute.

$$\varphi \circ \pi = \varphi \circ h \circ \vartheta^{|\mathrm{Im}(\varphi)} = i \circ \varphi^{|\mathrm{Im}(\varphi)} \circ h \circ \vartheta^{|\mathrm{Im}(\varphi)} = i \circ \mathrm{Id}_{\varphi(L)} \circ \vartheta^{|\mathrm{Im}(\varphi)} = \vartheta.$$

The last assertion follows directly from the injectivity of φ. □

Lemma 1.79. *Let $L \xrightarrow{f} M \xrightarrow{g} N$ be left R-module homomorphisms such that*

$$g \circ f = 0$$

and assume that there exists an R-module homomorphism $p : M \to L$ and an R-module homomorphism $s : N \to M$ such that

$$\mathrm{Id}_M = f \circ p + s \circ g.$$

In this case

1) *If f is injective, then*

$$p \circ f = \mathrm{Id}_L.$$

2) *If g is surjective, then*

$$g \circ s = \mathrm{Id}_N.$$

Proof. 1) We compute

$$f = \mathrm{Id}_M \circ f = (f \circ p + s \circ g) \circ f = f \circ p \circ f + s \circ g \circ f = f \circ p \circ f$$

and we deduce that

$$f \circ \mathrm{Id}_L = f = f \circ p \circ f.$$

Since f is injective, we get that $p \circ f = \mathrm{Id}_L$.

2) We compute

$$g = g \circ \mathrm{Id}_M = g \circ (f \circ p + s \circ g) = g \circ f \circ p + g \circ s \circ g = g \circ s \circ g$$

and we deduce that

$$\mathrm{Id}_N \circ g = g \circ s \circ g.$$

Since g is surjective, we get $g \circ s = \mathrm{Id}_N$. □

Proposition 1.80. *Let* $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ *be a short exact sequence. Then*

1) *For every R-module homomorphism $p : M \to L$ such that $p \circ f = \mathrm{Id}_L$, there exists a homomorphism $s : N \to M$ such that $\mathrm{Id}_M = f \circ p + s \circ g$. Moreover this s is unique.*

2) *For every R-module homomorphism $s : N \to M$ such that $g \circ s = \mathrm{Id}_N$, there exists a homomorphism $p : M \to L$ such that $\mathrm{Id}_M = f \circ p + s \circ g$. Moreover this p is unique.*

Proof. 1) Let $p : M \to L$ be such that $p \circ f = \mathrm{Id}_L$. Let $\xi = \mathrm{Id}_M - f \circ p$. We calculate

$$\xi \circ f = (\mathrm{Id}_M - f \circ p) \circ f = f - f = 0.$$

This implies that $\mathrm{Ker}\,(g) = \mathrm{Im}\,(f) \subseteq \mathrm{Ker}\,(\xi)$. Since g is surjective, we can apply Proposition 1.77 to deduce that there exists an homomorphism $s : N \to M$ such that $\xi = s \circ g$. Thus we get $\mathrm{Id}_M - f \circ p = s \circ g$ and hence $\mathrm{Id}_M = f \circ p + s \circ g$. Let $s' : N \to M$ such that $\mathrm{Id}_M = f \circ p + s' \circ g$. Then

$$f \circ p + s \circ g = f \circ p + s' \circ g$$

implies

$$s \circ g = s' \circ g$$

and from the surjectivity of g, we conclude.

2) Let $s : N \to M$ be such that $g \circ s = \mathrm{Id}_M$. Let $\vartheta = \mathrm{Id}_M - s \circ g$. We calculate

$$g \circ \vartheta = g \circ (\mathrm{Id}_M - s \circ g) = g - g = 0.$$

This implies that $\mathrm{Im}\,(\vartheta) \subseteq \mathrm{Ker}\,(g) = \mathrm{Im}\,(f)$. Since f is injective, we can apply Proposition 1.78 to deduce that there exists an homomorphism $p : M \to L$ such that $f \circ p = \vartheta$. Thus we get $\mathrm{Id}_M - s \circ g = f \circ p$ and hence $\mathrm{Id}_M = f \circ p + s \circ g$. Let $p' : M \to L$ such that $\mathrm{Id}_M = f \circ p' + s \circ g$. Then

$$f \circ p + s \circ g = f \circ p' + s \circ g$$

implies

$$f \circ p = f \circ p'$$

and from the injectivity of f, we conclude. □

Definitions 1.81. *Let* $L \xrightarrow{f} M$ *be a left R-module homomorphism. We say that*

1) f splits *if there exists a left R-module homomorphism $p : M \to L$ such that $p \circ f = \mathrm{Id}_L$.*

2) f cosplits *if there exists a left R-module homomorphism $s : M \to L$ such that $f \circ s = \mathrm{Id}_M$.*

Definition 1.82. *Let* $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ *be a short exact sequence. We say that this exact sequence* splits *if there exist* R*-module homomorphisms* $p : M \to L$ *and* $s : N \to M$ *such that* $\mathrm{Id}_M = f \circ p + s \circ g$*. In this case we also say the the given sequence is* split exact.

Lemma 1.83. *Let* $\alpha : U \to V$ *and* $\beta : V \to U$ *be left* R*-module homomorphisms such that* $\beta \circ \alpha = \mathrm{Id}_U$*. Then* $V = \mathrm{Im}\,(\alpha) \dot{\oplus} \mathrm{Ker}\,(\beta)$.

Proof. Let $x \in \mathrm{Im}\,(\alpha) \cap \mathrm{Ker}\,(\beta)$. Then there exists an element $u \in U$ such that $x = \alpha(u)$. Then we have

$$0 = \beta(x) = \beta(\alpha(u)) = (\beta \circ \alpha)(u) = u$$

and hence $x = \alpha(u) = \alpha(0) = 0$.

Let $x \in V$. Then

$$x = \alpha(\beta(x)) + [x - \alpha(\beta(x))]$$

where $\alpha(\beta(x)) \in \mathrm{Im}\,(\alpha)$ and $[x - \alpha(\beta(x))] \in \mathrm{Ker}\,(\beta)$. In fact we have

$$\beta([x - \alpha(\beta(x))]) = \beta(x) - (\beta \circ \alpha)(\beta(x)) = \beta(x) - \beta(x) = 0.$$

□

Theorem 1.84. *Let* $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ *be a short exact sequence. The following assertions are equivalent:*

(a) *f splits i.e. there exists an* R*-module homomorphism* $p : M \to L$ *such that* $p \circ f = \mathrm{Id}_L$.

(b) *g cosplits i.e. there exists an* R*-module homomorphism* $s : N \to M$ *such that* $g \circ s = \mathrm{Id}_N$.

(c) *The given exact sequence splits i.e. there exist* R*-module homomorphisms* $p : M \to L$ *and* $s : N \to M$ *such that* $\mathrm{Id}_M = f \circ p + s \circ g$.

(d) $f(L)$ *is a direct summand of* M *i.e. there exists an* R*-submodule* H *of* M *such that* $M = f(L) \dot{\oplus} H$.

Moreover

1) *if* (a) *holds then* $M = f(L) \dot{\oplus} \mathrm{Ker}\,(p)$;

2) *if* (b) *holds then* $M = \mathrm{Ker}\,(g) \dot{\oplus} \mathrm{Im}\,(s)$.

Proof. (a) $\Rightarrow$ (c) It follows by Proposition 1.80.

(c) $\Rightarrow$ (a). It follows by Lemma 1.79.

(b) $\Rightarrow$ (c) It follows by Proposition 1.80.

(c) $\Rightarrow$ (b). It follows by Lemma 1.79.

$(a) \Rightarrow (d)$. Apply Lemma 1.83 to $\alpha = f : L \to M$ and $\beta = p : M \to L$ to get that $M = f(L) \dot{\oplus} \mathrm{Ker}(p)$.

$(d) \Rightarrow (a)$. Let $v : f(L) \oplus H \to f(L) \dot{\oplus} H$ be the isomorphism defining this internal direct sum and let $\pi : f(L) \oplus H \to f(L)$ be the canonical projection. Since f is injective, we know that $f^{|\mathrm{Im}(f)}$ is bijective. Let $h : f(L) \to L$ be the two-sided inverse of $f^{|\mathrm{Im}(f)}$. Set $p = h \circ \pi \circ v^{-1}$. Then, for every $x \in L$ we have

$$(p \circ f)(x) = \left(h \circ \pi \circ v^{-1} \circ f\right)(x) = (h \circ \pi)((f(x), 0)) = h(f(x)) = x$$

and hence we deduce that $p \circ f = \mathrm{Id}_L$.

$(c) \Rightarrow 2)$ Apply Lemma 1.83 to $\alpha = s : N \to M$ and $\beta = g : M \to N$ to get that $M = \mathrm{Im}(s) \dot{\oplus} \mathrm{Ker}(g)$. □

1.6 $\mathbf{Hom}_R(M, N)$

Notation 1.85. *Let M and N be left R-modules. We set*

$$\mathrm{Hom}_R({}_RM, {}_RN) = \mathrm{Hom}_R(M, N) = \{f : M \to N \mid f \textit{ is an } R\textit{-module homomorphism}\}.$$

Proposition 1.86. *Let M and N be left R-modules. Then $\mathrm{Hom}_R(M, N)$ is a subgroup of the abelian group N^M. In particular $\mathrm{Hom}_R(M, N)$ is an abelian group.*

Proof. Exercise. □

Notations 1.87. *Let $f : L \to M$ and $f' : M' \to L'$ be left R-module homomorphisms . Then,we can consider the map*

$$\mathrm{Hom}_R(f', f) : \mathrm{Hom}_R(L', L) \to \mathrm{Hom}_R(M', M) \quad \textit{defined by setting}$$

$$\mathrm{Hom}_R(f', f)(\xi) = f \circ \xi \circ f' \qquad \textit{for every } \xi \in \mathrm{Hom}_R(L', L)$$

$$M' \overset{f'}{\to} L' \overset{\xi}{\to} L \overset{f}{\to} M \cdot$$

Whenever $L' = M' = U$ and $f = \mathrm{Id}_U$ we will simply write $\mathrm{Hom}_R(U, f)$ instead of $\mathrm{Hom}_R(\mathrm{Id}_U, f)$. Thus we have that

$$\mathrm{Hom}_R(U, f) : \mathrm{Hom}_R(U, L) \to \mathrm{Hom}_R(U, M) \quad \textit{is defined by setting}$$

$$\mathrm{Hom}_R(U, f)(\xi) = f \circ \xi \qquad \textit{for every } \xi \in \mathrm{Hom}_R(U, L)$$

Analogously whenever $L = M = U$ and $f = \mathrm{Id}_U$ we will simply write $\mathrm{Hom}_R(f', U)$ instead of $\mathrm{Hom}_R(f', \mathrm{Id}_U)$ Thus we have that

$$\mathrm{Hom}_R(f', U) : \mathrm{Hom}_R(L', U) \to \mathrm{Hom}_R(M', U) \quad \textit{is defined by setting}$$

$$\mathrm{Hom}_R(f', U)(\zeta) = \zeta \circ f' \qquad \textit{for every } \zeta \in \mathrm{Hom}_R(L', U).$$

Proposition 1.88. *Let $f : L \to M$ and $f' : M' \to L'$ be left R-module homomorphisms . Then, the map*

$$\operatorname{Hom}_R(f', f) : \operatorname{Hom}_R(L', L) \to \operatorname{Hom}_R(M', M)$$

is a group homomorphism.

Proof. Exercise. □

Theorem 1.89. ***(Universal Property of the Direct Product of a family of Modules)*** *Let R be a ring and let $(M_i)_{i\in I}$ be a family of left R-modules. For every $j \in I$, let $\pi_j : \prod_{i\in I} M_i \to M_j$ be the jth canonical projection. Let M be a left R-module. Then we can consider the family of group homomorphisms $(\operatorname{Hom}_R(M, \pi_i))_{i\in I}$ where, for each $i \in I$, we have that*

$$\operatorname{Hom}_R(M,\pi_i) : \operatorname{Hom}_R\left(M, \prod_{i\in I} M_i\right) \longrightarrow \operatorname{Hom}_R(M, M_i) \text{ and } \operatorname{Hom}_R(M,\pi_i)(f) = \pi_i \circ f$$

$$\text{for every } f \in \operatorname{Hom}_R\left(M, \prod_{i\in I} M_i\right).$$

Let

$$F = \Delta((\operatorname{Hom}_R(M,\pi_i))_{i\in I}) : \operatorname{Hom}_R\left(M, \prod_{i\in I} M_i\right) \longrightarrow \prod_{i\in I} \operatorname{Hom}_R(M, M_i).$$

$$\text{Then } F(f) = (\pi_i \circ f)_{i\in I} \text{ for every } f \in \operatorname{Hom}_R\left(M, \prod_{i\in I} M_i\right)$$

The group homomorphism F is bijective.

Theorem 1.90. ***(Universal Property of the Direct Sum of a family of Modules)*** *Let R be a ring, let $(M_i)_{i\in I}$ be a family of left R-modules. For every $j \in I$, let $\varepsilon_j : M_j \to \bigoplus_{i\in I} M_i$ be the jth canonical injection. Let M be a left R-module. Then we can consider the family of group homomorphisms $(\operatorname{Hom}_R(\varepsilon_i, M))_{i\in I}$ where, for each $i \in I$, we have that*

$$\operatorname{Hom}_R(\varepsilon_i, M) : \operatorname{Hom}_R\left(\bigoplus_{i\in I} M_i, M\right) \longrightarrow \operatorname{Hom}_R(M_i, M) \text{ and } \operatorname{Hom}_R(\varepsilon_i, M)(f) = f \circ \varepsilon_i$$

$$\text{for every } f \in \operatorname{Hom}_R\left(\bigoplus_{i\in I} M_i, M\right).$$

Let

$$G = \Delta((\operatorname{Hom}_R(\varepsilon_i, M))_{i\in I}) : \operatorname{Hom}_R\left(\bigoplus_{i\in I} M_i, M\right) \longrightarrow \prod_{i\in I} \operatorname{Hom}_R(M_i, M).$$

$$\text{Then } G(f) = (f \circ \varepsilon_i)_{i\in I} \text{ for every } f \in \operatorname{Hom}_R\left(\bigoplus_{i\in I} M_i, M\right)$$

The group homomorphism G is bijective.

Proposition 1.91. *1) Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N$ be an exact sequence. Then, for every left R-module U, the sequence*

$$0 \to \mathrm{Hom}_R(U, L) \overset{\mathrm{Hom}_R(U,f)}{\longrightarrow} \mathrm{Hom}_R(U, M) \overset{\mathrm{Hom}_R(U,g)}{\longrightarrow} \mathrm{Hom}_R(U, N) \tag{1.1}$$

is exact

2) Let $L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be an exact sequence. Then, for every left R-module U, the sequence

$$0 \to \mathrm{Hom}_R(N, U) \overset{\mathrm{Hom}_R(g,U)}{\longrightarrow} \mathrm{Hom}_R(M, U) \overset{\mathrm{Hom}_R(f,U)}{\longrightarrow} \mathrm{Hom}_R(L, U) \tag{1.2}$$

is exact.

Proof. 1 a)$\mathrm{Hom}_R(U, f)$ is injective. In fact, let $\zeta \in \mathrm{Hom}_R(U, L)$ be such that $0 = \mathrm{Hom}_R(U, f)(\zeta) = f \circ \zeta$. Since f is injective, from $f \circ \zeta = 0$ we deduce that $\zeta = 0$.

1b) $\mathrm{Im}(\mathrm{Hom}_R(U, f)) \subseteq \mathrm{Ker}(\mathrm{Hom}_R(U, g))$. Let $\zeta \in \mathrm{Hom}_R(U, L)$. Then
$\mathrm{Hom}_R(U, g)(\mathrm{Hom}_R(U, f)(\zeta)) = \mathrm{Hom}_R(U, g)(f \circ \zeta) = g \circ f \circ \zeta = 0$ since $g \circ f = 0$.

1c) $\mathrm{Ker}(\mathrm{Hom}_R(U, g)) \subseteq \mathrm{Im}(\mathrm{Hom}_R(U, f))$. Let $\vartheta \in \mathrm{Ker}(\mathrm{Hom}_R(U, g))$. This means that $0 = \mathrm{Hom}_R(U, g)(\vartheta) = g \circ \vartheta$. From $g \circ \vartheta = 0$ we deduce that $\mathrm{Im}(\vartheta) \subseteq \mathrm{Ker}(g) = \mathrm{Im}(f)$.

Since f is injective, by Proposition 1.78, there exists $p : M \to L$ such that $f \circ p = \vartheta$. Thus $\vartheta = \mathrm{Hom}_R(U, f)(p)$.

2a) $\mathrm{Hom}_R(g, U)$ is injective. Exercise.

2b) $\mathrm{Im}(\mathrm{Hom}_R(g, U)) \subseteq \mathrm{Ker}(\mathrm{Hom}_R(f, U))$. Exercise.

2c) $\mathrm{Ker}(\mathrm{Hom}_R(f, U)) \subseteq \mathrm{Im}(\mathrm{Hom}_R(g, U))$. Let $\xi \in \mathrm{Ker}(\mathrm{Hom}_R(f, U))$. This means that $0 = \mathrm{Hom}_R(f, U)(\xi) = \xi \circ f$. From $\xi \circ f = 0$ we deduce that $\mathrm{Ker}(g) = \mathrm{Im}(f) \subseteq \mathrm{Ker}(\xi)$.

Since g is surjective, by Proposition 1.77, There exists $s : N \to M$ such that $s \circ g = \xi$. Thus $\xi = \mathrm{Hom}_R(g, U)(s)$ □

Chapter 2

Free and projective modules

Definition 2.1. *Let R be a ring and let X be a nonempty set. A* free left R-module with basis X *is a pair (F, i) where*

- *F is a left R-module and*
- *$i : X \to F$ is a map*

such that the following universal property is satisfied.

For every map $f : X \to M$, where M is a left R-module, there exists a left R-module homomorphism $\overline{f}$ such that $\overline{f} \circ i = f$ and moreover this homomorphism is unique..

Proposition 2.2. *Let R be a ring and let M be a left R-module.*

1) *Then, for every $x \in M$, the map*

$$\mu_x : {}_RR \to {}_RM \text{ defined by setting } \mu_x(a) = ax \text{ for every } a \in R$$

is a left R-module homomorphism.

2) *The homomorphism $\mu = \nabla(\mu_x)_{x \in X} : {}_RR^{(X)} \to M$ is surjective if and only if X be a system of generators of M.*

Proof. 1) Follows by 1.60.

2) Always by 1.60 we know that $\mathrm{Im}(\mu_x) = Rx$. By Proposition 1.64 we have

$$\mathrm{Im}(\mu) = \sum_{x \in X} \mathrm{Im}(\mu_x) = \sum_{x \in X} Rx.$$

□

Proposition 2.3. *Let R be a ring and let X be a nonempty set. Let*

$$F = R^{(X)} = \bigoplus_{x \in X} R_x \text{ where, for each } x \in X,\ R_x = {}_RR$$

and, for every $y \in X$, let $\varepsilon_y : R_y \to \bigoplus_{x \in X} R_x$ be the canonical injection. Let $i : X \to F$ be the map defined by setting $i(x) = \varepsilon_x(1_R)$. Then (F, i) is a free module with basis X.

Proof. Let M be a left R-module and let $f : X \to M$ be a map. By Proposition 2.2, the map $f_x : R_x = R \to M$ defined by setting $f_x(a) = af(x)$ is a left R-module homomorphism. We set

$$\overline{f} = \nabla (f_x)_{x \in X}\,.$$

Recall that, for every $x \in X$, we have $\overline{f} \circ \varepsilon_x = f_x$. For every $x \in X$, we compute

$$\left(\overline{f} \circ i\right)(x) = (i(x)) = \overline{f}(\varepsilon_x(1_R)) = \overline{f} \circ \varepsilon_x(1_R) = f_x(1_R) = f(x)\,.$$

Therefore we get that $\overline{f} \circ i = f$. Let now $g : \bigoplus_{x \in X} R_x \to M$ be another morphism such that

$$g \cdot i = f.$$

Let us prove that $\overline{f} = g$ or equivalently that $\overline{f} \circ \varepsilon_x = g \circ \varepsilon_x$ for every $x \in X$. We have

$$g \circ \varepsilon_x(a_x) = a_x[g \circ \varepsilon_x(1_R)] = a_x[g \circ i(x)] = a_x f(x) = a_x\left[\overline{f} \circ i(x)\right] = a_x\left[\overline{f} \circ \varepsilon_x(1_R)\right] = \overline{f} \circ \varepsilon_x(a_x)$$

for every $a_x \in R_x$. □

Theorem 2.4. *Let R be a ring and let X be a nonempty set. Then*

1) *There exists a free left R-module with basis X.*

2) *Let (F, i) and (F', i') be free left R-modules with basis X. Then there exists a left R-module homomorphism $\varphi : F \to F'$ such that $\varphi \circ i = i'$. Moreover*

- *φ is unique with respect to this property.*
- *φ is an isomorphism.*

Proof. 1) follows by Proposition 2.3.

2) Since (F, i) is a free module with basis X, there exists a left R-module homomorphism $\varphi : F \to F'$ such that $\varphi \circ i = i'$. Since (F', i') is a free module with basis X, there exists a left R-module homomorphism $\varphi' : F' \to F$ such that $\varphi' \circ i' = i$. We compute

$$\varphi' \circ \varphi \circ i = \varphi' \circ i' = i.$$

On the other hand we also have

$$\mathrm{Id}_F \circ i = i.$$

In view of the definition of free module, there exists only one homomorphism which composed with i is equal to i. Therefore we get that $\varphi' \circ \varphi = i$. By interchanging the role of (F, i) with that of (F', i') we also get $\varphi \circ \varphi' = \mathrm{Id}_{F'}$. Therefore φ is bijective. □

Exercise 2.5. *Let (F, i) be a free module with basis X. Prove that i is injective.*

Definition 2.6. *Let X be a non-empty set and let M be a left R-module. Let (F, i) be a free module with basis X. Let $f = (m_x)_{x \in X} \in M^X$ and consider the only homomorphism $\varphi : F \to M$ such that $\varphi \circ i = f$. f is called* linearly independent *whenever φ is injective.*

$$\begin{array}{ccc} X & \overset{i}{\to} & F \\ & f \searrow & \downarrow \varphi \\ & & M \end{array}$$

Proposition 2.7. *Let X be a nonempty set, let M be a left R-module and let $f = (m_x)_{x \in X} \in M^X$. The following assertions are equivalent*

(a) *f is linearly independent.*

(b) *For every nonempty finite subset H of X and $(r_x)_{x \in X} \in R^{(X)}$*

$$\sum_{x \in H} r_x m_x = 0 \Rightarrow r_x = 0 \text{ for every } x \in H.$$

Proof. By Theorem 2.4 and by Proposition 2.3 we can assume that

$$F = R^{(X)} = \bigoplus_{x \in X} R_x \text{ where, for each } x \in X,\ R_x = {}_R R$$

and $i : X \to F$ be the map defined by setting $i(x) = \varepsilon_x(1_R)$ where, for every $y \in X$, $\varepsilon_y : R_y \to \bigoplus_{x \in X} R_x$ denote the canonical injection. Let $a = (r_x)_{x \in X} \in R^{(X)}$ and let $Supp(a) \subseteq H$ where H is a nonempty finite subset of X. Then

$$a = \sum_{x \in H} \varepsilon_x(r_x)$$

and

$$\begin{aligned} \varphi(a) &= \varphi\left(\sum_{x \in H} \varepsilon_x(r_x)\right) = \sum_{x \in H} (\varphi \circ \varepsilon_x)(r_x) = \sum_{x \in H} r_x [(\varphi \circ \varepsilon_x)(1_R)] \\ &= \sum_{x \in H} r_x [\varphi(\varepsilon_x(1_R))] = \sum_{x \in H} r_x [\varphi(i(x))] = \sum_{x \in H} r_x f(x) = \sum_{x \in H} r_x m_x \end{aligned}$$

$(a) \Rightarrow (b)$. Let H be a nonempty finite subset of X, let $(r_x)_{x \in X} \in R^{(X)}$ and assume that $\sum_{x \in H} r_x m_x = 0$. Set

$$a = \sum_{x \in H} \varepsilon_x(r_x)$$

Then, by the foregoing, we have

$$\varphi(a) = \sum_{x \in H} r_x m_x = 0.$$

Since φ is injective, we get $a = 0$ i.e. $r_x = 0$ for every $x \in X$.

$(b) \Rightarrow (a)$. Let $a = (r_x)_{x\in X} \in R^{(X)}$ and assume that $\varphi(a) = 0$. Let $H = Supp(a)$ and assume $H \neq \varnothing$. Then, by the foregoing we have

$$0 = \varphi(a) = \sum_{x\in H} r_x m_x.$$

In view of our assumption (b) this implies that $r_x = 0$ for every $x \in H$ i.e. $H = \varnothing$. Contradiction. □

Definition 2.8. *Let X be a non-empty set and let M be a left R-module. An element $(m_x)_{x\in X} \in M^X$ is called a* ***basis*** *of M if $(m_x)_{x\in X}$ is a linearly independent element and the set $\{m_x \mid x \in X\}$ is a set of generators of M.*

Proposition 2.9. *Let X be a non-empty set, let M be a left R-module and let $(m_x)_{x\in X} \in M^X$. Let $\varphi : R^{(X)} \to M$ be the only morphism of left R-modules such that $\varphi(\varepsilon_x(1_R)) = m_x$ for every $x \in X$. Then the following assertionsare equivalent:*

(a) *$(m_x)_{x\in X}$ is a basis of M.*

(b) *$\varphi : R^{(X)} \to M$ is an isomorphism.*

Proof. Note that

$$\varphi = \nabla(\varphi \circ \varepsilon_x)_{x\in X} = \nabla(\mu_x)_{x\in X}.$$

The conclusion follows in view of Propositions 2.7 and 2.2 □

Exercise 2.10. *$(e_x = \varepsilon_x(1_R) \mid x \in X)$ is a basis of $R^{(X)}$.*

Definition 2.11. *Let ${}_RP$ be a left R-module. ${}_RP$ is said to be projective if, for every surjective left R-module homomorphism*

$$M \xrightarrow{g} N \to 0$$

and for every left R-module homomorphism $h : P \to N$, there exists a left R-module homomorphism $\overline{h} : P \to M$ such that $g \circ \overline{h} = h$.

Proposition 2.12. *Let ${}_RP$ be a left R-module. Then the following assertions are equivalent.*

(a) *${}_RP$ is projective.*

(b) *For every short exact sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ of left R-module homomorphisms, the sequence*

$$0 \to \mathrm{Hom}_R(P, L) \xrightarrow{\mathrm{Hom}_R(P,f)} \mathrm{Hom}_R(P, M) \xrightarrow{\mathrm{Hom}_R(P,g)} \mathrm{Hom}_R(P, N) \to 0$$

is exact.

Proof. $(a) \Rightarrow (b)$. By Proposition 1.91, we have only to prove that $\mathrm{Hom}_R(P,g)$ is surjective. Thus let $h \in \mathrm{Hom}_R(P,N)$. Then $h : {}_RP \to {}_RN$ is an homomorphism. Since ${}_RP$ is projective, there exists an homomorphism $\overline{h} : P \to M$ such that $h = g \circ \overline{h} = \mathrm{Hom}_R(P,g)\left(\overline{h}\right)$.

$(b) \Rightarrow (a)$. Let $M \xrightarrow{g} N \to 0$ be a surjective homomorphism and le $h : P \to N$ be a left R-module homomorphism. Then $h \in \mathrm{Hom}_R(P,N)$. Since the sequence

$$0 \to \mathrm{Ker}(g) \xrightarrow{i} M \xrightarrow{g} N \to 0,$$

is exact, we deduce from (b) that $\mathrm{Hom}_R(P,g)$ is surjective so that there exists an homomorphism $\overline{h} \in \mathrm{Hom}_R(P,M)$ such that $h = \mathrm{Hom}_R(P,g)\left(\overline{h}\right) = g \circ \overline{h}$. □

Proposition 2.13. *Let $(P_i)_{i\in I}$ be a family of left R-modules. Then the following assertions are equivalent:*

(a) *Each P_i is projective, for every $i \in I$.*

(b) $\bigoplus_{i\in I} P_i$ *is projective.*

Proof. $(a) \Rightarrow (b)$. Let $M \xrightarrow{g} N \to 0$ be a surjective homomorphism and let $h : \bigoplus_{i\in I} P_i \to N$ be an homomorphism. For every $i \in I$ let $\varepsilon_i : P_i \to \bigoplus_{i\in I} P_i$ be the canonical injection. Since P_i is projective, for every $i \in I$, there exists an homomorphism $h_i : P_i \to M$ such that $g \circ h_i = h \circ \varepsilon_i$. Set $\overline{h} = \nabla(h_i)_{i\in I}$ and recall that $\overline{h} \circ \varepsilon_i = h_i$ for every $i \in I$. We compute

$$g \circ \overline{h} \circ \varepsilon_i = g \circ h_i = h \circ \varepsilon_i.$$

By the universal property of the direct sum, there exists only one homomorphism which composed with every ε_i is equal to $h \circ \varepsilon_i$. Therefore we deduce that $g \circ \overline{h} = h$.

$(b) \Rightarrow (a)$. Fix an $i_0 \in I$. Let $M \xrightarrow{g} N \to 0$ be a surjective homomorphism and let $h : P_{i_0} \to N$ be an homomorphism. Consider the family of left R-module homomorphisms $(h_i)_{i\in I}$ where $h_{i_0} = h$ and $h_i = 0$ for every $i \in I$, $i \neq i_0$. Let $f = \nabla(h_i)_{i\in I} : \bigoplus_{i\in I} P_i \to N$. Since $\bigoplus_{i\in I} P_i$ is projective, there exists an homomorphism $\overline{f} : \bigoplus_{i\in I} P_i \to M$ such that $g \circ \overline{f} = f$. Let $\overline{h} = \overline{f} \circ \varepsilon_{i_0}$. Then we get

$$g \circ \overline{h} = g \circ \overline{f} \circ \varepsilon_{i_0} = f \circ \varepsilon_{i_0} = h_{i_0} = h.$$

□

Corollary 2.14. *Every direct summand L of a projective left R-module P is projective.*

Proof. Since L is a direct summand of P, there exists a left submodule H of P such that

$$P = L \dot{\oplus} H.$$

Let $\varphi : L \oplus H \to L \dot{\oplus} H = P$ be the usual isomorphism. Since P is projective, also $L \oplus H$ is projective and hence, by Proposition 2.13, L is projective. □

Proposition 2.15. *Let R be a ring and let X be a nonempty set. Then the left R-module ${}_RR^{(X)}$ is projective.*

Proof. In view of Proposition 2.13, we will show that ${}_RR$ is projective. Thus let $M \xrightarrow{g} N \to 0$ be a surjective homomorphism and let $h : {}_RR \to N$ be an homomorphism. Since g is surjective, there exists an $x \in M$ such that $g(x) = h(1_R)$. By Proposition 2.2, there exists an homomorphism $\overline{h} : {}_RR \to M$ such that $\overline{h}(a) = ax$ for every $a \in R$. For every $a \in R$, we compute

$$\left(g \circ \overline{h}\right)(a) = g(ax) = ag(x) = ah(1_R) = h(a1_R) = h(a).$$

Thus we get that $g \circ \overline{h} = h$. □

Proposition 2.16. *Let (F, i) be a free left R-module with basis X. Then F is projective.*

Proof. It follows by Proposition 2.15, in view of Proposition 2.3 and Theorem 2.4. □

Proposition 2.17. *Let P be a left R-module. Then the following statements are equivalent*

(a) ${}_RP$ *is projective.*

(b) *Every short exact sequence of the form* $0 \to L \xrightarrow{f} M \xrightarrow{g} P \to 0$ *splits.*

(c) ${}_RP$ *is a direct summand of a free left R-module.*

(d) ${}_RP$ *is a direct summand of a projective left R-module.*

Proof. $(a) \Rightarrow (b)$. Since ${}_RP$ is projective, there exists a left R-module homomorphism $s : P \to M$ such that $s \circ g = \mathrm{Id}_P$.

$(b) \Rightarrow (c)$. By Proposition 2.2, we have a surjective homomorphism $g : {}_RR^{(P)} \to {}_RP$. By 2) in Theorem 1.84, there exists an R-submodule H of ${}_RR^{(P)}$ such that ${}_RR^{(P)} = \mathrm{Ker}(g) \dot{\oplus} H$. Then $H \cong {}_RR^{(P)}/\mathrm{Ker}(g) \cong P$ so that

P is a direct summand of $\mathrm{Ker}(g) \oplus P \cong \mathrm{Ker}(g) \oplus H \cong \mathrm{Ker}(g) \dot{\oplus} H = {}_RR^{(P)}$.

$(c) \Rightarrow (d)$ is trivial,

$(d) \Rightarrow (a)$ follows by Corollary 2.14, □

Chapter 3

Injective Modules and Injective Envelopes

Definition 3.1. *Let ${}_RE$ be a left R-module. ${}_RE$ is said to be injective if, for every injective left R-module homomorphism*

$$0 \to L \xrightarrow{j} M$$

and for every left R-module homomorphism $f : L \to E$, there exists a left R-module homomorphism $\overline{f} : M \to E$ such that $\overline{f} \circ j = f$.

Proposition 3.2. *Let ${}_RE$ be a left R-module. Then the following assertions are equivalent.*

(*a*) *${}_RE$ is injective.*

(*b*) *For every short exact sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ of left R-module homomorphisms, the sequence*

$$0 \to \mathrm{Hom}_R(N, E) \xrightarrow{\mathrm{Hom}_R(g,E)} \mathrm{Hom}_R(M, E) \xrightarrow{\mathrm{Hom}_R(f,E)} \mathrm{Hom}_R(L, E) \to 0$$

is exact.

Proof. Is analogous to the prove of Proposition 2.12 and it is left as an exercise to the reader. □

Proposition 3.3. *Let $(E_i)_{i\in I}$ be a family of left R-modules. Then the following assertions are equivalent:*

(*a*) *Each E_i is injective, for every $i \in I$.*

(*b*) *$\prod_{i\in I} E_i$ is injective.*

Proof. $(a) \Rightarrow (b)$. Let $j : L \to M$ be an injective left R-module homomorphism and let $f : L \to \prod_{i \in I} E_i$ be a left R-module homomorphism. Then

$$f = \Delta\left(\pi_i \circ f\right)_{i \in I}.$$

Let $i \in I$. Since E_i is injective, there exists a morphism $\overline{f_i} : M \to E_i$ such that $\overline{f_i} \circ j = \pi_i \circ f$. Let $\overline{f} = \Delta\left(\overline{f_i}\right)_{i \in I}$ and, for every $i \in I$, let us compute

$$\pi_i \circ \overline{f} \circ j = \overline{f_i} \circ j = \pi_i \circ f.$$

By the Universal Property of the Direct Product, we deduce that $\overline{f} \circ j = f$.

$(b) \Rightarrow (a)$. Let $j : L \to M$ be an injective left R-module homomorphism and let $f : L \to E_{i_0}$ be a left R-module homomorphism. For every $i \in I$ set $h_i : L \to E_i$ equal to the zero map if $i \neq i_0$ and $h_{i_0} = f$. Let $h = \Delta\left(h_i\right)_{i \in I} : L \to \prod_{i \in I} E_i$. Since $\prod_{i \in I} E_i$ is injective, there exists an homomorphism $\overline{h} : M \to \prod_{i \in I} E_i$ such that $\overline{h} \circ j = h$. Set $\overline{f} = \pi_{i_0} \circ \overline{h}$ and let us compute

$$\overline{f} \circ j = \pi_{i_0} \circ \overline{h} \circ j = \pi_{i_0} \circ h = h_{i_0} = f.$$

□

Corollary 3.4. *Let E_1 and E_2 be left R-modules. Then $E_1 \oplus E_2$ is injective if and only if each E_i is injective for $i = 1, 2$.*

Corollary 3.5. *Every direct summand L of an injective left R-module E is injective.*

Proof. Since L is a direct summand of E, there exists a left submodule H of E such that

$$E = L \dot{\oplus} H.$$

Let $\varphi : L \oplus H \to L \dot{\oplus} H = E$ be the usual isomorphism. Since E is injective, also $L \oplus H$ is injective and hence, by Corollary 3.4, L is injective. □

Theorem 3.6. *(Baer's Criterion for injectivity). Let E be a left R-module. The following assertions are equivalent.*

(a) *E is injective.*

(b) *For any left ideal I of R and for every homomorphism of left R-modules $f : I \to E$, there exists an homomorphism $h : R \to E$ such that $h \circ i = f$, where $i : I \to R$ is the canonical inclusion.*

Proof. $(a) \Rightarrow (b)$. It is trivial.

$(b) \Rightarrow (a)$. Let $j : L \to M$ be an injective left R-module homomorphism and let $f : L \to E$ be a left R-module homomorphism. We set

$$\mathcal{H} = \left\{ \begin{array}{c} (H, \psi) \mid j(L) \subseteq H \subseteq M \text{ and} \\ \psi : H \to E \text{ is a left } R\text{-module homomorphism such that } \psi \circ j^{|H} = f \end{array} \right\}.$$

Clearly $\mathcal{H} \neq \varnothing$ since $(L, \zeta) \in \mathcal{H}$ where $\zeta = f \circ \vartheta$ and ϑ is the two-sided inverse of $j^{|j(L)}$. In fact $\zeta \circ j^{|j(L)} = f \circ \vartheta \circ j^{|j(L)} = f$.
We define a partial order on $\mathcal{H}$ by setting

$$(H, \psi) \leq (H', \psi') \text{ if and only if } H \subseteq H' \text{ and } \psi'_{|H} = \psi.$$

It is easy to check that $(\mathcal{H}, \leq)$ is an inductive set. Hence, by Zorn's Lemma, it has a maximal element, say (H_0, ψ_0). We will prove that $H_0 = M$. Assume that $H_0 \subsetneqq M$ and let $x \in M \setminus H_0$ so that $H_0 \subsetneqq H_0 + Rx$. Set

$$J = \{a \in R \mid ax \in H_0\}.$$

J is a left ideal of R. In fact, let $a, a_1, a_2 \in J$ and let $r \in R$. Since $a_1, a_2 \in J$, we have that $a_1 x \in H_0$ and $a_2 x \in H_0$, from which we deduce that

$$(a_1 - a_2)\, x = a_1 x - a_2 x \in H_0$$

and hence $a_1 - a_2 \in J$. Moreover $a \in J$ means that $ax \in H_0$, from which we get

$$(ra)\, x = r\,(ax) \in H_0$$

which means that $ra \in J$. Let us consider the map $\chi : J \to E$ defined by setting

$$\chi\,(a) = \psi_0\,(ax) \text{ for every } a \in J. \tag{3.1}$$

χ is an R-module homomorphism. In fact let $a, a_1, a_2 \in J$ and let $r \in R$. We compute

$$\chi\,((a_1 + a_2)) = \psi_0\,((a_1 + a_2)\, x) = \psi_0\,(a_1 x + a_2 x) = \psi_0\,(a_1 x) + \psi_0\,(a_2 x) = \chi\,(a_1) + \chi\,(a_2)$$

and

$$\chi\,(ra) = \psi_0\,((ra)\, x) = \psi_0\,(r\,(ax)) \overset{\psi_0 \text{is} R\text{-homo}}{=} r\psi_0\,(ax) = r\chi\,(a).$$

By assumption there exists a left R-module homomorphism $\lambda : R \to E$ such that $\lambda \circ \alpha = \chi$ where $\alpha : J \to R$ is the canonical inclusion.

Let us define a map $\widehat{\psi_0} : H_0 + Rx \to E$ by setting

$$\widehat{\psi_0}\,(h + rx) = \psi_0\,(h) + \lambda\,(r).$$

$\widehat{\psi_0}$ is well defined. In fact, assume that $h + rx = h' + r'x$. Then

$$h - h' = (r' - r)\, x \in H_0 \cap Rx.$$

This means that $(r' - r) \in J$ so that

$$\psi_0\,(h) - \psi_0\,(h') = \psi_0\,(h - h') = \psi_0\,((r' - r)\, x) =$$
$$\overset{(3.1)}{=} \chi\,(r' - r) = \lambda \circ \alpha\,(r' - r) = \lambda\,(r' - r) = \lambda\,(r') - \lambda\,(r)$$

Thus $\widehat{\psi_0}$ is well defined. It is easy to check that $\widehat{\psi_0}$ is a left R-module homomorphism. Since $\widehat{\psi_0}_{|H_0} = \psi_0$ this contradicts the maximality of (H_0, ψ_0). $\square$

Definition 3.7. *Let R be a commutative ring. An element $a \in R$ is said to be a* zero-divisor *if there exists an element $b \in R, b \neq 0$ such that $a \cdot b = 0$.*

Remark 3.8. *The element 0 is always a zero-divisor. Any zero-divisor different from 0 is called* non trivial zero-divisor.

Examples 3.9. *1) In the commutative ring $\mathbb{Z}_6$ the unique non trivial zero-divisors are $2 + 6\mathbb{Z}$, $3 + 6\mathbb{Z}$ and $4 + 6\mathbb{Z}$.*

2) A commutative ring D is a domain if and only if it has no non trivial zero-divisor.

Definition 3.10. *Let E be a module over a commutative ring R. E is said to be divisible if, for any $r \in R$, r not a zero-divisor, we have $rE = E$ i.e. for every $x \in E$ there is an element $x' \in E$ such that $rx' = x$.*

Example 3.11. *Let D be a commutative domain and let $Q = Q(D)$ be its ring of quotients. Then Q is a divisible D-module. In fact, for every $d \in D, d \neq 0$ and for every $q \in Q$, one has*

$$q = d\left(\frac{1}{d}q\right).$$

Proposition 3.12. *Let R be a commutative ring, Let E be an R-module and let $(E_i)_{i\in I}$ be a family of R-modules. Then*

1) *E is divisible if and only if any quotient of E is divisible.*

2) *$\bigoplus_{i\in I} E_i$ is divisible $\Leftrightarrow$ E_i is divisible for any $i \in I$ $\Leftrightarrow$ $\prod_{i\in I} E_i$ is divisible.*

Proof. 1) Let L be a submodule of E and let $r \in R$ be a non-zero divisor. Let $y \in E/L$. Then there exists an element $x \in E$ such that $y = x + L$. Since E is divisible, there exists an element $x' \in E$ such that $rx' = x$. Then we have

$$r(x' + L) = (rx') + L = x + L.$$

2) Assume that E_i is divisible for any $i \in I$, let $x \in \prod_{i\in I} E_i$ and let $r \in R$ be a non-zero divisor. Then, for every $i \in I$, there is an element $x_i \in E_i$ such that $x = (x_i)_{i\in I}$. Since E_i is divisible, for every $i \in I$ there exists an element $x'_i \in E_i$ such that $rx'_i = x_i$. Let $x' = (x'_i)_{i\in I}$. Then $rx' = r(x'_i)_{i\in I} = (rx'_i)_{i\in I} = (x_i)_{i\in I} = x$.

Assume now that $x \in \bigoplus_{i\in I} E_i$ and set $x'_i = 0$ if $i \notin Supp(x)$ while, if $i \in Supp(x)$, let $x'_i \in E_i$ be such that $rx'_i = x_i$. Let $x' = (x'_i)_{i\in I}$. Then $Supp(x') = Supp(x)$ and hence

$x' \in \bigoplus_{i\in I} E_i$. Moreover we have $rx' = r(x'_i)_{i\in I} = (rx'_i)_{i\in I} = (x_i)_{i\in I} = x$. . This shows that also $\bigoplus_{i\in I} E_i$ is divisible.

Assume now that $\prod_{i\in I} E_i$ is divisible and consider the canonical projection $\pi_j : \prod_{i\in I} E_i \to E_j$. Since π_j is surjective, we deduce that E_j is isomorphic to a quotient of $\prod_{i\in I} E_i$ and hence, by 1), we get that E_j is divisible.

In the case when $\bigoplus_{i\in I} E_i$ is divisible, since the canonical projection $\pi'_j : \bigoplus_{i\in I} E_i \to E_j$ is still surjective, the same proof applies. □

Definitions 3.13. *Let D be a commutative domain and let M be a D-module. An element $x \in M$ is called a* torsion element *if there exists an $d \in D, d \neq 0$ such that $dx = 0$.*

We set

$$t(M) = \{x \in M \mid x \text{ is a torsion element}\}.$$

We say that M is a torsion module *if $t(M) = M$ and that M is a* torsion-free *module if $t(M) = \{0\}$.*

Exercise 3.14. *Let D be a commutative domain and let M be an D-module. Show that*

1) *$t(M)$ is a submodule of M.*

2) *$t(M)$ is the largest torsion submodule of M.*

3) *$M/t(M)$ is a torsion-free module.*

Proposition 3.15. *Let T be a torsion abelian group and let P be the set of prime natural numbers. For each $p \in P$ set*

$$T_p = \left\{x \in T \mid \text{there is an } h \in \mathbb{N} \text{ such that } p^h x = 0\right\}.$$

Then T_p is a subgroup of T and

$$T = \dot{\bigoplus_{p\in P}} T_p.$$

Proof. Let $p \in P$ and let $x, x' \in T_p$. Then there exist $h, h' \in \mathbb{N}$ such that $p^h x = 0$ and $p^{h'} x' = 0$. Then we get

$$p^{h+h'}(x - x') = p^{h+h'}x - p^{h+h'}x' = p^{h'}(p^h x) - p^h\left(p^{h'}x'\right) = 0.$$

Since $0 \in T_p$, we conclude that T_p is a subgroup of T.

Let us prove that

$$T = \sum_{p\in P} T_p.$$

Let $x \in T$. Then there is an element $n \in \mathbb{N}, n \neq 0$, such that $nx = 0$. If $n = 1$, then $x = 0$ and there is nothing to prove. Otherwise can write

$$\begin{aligned} n &= p_1^{h_1} \cdot \dots \cdot p_s^{h_s} \text{ for suitable } s \in \mathbb{N}, s \geq 1,\ h_1, \dots h_s \in \mathbb{N}, \\ \text{where } h_1, \dots, h_s &\geq 1 \text{ and } p_1, \dots, p_s \in P \text{ are distinct prime numbers.} \end{aligned}$$

For each $i = 1, \dots s$, we set $q_i = \frac{n}{p_i^{h_i}}$. Then we get that $MCD\,(q_1, \dots, q_s) = 1$ and hence, there exist $\lambda_1, \dots \lambda_s \in \mathbb{Z}$ such that

$$1 = \lambda_1 q_1 + \ldots + \lambda_s q_s.$$

Note that, for each $i = 1, \dots s$, we have

$$p_i^{h_i}\,(\lambda_i q_i x) = \lambda_i p_i^{h_i} \frac{n}{p_i^{h_i}} x = \lambda_i n x = 0$$

and hence we deduce that $\lambda_i q_i x \in T_{p_i}$. Moreover we get

$$x = 1 \cdot x = (\lambda_1 q_1 + \ldots + \lambda_s q_s)\, x = \lambda_1 q_1 x + \ldots + \lambda_s q_s x \in \sum_{p \in P} T_p.$$

Let us prove that, for each $q \in P$,

$$T_q \cap \sum_{p \in P \setminus \{q\}} T_p = \{0\}\,.$$

Let $x \in T_q \cap \sum_{p \in P \setminus \{q\}} T_p$. Then there exists an $s \in \mathbb{N}, s \geq 1$ and, for each $i = 1, \dots s$, an element $p_i \in P \setminus \{q\}$ and an element $x_i \in T_{p_i}$ such that

$$x = x_1 + \ldots + x_s.$$

Since $x_i \in T_{p_i}$, there exists an $h_i \in \mathbb{N}$ such that $p_i^{h_i} x_i = 0$. Moreover, since $x \in T_q$, there exists an $h \in \mathbb{N}$ such that $q^h x = 0$. Let $n = p_1^{h_1} \cdot \dots \cdot p_s^{h_s}$ and, for each i, let $q_i = \frac{n}{p_i^{h_i}}$ Then we get that

$$nx = n\,(x_1 + \ldots + x_s) = q_1 p_1^{h_1} x_1 + \ldots + q_s p_s^{h_s} x_s = 0.$$

Moreover, since each $p_i \in P \setminus \{q\}$ we have that $MCD\left(n, q^h\right) = 1$. Therefore there exist $\lambda, \mu \in \mathbb{Z}$ such that $1 = \lambda n + \mu q^h$. We obtain that

$$x = 1 \cdot x = \left(\lambda n + \mu q^h\right) \cdot x = \lambda n x + \mu q^h x = 0.$$

□

Example 3.16. *$\mathbb{Q}/\mathbb{Z}$ is a torsion abelian group. In fact, for every $q \in \mathbb{Q}$, there exist $m, n \in \mathbb{Z}, n > 0$ such that $q = \frac{m}{n}$. Then*

$$n\,(q + \mathbb{Z}) = \left(n\frac{m}{n}\right) + \mathbb{Z} = m + \mathbb{Z} = \mathbb{Z}.$$

Moreover, for each $p \in P$, we have

$$\begin{aligned}(\mathbb{Q}/\mathbb{Z})_p &= \left\{q+\mathbb{Z} \in \mathbb{Q}/\mathbb{Z} \mid \text{there exists an } h \in \mathbb{N} \text{ such that } p^h(q+\mathbb{Z}) = 0+\mathbb{Z}\right\} = \\ &\overset{Exercise}{=} \left\{\frac{m}{p^h}+\mathbb{Z} \mid m \in \mathbb{Z} \text{ and } h \in \mathbb{N}\right\}.\end{aligned}$$

The group $(\mathbb{Q}/\mathbb{Z})_p$ is usually denoted by $\mathbb{Z}(p^\infty)$ and it is called the Prufer p-group. By Proposition 3.15, we have that

$$\mathbb{Q}/\mathbb{Z} = \bigoplus_{p \in P} \mathbb{Z}(p^\infty).$$

Exercise 3.17. *Let $p \in P$. For each $h \in \mathbb{N}, h \geq 1$,let $\left\langle \frac{1}{p^h}+\mathbb{Z}\right\rangle$ be the cyclic subgroup of $\mathbb{Z}(p^\infty)$ spanned by $\frac{1}{p^h}+\mathbb{Z}$. Show that*

$$\left\langle \frac{1}{p}+\mathbb{Z}\right\rangle \subseteq \ldots \subseteq \left\langle \frac{1}{p^h}+\mathbb{Z}\right\rangle \subseteq \left\langle \frac{1}{p^{h+1}}+\mathbb{Z}\right\rangle \subseteq \ldots \subseteq$$

and that

$$\mathbb{Z}(p^\infty) = \bigcup_{h \in \mathbb{N}, h \geq 1} \left\langle \frac{1}{p^h}+\mathbb{Z}\right\rangle.$$

Proposition 3.18. *Let D be a commutative domain and let E be a torsion-free divisible R-module. Then E is an injective module.*

Proof. We will apply Theorem 3.6. Thus let I be an ideal of D and let $i : I \to D$ be the canonical inclusion. Let $f : I \to E$ be an homomorphism. We seek an homomorphism $\overline{f} : D \to E$ such that $\overline{f} \circ i = f$. If $f = 0$ we just set $\overline{f} = 0$. If $f \neq 0$, there exists an element $a \in I$ such that $f(a) \neq 0$. Then we get that $a \neq 0$ and hence, since E is divisible, there exists an element $x \in E$ such that $f(a) = ax$. Let $\overline{f} = \mu_x : D \to E$ i.e. $\overline{f}(d) = dx$ for every $d \in D$. Let us check that $\overline{f} \circ i = f$. Thus let $b \in I$ and let us prove that

$$\left(\overline{f} \circ i\right)(b) = f(b).$$

If $b = 0$, there is nothing to prove. Thus let us assume that $b \neq 0$. Then $f(b) \in E = bE$ and hence there is an element $x_b \in E$ such that $f(b) = bx_b$. We compute

$$bf(a) = f(ba) = f(ab) = af(b) = abx_b.$$

Therefore we obtain that $bf(a) = bax_b$ i.e.

$$b(f(a) - ax_b) = 0.$$

Since $b \neq 0$ and D is a domain, this implies that $f(a) - ax_b = 0$ i.e. that $f(a) = ax_b$. Since we have also that $f(a) = ax$, we deduce that

$$ax = ax_b$$

and since $a \neq 0$ and D is a domain, we infer that $x = x_b$. Then we finally obtain that

$$\left(\overline{f} \circ i\right)(b) = \overline{f}(b) = bx = bx_b = f(b).$$

□

Corollary 3.19. *Let D be a domain. Then the ring of quotient $Q(D)$ of D is an injective D-module.*

Proof. By Example 3.11, we have that $Q(D)$ is a divisible D-module. Since $Q(D)$ is a domain, it is in particular a torsion-free D-module. Thus,by Proposition 3.18, $Q(D)$ is an injective D-module. □

Proposition 3.20. *Let R be commutative ring. Every injective R-module is divisible.*

Proof. Let E be an injective R-module and let $a \in R$ be a non-zero divisor. We have to prove that $aE = E$. Thus let $x \in E$ and let us define a map $\varphi : (a) = Ra \to E$ by setting $\varphi(ra) = rx$. Let us check that φ is well-defined. Assume that $r, r' \in R$ and that $ra = r'a$. This implies that $(r - r')\,a = 0$ and hence, since a is not a zero-divisor, that $(r - r') = 0$ i.e. $r = r'$ so that $rx = r'x$. It is easy to check that φ is an R-module homomorphism. Since E is injective, φ extends to an homomorphism $\overline{\varphi} : R \to E$. Let $y = \overline{\varphi}(1)$. We have

$$ay = a\overline{\varphi}(1) = \overline{\varphi}(a) = \varphi(a) = x.$$

□

Proposition 3.21. *Let D be a principal ideal domain and let E be an D-module. Then E is injective if and only if E is divisible.*

Proof. In view of Proposition 3.20 we have only to prove that every divisible module is injective. Thus let E be a divisible D-module. We will prove that E is injective by using Baer's criterion (3.6). Thus let I be an ideal of D and let $f : I \to E$ be an D-module homomorphism. Since D is a principal ideal domain, there exists an $a \in D$ such that $I = (a) = Ra$. If $a = 0$, then f is the zero homomorphism and hence can be trivially extended to R. If $a \neq 0$ then a is not a zero-divisor in D. Since E is divisible, there exists an $y \in E$ such that

$$ay = f(a).$$

Let us consider the homomorphism $\mu_y : D \to E$ which is defined by setting $\mu_y(d) = dy$ for every $r \in D$. Then, for every $r \in D$ we have:

$$\mu_y(ra) = ray = rf(a) = f(ra).$$

Therefore $\mu_y : R \to E$ is an homomorphism which extends f. □

Example 3.22. *The abelian groups* $\mathbb{Q}, \mathbb{Q}/\mathbb{Z}, \mathbb{Z}(p^\infty)$ *are all divisible groups and hence injectives. In fact* $\mathbb{Q}$ *is divisible by Example 3.11. Hence* $\mathbb{Q}/\mathbb{Z}$ *is divisible by Proposition 3.12 and* $\mathbb{Z}(p^\infty)$ *is divisible by Propositions 3.15 and 3.12.*

Exercise 3.23. *Prove that the abelian groups* $\mathbb{R}$ *and* $\mathbb{R}/\mathbb{Z}$ *are injectives. Prove also that* $t(\mathbb{R}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}$*. Deduce that there exists a subgroup* H *of* $\mathbb{R}$ *which contains* $\mathbb{Z}$ *such that*

$$\mathbb{R}/\mathbb{Z} = \mathbb{Q}/\mathbb{Z} \dot{\oplus} H/\mathbb{Z}$$

and that $H/\mathbb{Z}$ *is torsion free.*

Theorem 3.24. *Every abelian group can be embedded in an injective abelian group.*

Proof. Let G be an abelian group. Then, by Proposition 2.2, there is a surjective homomorphism $h : \mathbb{Z}^{(G)} \to G$ and hence we have that

$$G \cong \mathbb{Z}^{(G)}/L.$$

. Let $L = \mathrm{Ker}(h)$. Then the canonical inclusion $i : \mathbb{Z}^{(G)} \to \mathbb{Q}^{(G)}$ induces an injective homomorphism $\widetilde{h} : \mathbb{Z}^{(G)}/L \to \mathbb{Q}^{(G)}/L$ and hence we get an injective homomorphism $\varphi : G \to \mathbb{Q}^{(G)}/L$. By Example 3.11, $\mathbb{Q}$ is divisible and hence, by Proposition 3.12 also $\mathbb{Q}^{(G)}$ and $\mathbb{Q}^{(G)}/L$ are divisible. Then we can apply Proposition 3.21 to infer that $\mathbb{Q}^{(G)}/L$ is an injective abelian group. □

3.25. *Let* R *be* ***any*** *ring and let* G *be an abelian group. Then we can consider the abelian group* $\mathrm{Hom}_{\mathbb{Z}}(R, G)$*. This abelian group can be endowed with a left* R*-* ***module structure*** *as follows. For every* $a \in R$ *and* $f \in \mathrm{Hom}_{\mathbb{Z}}(R, G)$*, consider the map*

$$g_a : R \to G \text{ defined by setting } g_a(r) = f(ra) \text{ for every } r \in R.$$

Let us check that $g_a \in \mathrm{Hom}_{\mathbb{Z}}(R, G)$*. Let* r_1 *and* $r_2 \in R$ *and let us compute*

$$g_a(r_1 + r_2) = f((r_1 + r_2)a) = f(r_1 a + r_2 a) = f(r_1 a) + f(r_2 a) = g_a(r_1) + g_a(r_2).$$

Then we set

$$a \cdot f = g_a \text{ which means that } (a \cdot f)(r) = f(ra) \text{ for every } r \in R. \tag{3.2}$$

Let us check that this defines a left R*-module structure on* $\mathrm{Hom}_{\mathbb{Z}}(R, G)$*. Thus let* $a, b, a_1, a_2 \in R$ *and* $f,\ f_1, f_2 \in \mathrm{Hom}_{\mathbb{Z}}(R, G)$*. For every* $r \in R$ *we compute*

$$\begin{aligned}[a \cdot (f_1 + f_2)](r) = (f_1 + f_2)(ra) = f_1(ra) + f_2(ra) = \\ = (a \cdot f_1)(r) + (a \cdot f_2)(r) = [(a \cdot f_1 + (a \cdot f_2))](r),\end{aligned} \tag{3.3}$$

$$\begin{aligned}[(a_1 + a_2) \cdot f](r) = f(r(a_1 + a_2)) = f(ra_1 + ra_2) = \\ = f(ra_1) + f(ra_2) = (a_1 \cdot f)(r) + (a_2 \cdot f)(r) = [(a_1 \cdot f) + (a_2 \cdot f)](r)\end{aligned} \tag{3.4}$$

and

$$[(ab) \cdot f](r) = f(rab) = (b \cdot f)(ra) = [a \cdot (b \cdot f)](r). \tag{3.5}$$

(3.3) *entails that* $a \cdot (f_1 + f_2) = (a \cdot f_1 + (a \cdot f_2))$, (3.4) *entails that* $(a_1 + a_2) \cdot f = (a_1 \cdot f) + (a_2 \cdot f)$ *and finally* (3.5) *entails that* $(ab) \cdot f = a \cdot (b \cdot f)$.

Proposition 3.26. *Let R be a ring and let E be an injective abelian group. Then* $\mathrm{Hom}_{\mathbb{Z}}(R, E)$ *is an injective left R-module.*

Proof. Let $0 \to L \xrightarrow{j} M$ be an injective R-module homomorphism and let $f : L \to \mathrm{Hom}_{\mathbb{Z}}(R, E)$ be a left R-module homomorphism. We seek a left R-module homomorphism $\overline{f} : M \to \mathrm{Hom}_{\mathbb{Z}}(R, E)$ such that $\overline{f} \circ j = f$. First of all we consider the map $\varphi : L \to E$ defined by setting $\varphi(a) = f(a)(1_R)$ for every $a \in L$. Let us check that φ is an abelian group homomorphism. Let $a_1, a_2 \in L$ and let us compute

$$\varphi(a_1 + a_2) = f(a_1)(1_R) + f(a_2)(1_R) \overset{def+\text{inHom}_{\mathbb{Z}}(R,E)}{=} [f(a_1) + f(a_2)](1_R) \overset{f\text{isanhomo}}{=} \\ = f(a_1 + a_2)(1_R) = \varphi(a_1 + a_2).$$

Since E is an injective abelian group,there is an abelian group homomorphism $\overline{\varphi} : M \to E$ such that $\overline{\varphi} \circ j = \varphi$. Now, for every $m \in M$ let us consider the map

$$f_m : R \to E \text{ defined by setting } f_m(a) = \overline{\varphi}(am) \text{ for every } a \in R.$$

Let us check that $f_m \in \mathrm{Hom}_{\mathbb{Z}}(R, E)$. Let $a_1, a_2 \in R$. We have

$$f_m(a_1 + a_2) = \overline{\varphi}((a_1 + a_2)m) = \overline{\varphi}(a_1 m + a_2 m) \overset{\overline{\varphi}\text{isgrouphomo}}{=} \\ = \overline{\varphi}(a_1 m) + \overline{\varphi}(a_2 m) = f_m(a_1) + f_m(a_2).$$

Hence $f_m \in \mathrm{Hom}_{\mathbb{Z}}(R, E)$. Now we consider the map

$$\overline{f} : M \to \mathrm{Hom}_{\mathbb{Z}}(R, E) \text{ defined by setting } \overline{f}(m) = f_m \text{ for every } m \in M.$$

This means that, for every $m \in M$ and $a \in R$, we have

$$\left[\overline{f}(m)\right](a) = \overline{\varphi}(am).$$

Let us check that $\overline{f}$ is a left R-module homomorphism. Let $x, x_1, x_2 \in M$ and let $r \in R$. For every $a \in R$ we compute

$$\left[\overline{f}(x_1 + x_2)\right](a) = \overline{\varphi}(a(x_1 + x_2)) = \overline{\varphi}(ax_1 + ax_2) \overset{\overline{\varphi}\text{isgrouphomo}}{=} \\ = \overline{\varphi}(ax_1) + \overline{\varphi}(ax_2) = \overline{f}(x_1)(a) + \overline{f}(x_2)(a) \overset{def+\text{inHom}_{\mathbb{Z}}(R,E)}{=} \\ = \left[\overline{f}(x_1) + \overline{f}(x_2)\right](a) \tag{3.6}$$

and

$$\left[\overline{f}(rx)\right](a) = \overline{\varphi}(a(rx)) = \overline{\varphi}((ar)x) = \left[\overline{f}(x)\right](ar) \overset{(3.2)}{=} \left[r \cdot \overline{f}(x)\right](a). \tag{3.7}$$

(3.6) entails that $\overline{f}(x_1 + x_2) = \overline{f}(x_1) + \overline{f}(x_2)$, while (3.7) entails $\overline{f}(rx) = r \cdot \overline{f}(x)$. Therefore we deduce that $\overline{f}$ is a left R-module homomorphism.

It remains to check that $\overline{f} \circ j = f$. Thus let $y \in L$ and, for every $a \in R$, let us compute

$$\left[\left(\overline{f} \circ j\right)(y)\right](a) = \overline{f}(j(y))(a) = \overline{\varphi}(aj(y)) = \overline{\varphi}(j(ay)) = (\overline{\varphi} \circ j)(ay) = \varphi(y) = [f(ay)](1_R) \\ \overset{f\text{is}R\text{modmorph}}{=} [a \cdot f(y)](1_R) \overset{(3.2)}{=} f(y)(a1_R) = f(y)(a).$$

This implies that $\left(\overline{f} \circ j\right)(y) = f(y)$ for every $y \in L$ and hence that $\overline{f} \circ j = f$. $\square$

Lemma 3.27. *Let M be a left R-module. Then the map $\chi : M \to \mathrm{Hom}_{\mathbb{Z}}(R, M)$, defined by setting, using the notations of Proposition 2.2,*

$$\chi(x) = \mu_x \text{ for every } x \in M,$$

is an injective left R-module homomorphism.

Proof. Let $x, x_1, x_2 \in M$ and $a \in R$. For every $r \in R$ we have

$$\begin{aligned} [\chi(x_1 + x_2)](r) &= h_{x_1+x_2}(r) = r(x_1 + x_2) = rx_1 + rx_2 = h_{x_1}(r) + h_{x_2}(r) \\ &= [h_{x_1} + h_{x_2}](r) = [\chi(x_1) + \chi(x_2)](r) \end{aligned}$$

and

$$[\chi(ax)](r) = h_{ax}(r) = r(ax) = (ra)x = h_x(ar) = [a \cdot h_x](r) = [a \cdot \chi(x)](r).$$

Moreover we have

$$\chi(x)(1_R) = h_x(1_R) = 1_R x = x$$

so that, if $x \neq 0$, we infer that $\chi(x) \neq 0$. □

Theorem 3.28. *Let R be a ring. Then any left R-module can be embedded in an injective left R-module.*

Proof. Let M be a left R-module. We seek an injective left R-homomorphism $\varphi : M \to H$ where H is an injective left R-module. By Theorem 3.24, there is an injective abelian group homomorphism i from the abelian group M to an injective abelian group E:

$$0 \to M \xrightarrow{i} E.$$

By Proposition 1.91, we know that $\mathrm{Hom}_{\mathbb{Z}}(R, i) : \mathrm{Hom}_{\mathbb{Z}}(R, M) \to \mathrm{Hom}_{\mathbb{Z}}(R, E)$ is an injective group homomorphism. Let us check that $\varphi = \mathrm{Hom}_{\mathbb{Z}}(R, i)$ is a left R-module homomorphism. Thus let $r \in R$ and $f \in \mathrm{Hom}_{\mathbb{Z}}(R, M)$. For every $a \in R$ we compute

$$[\varphi(rf)](a) = (i \circ rf)(a) = i[(r \cdot f)(a)] \overset{(3.2)}{=} i(f(ar)) = (i \circ f)(ar) = [\varphi(f)](ar) = \overset{(3.2)}{=} [r \cdot \varphi(f)](a).$$

This implies that $\varphi(rf) = r \cdot \varphi(f)$ and hence φ is a left R-module homomorphism. By Proposition 3.26, $\mathrm{Hom}_{\mathbb{Z}}(R, E)$ is an injective left R-module. By Lemma 3.27, we conclude. □

Proposition 3.29. *Let E be a left R-module. Then the following statements are equivalent*

(a) *${}_RE$ is injective.*

(b) *Every short exact sequence of the form $0 \to E \xrightarrow{f} M \xrightarrow{g} N \to 0$ splits.*

(c) *For every injective left R-module homomorphism $f : E \to M$, $f(E)$ is a direct summand of M.*

Proof. $(a) \Rightarrow (b)$. Since E is injective, there exists an homomorphism $p : M \to E$ such that $p \circ f = \mathrm{Id}_E$ and hence, by Theorem 1.84, the given short exact sequence splits.

$(b) \Rightarrow (c)$. Let $f : E \to M$ be an injective left R-module homomorphism. Then we can consider the short exact sequence

$$0 \to E \xrightarrow{f} M \xrightarrow{p_{f(E)}} M/f(E) \to 0.$$

By assumption (b), this sequence splits and hence, by Theorem 1.84, there is a submodule X of M such that

$$M = f(E) \dot{\oplus} X.$$

$(c) \Rightarrow (a)$. By Theorem 3.28, there is an injective left R-module homomorphism $\varphi : E \to H$ where H is an injective left R-module. In view of assumption (c), there is a submodule X of H such that

$$H = \varphi(E) \dot{\oplus} X.$$

By Corollary 3.5, we deduce that $\varphi(E)$ is an injective left R-module. Since φ is an injective homomorphism, we deduce that $E \cong \varphi(E)$ and hence E is injective. □

Definition 3.30. *Let L be a submodule of a left R-module M. We say that L is* essential in M *if, for every non-zero submodule H of M, $H \cap L \neq \{0\}$.*

Proposition 3.31. *Let L be a submodule of a left R-module M. Then L is essential in M if and only if, for every $x \in M$, $x \neq 0$, there is an $r \in R$ such that $0 \neq rx \in L$.*

Proof. Exercise. □

Examples 3.32. *$\mathbb{Z}$ is essenzial in the $\mathbb{Z}$-module $\mathbb{Q}$ and $\left\langle \frac{1}{p} + \mathbb{Z} \right\rangle$ is essential in the $\mathbb{Z}$-module $\mathbb{Z}(p^\infty)$.*

Proposition 3.33. *Let $(M_\lambda)_{\lambda \in \Lambda}$ be a family of left R-modules and assume that, for every $\lambda \in \Lambda$, L_λ is an essential submodule of M_λ. Then*

$$\bigoplus_{\lambda \in \Lambda} L_\lambda \text{ is an essential submodule of } \bigoplus_{\lambda \in \Lambda} M_\lambda.$$

Proof. Let $x \in \bigoplus_{\lambda \in \Lambda} M_\lambda$, $x \neq 0$. Then $Supp(x)$ is a finite nonempty subset F of Λ. By induction on $n = |F|$, we will prove that there is an $r \in R$ such that $0 \neq rx \in \bigoplus_{\lambda \in \Lambda} L_\lambda$. If $n = 1$, then $F = \{\lambda_1\}$ for some $\lambda_1 \in \Lambda$. Then $x = \varepsilon_{\lambda_1}(x_{\lambda_1})$

where $x_{\lambda_1} \in M_{\lambda_1}$. Since $0 \neq x_{\lambda_1}$ and L_{λ_1} is essential in M_{λ_1} there exists an $r \in R$ such that $0 \neq r \cdot x_{\lambda_1} \in L_{\lambda_1}$. Hence we get

$$0 \neq \varepsilon_{\lambda_1}(r \cdot x_{\lambda_1}) \in \varepsilon_{\lambda_1}(L_{\lambda_1}) \in \bigoplus_{\lambda \in \Lambda} L_\lambda$$

and since

$$r \cdot x = r \cdot \varepsilon_{\lambda_1}(x_{\lambda_1}) = \varepsilon_{\lambda_1}(r \cdot x_{\lambda_1})$$

we conclude.

Let us assume that the statement hold for all $k \in \mathbb{N}$, $k \geq 1$ and $k \leq n$ for some $n \in \mathbb{N}$, $n \geq 1$, and let us prove it for $n+1$. Let $\lambda_1 \in F$. Then there exists an $r \in R$ such that $0 \neq rx_{\lambda_1} \in L_{\lambda_1}$. Let us consider $rx - r\varepsilon_{\lambda_1}(x_{\lambda_1})$. If $rx - r\varepsilon_{\lambda_1}(x_{\lambda_1}) = 0$, then $0 \neq rx = r\varepsilon_{\lambda_1}(x_{\lambda_1}) \in \varepsilon_{\lambda_1}(L_{\lambda_1}) \in \bigoplus_{\lambda \in \Lambda} L_\lambda$. Otherwise $0 \neq rx - rx_{\lambda_1}$ and $Supp(rx - r\varepsilon_{\lambda_1}(x_{\lambda_1})) \subseteq Supp(x) \setminus \{\lambda_1\}$ so that $|Supp(rx - rx_{\lambda_1})| < |F| = n+1$. Thus there exists an $s \in R$ such that

$$0 \neq s(rx - r\varepsilon_{\lambda_1}(x_{\lambda_1})) \in \bigoplus_{\lambda \in Supp(x) \setminus \{\lambda_1\}} L_\lambda. \tag{3.8}$$

Then

$$srx = srx - sr\varepsilon_{\lambda_1}(x_{\lambda_1}) + sr\varepsilon_{\lambda_1}(x_{\lambda_1}) \in \bigoplus_{\lambda \in Supp(x)} L_\lambda.$$

Assume that $srx = 0$. Then from (3.8) we would get

$$0 \neq -sr\varepsilon_{\lambda_1}(x_{\lambda_1}) \in \bigoplus_{\lambda \in Supp(x) \setminus \{\lambda_1\}} L_\lambda$$

which is a contradiction. Therefore $0 \neq rx \in \bigoplus_{\lambda \in Supp(x)} L_\lambda$. □

Proposition 3.34. *Let L be a submodule of a left R-module M. Let H be a submodule of M maximal with respect to the property $L \cap H = \{0\}$. Then $L + H = L \dot{\oplus} H$ is essential in M.*

Proof. Let $x \in M$ such that $(L + H) \cap Rx = \{0\}$. Let $y \in L \cap (H + Rx)$. Then there exists an element $h \in H$ and an element $r \in R$ such that $y = h + rx$. Then we get

$$rx = y - h \in (L + H) \cap Rx = \{0\}$$

and hence we deduce that $rx = y - h = 0$ so that $y = h \in L \cap H = \{0\}$. Thus we obtain that $L \cap (H + Rx) = \{0\}$. By the maximality property of H we deduce that $Rx \subseteq H$. Hence we obtain

$$Rx \subseteq H \subseteq (L + H) \cap Rx = \{0\}$$

and we deduce that $x = 0$. □

Proposition 3.35. *Let A, B be submodules of a left R-module M and assume that $A \subseteq B$. Then the following assertions are equivalent.*

(a) *A is an essential submodule of B and B is an essential submodule of M.*

(b) *A is an essential submodule of M.*

Proof. $(a) \Rightarrow (b)$. Let $x \in M$, $x \neq 0$. Since B is essential in M, there is an $r \in R$ such that $0 \neq rx \in B$. Since A is essential in B, there exists an $s \in R$ such that $0 \neq srx \in A$.

$(b) \Rightarrow (a)$. It is trivial. □

Definitions 3.36. *Let M be a left R-module. An* extension *of M is a pair (H, j) where H is a left R-module and $j : M \to H$ is an injective left R-module homomorphism.*

- *An extension (H, j) of M is called* proper *whenever $j(M) \subsetneq H$.*
- *An extension (H, j) of M is called* injective *whenever H is an injective left R-module.*
- *An extension (H, j) of M is called* essential *whenever $j(M)$ is essential in H.*

Exercise 3.37. *Let L be a submodule of a left R-module M and let $f : M \to M'$ be an inective homomorphism. Show that L is an essential submodule of M if and only if $f(L)$ is an essential submodule of $f(M)$.*

Proposition 3.38. *Let $j : M \to H$ and $\eta : H \to H'$ be injective homomorphisms of left R-modules. Assume that $j(M)$ is an essential submodule of H. Then the following assertions are equivalent:*

(a) *$\eta \circ j(M)$ is an essential submodule of H'.*

(c) *$\eta(H)$ is an essential submodule of H'.*

Proof. Since η is injective, $\eta \circ j(M)$ is essential in $\eta(H)$. The conclusion follows by Proposition 3.35. □

Definition 3.39. *Let $j : M \to H$ be an injective homomorphism of left R-modules. (H, j) is said to be a* maximal essential extension *of M if*

1) *(H, j) is an essential extension of M i.e. $j(M)$ is an essential submodule of H,*

2) *if (H', η) is an essential extension of H, i.e. if $\eta : H \to H'$ is an injective homomorphism of left R-modules such that $\eta(H)$ is an essential submodule of H', then $\eta(H) = H'$.*

Remark 3.40. *Let (H, j) be an essential extension of M and let (H', η)be an extension of H. In view of Proposition 3.38 (H', η) is an essential extension of H if and ony if $\eta \circ j(M)$ is an essential submodule of H'.*

Proposition 3.41. *Let M be a left R-module, let (N, j) be an essential extension of M and let (E, i) be an injective extension of M. Then there exists an injective homomorphism $\alpha : N \to E$ such that $\alpha \circ j = i$.*

Proof. Since E is injective, there is a left R-module homomorphism $\alpha : N \to E$ such that $\alpha \circ j = i$. Let $y \in \mathrm{Ker}(\alpha) \cap j(M)$. Then there is an $x \in M$ such that $j(x) = y$ and from $y \in \mathrm{Ker}(\alpha)$ we infer that

$$0 = \alpha(y) = \alpha(j(x)) = (\alpha \circ j)(x) = i(x).$$

Since i is injective this implies that $x = 0$ and hence $y = j(x) = j(0) = 0$. Thus we deduce that $\mathrm{Ker}(\alpha) \cap j(M) = \{0\}$. Since $j(M)$ is an essential submodule of N, this implies that $\mathrm{Ker}(\alpha) = \{0\}$ i.e. α is injective. □

Proposition 3.42. *Let M be a left R-module and and let (E, j) be an injective extension of M. Then E contains a submodule H such that $j(M) \subseteq H$ and $\left(H, j^{|H}\right)$ is a maximal essential extension of M.*

Proof. Let $\Omega = \{K \mid j(M)$ is an essential submodule of K and $K \leq {}_R E\}$. Clearly $\Omega \neq \varnothing$ since $j(M) \in \Omega$. Now $(\Omega, \subseteq)$ is an inductive partially ordered set. Hence, by Zorn's Lemma, it has a maximal element. Let H be a maximal element for $(\Omega, \subseteq)$. Then $\left(H, j^{|H}\right)$ is an essential extension of M. Let us prove that $\left(H, j^{|H}\right)$ is a maximal essential extension of M. Let $i : H \to E$ be the canonical inclusion. Then $i(H) = H$ and $i \circ j^{|H} = j$. Hence we have

$$i\left(j^{|H}(M)\right) = j(M) \text{ is an essential in } i(H) = H. \tag{3.9}$$

Let $\eta : H \to H'$ be an injective homomorphism of left R-modules such that $\eta(H)$ is an essential submodule of H'. We have to prove that $\eta(H) = H'$.

By Proposition 3.41, there is an injective homomorphism $\alpha : H' \to E$ such that $\alpha \circ \eta = i$. Therefore since $\eta(H)$ is an essential submodule of H' and α is injective, we deduce that $\alpha(\eta(H))$ is an essential in $\alpha(H')$. From (3.9), we know that $j(M) = i\left(j^{|H}(M)\right)$ is an essential in $H = i(H) = \alpha \circ \eta(H) = \alpha(\eta(H))$. Since $H = \alpha(\eta(H))$ is an essential in $\alpha(H')$, by Proposition 3.35, we get that $j(M)$ is essential in $\alpha(H')$ so that $\alpha(H') \in \Omega$. From $H = \alpha(\eta(H)) \subseteq \alpha(H')$, by the maximality of H we get that $H = \alpha(H')$. As $H = \alpha(\eta(H))$, we obtain that $\alpha(\eta(H)) = \alpha(H')$ which implies, in view of the injectivity of α, that $\eta(H) = H'$. □

Theorem 3.43. *Let E be a left R-module. Then E is injective if and only if E has no proper essential extension.*

Proof. Assume that E is injective. Let $j : E \to H$ be an injective homomorphism of left R-modules and suppose that $j(E)$ is essential in H. We will prove that

$j(E) = H$. Since E is injective, by Proposition 3.29, there is a submodule L of H such that $H = j(E) \oplus L$. Since $j(E)$ is essential in H and $j(E) \cap L = \{0\}$, we deduce that $L = \{0\}$. Hence $H = j(E)$.

Conversely, assume that E has no proper essential extension. Let $i : E \to M$ be an injective homomorphism. We will prove that i splits. Assume that $i(E) \subsetneqq M$. By Zorn's Lemma, there exists a submodule H of M maximal with respect to the property $i(E) \cap H = \{0\}$. If $H = \{0\}$ then, for any $L \leq M$ with $L \neq \{0\}$, we would get that $i(E) \cap L \neq \{0\}$ and hence $i(E)$ would be essential in M which is a contradiction since $i(E) \subsetneqq M$. Thus $H \neq \{0\}$. If $i(E) + H = M$ we would get $i(E) \oplus H = M$. Therefore we can assume that $i(E) + H \subsetneqq M$. We deduce that

$$i(E) \cong \frac{i(E)}{i(E) \cap H} \cong \frac{i(E) + H}{H} \subsetneqq \frac{M}{H}.$$

Let $j : i(E) \to \frac{i(E)+H}{H}$ be the composition of the displayed isomorphisms . Then $j \circ i(E) \subsetneqq \frac{M}{H}$. Thus there exists a submodule Y of M such that $H \subsetneqq Y \subseteq M$ and

$$\begin{aligned} \left(\frac{i(E) + H}{H}\right) \cap \frac{Y}{H} &= \{0\} = \frac{H}{H} \text{ i.e.} \\ (i(E) + H) \cap Y &= H. \end{aligned}$$

Thus we infer that $(i(E) \cap Y) \subseteq (i(E) + H) \cap Y = H$ and hence $(i(E) \cap Y) \subseteq (i(E) \cap H) = \{0\}$. Since $H \subsetneqq Y \subseteq M$ this contradicts the maximality of H. Therefore we get that $i(E) + H = M$ and hence $i(E) \oplus H = M$. □

Definition 3.44. *Let $i : M \to E$ be an injective homomorphism of left R-modules. (E, i) is said to be a* minimal injective extension *of M if*

1) *E is an injective left R-module,*

2) *for any injective homomorphism $i' : M \to E'$ where E' is an injective left R-module, there exists an injective homomorphism $\chi : E \to E'$ such that $\chi \circ i = i'$.*

Proposition 3.45. *Let $i : M \to E$ be an injective left R-module homomorphism. Then the following assertions are equivalent.*

(a) (E, i) is an injective and essential extension of M.

(b) (E, i) is a maximal essential extension of M.

(c) (E, i) is a minimal injective extension of M.

Proof. $(a) \Rightarrow (b)$. Let $\eta : E \to H$ be an injective homomorphism of left R-modules and assume that $\eta(E)$ is essential in H. Then, by Theorem 3.43, we have that η is an isomorphism.

$(b) \Rightarrow (a)$. Let us prove that E is injective. By Theorem 3.43, this is equivalent to prove that E has no proper essential extension. Let $\eta : E \to E'$ be an injective homomorphism and assume that $\eta(E)$ is essential in E'. Since (E,i) is a maximal essential extension of M, we deduce that η is an isomorphism.

$(a) \Rightarrow (c)$. Let $i' : M \to E'$ an injective homomorphism and assume that E' is injective. Then, by Proposition 3.41, there exists an injective left R-module homomorphism $\chi : E \to E'$ such that $\chi \circ i' = i$.

$(c) \Rightarrow (a)$. By Proposition 3.42, E contains a submodule H such that $i(M) \subseteq H$ and $\left(H, i^{|H}\right)$ is a maximal essential extension of M. Since we already proved that $(b) \Rightarrow (a)$, we know that H is injective and hence $\left(H, i^{|H}\right)$ is an injective (and essential) extension of M. Then, by (c), there exists an injective homomorphism $\chi : E \to H$ such that $\chi \circ i = .i^{|H}$. Since $i^{|H}(M)$ is essential in H and $i^{|H}(M) = \chi \circ i(M) \subseteq \chi(E)$, by Proposition 3.35 we deduce that $\chi \circ i(M)$ is essential in $\chi(E)$.

Since χ is injective, we deduce that $i(M)$ is essential in E. □

Theorem 3.46. *Let M be a left R-module. Then there exists an injective homomorphism of left R-modules $i : M \to E$ such that (E,i) fulfills the following equivalent conditions:*

(a) *(E,i) is an injective and essential extension of M.*

(b) *(E,i) is a maximal essential extension of M.*

(c) *(E,i) is a minimal injective extension of M.*

Moreover if both (E,i) and (E',i') fulfill these conditions, then there exists an homomorphism $\alpha : E \to E'$ such that $\alpha \circ i = i'$. Furthermore α is an isomorphism.

Proof. In view of Proposition 3.45, we know that conditions $(a),(b)$ and (c) are equivalent. By Theorem 3.28, there exists an injective left R-module homomorphism $i : M \to I$ where I is injective. By Proposition 3.42, I contains a submodule H such that $i(M) \subseteq H$ and $\left(i^{|H}, H\right)$ is a maximal essential extension of M.

Assume now that both (E,i) and (E',i') fulfill above conditions. Since (E,i) is a minimal injective extension of M, there exists an injective homomorphism $\alpha : E \to E'$ such that $\alpha \circ i = i'$. Then $\alpha \circ i(M) = i'(M)$ is essential in E' and being (E,i) a maximal essential extension of M, we get that $\alpha(E) = E'$. □

Definition 3.47. *Let M be a left R-module. A pair (E,i) which satisfies the equivalent conditions of Theorem 3.46 is called an* injective envelope *of M. An injective envelope of M will also be denoted simply by $E_R(M)$ or even by $E(M)$.*

Exercise 3.48. *Let L be an essential submodule of a left R-module M. Show that $E(L) = E(M)$.*

Examples 3.49.

1) $E_{\mathbb{Z}}(\mathbb{Z}) = \mathbb{Q}$. *In fact, by Example 3.22, $\mathbb{Q}$ is an injective abelian group. Let us prove that $\mathbb{Z}$ is essential in $\mathbb{Q}$. Let $q \in \mathbb{Q}, q \neq 0$. Write $q = \frac{m}{n}$ where $m,n \in \mathbb{Z}$ and $m,n \neq 0$. Then $nq = m \in \mathbb{Z}$ and $m \neq 0$.*

2) $E_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}(p^\infty)$. *In fact, by Example 3.22,* $\mathbb{Z}(p^\infty)$ *is an injective abelian group. Let* $H = \left\langle \frac{1}{p} + \mathbb{Z} \right\rangle$. *Then* H *is an essential submodule of* $\mathbb{Z}(p^\infty)$. *In fact, if* $x \in \mathbb{Z}(p^\infty)$ *and* $x \neq 0$ *there exist* $m \in \mathbb{Z}, h \in \mathbb{N}$ *such that*

$$x = \frac{m}{p^h} + \mathbb{Z} \text{ where } h > 0 \text{ and } (m,p) = 1.$$

Then

$$\left(p^{h-1}\right) x = \frac{m}{p} + \mathbb{Z} \neq 0 + \mathbb{Z}.$$

In fact if $\frac{m}{p} \in \mathbb{Z}$, *then there is an* $a \in \mathbb{Z}$ *such that* $m = ap$ *which contradicts that* $(m,p) = 1$. *Since* $o\left(\frac{1}{p} + \mathbb{Z}\right) = p$ *we get that* $\mathbb{Z}/p\mathbb{Z} \cong H$.

Exercise 3.50. *Let* D *be a commutative domain. Show that* $E_D(D) = Q(D)$.

Chapter 4

Generators and Cogenerators

Notation 4.1. *In the following we will denote by R-Mod the class of all left R-modules.*

Definition 4.2. *Let R be a ring. A left R-module ${}_RQ$ is called a* generator of R-Mod *if, given R-module homomorphisms $f, g : M \to N$ with $f \neq g$, there is a left R-module homomorphism $h : Q \to M$ such that*

$$f \circ h \neq g \circ h.$$

Proposition 4.3. *Let Q be a left R-module. The following assertions are equivalent:*

(a) *Q is a generator of R-Mod.*

(b) *For every left R-module M we have that*

$$M = \sum_{h \in \mathrm{Hom}_R(Q,M)} \mathrm{Im}\,(h)$$

(c) *For every left R-module M, there exists a nonempty set I and a surjective R-module homomorphism*

$$Q^{(I)} \to M \to 0.$$

Proof. Let us consider $Q^{(\mathrm{Hom}_R(Q,M))} = \bigoplus_{h \in \mathrm{Hom}_R(Q,M)} Q_h$ where $Q_h = Q$ for every $h \in \mathrm{Hom}_R(Q, M)$. Let $\varphi = \nabla\,(h)_{h \in \mathrm{Hom}_R(Q,M)} : Q^{(\mathrm{Hom}_R(Q,M))} \to M$. We know (cf. Proposition 1.64) that

$$\mathrm{Im}\,(\varphi) = \sum_{h \in \mathrm{Hom}_R(Q,M)} \mathrm{Im}\,(h)$$

$(a) \Rightarrow (b)$. Let us prove that φ is surjective. Let $T = \mathrm{Im}\,(\varphi)$ and let us assume that $T \subsetneqq M$. Then $M/T \neq \{0\}$. Let $p = p_T : M \to M/T$ be the canonical projection. Then $p \neq 0$ and hence there exists a left R-module homomorphism $\chi : Q \to M$ such that $p \circ \chi \neq 0 \circ \chi = 0$. Since $p \circ \chi \neq 0$ we get that

$$\mathrm{Im}\,(\chi) \nsubseteq \mathrm{Ker}\,(p) = T = \mathrm{Im}\,(\varphi) = \sum_{h \in \mathrm{Hom}_R(Q,M)} \mathrm{Im}\,(h)$$

which is a contradiction.

$(b) \Rightarrow (c)$. Let $I = \mathrm{Hom}_R(Q, M)$ and let $\varphi = \nabla(h)_{h \in \mathrm{Hom}_R(Q,M)}$. Then

$$\mathrm{Im}(\varphi) = \sum_{h \in \mathrm{Hom}_R(Q,M)} \mathrm{Im}(h) = M.$$

$(c) \Rightarrow (a)$.Let $f, g : M \to N$ be homomorphisms of left R-modules with $f \neq g$. By assumption (c), there exists a nonempty set I and a surjective R-module homomorphism

$$p : Q^{(I)} \to M.$$

Since p is surjective, from $f \neq g$ we infer that $f \circ p \neq g \circ p$ and hence, there exists an $i_0 \in I$ such that

$$f \circ p \circ \varepsilon_{i_0} \neq g \circ p \circ \varepsilon_{i_0}.$$

Set $h = p \circ \varepsilon_{i_0} : Q \to M$. Then $f \circ h \neq g \circ h$. □

Corollary 4.4. *${}_RR$ is a generator of R-Mod.*

Proof. It follows by Propositions 2.2 and 4.3. □

Exercise 4.5. *Let ${}_RQ$ be a left R-module and assume that there is a surjective left R-module homomorphism $p : {}_RQ \to {}_RR$. Show that ${}_RQ$ is a generator of R-Mod. Deduce from this, that if ${}_RL$ is a left R-module, then the left R-module ${}_RR \oplus {}_RL$ is a generator of R-Mod.*

Exercise 4.6. *Let ${}_RQ$ be a generator of R-Mod. Show that there is an $n \in \mathbb{N}, n \geq 1$ and a a surjective left R-module homomorphism $p : {}_RQ^n \to {}_RR$.*

Definition 4.7. *Let R be a ring. A left R-module ${}_RK$ is called a* cogenerator of R-Mod *if, given R-module homomorphisms $f, g : M \to N$ with $f \neq g$, there is a left R-module homomorphism $h : N \to K$ such that*

$$h \circ f \neq h \circ g.$$

Proposition 4.8. *Let K be a left R-module. The following assertions are equivalent:*

(a) *K is a cogenerator of R-Mod.*

(b) *For every left R-module M we have that*

$$\bigcap_{f \in \mathrm{Hom}_R(M,K)} \mathrm{Ker}(f) = \{0\}.$$

.

(c) *For every left R-module M, there exists a nonempty set I and an injective R-module homomorphism*

$$0 \to M \to K^I.$$

Proof. Let us consider $K^{\mathrm{Hom}_R(M,K)} = \bigoplus_{h\in\mathrm{Hom}_R(M,K)} K_h$ where $K_h = K$ for every $h \in \mathrm{Hom}_R(M,K)$. Let $\psi = \Delta(h)_{h\in\mathrm{Hom}_R(M,K)} : M \to K^{\mathrm{Hom}_R(M,K)}$. We know (cf. 1.46) that

$$\mathrm{Ker}(\psi) = \bigcap_{f\in\mathrm{Hom}_R(M,K)} \mathrm{Ker}(f)$$

$(a) \Rightarrow (b)$. Let M be a left R-module and let $x \in M, x \neq 0$. Let $i : Rx \to M$ be the canonical inclusion. Then $i \neq 0$. Hence there exists a morphism $h : M \to K$ such that $h \circ i \neq h \circ 0 = 0$. Clearly $h \circ i \neq 0$ infers that $h(x) \neq 0$. We deduce that

$$\bigcap_{f\in\mathrm{Hom}_R(M,K)} \mathrm{Ker}(f) = \{0\}.$$

$(b) \Rightarrow (c)$. Since

$$\mathrm{Ker}(\psi) = \bigcap_{x\in M} \mathrm{Ker}(f_x) = \{0\},$$

$\psi : M \to K^{\mathrm{Hom}_R(M,K)}$ is injective.

$(c) \Rightarrow (a)$. Let $f, g : M \to N$ with $f \neq g$ be left R-module homomorphisms and let $\varphi : N \to K^I$ be an injective R-module homomorphism. Since φ is injective, from $f \neq g$ we get that $\varphi \circ f \neq \varphi \circ g$. This implies that there is an $i_0 \in I$ such that $\pi_{i_0} \circ \varphi \circ f \neq \pi_{i_0} \circ \varphi \circ g$ where $\pi_{i_0} : K^I \to K$ denotes the i_0-th canonical projection. Let $h = \pi_{i_0} \circ \varphi : N \to K$. Then $h \circ f \neq h \circ g$. □

Definition 4.9. *Let ${}_RS$ be a left R-module. We say that ${}_RS$ is a simple left R-module if*

1) $S \neq \{0\}$,

2) *the only submodules of ${}_RS$ are S and $\{0\}$.*

Proposition 4.10. *Let ${}_RS$ be a left R-module. Then the following statement are equivalent.*

(a) *${}_RS$ is simple.*

(b) *$S \neq \{0\}$ and, for any $x \in S$, $x \neq 0$, $Rx = S$.*

Proof. $(a) \Rightarrow (b)$. Let $x \in S$, $x \neq 0$. Then $0 \neq x \in Rx$ so that $Rx \neq \{0\}$. Therefore we infer that $Rx = S$.

$(b) \Rightarrow (a)$. Let L be a non-zero submodule of S. Then there is an $x \in L$ such that $x \neq 0$ and hence we get that $S = Rx \subseteq L$ so that $L = S$. □

Proposition 4.11. *A cyclic left R-module Rx is simple if and only if* $\mathrm{Ann}_R(x)$ *is a left maximal ideal of R.*

Proof. We know that the map $h_x : R \to Rx$ defined by setting $h_x(a) = ax$ for every $r \in R$, is a surjective left R-module homomorphism and $\operatorname{Ker}(h_x) = \operatorname{Ann}_R(x)$ so that we have that $\varphi = \widehat{h}_x : \frac{R}{\operatorname{Ann}_R(x)} \to Rx$ is an isomorphism. Therefore Rx is simple if and only if $\frac{R}{\operatorname{Ann}_R(x)}$ is simple i.e. there are no proper left ideals I of R which properly contain $\operatorname{Ann}_R(x)$. □

Corollary 4.12. *Let $_RS$ be a left R-module. Then $_RS$ is simple if and only if $_RS$ is isomorphic to $\frac{R}{\mathfrak{m}}$ where $\mathfrak{m}$ is a left maximal ideal of R.*

Proof. Assume that $_RS$ is simple and let $x \in S$, $x \neq 0$. Then, by Proposition 4.10, $Rx = S$ is simple so that, by Proposition 4.11, $\operatorname{Ann}_R(x)$ is a left maximal ideal of R. Conversely assume that $_RS$ is isomorphic to $\frac{R}{\mathfrak{m}}$ where $\mathfrak{m}$ is a left maximal ideal of R and let $x = 1 + \mathfrak{m}$. Then $Rx = \frac{R}{\mathfrak{m}}$ and $\operatorname{Ann}_R(x) = \mathfrak{m}$. Thus, by Proposition 4.11, Rx is simple. □

4.13. *Let R be a ring and let $\mathfrak{M}$ be the set of maximal left ideals of R. We define an equivalence relation on $\mathfrak{M}$ by setting*

$$\mathfrak{m}_1 \sim \mathfrak{m}_2 \Leftrightarrow \frac{R}{\mathfrak{m}_1} \cong \frac{R}{\mathfrak{m}_2} \text{ as left } R\text{-modules.}$$

We denote by Ω a set of representatives of the equivalence classes of $\mathfrak{M}$ with respect to $\sim$. Clearly, by Corollary 4.12,

$$\mathcal{S} = \left\{ \frac{R}{\mathfrak{m}} \mid \mathfrak{m} \in \Omega \right\}$$

is a set of representatives of the isomorphism classes of simple left R-modules.

Theorem 4.14. *Let R be a ring. Then*

$$K = \bigoplus_{\mathfrak{m} \in \Omega} E\left(\frac{R}{\mathfrak{m}}\right)$$

is a cogenerator of R-Mod.

Proof. Let M be a left R-module and let $0 \neq x \in M$. Let

$$\mathcal{E} = \{L \mid L \leq_R M \text{ and } x \notin L\}.$$

Since $0 \neq x \in M$ we have that $\{0\} \in \mathcal{E}$ and hence $\mathcal{E} \neq \varnothing$. It is easy to prove that $(\mathcal{E}, \subseteq)$ is an inductive set. Let L_0 be a maximal element in $(\mathcal{E}, \subseteq)$. Set

$$\overline{x} = x + L_0 \in \frac{Rx + L_0}{L_0}.$$

Then

$$R\overline{x} = \frac{Rx + L_0}{L_0}.$$

$R\overline{x}$ **is a simple left** R**-module**.

Let $\overline{H} \subsetneqq R\overline{x}$ be a proper submodule of $R\overline{x}$. Then there is a submodule H of $Rx + L_0$ such that

$$L_0 \subseteq H \subsetneqq Rx + L_0 \text{ and } \overline{H} = \frac{H}{L_0}.$$

Now $L_0 \subseteq H \subsetneqq Rx + L_0$ implies that $x \notin H$. Hence, by the maximality property of L_0, we deduce that $L_0 = H$ so that $\overline{H} = \{0\}$.

By 1) and Proposition 4.11, there is an $\mathfrak{m} \in \Omega$ such that $\mathfrak{m} = \mathrm{Ann}_R(\overline{x})$. Hence we have an injective left R-module homomorphism $\chi : R\overline{x} \to E\left(\frac{R}{\mathfrak{m}}\right)$.

Let $i : R\overline{x} \to \frac{M}{L_0}$ be the canonical inclusion. Since $E\left(\frac{R}{\mathfrak{m}}\right)$ is injective, χ extends to a left R-module homomorphism $\eta : \frac{M}{L_0} \to E\left(\frac{R}{\mathfrak{m}}\right)$.

Let $p = p_{L_0} : M \to \frac{M}{L_0}$ be the canonical projection and let $i_{\mathfrak{m}} : E\left(\frac{R}{\mathfrak{m}}\right) \to K$ be the canonical injection and set

$$f = i_{\mathfrak{m}} \circ \eta \circ p : M \to K.$$

Then

$$f(x) = i_{\mathfrak{m}}(\eta(x + L_0)) = i_{\mathfrak{m}}(\chi(x + L_0)) \neq 0.$$

The conclusion now follows in view of Proposition 4.8. □

Lemma 4.15. *Let K be a cogenerator of R-Mod and let $\chi : K \to U$ be an injective R-module homomorphism. Then U is a cogenerator of R-Mod.*

Proof. Let M be a left R-module and let $x \in M, x \neq 0$. Since K is a cogenerator of R-Mod, By Proposition 4.8, there exists a left R-module homomorphism $f_x : M \to K$ such that $f_x(x) \neq 0$. Since $\chi : K \to U$ is an injective R-module homomorphism, we have that $(\chi \circ f_x)(x) \neq 0$ and $\chi \circ f_x : M \to U$ is a left R-module homomorphism. We conclude by Proposition 4.8. □

Proposition 4.16. *The left R-module $E = E\left(\bigoplus_{\mathfrak{m}\in\Omega} \frac{R}{\mathfrak{m}}\right)$ is an injective cogenerator of R-mod.*

Proof. Let $i : \bigoplus_{\mathfrak{m}\in\Omega} \frac{R}{\mathfrak{m}} \to \bigoplus_{\mathfrak{m}\in\Omega} E\left(\frac{R}{\mathfrak{m}}\right)$ and $j : \bigoplus_{\mathfrak{m}\in\Omega} \frac{R}{\mathfrak{m}} \to E\left(\bigoplus_{\mathfrak{m}\in\Omega} \frac{R}{\mathfrak{m}}\right)$ be the canonical inclusions. Since E is injective, there is a left R-module homomorphism $\chi : \bigoplus_{\mathfrak{m}\in\Omega} E\left(\frac{R}{\mathfrak{m}}\right) \to E$ such that $\chi \circ i = j$. Thus $\mathrm{Ker}(\chi) \cap \mathrm{Im}(i) = \{0\}$. By Proposition 3.33, $\mathrm{Im}(i)$ is essential in $\bigoplus_{\mathfrak{m}\in\Omega} E\left(\frac{R}{\mathfrak{m}}\right)$ so that χ is injective. Apply now Lemma 4.15. □

Remark 4.17. *It is very well known that there exists a unique minimal injective cogenerator M in the category of modules over a ring R with 1. It is very tempting to think that the uniqueness holds in general when the injectivity property is dropped [see, e.g., C. C. Faith, Algebra, I. Rings, modules and categories, corrected reprint, Proposition 3.55, Springer, Berlin, 1981; F. W. Anderson and K. R. Fuller, Rings* *and categories of modules, see pp. 211, 216, Exercise 14, Springer, New York, 1974.].*

In the paper by Barbara Osofsky, "Minimal cogenerators need not be unique", Comm. Algebra 19 (1991), no. 7, 2071–2080, two counterexamples are presented. In the first one, an arbitrarily large cardinal number of nonisomorphic cogenerators which embed in every cogenerator is obtained. In the second one it is shown that even a commutative ring need not have a unique minimal cogenerator.

Chapter 5

2×2 Matrix Ring

Let k be a field and let $R = M_2(k)$ be the ring of 2×2 matrices over R. Let e_{ij} be the matrix with all zero entries except for (i, j) where the entry is 1_k. A simple calculation show that

$$Re_{11} = ke_{11} + ke_{21} = Re_{21} \text{ and } Re_{12} = ke_{12} + ke_{22} = Re_{22}.$$

Set

$$I_1 = Re_{11} \text{ and } I_2 = Re_{22}$$

$$\begin{aligned}
\mathrm{Ann}_R(e_{11}) &= ke_{12} + ke_{22} = I_2 \\
\mathrm{Ann}_R(e_{12}) &= ke_{12} + ke_{22} = I_2 \\
\mathrm{Ann}_R(e_{21}) &= ke_{11} + ke_{21} = I_1 \\
\mathrm{Ann}_R(e_{22}) &= ke_{11} + ke_{21} = I_1.
\end{aligned}$$

Therefore we have

$$\frac{R}{I_2} = \frac{R}{\mathrm{Ann}_R(e_{11})} \cong Re_{11} = I_1 \tag{5.1}$$

$$\frac{R}{I_2} = \frac{R}{\mathrm{Ann}_R(e_{12})} \cong Re_{12} = I_2 \tag{5.2}$$

$$\frac{R}{I_1} = \frac{R}{\mathrm{Ann}_R(e_{21})} \cong Re_{21} = I_1 \tag{5.3}$$

$$\frac{R}{I_1} = \frac{R}{\mathrm{Ann}_R(e_{22})} \cong Re_{22} = I_2. \tag{5.4}$$

This implies that

$$I_1 = Re_{11} \cong \frac{R}{I_2} \cong Re_{12} = I_2. \tag{5.5}$$

Proposition 5.1.

1) *Both I_1 and I_2 are left simple modules and $I_1 \cong I_2$.*

2) *Both I_1 and I_2 are left maximal ideals of R.*

3) *Let $\lambda \in k, \lambda \neq 0$. Set $x_\lambda = e_{11} + \lambda e_{12}$ and $I_\lambda = Rx_\lambda$. Then I_λ is a left maximal ideal of R and $\frac{R}{I_\lambda} \cong I_2$.*

Moreover I_λ is also a simple left R-module and $I_\lambda \cong \frac{R}{I_2} \cong I_1$.

4) *For every maximal ideal M of R with $M \neq I_1$ and $M \neq I_2$, there exists a $\lambda \in K, \lambda \neq 0$, such that $M = I_\lambda$.*

5) *If $\lambda, \lambda' \in k, \lambda \neq 0 \neq \lambda'$and $\lambda \neq \lambda'$ then $I_\lambda \neq I_{\lambda'}$.*

6) *Every simple left module is isomorphic to I_1.*

Proof. 1) Let $x \in I_1, x \neq 0$. Then

$$x = \lambda_{11}e_{11} + \lambda_{21}e_{21} \text{ where } \lambda_{11}, \lambda_{21} \in k.$$

Case $\lambda_{11} \neq 0$. Then

$$e_{11} = (\lambda_{11})^{-1} e_{11}x \in Rx$$

and hence $I_1 = Re_{11} \subseteq Rx$ so that $Rx = I_1$.

Case $\lambda_{11} = 0$ **and** $\lambda_{21} \neq 0$. Then

$$e_{21} = (\lambda_{21})^{-1} e_{22}x \in Rx$$

and hence $I_1 = Re_{21} \subseteq Rx$ so that $Rx = I_1$.

2). It follows from 1) in view of Proposition 4.11.

3) Let $\lambda \in k, \lambda \neq 0$. Let us prove that I_λ a left maximal ideal of R. We have

$$y_\lambda = e_{21} + \lambda e_{22} = e_{21}x_\lambda \in Rx_\lambda \text{ and } x_\lambda = e_{11} + \lambda e_{12} = e_{12}y_\lambda \in Ry_\lambda$$

and hence

$$Rx_\lambda = Ry_\lambda.$$

Now $I_2 \nsubseteq I_\lambda$, otherwise $e_{22} \in I_\lambda$ and hence also $e_{21} = y_\lambda - \lambda e_{22} \in I_\lambda$. Thus $R = Re_{21} + Re_{22} \subseteq I_\lambda$ so that $R = Rx_\lambda$ and hence $\det(x_\lambda) \neq 0$. Since $\det(x_\lambda) = 0$, this is a contradiction. Since I_2 is simple, we get that $I_\lambda \cap I_2 = \{0\}$. On the other hand x_λ and y_λ are linearly independent. In fact $\alpha x_\lambda = \beta y_\lambda$ writes as

$$\alpha e_{11} + \alpha\lambda e_{12} = \beta e_{21} + \beta\lambda e_{22}$$

from which it follows that $\alpha = 0 = \beta$. Hence $\dim_k(I_\lambda + I_2) = \dim_k\left(I_\lambda \dot{\oplus} I_2\right) = \dim_k(I_\lambda) + \dim_k(I_2) \geq 4$ which implies that $I_\lambda \dot{\oplus} I_2 = R$ and $\dim_k(I_\lambda) = 2$. In particular we get that (x_λ, y_λ) is a basis for I_λ. Moreover we have

$$\frac{R}{I_\lambda} \cong I_2$$

is a simple left R-module whence I_λ is a left maximal ideal of R. Furthermore, since

$$I_\lambda \cong \frac{R}{I_2} \cong I_1$$

we obtain that I_λ is also a simple left R-module.

4) Let M be a left maximal ideal of R and assume that $M \neq I_1$ and also $M \neq I_2$. Then $I_2 \nsubseteq M$ and hence, since I_2 is a simple left R-module, we deduce that $M \cap I_2 = \{0\}$. Clearly we also have $M + I_2 = R$. Thus we deduce that

$$R = M \dot{\oplus} I_2.$$

Therefore there exist $m \in M, \lambda_{11}, \lambda_{22} \in k$ such that

$$e_{11} = m + \lambda_{12}e_{12} + \lambda_{22}e_{22}. \tag{5.6}$$

By multiplying (5.6) on the left by e_{12} we get

$$0 = e_{12}m + \lambda_{22}e_{12}.$$

Assume that $\lambda_{22} \neq 0$. Then we obtain

$$e_{12} = -\lambda_{22}^{-1}e_{12}m \in M$$

so that $I_2 = Re_{12} \subseteq M$, a contradiction. Therefore $\lambda_{22} = 0$ and (5.6) rewrites as

$$e_{11} = m + \lambda_{12}e_{12}.$$

Clearly $\lambda_{12} \neq 0$ otherwise we would have $e_{11} = m \in M$ and hence $I_1 = Re_{11} \subseteq M$, a contradiction. Hence we obtain

$$m = e_{11} - \lambda_{12}e_{12} = x_\lambda \text{ where } \lambda = -\lambda_{12} \neq 0.$$

From 3) we know that $Rm = Rx_\lambda$ is a left maximal ideal of R. Since $Rx_\lambda = Rm \subseteq M$, we conclude that $M = Rx_\lambda$.

5) Assume that $I_\lambda = I_{\lambda'}$. Then there exists $t, s \in k$ such that

$$\begin{aligned} x_\lambda &= tx_{\lambda'} + sy_{\lambda'} \text{ i.e.} \\ e_{11} + \lambda e_{12} &= te_{11} + t\lambda' e_{12} + se_{21} + s\lambda e_{22} \end{aligned}$$

which implies $s = 0, t = 1$ and $\lambda = \lambda'$.

6) Let S be a left simple module and let $0 \neq x \in S$. Then $S = Rx \cong R/\mathrm{Ann}_R(x)$ and $\mathrm{Ann}_R(x)$ is a left maximal ideal of R. Hence, in view of 2) and 4) we have $\mathrm{Ann}_R(x) \in \{I_1, I_2, I_\lambda \mid \lambda \in k, \lambda \neq 0\}$. If $\mathrm{Ann}_R(x) = I_\lambda$ for some $\lambda \in k, \lambda \neq 0$, then, in view of 3) and (5.5)we have that $\frac{R}{I_\lambda} \cong I_2 \cong I_1$. If $\mathrm{Ann}_R(x) = I_1$, then, by (5.3) $R/I_1 \cong I_1$.If $\mathrm{Ann}_R(x) = I_2$, then, by (5.1) $R/I_2 \cong I_1$. □

Chapter 6

Tensor Product and bimodules

6.1 Tensor Product 1

Definition 6.1. *Let R be a ring. Let M_R be a right R-module and let ${}_RN$ be a left R-module. Given an abelian group G, a map $\beta : M \times N \to G$ is said to be* R-balanced *if*

1) $\beta((x_1 + x_2, y)) = \beta((x_1, y)) + \beta((x_2, y))$ *for every* $x_1, x_2 \in M$ *and* $y \in N$;

2) $\beta((x, y_1 + y_2)) = \beta((x, y_1)) + \beta((x, y_2))$ *for every* $x \in M$ *and* $y_1, y_2 \in N$;

3) $\beta((xr, y)) = \beta((x, ry))$ *for every* $x \in M, r \in R, y \in N$.

Definition 6.2. *Let R be a ring. Let M_R be a right R-module and let ${}_RN$ be a left R-module. A pair (T, τ) is called a tensor product of M_R and ${}_RN$ if*

T1) *T is an abelian group;*

T2) *$\tau : M \times N \to T$ is an R-balanced map;*

T3) *for every abelian group G and every R-balanced map $\beta : M \times N \to G$ there exists a unique abelian group homomorphism $f : T \to G$ such that $f \circ \tau = \beta$.*

Theorem 6.3. *Let R be a ring. Let M_R be a right R-module and let ${}_RN$ be a left R-module. Assume that both (T, τ) and (T', τ') are tensor products of M_R and ${}_RN$. Then there is a unique abelian group homomorphism $\alpha : T \to T'$ such that $\alpha \circ \tau = \tau'$. Moreover α is an isomorphism.*

Proof. Since (T, τ) is a tensor product of M_R and ${}_RN$ and $\tau' : M \times N \to T'$ is an R-balanced map, there is a unique abelian group homomorphism $\alpha : T \to T'$ such that $\alpha \circ \tau = \tau'$.

Since (T', τ') is a tensor product of M_R and ${}_RN$ and $\tau : M \times N \to T$ is an R-balanced map, there is a unique abelian group homomorphism $\alpha' : T' \to T$ such that $\alpha' \circ \tau' = \tau$. Therefore we obtain that

$$\alpha' \circ \alpha \circ \tau = \alpha' \circ \tau' = \tau \text{ and } \alpha \circ \alpha' \circ \tau' = \alpha \circ \tau = \tau'.$$

Since both $\mathrm{Id}_T : T \to T$ and $(\alpha' \circ \alpha) : T \to T$ are abelian group homomorphisms such that

$$\mathrm{Id}_T \circ \tau = \tau \text{ and } (\alpha' \circ \alpha) \circ \tau = \tau,$$

and since (T, τ) is a tensor product of M_R and ${}_RN$, in view of property T3) we deduce that $\mathrm{Id}_T = \alpha' \circ \alpha$.

Since both $\mathrm{Id}_{T'} : T' \to T'$ and $(\alpha \circ \alpha') : T' \to T'$ are abelian group homomorphisms such that

$$\mathrm{Id}_{T'} \circ \tau' = \tau' \text{ and } (\alpha \circ \alpha') \circ \tau' = \tau',$$

and since (T', τ') is a tensor product of M_R and ${}_RN$, in view of property T3) we deduce that $\mathrm{Id}_{T'} = \alpha \circ \alpha'$. □

6.4. *Let us consider the abelian group*

$$\mathbb{Z}^{(M\times N)} = \bigoplus_{(x,y)\in M\times N} \mathbb{Z}_{(x,y)} \text{ where } \mathbb{Z}_{(x,y)} = \mathbb{Z} \text{ for every } (x,y) \in M \times N$$

and, for every $(x,y) \in M \times N$*, let* $\varepsilon_{(x,y)} : \mathbb{Z}_{(x,y)} \to \mathbb{Z}^{(M\times N)}$ *be the canonical injection. For every* $x \in M$ *and* $y \in N$ *let us set*

$$\widehat{(x,y)} = \varepsilon_{(x,y)}(1_{\mathbb{Z}})$$

so that

$$\widehat{(x,y)} : M \times N \to \mathbb{Z} \text{ and } \widehat{(x,y)}((t,s)) = \begin{cases} 1_{\mathbb{Z}} \text{ whenever } (x,y) = (t,s) \\ 0_{\mathbb{Z}} \text{ whenever } (x,y) \neq (t,s) \end{cases}.$$

Recall that $\mathbb{Z}^{(M\times N)}$ *is an abelian group where the addition is defined by setting*

$$(f+g)((m,n)) = f((m.n)) + g((m.n)) \text{ for every } (m,n) \in M \times N.$$

Let L *be the subgroup of* $\mathbb{Z}^{(M\times N)}$ *generated by all elements of the form*

$$\widehat{(x_1 + x_2, y)} - \widehat{(x_1, y)} - \widehat{(x_2, y)} \text{ for all } x_1, x_2 \in M, y \in N;$$
$$\widehat{(x, y_1 + y_2)} - \widehat{(x, y_1)} - \widehat{(x, y_2)} \text{ for all } x \in M, y_1, y_2 \in N;$$
$$\widehat{(xr, y)} - \widehat{(x, ry)} \text{ for all } x \in M, r \in R, y \in N.$$

Then in $\frac{\mathbb{Z}^{(M\times N)}}{L}$ *we have the following equalities*

$$\left[\widehat{(x_1 + x_2, y)} + L\right] = \left[\widehat{(x_1, y)} + L\right] + \left[\widehat{(x_2, y)} + L\right] \text{ for all } x_1, x_2 \in M, y \in N; \tag{6.1}$$

$$\left[\widehat{(x, y_1 + y_2)} + L\right] = \left[\widehat{(x, y_1)} + L\right] + \left[\widehat{(x, y_2)} + L\right] \text{ for all } x \in M, y_1, y_2 \in N; \tag{6.2}$$

$$\left[\widehat{(xr, y)} + L\right] = \left[\widehat{(x, ry)} + L\right] \text{ for all } x \in M, r \in R, y \in N. \tag{6.3}$$

We set

$$x \otimes_R y = \widehat{(x,y)} + L \in \frac{\mathbb{Z}^{(M\times N)}}{L} \text{ for every } (x,y) \in M \times N.$$

With this notations, from (6.1), (6.2) *and* (6.3) *rewrite as*

$$\begin{aligned}
&(6.4) \qquad (x_1 + x_2) \otimes_R y &=& \; x_1 \otimes_R y + x_2 \otimes_R y \text{ for all } x_1, x_2 \in M, y \in N;\\
&(6.5) \qquad x \otimes_R (y_1 + y_2) &=& \; x \otimes_R y_1 + x \otimes_R y_2 \text{ for all } x \in M, y_1, y_2 \in N;\\
&(6.6) \qquad xr \otimes_R y &=& \; x \otimes_R ry \text{ for all } x \in M, r \in R, y \in N.
\end{aligned}$$

Set

$$T = \frac{\mathbb{Z}^{(M\times N)}}{L} \text{ and let } \tau : M \times N \to T \text{ be the map defined by setting}$$
$$\tau((x,y)) = x \otimes_R y \text{ for every } (x,y) \in M \times N.$$

Theorem 6.5. *Let R be a ring. Let M_R be a right R-module and let ${}_RN$ be a left R-module. Using the notations introduced in* (6.4)*,the pair (T,τ) is a tensor product of M_R and ${}_RN$.*

Proof. First of all let us prove that $\tau : M \times N \to T$ is an R-balanced map. We have

$$\tau((x_1 + x_2, y)) = (x_1 + x_2) \otimes_R y \overset{(6.4)}{=} x_1 \otimes_R y + x_2 \otimes_R y = \tau((x_1, y)) + \tau((x_2, y))$$

for all $x_1, x_2 \in M, y \in N$,

$$\tau((x, y_1 + y_2)) = x \otimes_R (y_1 + y_2) \overset{(6.5)}{=} x \otimes_R y_1 + x \otimes_R y_2 = \tau((x, y_1)) + \tau((x, y_2))$$

for all $x \in M, y_1, y_2 \in N$ and

$$\tau((xr, y)) = xr \otimes_R y = x \otimes_R ry = \tau((xr, ry))$$

for all $x \in M, r \in R, y \in N$.

Let $i : M \times N \to \mathbb{Z}^{(M\times N)}$ be the map defined by setting $i((x,y)) = \varepsilon_{(x,y)}(1_{\mathbb{Z}}) = \widehat{(x,y)}$. Recall that, by Proposition 2.3, $\left(\mathbb{Z}^{(M\times N)}, i\right)$ is a free $\mathbb{Z}$-module with basis $M \times N$.

Let now $\beta : M \times N \to G$ be an R-balanced map. Since $\left(\mathbb{Z}^{(M\times N)}, i\right)$ is a free $\mathbb{Z}$-module, there exists a unique abelian group homomorphism $h : \mathbb{Z}^{(M\times N)} \to G$ such that $h \circ i = \beta$. Let us compute

$$\begin{aligned}
h\left(\widehat{(x_1 + x_2, y)}\right) &= (h \circ i)((x_1 + x_2, y)) = \beta((x_1 + x_2, y)) \overset{\beta \text{isbalanc}}{=} \beta((x_1, y)) + \beta((x_2, y)) = \\
&= (h \circ i)((x_1, y)) + (h \circ i)((x_2, y)) = h\left(\widehat{(x_1, y)}\right) + h\left(\widehat{(x_2, y)}\right)
\end{aligned}$$

which means that

$$(6.7) \qquad \widehat{(x_1 + x_2, y)} - \widehat{(x_1, y)} - \widehat{(x_2, y)} \in \operatorname{Ker}(h);$$

$$\begin{aligned} h\left(\widehat{(x,y_1+y_2)}\right) &= (h\circ i)((x,y_1+y_2)) = \beta((x,y_1+y_2)) \overset{\beta \text{isbalanc}}{=} \beta((x,y_1))+\beta((x,y_2)) = \\ &= (h\circ i)((x,y_1)) + (h\circ i)((x,y_2)) = h\left(\widehat{(x,y_1)}\right) + h\left(\widehat{(x,y_2)}\right) \end{aligned}$$

which means that

$$\widehat{(x,y_1+y_2)} - \widehat{(x,y_1)} - \widehat{(x,y_2)} \in \operatorname{Ker}(h)\,; \tag{6.8}$$

$$\begin{aligned} h\left(\widehat{(xr,y)}\right) &= (h\circ i)((xr,y)) = \beta((xr,y)) \overset{\beta \text{isbalanc}}{=} \beta((x,ry)) \\ &= (h\circ i)((x,ry)) = h\left(\widehat{(x,ry)}\right) \end{aligned}$$

which means that

$$\widehat{(xr,y)} - \widehat{(x,ry)} \in \operatorname{Ker}(h)\,. \tag{6.9}$$

From $(6.7)\,,(6.8)$ and (6.9), we deduce that $L \subseteq \operatorname{Ker}(h)$. Hence, by the Fundamental Theorem of Quotient Groups, there exists a unique group homomorphism $\overline{h} : T = \frac{\mathbb{Z}^{(M\times N)}}{L} \to G$ such that $\overline{h}\circ p_L = h$. Note that $p_L \circ i = \tau$ so that

$$\overline{h}\circ\tau = \overline{h}\circ p_L\circ i = h\circ i = \beta.$$

Let us prove that $f = \overline{h}$ is unique. Let $f' : T \to G$ be a group homomorphism such that $f'\circ\tau = \beta$. Then we have

$$f'\circ p_L\circ i = f'\circ\tau = \beta = \overline{h}\circ p_L\circ i$$

Since there is a unique group homomorphism $h : \mathbb{Z}^{(M\times N)} \to G$ such that $h\circ i = \beta$ we infer that

$$f'\circ p_L = h = \overline{h}\circ p_L.$$

Since p_L is surjective, this implies that $f' = \overline{h}$. □

Notation 6.6. *In view of Theorem 6.5, we know that for*

$$T = \frac{\mathbb{Z}^{(M\times N)}}{L} \text{ and } \tau : M\times N \to T \text{ the map defined by setting}$$
$$\tau((x,y)) = x\otimes_R y \text{ for every } (x,y)\in M\times N.$$

(T,τ) is a tensor product of M_R and ${}_RN$. Moreover, by Theorem 6.3, such a pair is essentially unique. We will denote it by $(M\otimes_R N,\tau)$, or even by $M\otimes_R N$, if there is no risk of confusion. Given $(x,y)\in M\times N$, sometimes we will simply write $x\otimes y$ instead of $x\otimes_R y$.

Exercise 6.7. *Let M_R be a right R-module and let ${}_RN$ be a left R-module. Show that, for any $m\in\mathbb{Z}, x\in M, y\in N$ we have*

$$m(x\otimes y) = (mx)\otimes y = x\otimes(my)$$

Proposition 6.8. *Let M_R be a right R-module and let ${}_RN$ be a left R-module. Then any element of $M \otimes_R N$ can be written as*

$$\sum_{i=1}^{n} x_i \otimes y_i \text{ where } n \in \mathbb{N}, n \geq 1 \text{ and }, x_1, \ldots, x_n \in M, y_1, \ldots, y_n \in N.$$

In particular the elements of type $x \otimes_R y, x \in M, y \in N$, for a system of generators of the abelian group $M \otimes_R N$.

Proof. Let $w \in \mathbb{Z}^{(M \times N)} = \bigoplus_{(x,y) \in M \times N} \mathbb{Z}_{(x,y)}$. By Lemma 1.40 we have

$$w = \sum_{(x,y) \in M \times N} \varepsilon_{(x,y)} \left(\pi_{(x,y)} (w) \right).$$

For every $(x, y) \in M \times N$, set $m_{(x,y)} = \pi_{(x,y)} (w) \in \mathbb{Z}$. Then we have

$$w = \sum_{(x,y) \in Supp(w)} \varepsilon_{(x,y)} \left(m_{(x,y)} \right) = \sum_{(x,y) \in Supp(w)} m_{(x,y)} \varepsilon_{(x,y)} (1_{\mathbb{Z}}) = \sum_{(x,y) \in Supp(w)} m_{(x,y)} \widehat{(x, y)} .$$

Therefore there exist $n \in \mathbb{N}, n \geq 1$ and $x_1, \ldots, x_n \in M, y_1, \ldots, y_n \in N, m_1, \ldots, m_n \in \mathbb{Z}$ such that

$$w = \sum_{i=1}^{n} m_i \widehat{(x_i, y_i)} .$$

Hence in $\mathbb{Z}^{(M \times N)}/L$ we have

$$w + L = \sum_{i=1}^{n} m_i \widehat{(x_i, y_i)} + L = \sum_{i=1}^{n} m_i \left[\widehat{(x_i, y_i)} + L \right] = \sum_{i=1}^{n} m_i \left(x_i \otimes y_i \right) \overset{\text{Ex.6.7}}{=}$$
$$= \sum_{i=1}^{n} (m_i x_i) \otimes y_i = \sum_{i=1}^{n} t_i \otimes y_i \text{ where } t_i = m_i x_i \in M.$$

□

Remarks 6.9. *Let M_R be a right R-module and let ${}_RN$ be a left R-module.*

1) *Let G be an abelian group, To give an abelian group homomorphism $f : M \otimes_R N \to G$, it is enough to give an R-balanced map $\beta : M \times N \to G$.*

2) *In view of Proposition 6.8, if f and $g : M \otimes_R N \to G$ are group homomorphisms, we have that $f = g$ if and only if $f (x \otimes_R y) = g (x \otimes_R y)$ for all $x \in M$ and $y \in N$.*

Lemma 6.10. *Let M_R be a right R-module and let ${}_RN$ be a left R-module. Then, for every $x \in M$ and $y \in N$ we have*

$$x \otimes_R 0 = 0 \text{ and } 0 \otimes_R y = 0.$$

Proof. Let $x \in M$. We have:

$$x \otimes_R 0 = x \otimes_R (0+0) \overset{(6.5)}{=} x \otimes_R 0 + x \otimes_R 0$$

so that we get

$$x \otimes_R 0 = x \otimes_R 0 + x \otimes_R 0.$$

Since $M \otimes_R N$ is a group, we deduce that $x \otimes_R 0 = 0$. The other equality is proved in an analogous way. □

Lemma 6.11. *Let $f : L \to L'$ be a right R-module homomorphism and let $g : M \to M'$ a left R-module homomorphism. The map*

$\beta : L \times M \to L' \otimes_R M'$ defined by setting $\beta((x,y)) = f(x) \otimes_R g(y)$ for every $x \in L$ and $y \in M$.

is R-balanced.

Proof. Let $x, x_1, x_2 \in L, y, y_1, y_2 \in M$ and $r \in R$. We compute

$$\beta((x_1 + x_2, y)) = f(x_1 + x_2) \otimes_R g(y) = [f(x_1) + f(x_2)] \otimes_R g(y) \overset{(6.4)}{=}$$
$$= f(x_1) \otimes_R g(y) + f(x_2) \otimes_R g(y) = \beta((x_1, y)) + \beta((x_2, y)).$$
$$\beta((x, y_1 + y_2)) = f(x) \otimes_R g(y_1 + y_2) = f(x) \otimes_R [g(y_1) + g(y_2)]) \overset{(6.5)}{=}$$
$$= f(x) \otimes_R g(y_1) + f(x) \otimes_R g(y_2) = \beta((x, y_1)) + \beta((x, y_2))$$
$$\beta((xr, y))) = f(xr) \otimes g(y) = f(x) r \otimes g(y) \overset{(6.6)}{=} f(x) \otimes rg(y) = f(x) \otimes g(ry) =$$
$$= \beta((x, ry)).$$

□

Notation 6.12. *Let $f : L \to L'$ be a right R-module homomorphism and let $g : M \to M'$ a left R-module homomorphism. By Lemma 6.11, the map $\beta : L \times M \to L' \otimes_R M'$ defined by setting $\beta((x,y)) = f(x) \otimes_R g(y)$ is R-balanced. Therefore there is a unique group homomorphism, which will be denoted by $f \otimes_R g$, or simply by $f \otimes g$, such that*

$f \otimes_R g : L \otimes_R M \to L' \otimes_R M'$ and $(f \otimes_R g)(x \otimes y) = f(x) \otimes_R g(y)$ for every $x \in L$ and $y \in M$.

If $f = \mathrm{Id}_L$, the notation $L \otimes_R g$ will be also used. Similarly if $f = \mathrm{Id}_M$.

Lemma 6.13. *Let $f : L \to L'$ and $f' : L' \to L''$ be right R-module homomorphisms and let $g : M \to M'$ and $g' : M' \to M''$ be left R-module homomorphisms. Then*

$$(f' \circ f) \otimes_R (g' \circ g) = (f' \otimes_R g') \circ (f \otimes_R g).$$

Proof. Let $x \in L$ and $y \in M$. We compute

$$[(f' \circ f) \otimes_R (g' \circ g)](x \otimes y) = [(f' \circ f)(x)] \otimes [(g' \circ g)(y)] = f'(f(x)) \otimes g'(g(y)) =$$
$$= (f' \otimes_R g')(f(x) \otimes g(y)) = [(f' \otimes_R g') \circ (f \otimes_R g)](x \otimes y).$$

In view of 2) in Remarks 6.9, we conclude. □

Proposition 6.14. *Let*

$$_RM' \xrightarrow{f} {_RM} \xrightarrow{g} {_RM''} \to 0$$

be an exact sequence of left R-modules and left R-modules homomorphism. Then, for every right R-module L_R, the sequence of abelian groups

$$L \otimes_R M' \overset{L\otimes_R f}{\longrightarrow} L \otimes_R M \overset{L\otimes_R g}{\longrightarrow} L \otimes_R M'' \to 0$$

is exact.

Proof. Let $x \in L$ and $y'' \in M''$. Since g is surjective there exists an $y \in M$ such that $g(y) = y''$. Then

$$(L \otimes_R g)(x \otimes y) = x \otimes g(y) = x \otimes y''.$$

In view of Proposition 6.8, we conclude that $L \otimes_R g$ is surjective .By Lemma 6.13, we have that

$$(L \otimes_R g) \circ (L \otimes_R f) = L \otimes_R (f \circ g) = L \otimes_R 0 = 0.$$

Therefore $\operatorname{Im}(L \otimes_R f) \subseteq \operatorname{Ker}(L \otimes_R g)$. Let $p : L \otimes_R M \to \frac{L\otimes_R M}{\operatorname{Im}(L\otimes_R f)}$ be the canonical projection. Then, By the Fundamental Theorem for Quotient Modules 1.20, there exists a unique $\mathbb{Z}$-module homomorphism

$$\overline{g} : \frac{L \otimes_R M}{\operatorname{Im}(L \otimes_R f)} \to L \otimes_R M''$$

such that $\overline{g} \circ p = L \otimes_R g$. Moreover $\overline{g}$ is injective if and only if $\operatorname{Im}(L \otimes_R f) = \operatorname{Ker}(L \otimes_R g)$. To this aim, we will construct a group homomorphism $q : L \otimes_R M'' \to \frac{L\otimes_R M}{\operatorname{Im}(L\otimes_R f)}$ which will be a left inverse of $\overline{g}$. Let us consider the map

$$\beta : L \times M'' \to \frac{L \otimes_R M}{\operatorname{Im}(L \otimes_R f)} \text{ defined by setting , for every } (x, y'') \in L \times M''$$
$$\beta((x, y'')) = (x \otimes y) + \operatorname{Im}(L \otimes_R f) \text{ where } y \in M \text{ and } g(y) = y''.$$

β **is well defined**. In fact, assume that y_1,$y_2 \in M$ and $g(y_1) = g(y_2) = y''$. Then $y_1 - y_2 \in \operatorname{Ker}(g) = \operatorname{Im}(f)$ so that there is an $m \in M$ such that $f(m) = y_1 - y_2$. Thus we get

$$x \otimes y_1 - x \otimes y_2 = x \otimes (y_1 - y_2) = x \otimes f(m) =$$
$$= (L \otimes_R f)(x \otimes m) \in \operatorname{Im}(L \otimes_R f)$$

so that

$$x \otimes y_1 + \operatorname{Im}(L \otimes_R f) = x \otimes y_2 + \operatorname{Im}(L \otimes_R f).$$

β **is balanced**. Let $x, x_1, x_2 \in L, y'', y_1'', y_2'' \in M''$ and $r \in R$. Let $y, y_1, y_2 \in M$ such that $g(y) = y'', g(y_1) = y_1'', g(y_2) = y_2''$. Then $g(yr) = g(y)r = y''r$ and $g(y_1 + y_2) = g(y_1) + g(y_2)$ so that we have

$$\beta((x_1 + x_2, y'')) = (x_1 + x_2) \otimes y + \mathrm{Im}(L \otimes_R f) \overset{(6.4)}{=} [x_1 \otimes y + x_2 \otimes y] + \mathrm{Im}(L \otimes_R f)$$
$$= [x_1 \otimes y + \mathrm{Im}(L \otimes_R f)] + [x_2 \otimes y + \mathrm{Im}(L \otimes_R f)] = \beta((x_1, y)) + \beta((x_2, y)).$$
$$\beta((x, y_1'' + y_2'')) = [x \otimes (y_1 + y_2)] + \mathrm{Im}(L \otimes_R f) \overset{(6.5)}{=} [x \otimes y_1 + x \otimes y_2] + \mathrm{Im}(L \otimes_R f)$$
$$= [x \otimes y_1 + \mathrm{Im}(L \otimes_R f)] + [x \otimes y_2 + \mathrm{Im}(L \otimes_R f)] = \beta((x, y_1'')) + \beta((x, y_2''))$$
$$\beta((xr, y''))) = xr \otimes y + \mathrm{Im}(L \otimes_R f) \overset{(6.6)}{=} x \otimes ry + \mathrm{Im}(L \otimes_R f) = \beta((x, ry'')).$$

Therefore there is a group homomorphism

$$q \quad : \quad L \otimes M'' \to \frac{L \otimes_R M}{\mathrm{Im}(L \otimes_R f)} \text{ such that, for every } (x, y'') \in L \times M''$$
$$q(x \otimes y'') \quad = \quad (x \otimes y) + \mathrm{Im}(L \otimes_R f) \text{ where } y \in M \text{ and } g(y) = y''.$$

For every $x \in L$ and $y \in M$, we compute

$$(q \circ \overline{g})[(x \otimes y) + \mathrm{Im}(L \otimes_R f)] = (q \circ \overline{g} \circ p)(x \otimes y) = q(x \otimes g(y)) = x \otimes y.$$

□

Proposition 6.15. *Let L be a right R-module and let $(M_i)_{i\in I}$ be a family of left R-modules. Let $\tau : L \times \left(\bigoplus_{i\in I} M_i\right) \to \bigoplus_{i\in I} (L \otimes_R M_i)$ be the map defined by setting*

$$\tau\left(\left(x, (y_i)_{i\in I}\right)\right) = (x \otimes y_i)_{i\in I} \textit{ for every } x \in L \textit{ and } (y_i)_{i\in I} \in \bigoplus_{i\in I} M_i.$$

Then τ is R-balanced and $\left(\bigoplus_{i\in I} (L \otimes_R M_i), \tau\right) = L \otimes_R \left(\bigoplus_{i\in I} M_i\right)$.

Proof. Let $x, x_1, x_2 \in L, (y_i)_{i\in I}, (z_i)_{i\in I} \in \bigoplus_{i\in I} M_i$ and $r \in R$. We compute

$$\tau\left((x_1 + x_2), (y_i)_{i\in I}\right) = ((x_1 + x_2) \otimes y_i)_{i\in I} \overset{(6.4)}{=} (x_1 \otimes y_i + x_2 \otimes y_i)_{i\in I} =$$
$$= (x_1 \otimes y_i)_{i\in I} + (x_2 \otimes y_i)_{i\in I} = \tau\left(\left(x_1, (y_i)_{i\in I}\right)\right) + \tau\left(\left(x_2, (y_i)_{i\in I}\right)\right)$$
$$\tau\left(\left(x, (y_i)_{i\in I} + (z_i)_{i\in I}\right)\right) = \tau\left(\left(x, (y_i + z_i)_{i\in I}\right)\right) = (x \otimes (y_i + z_i))_{i\in I} \overset{(6.5)}{=}$$
$$= (x \otimes y_i + x \otimes z_i)_{i\in I} = (x \otimes y_i)_{i\in I} + (x \otimes z_i)_{i\in I} = \tau\left(\left(x, (y_i)_{i\in I}\right)\right) + \tau\left(\left(x, (z_i)_{i\in I}\right)\right)$$
$$\tau\left(\left(xr, (y_i)_{i\in I}\right)\right) = ((xr) \otimes y_i)_{i\in I} \overset{(6.6)}{=} (x \otimes ry_i)_{i\in I} = \tau\left(\left(x, (ry_i)_{i\in I}\right)\right) = \tau\left(\left(x, r(y_i)_{i\in I}\right)\right).$$

Hence τ is R-balanced. Let now $\beta : L \times \left(\bigoplus_{i\in I} M_i\right) \to G$ be an R-balanced map. We have to show that there exists a group homomorphism $f : \bigoplus_{i\in I} (L \otimes_R M_i) \to G$ such that $f \circ \tau = \beta$ and moreover this f is unique w.r.t. this property. Let $\varepsilon_j : M_j \to \bigoplus_{i \in I} M_i$ denote the jth canonical injection. First of all let us show that the map

$$\beta \circ (L \times \varepsilon_i) : L \times M_i \to G$$

is R-balanced. Let $x, x_1, x_2 \in L, y, y_1, y_2 \in M_i$ and $r \in R$. We compute

$$\begin{gathered}
[\beta \circ (L \times \varepsilon_i)]\left((x_1 + x_2, y)\right) = \beta\left((x_1 + x_2, \varepsilon_i(y))\right) = \beta\left((x_1, \varepsilon_i(y))\right) + \beta\left((x_2, \varepsilon_i(y))\right) = \\
= [\beta \circ (L \times \varepsilon_i)]\left[(x_1, y) + (x_2, y)\right]. \\
[\beta \circ (L \times \varepsilon_i)]\left((x, y_1 + y_2)\right) = \beta\left((x, \varepsilon_i(y_1 + y_2))\right) = \beta\left((x, \varepsilon_i(y_1) + \varepsilon_i(y_2))\right) \\
== \beta\left((x, \varepsilon_i(y_1))\right) + \beta\left((x, \varepsilon_i(y_2))\right). \\
= [\beta \circ (L \times \varepsilon_i)]\left((x, y_1)\right) + [\beta \circ (L \times \varepsilon_i)]\left((x, y_2)\right) \\
[\beta \circ (L \times \varepsilon_i)]\left(xr, y)\right) = \beta\left((xr, \varepsilon_i(y))\right) = \beta\left((x, r\varepsilon_i(y))\right) = \beta\left((x, r\varepsilon_i(ry))\right) = \\
= [\beta \circ (L \times \varepsilon_i)]\left((x, ry)\right).
\end{gathered}$$

Hence there exists a unique group homomorphism $f_i : L \otimes_R M_i \to G$ such that

$$f_i(x \otimes y) = \beta\left((x, \varepsilon_i(y))\right)$$

for every $x \in L$ and $y \in M_i$. By the universal property of the direct sum, we can consider $f = \nabla(f)_{i \in I} : \bigoplus_{i \in I}(L \otimes_R M_i) \to G$. We have

$$\begin{aligned}
(f \circ \tau)\left(\left(x, (y_i)_{i \in I}\right)\right) &= f\left((x \otimes y_i)_{i \in I}\right) = \sum_{i \in I} \beta\left((x, \varepsilon_i(y_i))\right) = \beta\left(\left(x, \sum_{i \in I} \varepsilon_i(y_i)\right)\right) = \\
&= \beta\left(\left(x, (y_i)_{i \in I}\right)\right)
\end{aligned}$$

Let now $f' : \bigoplus_{i \in I}(L \otimes_R M_i) \to G$ be another group homomorphism such that $f' \circ \tau = \beta$. For every $j \in I$, let $\varepsilon'_j : (L \otimes_R M_j) \to \bigoplus_{i \in I}(L \otimes_R M_i)$ denote the j-th canonical injection. Note that for every $j \in I$, $x \in L$ and $y_j \in M_j$

$$\begin{aligned}
\left((x \otimes (y_j \delta_{ij}))_{i \in I}\right)_j &= x \otimes y_j \text{ and} \\
\left((x \otimes (y_j \delta_{ij}))_{i \in I}\right)_i &= x \otimes 0 = 0 \text{ for } i \neq j.
\end{aligned}$$

Thus we deduce that

$$\varepsilon'_j(x \otimes y_j) = (x \otimes (y_j \delta_{ij}))_{i \in I}$$

and hence we get

$$[\tau \circ (L \times \varepsilon_j)]\left((x, y_j)\right) = \tau\left((x, \varepsilon_j(y_j))\right) = \tau\left(\left(x, (y_j \delta_{ij})_{i \in I}\right)\right) = (x \otimes (y_j \delta_{ij}))_{i \in I} = \varepsilon'_j(x \otimes y_j)$$

For every $x \in L$ and $y_j \in M_j$, we have

$$\begin{gathered}
\left(f' \circ \varepsilon'_j\right)(x \otimes y_j) = f'\left(\varepsilon'_j(x \otimes y_j)\right) = f'[\tau \circ (L \times \varepsilon_j)]\left((x, y_j)\right) = \\
= [f' \circ \tau \circ (L \times \varepsilon_j)]\left((x, y_j)\right) = [\beta \circ (L \times \varepsilon_j)]\left((x, y_j)\right) = [f \circ \tau \circ (L \times \varepsilon_j)]\left((x, y_j)\right) = \\
= f[\tau \circ (L \times \varepsilon_j)]\left((x, y_j)\right) == f\varepsilon'_j(x \otimes y_j) = \left(f \circ \varepsilon'_j\right)(x \otimes y_j).
\end{gathered}$$

We deduce that $f' \circ \varepsilon'_j = f \circ \varepsilon'_j$ for every $j \in I$. In view of the universal property of the direct sum, we conclude. □

Proposition 6.16. *Let L be a right R-module and let $(M_i)_{i\in I}$ be a family of left R-modules. Let $\tau : L \times \left(\bigoplus_{i\in I} M_i\right) \to \bigoplus_{i\in I} (L \otimes_R M_i)$ be the map defined by setting*

$$\tau\left(\left(x, (y_i)_{i\in I}\right)\right) = (x \otimes y_i)_{i\in I} \text{ for every } x \in L \text{ and } (y_i)_{i\in I} \in \bigoplus_{i\in I} M_i.$$

Then τ is R-balanced so that there is a group homomorphism $\varphi : L \otimes_R \left(\bigoplus_{i\in I} M_i\right) \to \bigoplus_{i\in I} (L \otimes_R M_i)$ such that

$$\varphi\left(x \otimes_R (y_i)_{i\in I}\right) = (x \otimes y_i)_{i\in I} \text{ for every } x \in L \text{ and } (y_i)_{i\in I} \in \bigoplus_{i\in I} M_i.$$

φ is an isomorphism.

Proof. By prposition 6.15, we know that τ is R-balanced. Let $\varepsilon_j : M_j \to \bigoplus_{i\in I} M_i$ denote the jth canonical injection and let $\psi_j = L \otimes_R \varepsilon_j : L \otimes_R M_j \to L \otimes_R \left(\bigoplus_{i\in I} M_i\right)$. Set $\psi = \nabla (L \otimes_R \varepsilon_i)_{i\in I} : \bigoplus_{i\in I} (L \otimes_R M_i) \to L \otimes_R \left(\bigoplus_{i\in I} M_i\right)$. Let us prove that ψ is a two-sided inverse of φ. We have

$$(\psi \circ \varphi)\left(x \otimes_R (y_i)_{i\in I}\right) = \psi\left((x \otimes y_i)_{i\in I}\right) = \sum_{i\in I} \psi_i (x \otimes y_i) = \sum_{i\in I} (x \otimes \varepsilon_i (y_i)) \overset{(6.5)}{=}$$
$$= x \otimes \sum_{i\in I} \varepsilon_i (y_i) = x \otimes_R (y_i)_{i\in I}.$$

By 2) in Remarks 6.9, we conclude that $\psi \circ \varphi = \mathrm{Id}_{L\otimes_R\left(\bigoplus_{i\in I} M_i\right)}$. Let now $j \in I$ and let $\varepsilon'_j : (L \otimes_R M_j) \to \bigoplus_{i\in I} (L \otimes_R M_i)$ denote the jth canonical injection. Note that for every $j \in I$, $x \in L$ and $y_j \in M_j$

$$\begin{aligned} \left((x \otimes (y_j \delta_{ij}))_{i\in I}\right)_j &= x \otimes y_j \text{ and} \\ \left((x \otimes (y_j \delta_{ij}))_{i\in I}\right)_i &= x \otimes 0 = 0 \text{ for } i \neq j. \end{aligned}$$

Thus we deduce that

$$\varepsilon'_j (x \otimes y_j) = (x \otimes (y_j \delta_{ij}))_{i\in I}$$

Let us compute

$$\left(\varphi \circ \psi \circ \varepsilon'_j\right)(x \otimes y_j) = (\varphi \circ \psi_j)(x \otimes y_j) = \varphi (x \otimes \varepsilon_j (y_j)) = \varphi\left(x \otimes (y_j \delta_{ij})_{i\in I}\right) =$$
$$= (x \otimes (y_j \delta_{ij}))_{i\in I} = \varepsilon'_j (x \otimes y_j).$$

By 2) in Remarks 6.9 we deduce that

$$\varphi \circ \psi \circ \varepsilon'_j = \varepsilon'_j$$

for every $j \in I$. By the universal property of the direct sum, this implies that $\varphi \circ \psi = \mathrm{Id}_{\bigoplus_{i \in I}(L \otimes_R M_i)}$. □

6.2 Bimodules

Definition 6.17. *Let A and R be rings. An A-R-bimodule (left A-module - right R-module) is a tern $\left(M, {}^A\mu_M, \mu_M^R\right)$ where $\left(M, {}^A\mu_M\right)$ is a left A-module, $\left(M, \mu_M^R\right)$ is a right R-module and*

$$a \cdot (x \cdot r) = (a \cdot x) \cdot r \qquad \text{for every } a \in A, x \in M, r \in R.$$

We will use the notation ${}_AM_R$ to denote the A-R bimodule $\left(M, {}^A\mu_M, \mu_M^R\right)$.

6.18. *We have seen in 1.7 that any abelian group M is a left $\mathrm{End}(M)$ module where*

$$f \cdot x = f(x) \qquad \text{for every } f \in \mathrm{End}(M) \text{ and } x \in M.$$

Also, M is a right $End(M)^{op}$-right module When we regard M as a right $\mathrm{End}(M)^{op}$-module, using the convention introduced in 1.2, we write

$$x \cdot f \qquad \text{for every } f \in \mathrm{End}(M)^{op} \text{ and } x \in M.$$

Now we have

$$(x \cdot f) \cdot g = x \cdot \left(f \cdot_{\mathrm{End(M)}^{op}} g\right) = x \cdot \left(g \cdot_{\mathrm{End(M)}} f\right) = x\,(g \circ f) = g\,(f\,(x))\,. \tag{6.10}$$

For this reason, when considering** $f \in \mathrm{End}(M)^{op}$**, we prefer to write

$$(x)\,f$$

***instead of** $f(x)$. In this way (6.10) rewrites as*

$$(x \cdot f) \cdot g = ((x)\,f)\,g = (x)\left(f \cdot_{\mathrm{End(M)}^{op}} g\right).$$

Let now M be a left module over a ring A and let $\overline{M}$ denote the abelian group underlying M. We denote by $\mathrm{End}\left({}_AM\right)$ or ${}_A\mathrm{End}\left(M\right)$ the subring of $\mathrm{End}\left(\overline{M}\right)^{op}$ defined by

$$\mathrm{End}\left({}_AM\right) = \left\{f \in \mathrm{End}\left(\overline{M}\right)^{op} \mid f \text{ is a left } A\text{-module homomorphism}\right\}.$$

Then M is an A-$\mathrm{End}_A\left(M\right)$-bimodule. In fact, for every $a \in A, x \in M$ and $f \in \mathrm{End}_A\left(M\right)$ we have

$$(a \cdot x) \cdot f = (a \cdot x)\,f \overset{f isomorph}{=} a \cdot [(x)\,f] = a \cdot (x \cdot f)\,.$$

Similarly, let M be a right module over a ring R and let $\overline{M}$ denote the abelian group underlying M. We denote by $\mathrm{End}(M_R)$ *or* $\mathrm{End}_R(M)$ *the subring of* $\mathrm{End}\left(\overline{M}\right)$ *defined by*

$$\mathrm{End}(M_R) = \left\{f \in \mathrm{End}\left(\overline{M}\right) \mid f \text{ is a right } R\text{-module homomorphism}\right\}.$$

Then M is an $\mathrm{End}(M_R)$*-R-bimodule. In fact, for every $f \in \mathrm{End}(M_R)$, $x \in M$ and $r \in R$ we have*

$$(f \cdot x) \cdot r = (f(x)) \cdot r \overset{f\,isomorph}{=} f(xr) = f(x \cdot r) = f \cdot (x \cdot r).$$

Notation 6.19. ***To be consistent with 6.18, from now on, if*** $f : M \to L$ ***is a left*** A***-module homomorphism, we will write***

$$(x)f$$

instead of $f(x)$***, for every*** $x \in M$***.***

6.20. *Let A be a commutative ring and let M be a left A-module. Then M has a right A-module structure defined by setting*

$$x \cdot a = a \cdot x \text{ for every } a \in A \text{ and } x \in M.$$

M endowed with its left A-module structure and with this right A-module structure becomes an A-A-bimodule. In fact we have

$$a \cdot (x \cdot b) = a \cdot (b \cdot x) = (a \cdot b) \cdot x = (b \cdot a) \cdot x = b \cdot (a \cdot x) = (a \cdot x) \cdot b \text{ for every } a, b \in A \text{ and } x \in M.$$

This particular A-bimodule structure will be called symmetrical A-bimodule structure. *In the particular case when A is a commutative ring, symmetric A-modules are often called just A-modules. If $A = k$ is a field, a symmetric k-k-bimodule is simply called a vector space.*

Exercise 6.21. *Let k be a field. Are all k-k-bimodule structure over k symmetrical?*

Exercise 6.22. *Let k be a field and let V be vector space over k of dimension 2. Let us consider V as a right k-module. Then V has a natural structure of* $\mathrm{End}(V_k)$*-k-bimodule. Fix a basis (e_1, e_2) of V_k. For each $\Lambda \in \mathrm{End}(V_k)$ write*

$$\begin{aligned} \Lambda(e_1) &= e_1\Lambda_{11} + e_2\Lambda_{21} \\ \Lambda(e_2) &= e_1\Lambda_{12} + e_2\Lambda_{22} \end{aligned}$$

and set

$$F(\Lambda) = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix}.$$

Show that the assignment $\Lambda \mapsto F(\Lambda)$ yields a ring isomorphism $F : \mathrm{End}(V_k) \to \mathrm{M}_2(k)$. Show also that ${}_{\mathrm{End}(V_k)}V$ is simple.

Remark 6.23. *By Remark 1.3 every abelian group is a left $\mathbb{Z}$-module and hence a symmetrical $\mathbb{Z}$-bimodule. Let now M_R be a right R-module. Since M is an abelian group, M can be considered as a left $\mathbb{Z}$-module. Let us check that indeed M is a $\mathbb{Z}$-R-bimodule. In fact, given $n \in \mathbb{Z}, x \in M, r \in R$, we have*

$$n \cdot (x \cdot r) = n\,(x \cdot r) \overset{By 6 in Proposition 1.6}{=} nx \cdot r = (n \cdot x) \cdot r.$$

Definition 6.24. *Let L and M be A-R-bimodules. An A-R-*bimodules homomorphism *from L to M is a map $f : L \to M$ which is both a left A-modules homomorphism and a right R-module homomorphism. In this case we write $f : {}_AL_R \to {}_AM_R$.*

Exercise 6.25. *$\left({}_A\left(M_i\right)_R\right)_{i\in I}$ be a family of A-R-bimodules. Show that $\prod\limits_{i\in I} M_i$ and $\bigoplus\limits_{i\in I} M_i$, endowed with their left A-module structure and their right R-module structure are A-R-bimodules.*

6.26. *Let ${}_AL_R$ be an A-R-bimodule and let ${}_BM_R$ be a B-R-bimodule. For every $a \in A, b \in B$ and $f \in \mathrm{Hom}_R\left(L_R, M_R\right)$ we can consider the maps*

$$\begin{aligned} f_a &: L \to M \text{ defined by setting } f_a(x) = f(a \cdot x) \text{ for every } x \in L, \\ {}_bf &: L \to M \text{ defined by setting } {}_bf(x) = b \cdot f(x) \text{ for every } x \in L. \end{aligned}$$

Proposition 6.27. *By means of the notations introduced in 6.26, for every $a \in A, b \in B$ and $f \in \mathrm{Hom}_R\left(L_R, M_R\right)$, the maps f_a and ${}_bf$ are right R-module homomorphism.*

Proof. Let $x, x_1, x_2 \in L$ and $r \in R$: We compute

$$\begin{gathered} f_a(x_1 + x_2) = f(a \cdot x_1 + a \cdot x_2) = f(a \cdot x_1) + f(a \cdot x_2) = f_a(x_1) + f_a(x_2) \\ f_a(x \cdot r) = f(a \cdot (x \cdot r)) = f((a \cdot x) \cdot r) = f(a \cdot x) \cdot r = f_a(x) \cdot r \\ {}_bf(x_1 + x_2) = f(x_1 + x_2) = b \cdot [f(x_1) + f(x_2)] = b \cdot f(x_1) + b \cdot f(x_2) = {}_bf(x_1) + {}_bf(x_2) \\ {}_bf(x \cdot r) = b \cdot f(x \cdot r) = b \cdot [f(x) \cdot r] = [b \cdot f(x)]\, r = {}_bf(x) \cdot r. \end{gathered}$$

□

Proposition 6.28. *Let ${}_AL_R$ be an A-R-bimodule and let ${}_BM_R$ be a B-R-bimodule. The abelian group $\mathrm{Hom}_R\left(L_R, M_R\right)$ has a natural structure of B-A-bimodule defined by setting, in the notations of Proposition 6.27,*

$$f \cdot a = f_a \text{ and } b \cdot f = {}_bf \text{ for every } a \in A, b \in B \text{ and } f \in \mathrm{Hom}_R\left(L_R, M_R\right).$$

Proof. Let $f, g \in \mathrm{Hom}_R\left(L_R, M_R\right), a, a' \in A, b, b' \in B$. For every $x \in L$, we compute

$$\begin{gathered} [(f+g) \cdot a](x) = (f+g)(a \cdot x) = f(a \cdot x) + g(a \cdot x) = (f \cdot a)(x) + (g \cdot a)(x) = \\ = [(f \cdot a) + (g \cdot a)](x) \\ [f \cdot (a + a')](x) = f((a + a') \cdot x) = f(a \cdot x + a' \cdot x) = f(a \cdot x) + f(a' \cdot x) = \\ = (f \cdot a)(x) + (f \cdot a')(x) == [(f \cdot a) + (f \cdot a')](x) \\ [(f \cdot a) \cdot a'](x) = (f \cdot a)(a' \cdot x) = f(a \cdot (a' \cdot x)) = f((a \cdot a')\, x) = [f \cdot (a \cdot a')](x) \\ (f \cdot 1_A)(x) = f(1_a \cdot x) = f(x). \end{gathered}$$

From this equalities we deduce that

$$(f+g)\cdot a = (f\cdot a) + (g\cdot a)$$
$$f\cdot (a+a') = (f\cdot a) + (f\cdot a')$$
$$f\cdot 1_A = f$$

and hence $\mathrm{Hom}_R(L_R, M_R)$ becomes a right A-module. Similarly, we calculate

$$\begin{gathered}[b\cdot (f+g)](x) = b\cdot [(f+g)(x)] = b\cdot f(x) + g(x) = b\cdot f(x) + b\cdot g(x) = \\ = (b\cdot f)(x) + (b\cdot g)(x) = [(b\cdot f) + (b\cdot g)](x) \\ [(b+b')\cdot f](x) = (b+b')\cdot f(x) = b\cdot f(x) + b'\cdot f(x) = (b\cdot f)(x) + (b'\cdot f)(x) = \\ = [(b\cdot f) + (b'\cdot f)](x) \\ [b\cdot (b'\cdot f)](x) = b\cdot [(b'\cdot f)(x)] = b\cdot [b'\cdot f(x)] = (b\cdot b') f(x) = [(b\cdot b')\cdot f](x).\end{gathered}$$

From this equalities we deduce that

$$\begin{aligned} b\cdot (f+g) &= b\cdot f + b\cdot g \\ (b+b')\cdot f &= (b\cdot f) + (b'\cdot f) \\ b\cdot (b'\cdot f) &= (b\cdot b')\cdot f \end{aligned}$$

and hence $\mathrm{Hom}_R(L_R, M_R)$ becomes a left B-module. Finally we have

$$[b\cdot (f\cdot a)](x) = b\cdot [(f\cdot a)(x)] = b\cdot f(a\cdot x) = (b\cdot f)(a\cdot x) = [(b\cdot f)\cdot a](x)$$

which implies that

$$b\cdot (f\cdot a) = (b\cdot f)\cdot a.$$

From this we deduce that $\mathrm{Hom}_R(L_R, M_R)$ is a B-A-bimodule. □

Proposition 6.29. *Let ${}_AM_R$ be an A-R-bimodule. The map*

$$\begin{array}{rccc} \rho_M : & \mathrm{Hom}_R(R, M) & \to & M \\ & f & \mapsto & f(1_R) \end{array}$$

is an isomorphism of A-R-bimodules whose inverse is the map

$$\begin{array}{rccc} \rho'_M : & M & \to & \mathrm{Hom}_R(R, M) \\ & x & \mapsto & (r \mapsto x\cdot r) \end{array}.$$

Proof. It is easy to check that ρ_M is a group homomorphism. Let $x \in M$ and let $\rho_x : R \to M$ be the map defined by setting

$$\rho_x(r) = x\cdot r.$$

Let $r, s \in R$. We compute

$$\rho_x(r\cdot s) = x\cdot (r\cdot s) = (x\cdot r)\cdot s = \rho_x(r)\cdot s.$$

Thus we deduce that ρ'_M is well defined. Let $f \in \mathrm{Hom}_R(R, M)$, $r \in R$ and $x \in M$. We have

$$[(\rho'_M \circ \rho_M)(f)](r) = [\rho'_M(\rho_M(f))](r) = \rho_M(f) \cdot r = f(1_R) \cdot r = f(r)$$

and

$$(\rho_M \circ \rho'_M)(x) = [\rho'_M(x)](1_R) = x \cdot 1_R = x.$$

Now let $r \in R, f \in \mathrm{Hom}_R(R, M), a \in A$. We have

$$\begin{aligned} \rho_M(a \cdot f \cdot r) &= (a \cdot f \cdot r)(1_R) = a \cdot f(r \cdot 1_R) = a \cdot f(r) = a \cdot f(1_R \cdot r) = \\ &= a \cdot f(1_R) \cdot r = a \cdot \rho_M(f) \cdot r. \end{aligned}$$

□

6.3 Tensor Product 2

6.30. *Let A and R be rings and let ${}_AM_R = \left(M, {}^A\mu_M, \mu_M^R\right)$ be an A-R-bimodule. Given a left R-module ${}_RN$, we want to endow the abelian group $M \otimes_R N$ with a left A-module structure. For this purpose, for any $a \in A$, we consider the map*

$$\alpha_a : M \times N \to M \otimes_R N$$

defined by setting

$$\alpha_a((x, y)) = (ax) \otimes y.$$

Lemma 6.31. *By using assumptions and notations of 6.30, the map $\alpha_a : M \times N \to M \otimes_R N$ is R-balanced.*

Proof. Let $x, x_1, x_2 \in M, y, y_1, y_2 \in N$ and $r \in R$. We compute

$$\begin{aligned} \alpha_a((x_1 + x_2, y)) = [a(x_1 + x_2)] \otimes y = (ax_1 + ax_2) \otimes y \overset{(6.4)}{=} (ax_1) \otimes y + (ax_2) \otimes y = \\ = \alpha_a((x_1, y)) + \alpha_a((x_2, y)). \end{aligned}$$

$$\alpha_a((x, y_1 + y_2)) = (ax) \otimes (y_1 + y_2) \overset{(6.5)}{=} (ax) \otimes y_1 + (ax) \otimes y_2 = \alpha_a((x, y_1)) + \alpha_a((x, y_2)).$$

$$\alpha_a((xr, y)) = [a(xr)] \otimes y \overset{\text{defbim}}{=} [(ax)r] \otimes y \overset{(6.6)}{=} (ax) \otimes ry = \alpha((x, ry)).$$

□

6.32. *In view of Lemma 6.31, for every $a \in A$, there is a group homomorphism $\sigma_a : M \otimes_R N \to M \otimes_R N$ such that $\sigma_a \circ \tau = \alpha_a$.*

Proposition 6.33. *By using assumptions and notations of 6.30 and of 6.32, the map*

$$\sigma : A \to End(M \otimes_R N)$$

defined by setting

$$\begin{aligned} \sigma(a) &= \sigma_a \text{ for every } a \in A, \text{ i.e.} \\ \sigma(a)(x \otimes y) &= (ax) \otimes y \text{ for every } x \in M \text{ and } y \in N, \end{aligned}$$

is a ring homomorphism.

Proof. Let $a, b \in A$. Then, for every $x \in M$ and $y \in N$ we have

$$\sigma(a+b)(x \otimes y) = [(a+b)x] \otimes y = (ax+bx) \otimes y \overset{(6.4)}{=} (ax) \otimes y + (bx) \otimes y =$$
$$= \sigma(a)(x \otimes y) + \sigma(b)(x \otimes y) \overset{\text{def+in}End}{=} [\sigma(a) + \sigma(b)](x \otimes y)$$

$$\sigma(1_A)(x \otimes y) = (1_A x) \otimes y = x \otimes y$$

$$\sigma(a \cdot_A b)(x \otimes y) = [(a \cdot_A b)x] \otimes y = [a(bx)] \otimes y = \sigma(a)(bx \otimes y) =$$
$$= \sigma(a)(\sigma(b)(x \otimes y)) = [\sigma(a) \circ \sigma(b)](x \otimes y).$$

In view of 2) in Remarks 6.9 we deduce that

$$\sigma(a+b) = \sigma(a) + \sigma(b), \sigma(1_A) = \mathrm{Id}_{M \otimes_R N}, \sigma(a \cdot_A b) = \sigma(a) \circ \sigma(b).$$

Hence σ is a ring homomorphism. □

6.34. *Let A and R be rings, let ${}_AM_R = \left(M, {}^A\mu_M, \mu_M^R\right)$ be an A-R-bimodule and let ${}_RN$ be a left R-module. By Proposition 6.33, in view of Theorem 1.8, the group $M \otimes_R N$ becomes a left A-module by setting*

$$a(x \otimes y) = (ax) \otimes y \text{ for every } a \in A \text{ and } x \in M, y \in N.$$

In an analogous way, one can prove that if ${}_RN_B = \left(N, {}^R\mu_N, \mu_N^B\right)$ is an R-B-bimodule, the group $M \otimes_R N$ becomes a right B-module by setting

$$(x \otimes y)b = x \otimes (yb) \text{ for every } a \in A \text{ and } x \in M, y \in N.$$

Proposition 6.35. *Let A and R be rings, let ${}_AM_R = \left(M, {}^A\mu_M, \mu_M^R\right)$ be an A-R-bimodule and let ${}_RN_B = \left(N, {}^R\mu_N, \mu_N^B\right)$ be an R-B-bimodule. With respect to the left A-module structure and to the right B-module structure described in 6.34, the abelian group $M \otimes_R N$ becomes an A-B-bimodule.*

Proof. Let $a \in A, b \in B$ and $z \in M \otimes_R N$. We have to prove that

$$(az)b = a(zb).$$

In view of Proposition 6.8, it is enough to prove that

$$[a(x \otimes y)]b = a[(x \otimes y)b].$$

We compute

$$[a(x \otimes y)]b = [(ax) \otimes y]b = (ax) \otimes (yb) = a[x(yb)] = a[(x \otimes y)b].$$

□

Proposition 6.36. *Let A and R be rings, let ${}_AM_R = \left(M, {}^A\mu_M, \mu_M^R\right)$ be an A-R-bimodule, let ${}_RN$ be a left R-module and let L be a right A-module. To give a left A-module homomorphism*

$$f : {}_A(M \otimes_R N) \to {}_AL$$

one has to give an R-balanced map $\beta : M \times N \to L$ such that

$$\beta((ax, y)) = a\beta((x, y)) \text{ for every } x \in M \text{ and } y \in N. \tag{6.11}$$

Proof. Let $\beta : M \times N \to L$ be an R-balanced map such that (6.11) is fulfilled. Then there exist a group homomorphism $f : M \otimes_R N \to L$ such that

$$(x \otimes y) f = \beta((x, y)).$$

Let us check that f is a right A-module homomorphism. Let $a \in A$ and $z \in M \otimes_R N$. We have to prove that

$$(az) f = a((z) f).$$

In view of Proposition 6.8, it is enough to prove that

$$(a(x \otimes y)) f = a[(x \otimes y) f] \text{ for every } x \in M \text{ and } y \in N.$$

We have

$$(a(x \otimes y)) f = ((ax) \otimes y) f = \beta((ax, y)) = a\beta((x, y)) = af(x \otimes y).$$

The converse is trivial. □

6.37. *In the particular case when A is a commutative ring and we consider (symmetric) A-bimodules, we have*

$$a(x \otimes_A y) = (ax) \otimes_A y = (xa) \otimes_A y = x \otimes_A ay = x \otimes_A (ya) = (x \otimes_A y) a$$
$$\text{for every } a \in A, x \in M, y \in N.$$

In this case (6.11) *rewrites as*

$$\beta((ax, y)) = \beta((x, ya)) = a\beta((x, y)) \text{ for every } a \in A, x \in M, y \in N.$$

In this case β is called A-bilinear map.

Definition 6.38. *Let A be a commutative ring and let M and N and L be (symmetric) A-bimodules. A map $\beta : M \times N \to L$ is said to be A*-bilinear *if*

1) $\beta((x_1 + x_2, y)) = \beta((x_1, y)) + \beta((x_2, y))$ *for every $x_1, x_2 \in M$ and $y \in N$;*

2) $\beta((x, y_1 + y_2)) = \beta((x, y_1)) + \beta((x, y_2))$ *for every $x \in M$ and $y_1, y_2 \in N$;*

3) $\beta((ax, y)) = \beta((x, ya)) = a\beta((x, y))$ *for every $x \in M, r \in A, y \in N$*

Proposition 6.39. *Let A be a commutative ring. Any A-bilinear map is A-balanced.*

Proof. Let M, N, L be symmetric A-bimodules and let $\beta : M \times N \to L$ be an A-bilinear map. Since we are considering symmetric A-bimodules, we have:

$$\beta\left((xa, y)\right) = \beta\left((ax, y)\right) = \beta\left((x, ya)\right) = \beta\left((x, ay)\right)$$

for every $x \in M, y \in N, a \in A$. □

Proposition 6.40. *Let $f : {}_AL_R \to {}_AM_R$ and $g : {}_RW_B \to {}_RZ_B$ be bimodule homomorphism. Then $f \otimes_R g : {}_A\left(L \otimes_R W\right)_B \to {}_A\left(M \otimes_R Z\right)_B$ is a bimodule homomorphism.*

Proof. For every $n \in \mathbb{N}, n \geq 1, x_1, \ldots, x_n \in L, w_1, \ldots, w_n \in W, a \in A, b \in B$ we have:

$$(f \otimes_R g)\left[a\left(\sum_{i=1}^{n} x_i \otimes w_i\right)b\right] = (f \otimes_R g)\left(\sum_{i=1}^{n} (ax_i) \otimes (w_i b)\right) =$$

$$= \sum_{i=1}^{n} f\left(ax_i\right) \otimes g\left(w_i b\right) = \sum_{i=1}^{n} \left[af\left(x_i\right)\right] \otimes \left[g\left(w_i\right)b\right] = a\left(\sum_{i=1}^{n} f\left(x_i\right) \otimes g\left(w_i\right)\right)b$$

□

6.41. *Let ${}_AL_R$ be an A- R-bimodule and let $\left({}_R\left(M_i\right)_B\right)_{i\in I}$ be a family of R-B-bimodules. Then, by Exercise 6.25, $\bigoplus_{i\in I} M_i$ is an R-B-bimodule and .$\bigoplus_{i\in I}\left(L \otimes_R M_i\right)$ is an A- B-bimodule. By Proposition 6.16,there is a group isomorphism $\varphi : L \otimes_R \left(\bigoplus_{i\in I} M_i\right) \to \bigoplus_{i\in I}\left(L \otimes_R M_i\right)$ such that*

$$\varphi\left(x \otimes_R (y_i)_{i\in I}\right) = (x \otimes_R y_i)_{i\in I} \text{ for every } x \in L \text{ and } (y_i)_{i\in I} \in \bigoplus_{i\in I} M_i.$$

is an isomorphism.

Proposition 6.42. *By means of the notations of 6.41, the map $\varphi : L \otimes_R \left(\bigoplus_{i\in I} M_i\right) \to \bigoplus_{i\in I}\left(L \otimes_R M_i\right)$ is an isomorphism of A- B-bimodules.*

Proof. For every $x \in L, (y_i)_{i\in I} \in \bigoplus_{i\in I} M_i, a \in A, b \in B$, we have:

$$\begin{aligned}
\varphi\left(a \cdot \left[x \otimes_R (y_i)_{i\in I}\right]\right) &= \varphi\left((a \cdot x) \otimes_R (y_i)_{i\in I}\right) = \left((a \cdot x) \otimes_R y_i\right)_{i\in I} = a \cdot (x \otimes_R y_i)_{i\in I} = \\
&= a\varphi\left(x \otimes_R (y_i)_{i\in I}\right) \\
\varphi\left(\left[x \otimes_R (y_i)_{i\in I}\right] \cdot b\right) &= \varphi\left(x \otimes_R \left[(y_i)_{i\in I} \cdot b\right]\right) = \varphi\left(x \otimes_R \left[(y_i \cdot b)_{i\in I}\right]\right) = \left(x \otimes_R (y_i \cdot b)\right)_{i\in I} = \\
&= (x \otimes_R y_i)_{i\in I} \cdot b = \varphi\left(x \otimes_R (y_i)_{i\in I}\right) \cdot b
\end{aligned}$$

In view of Proposition 6.8, we conclude. □

Proposition 6.43. *Let A be a ring and let ${}_AM$ be a left A-module. Then there is defines an isomorphism of left A-modules*

$$\mu = \mu^M : {}_A(A \otimes_A M) \to {}_AM$$

which satifies

$$\mu(a \otimes x) = a \cdot x \text{ for every } a \in A \text{ and } x \in M.$$

Proof. Let $\beta : A \times M \to M$ be the map defined by setting

$$\beta((a, x)) = a \cdot x \text{ for every } a \in A \text{ and } x \in M.$$

β is A-balanced. In fact, given $a, b, a_1, a_2 \in A, x, x_1, x_2 \in M$ we have

$$\begin{aligned} \beta(((a_1 + a_2), x)) &= (a_1 + a_2) \cdot x = a_1 \cdot x + a_2 \cdot x = \beta((a_1, x)) + \beta((a_2, x)) \\ \beta((a, x_1 + x_2)) &= a \cdot (x_1 + x_2) = a \cdot x_1 + a \cdot x_2 = \beta((a, x_1)) + \beta((a, x_2)) \\ \beta((ab, x)) &= (a \cdot b) \cdot x = a \cdot (b \cdot x) = \beta((a, bx)). \end{aligned}$$

Moreover β fulfills (6.11). In fact, we have

$$a \cdot \beta((b, x)) = a \cdot (b \cdot x) = \beta((a \cdot b, x)) \text{ for every } a, b \in A \text{ and } x \in M.$$

Let us prove that μ is an isomorphism. Since $((1_A \otimes x))\mu = x$, μ is clearly surjective. Let $x \in \mathrm{Ker}(\mu)$. Then there exists $n \in \mathbb{N}, n \geq 1, a_1, \ldots, a_n \in A$ and $x_1, \ldots, x_n \in M$ such that

$$x = \sum_{i=1}^{n} a_i \otimes x_i \text{ and } 0 = (x)\mu = \sum_{i=1}^{n} a_i x_i$$

so that

$$x = \sum_{i=1}^{n} a_i \otimes x_i = \sum_{i=1}^{n} 1_A \otimes a_i x_i = 1_A \otimes \sum_{i=1}^{n} a_i x_i = 1_A \otimes 0 = 0.$$

□

Definition 6.44. *Let A be a ring. A right A-module L_A is said to be* flat *if, for any short exact sequence of left A-module homomorphism*

$$0 \to {}_AM' \xrightarrow{f} {}_AM \xrightarrow{g} {}_AM'' \to 0$$

the sequence

$$0 \to L \otimes_A M' \xrightarrow{L \otimes_A f} L \otimes_A M \xrightarrow{L \otimes_A g} L \otimes_A M'' \to 0$$

is exact.

In view of Proposition 6.14, we have:

Proposition 6.45. *A right A-module L_A is flat if and only if, for every injective left A-module homomorphism $f : {}_AM' \to {}_AM$, the homomorphism $L \otimes_A f$ is injective.*

Remark 6.46. *Not every right A-module is, in general, flat. In fact consider the exact sequence of $\mathbb{Z}$-modules:*

$$0 \to 2\mathbb{Z} \xrightarrow{i} \mathbb{Z} \xrightarrow{p} \mathbb{Z}/2\mathbb{Z} \to 0$$

where i is the canonical injection and p is the canonical projection. Then

$$\mathbb{Z}/2\mathbb{Z} \otimes i : \mathbb{Z}/2\mathbb{Z} \otimes 2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}$$

is not injective. In fact, for every $a, b \in \mathbb{Z}$, we have

$$(\mathbb{Z}/2\mathbb{Z} \otimes i)\left((a + 2\mathbb{Z}) \otimes 2b\right) = (a + 2\mathbb{Z}) \otimes 2b = (a + 2\mathbb{Z})\, 2 \otimes b = (2a + 2\mathbb{Z}) \otimes b = 0$$

and hence $(\mathbb{Z}/2\mathbb{Z} \otimes i) = 0$. On the other hand $2\mathbb{Z} \cong \mathbb{Z}$ and hence $\mathbb{Z}/2\mathbb{Z} \otimes 2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \neq 0$.

Lemma 6.47. *Let $(f_i : N_i' \to N_i)_{i \in I}$ be a family of right A-module homomorphisms. Then the homomorphism*

$$\begin{array}{rccc} \bigoplus_{i \in I} f_i : & \bigoplus_{i \in I} N_i' & \to & \bigoplus_{i \in I} N_i \\ & (x_i')_{i \in I} & \longmapsto & (f(x_i'))_{i \in I} \end{array}$$

is injective if and only if $f_i : N_i' \to N_i$ is injective for every $i \in I$.

Proof. Exercise. □

Proposition 6.48. *Let $(L_i)_{i \in I}$ be a family of right A-modules. Then $\bigoplus_{i \in I} L_i$ is flat if and only if L_i is flat, for every $i \in I$.*

Proof. Let $f : {}_A M' \to {}_A M$ be an injective left A-module homomorphism. Let us consider the isomorphism of Proposition 6.16, $\varphi^M : \left(\bigoplus_{i \in I} L_i\right) \otimes_A M \to \bigoplus_{i \in I} (L_i \otimes_A M)$ where

$$\varphi\left((y_i)_{i \in I} \otimes_A x\right) = \text{ for every } (y_i)_{i \in I} \in \bigoplus_{i \in I} L_i \text{ and } x \in M.$$

Then the diagram

$$\begin{array}{ccc} \left(\bigoplus_{i \in I} L_i\right) \otimes_A M' & \xrightarrow{\varphi^{M'}} & \bigoplus_{i \in I} (L_i \otimes_A M') \\ \left(\bigoplus_{i \in I} L_i\right) \otimes_A f \downarrow & & \downarrow \bigoplus_{i \in I} (L_i \otimes_A f) \\ \left(\bigoplus_{i \in I} L_i\right) \otimes_A M & \xrightarrow{\varphi^M} & \bigoplus_{i \in I} (L_i \otimes_A M) \end{array}$$

is commutative. In fact for every $(y_i)_{i\in I} \in \bigoplus_{i\in I} L_i$ and $x' \in M'$ we have

$$\left[\bigoplus_{i\in I}(L_i \otimes_A f)\right] \circ \varphi^{M'}\left((y_i)_{i\in I} \otimes_A x'\right) = \left[\bigoplus_{i\in I}(L_i \otimes_A f)\right]\left((y_i \otimes x')_{i\in I}\right) =$$
$$= (y_i \otimes f(x'))_{i\in I} = \varphi^M\left((y_i)_{i\in I} \otimes_A f(x')\right) = \varphi^M\left(\left[\left(\bigoplus_{i\in I} L_i\right) \otimes_A f\right]\left((y_i)_{i\in I} \otimes_A x'\right)\right) =$$
$$= \left[\varphi^M \circ \left(\left(\bigoplus_{i\in I} L_i\right) \otimes_A f\right)\right]\left((y_i)_{i\in I} \otimes_A x'\right).$$

Hence $\left(\bigoplus_{i\in I} L_i\right) \otimes_A f$ is injective if and only if $\bigoplus_{i\in I}(L_i \otimes_A f)$ is injective. By Lemma 6.47 $\bigoplus_{i\in I}(L_i \otimes_A f)$ is injective if and only if $L_i \otimes_A f$ is injective, for every $i \in I$. □

Lemma 6.49. *Let A be a ring. Then the right module A_A is flat.*

Proof. Let $f : {}_AM' \to {}_AM$ be an injective left A-module homomorphism. Let us consider the isomorphism of Proposition 6.43

$$\begin{array}{rccc} \mu^M: & A \otimes_A M & \to & M \\ & a \otimes_A x & \longmapsto & a \cdot x \end{array}.$$

Then the diagram

$$\begin{array}{ccc} A \otimes_A M' & \overset{A\otimes_A f}{\longrightarrow} & A \otimes_A M \\ \mu^{M'} \downarrow & & \downarrow \mu^M \\ M' & \overset{f}{\longrightarrow} & M \end{array}$$

is commutative. In fact, for every $a \in A, x' \in M'$ we have

$$\left(f \circ \mu^{M'}\right)(a \otimes x') = f(a \cdot x') = a \cdot f(x') = \mu^M(a \otimes f(x')) = \left(\mu^M \circ (A \otimes_A f)\right)(a \otimes x').$$

Since f is injective, we deduce that also $A \otimes_A f$ is injective. □

Proposition 6.50. *Every projective right A-module P_A is flat.*

Proof. By Proposition 2.17, P_A is a direct summand of a free right A module $A_A^{(X)}$. By Lemma 6.49 and Proposition 6.48, the right A module $A_A^{(X)}$ is flat so that, by Proposition 6.48, P_A is flat. □

Corollary 6.51. *Every vector space over a field k is flat.*

Lemma 6.52. *Let us consider the commutative diagram*

$$\begin{array}{ccccccc} M' & \overset{f}{\to} & M & \overset{g}{\to} & M'' & \to & 0 \\ \downarrow \varphi & & \downarrow\cong \psi & & & & \\ N' & \overset{j}{\to} & N & \overset{h}{\to} & N'' & \to & 0 \\ \downarrow & & & & & & \\ 0 & & & & & & \end{array}$$

where φ is a surjective homomorphism and ψ is an isomorphism of right A-modules. Then there exists a right A-module homomorphism $\zeta : M'' \to N''$ such that the diagram

$$\begin{array}{ccccccc} M' & \overset{f}{\to} & M & \overset{g}{\to} & M'' & \to & 0 \\ \downarrow \varphi & & \downarrow\cong \psi & & \downarrow \zeta & & \\ N' & \overset{j}{\to} & N & \overset{h}{\to} & N'' & \to & 0 \\ \downarrow & & & & & & \\ 0 & & & & & & \end{array}$$

is commutative. Moreover ζ is an isomorphism.

Proof. Let us define $\zeta : M'' \to N''$ by setting

$$\zeta\left(x''\right) = h\left(\psi\left(x\right)\right) \text{ where } x \in M \text{ and } g\left(x\right) = x''.$$

Let us check that ζ is well-defined. Let x and $\overline{x} \in M$ such that $g\left(x\right) = x'' = g\left(\overline{x}\right)$. Then $x - \overline{x} \in \operatorname{Ker}\left(g\right) = \operatorname{Im}\left(f\right)$ and hence there exists an element $x' \in M'$ such that $x - \overline{x} = f\left(x'\right)$. We compute

$$\left(h \circ \psi\right)\left(x - \overline{x}\right) = \left(h \circ \psi\right)\left(f\left(x'\right)\right) = \left(h \circ \psi \circ f\right)\left(x'\right) = \left(h \circ j \circ \psi\right)\left(x'\right) = \left(h \circ 0\right)\left(x'\right) = 0.$$

We deduce that $\left(h \circ \psi\right)\left(x\right) = \left(h \circ \psi\right)\left(\overline{x}\right)$ and hence ζ is well defined. Moreover, by construction we have

$$\zeta \circ g = h \circ \psi.$$

Since g is surjective and $h \circ \psi$ is a right A-module homomorphism, we deduce (exercise) that ζ is a right A-module homomorphism. Moreover since $h \circ \psi$ is surjective, also ζ is surjective. Let us prove that ζ is injective. Let $x'' \in M''$ be such that $\zeta\left(x''\right) = 0$. Then there exists an $x \in M$ such that $g\left(x\right) = x''$ so that

$$0 = \zeta\left(g\left(x\right)\right) = h \circ \psi\left(x\right).$$

Hence $\psi\left(x\right) \in \operatorname{Ker}\left(h\right) = \operatorname{Im}\left(j\right)$ so that there exists an $y' \in N'$ such that $\psi\left(x\right) = j\left(y'\right)$. Since φ is surjective, there is an $x' \in M'$ such that $y' = \varphi\left(x'\right)$. We deduce that

$$\psi\left(x\right) = j\left(y'\right) = j\left(\varphi\left(x'\right)\right) = \psi\left(f\left(x'\right)\right).$$

Since ψ is injective, this implies that $x = f\left(x'\right)$ so that $x'' = g\left(x\right) = g\left(f\left(x'\right)\right) = 0.$ □

6.53. *Let L_A be a right A-module and let I be a right ideal of A. We set*

$$L \cdot I = \{\sum_{i=1}^{n} x_i a_i \mid n \in \mathbb{N}, n \geq 1, x_1, \ldots, x_n \in L, a_1, \ldots, a_n \in I\}.$$

Clearly $L \cdot I$ is a right A-submodule of L.

Proposition 6.54. *Let L_A be a right A-module and let I be a two-sided ideal of A. Then the map*

$$\begin{array}{cccc} \zeta: & L\otimes_A \frac{A}{I} & \to & \frac{L}{L\cdot I} \\ & x\otimes(a+I) & \mapsto & xa+L\cdot I \end{array}$$

is well-defined and is an isomorphism of right A-modules.

Proof. Let us consider the isomorphism $\mu^L : L\otimes_A A \to L$ of Proposition 6.43. Let $i : I \to A$ be the canonical inclusion and $p : A \to A/I$ the canonical projection. Then we have

$$\operatorname{Im}\left(\mu^L \circ (L\otimes_A i)\right) = \{\sum_{i=1}^{n} x_i a_i \mid n\in\mathbb{N}, n\geq 1, x_1,\ldots,x_n\in L, a_1,\ldots,a_n\in I\} = L\cdot I.$$

Let φ be the corestriction of $\mu^L\circ(L\otimes_A i)$ to $L\cdot I$ and let $j : LI\to L$ and be the canonical inclusion. Then we have a commutative diagram $h : L\to L/LI$ is the canonical projection. By Lemma 6.52, there exists an isomorphism $\zeta : L\otimes_A A/L\otimes_A I \to L/LI$ such that the diagram

$$\begin{array}{ccccccc} L\otimes_A I & \overset{L\otimes_A i}{\to} & L\otimes_A A & \overset{L\otimes_A p}{\to} & L\otimes_A (A/I) & \to & 0 \\ \downarrow\varphi & & \downarrow\cong\mu^L & & & \downarrow\zeta & \\ LI & \overset{j}{\to} & L & \overset{h}{\to} & L/LI & \to & 0 \\ \downarrow & & & & & & \\ 0 & & & & & & \end{array}$$

is commutative so that we have

$$\zeta\left(x\otimes(a+I)\right) = \zeta\left(L\otimes_A p\right)(x\otimes a) = h\mu^L(x\otimes a) = xa+LI.$$

□

6.55. *Let ${}_AM_R$ be a bimodule. Let L be a right A-module and let N be a right R-module. For every $\xi\in\operatorname{Hom}_R(L\otimes_A M, N)$ and for every $x\in L$ we consider the map*

$$\begin{array}{cccc} \xi_x: & M & \longrightarrow & N \\ & m & \longmapsto & \xi(x\otimes m) \end{array}.$$

Proposition 6.56. *In the notations of 6.55,*

1) *the map $\xi_x : M\to N$ is a right R-module homomorphism.*

2) *For every $x, x'\in L$ and $a\in A$ we have that, in the right A-module* $\operatorname{Hom}_R({}_AM_R, N_R)$:

$$\xi_{x+x'} = \xi_x + \xi_{x'} \text{ and } \xi_{x\cdot a} = \xi_x\cdot a. \tag{6.12}$$

3) *the map*

$$\begin{array}{cccc} \Lambda_\xi: & L & \longrightarrow & \operatorname{Hom}_R(M,N) \\ & x & \longmapsto & \xi_x \end{array}$$

is a right A-module homomorphism.

4) *given* $\xi' \in \mathrm{Hom}_R(L \otimes_A M, N)$ *we have that, in the abelian group* $\mathrm{Hom}_R(M, N)$

$$(\xi + \xi')_x = \xi_x + \xi'_x. \tag{6.13}$$

5) *given* $\xi' \in \mathrm{Hom}_R(L \otimes_A M, N)$ *we have that, in the abelian group* $\mathrm{Hom}_A(L, \mathrm{Hom}_R(M, N))$

$$\Lambda_{\xi+\xi'} = \Lambda_\xi + \Lambda_{\xi'}. \tag{6.14}$$

Proof. 1) Let $m, m' \in M$ and let $r \in R$. We compute

$$\begin{aligned} \xi_x(m + m') &= \xi(x \otimes (m + m')) \overset{(6.5)}{=} \xi(x \otimes m + x \otimes m') = \\ &= \xi(x \otimes m) + \xi(x \otimes m') = \xi_x(m) + \xi_x(m'), \\ \xi_x(m \cdot r) &= \xi(x \otimes m \cdot r) \overset{6.34}{=} \xi((x \otimes m) \cdot r) \overset{\xi \mathrm{morph} R\text{-mod}}{=} \xi(x \otimes m) \cdot r = \xi_x(m) \cdot r. \end{aligned}$$

2) Let $x, x' \in L$ and let $a \in A$. For every $m \in M$ we compute

$$\begin{gathered} \xi_{x+x'}(m) = \xi(x + x' \otimes m) \overset{(6.4)}{=} \xi(x \otimes m + x' \otimes m) = \xi(x \otimes m) + \xi(x' \otimes m) = \\ = \xi_x(m) + \xi_{x'}(m) = (\xi_x + \xi_{x'})(m), \\ \xi_{x \cdot a}(m) = \xi(x \cdot a \otimes m) \overset{(6.3)}{=} \xi(x \otimes a \cdot m) = \xi_x(a \cdot m) \overset{\mathrm{Prop}6.28}{=} (\xi_x \cdot a)(m). \end{gathered}$$

3) Let $x, x' \in L$ and let $a \in A$. In view of (6.12) we have

$$\begin{aligned} \Lambda_\xi(x + x') &= \xi_{x+x'} = \xi_x + \xi_{x'} = \Lambda_\xi(x) + \Lambda_\xi(x'), \\ \Lambda_\xi(x \cdot a) &= \xi_{x \cdot a} = \xi_x \cdot a = \Lambda_\xi(x) \cdot a. \end{aligned}$$

4) For every $m \in M$, we compute

$$(\xi + \xi')_x(m) = (\xi + \xi')(x \otimes m) = \xi(x \otimes m) + \xi'(x \otimes m) = \xi_x(m) + \xi'_x(m) = (\xi_x + \xi'_x)(m).$$

5) For every $x \in L$, we compute

$$\Lambda_{\xi+\xi'}(x) = (\xi + \xi')_x \overset{(6.13)}{=} \xi_x + \xi'_x = \Lambda_\xi(x) + \Lambda_{\xi'}(x) = (\Lambda_\xi + \Lambda_{\xi'})(x).$$

□

6.57. *Let* ${}_AM_R$ *be a bimodule. Let* L *be a right* A*-module and let* N *be a right* R*-module. For every* $\zeta \in \mathrm{Hom}_A(L_A, \mathrm{Hom}_R({}_AM_R, N))$*, we consider the map*

$$\begin{array}{rccc} \beta_\zeta : & L \times M & \longrightarrow & N \\ & (x, m) & \longmapsto & \zeta(x)(m) \end{array}.$$

Proposition 6.58. *In the notations of 6.57, the map* $\beta_\zeta : L \times M \longrightarrow N$ *is* A*-balanced and it satisfies* $\beta_\zeta((x, m \cdot r)) = \beta_\zeta((x, m)) \cdot r$ *for every* $x \in L$*,* $m \in M$ *and* $r \in R$*. Therefore by Proposition 6.36, there exists a left* R*-module hoomorphism* $\Gamma_\zeta : L \otimes_A M \to N$ *such that*

$$\Gamma_\zeta(x \otimes m) = \zeta(x)(m) \text{ for every } x \in L \text{ and } m \in M.$$

Proof. Let $x, x' \in L, m, m' \in M, a \in A, r \in R$. We compute

$$\begin{aligned}
\beta_\zeta\left((x+x',m)\right) &= \zeta\left(x+x'\right)(m) \overset{\zeta \text{isgrouphom}}{=} \left[\zeta(x)+\zeta(x')\right](m) = \\
&= \zeta(x)(m)+\zeta(x')(m) = \beta_\zeta\left((x,m)\right)+\beta_\zeta\left((x',m)\right) \\
\beta_\zeta\left((x,m+m')\right) &= \zeta(x)(m+m') \overset{\zeta(x)\text{isanhomom}}{=} \zeta(x)(m)+\zeta(x)(m') = \\
&= \beta_\zeta\left((x,m)\right)+\beta_\zeta\left((x,m')\right) \\
\beta_\zeta\left((x\cdot a,m)\right) &= \zeta(x\cdot a)(m) \overset{\zeta \text{is} A\text{-lin}}{=} \left[\zeta(x)\cdot a\right](m) \overset{\text{Prop6.28}}{=} \\
&= \zeta(x)(a\cdot m) = \beta_\zeta\left((x,a\cdot m)\right) \\
\beta_\zeta\left((x,m\cdot r)\right) &= \zeta(x)(m\cdot r) \overset{\zeta(x)\in \mathrm{Hom}_R({}_AM_R,N)}{=} \left[\zeta(x)(m)\right]\cdot r = \\
&= \beta_\zeta\left((x,m)\right)\cdot r.
\end{aligned}$$

□

Theorem 6.59. *Let ${}_AM_R$ be a bimodule. For every right A-module L and every right R-module N, we set*

$$\begin{array}{rccc}
\Lambda_N^L: & \mathrm{Hom}_R\left(L\otimes_A M,N\right) & \longrightarrow & \mathrm{Hom}_A\left(L,\mathrm{Hom}_R\left(M,N\right)\right) \\
& \left(L\otimes_A M \xrightarrow{\xi} N\right) & \longmapsto & \Lambda_\xi
\end{array}$$

and

$$\begin{array}{rccc}
\Gamma_N^L: & \mathrm{Hom}_A\left(L,\mathrm{Hom}_R\left(M,N\right)\right) & \longrightarrow & \mathrm{Hom}_R\left(L\otimes_A M,N\right) \\
& \left(L \xrightarrow{\zeta} \mathrm{Hom}_R\left(M,N\right)\right) & \longmapsto & \Gamma_\zeta.
\end{array}$$

Then $\xi \in \mathrm{Hom}_R\left(L\otimes_A M,N\right), \zeta \in \mathrm{Hom}_A\left(L,\mathrm{Hom}_R\left(M,N\right)\right)$, for every $x \in L$ and $m \in M$ we have

$$\begin{aligned}
\left[\Lambda_N^L(\xi)(x)\right][m] &= \left[\Lambda_\xi(x)\right](m) = \xi_x(m) = \xi(x\otimes m) \\
\left[\Gamma_N^L(\zeta)\right](x\otimes m) &= \Gamma_\zeta(x\otimes m) = \zeta(x)(m)
\end{aligned}$$

Λ_N^L is a group isomorphism with inverse Γ_N^L.

Proof. Let $\xi, \xi' \in \mathrm{Hom}_R\left(L\otimes_A M,N\right)$. We have

$$\Lambda_N^L\left(\xi+\xi'\right) = \Lambda_{\xi+\xi'} \overset{(6.14)}{=} \Lambda_\xi + \Lambda_{\xi'} = \Lambda_N^L(\xi)+\Lambda_N^L(\xi').$$

Moreover for every $x \in L$ and $m \in M$ we have

$$\begin{aligned}
\left[\left(\Gamma_N^L\circ\Lambda_N^L\right)(\xi)\right](x\otimes m) &= \left[\Gamma_N^L(\Lambda_\xi)\right](x\otimes m) = \Gamma_{\Lambda_\xi}(x\otimes m) = \Lambda_\xi(x)(m) = \\
&= \xi_x(m) = \xi(x\otimes m).
\end{aligned}$$

By 2) in Remarks 6.9, we conclude that $\left(\Gamma_N^L\circ\Lambda_N^L\right)(\xi)=\xi$.

Let now $\zeta \in \mathrm{Hom}_A\left(L,\mathrm{Hom}_R\left(M,N\right)\right)$. For every $x \in L$ and $m \in M$, we compute

$$\begin{aligned}
&\left\{\left[\left(\Lambda_N^L\circ\Gamma_N^L\right)(\zeta)\right](x)\right\}(m) = \left\{\left[\left(\Lambda_N^L\right)(\Gamma_\zeta)\right](x)\right\}(m) = \left[\Lambda_{\Gamma_\zeta}(x)\right](f) = \\
&= \left(\Gamma_\zeta\right)_x(m) = \Gamma_\zeta(x\otimes m) = \zeta(x)(m).
\end{aligned}$$

This yields that $\left(\Lambda_N^L\circ\Gamma_N^L\right)(\zeta)=\zeta$. □

Exercise 6.60. *In the assumptions and notations of Theorem 6.59, Assume that L is B-A-bimodule and that N is an S-R-bimodule. Prove that Λ_N^L is an S-B-bimodule homomorphism.*

Theorem 6.61. *In the assumptions and notations of Theorem 6.59, let $f \in \operatorname{Hom}_A(L_2, L_1)$ and $g \in \operatorname{Hom}_R(N_1, N_2)$. Then the following diagram is commutative.*

$$\begin{array}{ccc} \operatorname{Hom}_R(L_1 \otimes_A M, N_1) & \overset{\Lambda_{N_1}^{L_1}}{\longrightarrow} & \operatorname{Hom}_A(L_1, \operatorname{Hom}_R(M, N_1)) \\ \operatorname{Hom}_R(f \otimes_A M, g) \downarrow & & \downarrow \operatorname{Hom}_A(f, \operatorname{Hom}_R(M, g)) \\ \operatorname{Hom}_R(L_2 \otimes_A M, N_2) & \overset{\Lambda_{N_2}^{L_2}}{\longrightarrow} & \operatorname{Hom}_A(L_2, \operatorname{Hom}_R(M, N_2)) \end{array}$$

Proof. Let $\xi \in \operatorname{Hom}_R(L_1 \otimes_A M, N_1)$. Note that $\operatorname{Hom}_R(f \otimes_A M, g)(\xi) = g \circ \xi \circ (f \otimes_A M)$. Also if $\zeta \in \operatorname{Hom}_A(L_1, \operatorname{Hom}_R(M, N_1))$, we have that $\operatorname{Hom}_A(f, \operatorname{Hom}_R(M, g))(\zeta) = \operatorname{Hom}_R(M, g) \circ \zeta \circ f$.

For every $x \in L_2$ and $m \in M$, we calculate:

$$\left[\left\{\left[\Lambda_{N_2}^{L_2} \circ \operatorname{Hom}_R(f \otimes_A M, g)\right](\xi)\right\}(x)\right](m) = \left\{\left[\Lambda_{N_2}^{L_2}(g \circ \xi \circ (f \otimes_A M))\right](x)\right\}(m) =$$
$$= [g \circ \xi \circ (f \otimes_A M)](x \otimes m) = [g \circ \xi \circ (f \otimes_A M)](x \otimes m) = (g \circ \xi)(f(x) \otimes m)$$
$$\left[\left\{\left[\operatorname{Hom}_A(f, \operatorname{Hom}_R(M, g)) \circ \Lambda_{N_1}^{L_1}\right](\xi)\right\}(x)\right](m) = \left\{\left[\operatorname{Hom}_R(M, g) \circ \Lambda_{N_1}^{L_1}(\xi) \circ f\right](x)\right\}(m) =$$
$$= \left[\operatorname{Hom}_R(M, g)\left(\xi_{f(x)}\right)\right](m) = \left(g \circ \xi_{f(x)}\right)(m) = g(\xi(f(x) \otimes m)) = (g \circ \xi)(f(x) \otimes m).$$

□

Chapter 7

Homology

7.1 Categories and Functors

Definition 7.1. *A* category $\mathcal{C}$ *consists of:*

1) a class of objects denoted by $\mathrm{Ob}(\mathcal{C})$*.*

2) for every $C_1, C_2 \in \mathrm{Ob}(\mathcal{C})$ *a set* $Hom_{\mathcal{C}}(C_1, C_2)$*, called the* set of morphisms from C_1 to C_2

3) for every $C_1, C_2, C_3 \in \mathrm{Ob}(\mathcal{C})$ *there is a map*

$$\begin{array}{rccc} \circ: & \mathrm{Hom}_{\mathcal{C}}(C_1, C_2) \times \mathrm{Hom}_{\mathcal{C}}(C_2, C_3) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(C_1, C_3) \\ & (f, g) & \longmapsto & g \circ f \text{ called the composite of } g \text{ and } f \end{array}$$

satisfying the following conditions:

1) *if* $(C_1, C_2) \neq (C_3, C_4)$*,* $\mathrm{Hom}_{\mathcal{C}}(C_1, C_2) \cap \mathrm{Hom}_{\mathcal{C}}(C_3, C_4) = \varnothing$*;*

2) *if* $h \in \mathrm{Hom}_{\mathcal{C}}(C_3, C_4)$*,* $h \circ (g \circ f) = (h \circ g) \circ f$*;*

3) *for every* $C \in \mathcal{C}$*, there exists* $\mathrm{Id}_C \in \mathrm{Hom}_C(C, C)$ *such that for every* $f \in \mathrm{Hom}_{\mathcal{C}}(C, C')$*,* $f \circ \mathrm{Id}_C = f = \mathrm{Id}_{C'} \circ f$*.*

We also write $f : C_1 \to C_2$ *or* $C_1 \xrightarrow{f} C_2$ *instead of* $f \in \mathrm{Hom}_{\mathcal{C}}(C_1, C_2)$*.*
Moreover if $C \in \mathrm{Ob}(\mathcal{C})$*, we will simply write* $C \in \mathcal{C}$*.*

Example 7.2. *Sets, together with functions between sets, form the category Sets. For every algebraic structure you can consider its category: take sets endowed with that algebraic structure as objects and take morphisms between two objects as morphisms. In this way, you obtain the category of groups, Grps, of rings, Rings, of right R-modules, Mod-R and so on.*

Definition 7.3. *A category is called* small *if the class of its objects is a set;* discrete *if, given two objects* C_1, C_2*, if* $C_1 = C_2$ *then* $\mathrm{Hom}_{\mathcal{C}}(C_1, C_2) = \{\mathrm{Id}_{C_1}\}$*, if* $C_1 \neq C_2$ *then* $\mathrm{Hom}_{\mathcal{C}}(C_1, C_2) = \varnothing$*. Let* $\mathcal{C}$ *be a category. The* opposite *category of a category* $\mathcal{C}$ *is the category* $\mathcal{C}^{\circ}$ *where* $\mathrm{Ob}(\mathcal{C}^{\circ}) = Ob(\mathcal{C})$ *and* $\mathrm{Hom}_{\mathcal{C}^{\circ}}(C_1, C_2) = \mathrm{Hom}_{\mathcal{C}}(C_2, C_1)$*.*

Definition 7.4. *A* subcategory $\mathcal{D}$ *of a category* $\mathcal{C}$ *is a category such that* $\mathrm{Ob}(\mathcal{D}) \subseteq \mathrm{Ob}(\mathcal{C})$ *and for every* $D_1, D_2 \in \mathcal{D}$*,* $\mathrm{Hom}_{\mathcal{D}}(D_1, D_2) \subseteq \mathrm{Hom}_{\mathcal{C}}(D_1, D_2)$*. When the inclusion is an equality,* $\mathcal{D}$ *is called* full subcategory *of* $\mathcal{C}$*.*

Definition 7.5. *Let* $\mathcal{C}$ *be a category. A morphism* $C_1 \xrightarrow{f} C_2$ *is an* isomorphism *if there exists a morphism* $C_2 \xrightarrow{g} C_1$ *such that* $f \circ g = \mathrm{Id}_{C_2}$ *and* $g \circ f = \mathrm{Id}_{C_1}$*.*

Remark 7.6. *Let* $f : C_1 \to C_2$ *be an isomorphism in a category* $\mathcal{C}$ *and let* $g, g' : C_2 \to C_1$ *be such that* $f \circ g = \mathrm{Id}_{C_2} = f \circ g'$ *and* $g \circ f = \mathrm{Id}_{C_1} = g' \circ f$*. Then we have*

$$g' = g' \circ \mathrm{Id}_{C_2} = g' \circ (f \circ g) = (g' \circ f) \circ g = \mathrm{Id}_{C_1} \circ g = g.$$

Hence there exists a ***unique morphism*** $g : C_2 \to C_1$ *be such that* $f \circ g = \mathrm{Id}_{C_2}$ *and* $g \circ f = \mathrm{Id}_{C_1}$*. This unique morphism will be denoted by* f^{-1}*.*

Definition 7.7. *Let* $A, B \in \mathcal{C}$ *and* $f : A \longrightarrow B$*, then*

- f *is a* monomorphism *if, for every* $g_1, g_2 : C \longrightarrow A$ *such that* $f \circ g_1 = f \circ g_2$*, we have* $g_1 = g_2$*;*

- f *is an* epimorphism *if, for every* $g_1, g_2 : B \longrightarrow C$ *such that* $g_1 \circ f = g_2 \circ f$*, we have* $g_1 = g_2$*.*

Proposition 7.8. *Let* $A, B \in \mathcal{C}$ *and let* $f : A \longrightarrow B$*. If* f *is an isomorphism then* f *is a monomorphism and an epimorphism.*

Proof. Since f is an isomorphism, there exists a morphism f^{-1} which is a two-sided inverse of f. First we prove that f is a monomorphism. Let $g_1, g_2 : C \longrightarrow A$ be a morphism such that $f \circ g_1 = f \circ g_2$. Then, by composing to the left with f^{-1} we get $f^{-1} \circ f \circ g_1 = f^{-1} \circ f \circ g_2$ and thus $g_1 = g_2$, i.e. f is a monomorphism. Now we want to prove that f is an epimorphism. Let $g_1, g_2 : B \longrightarrow C$ such that $g_1 \circ f = g_2 \circ f$. By composing to the right with f^{-1} we get $g_1 \circ f \circ f^{-1} = g_2 \circ f \circ f^{-1}$ from which follows $g_1 = g_2$, i.e. f is an epimorphism. □

Exercise 7.9. *Let* $f : A \longrightarrow B$ *and* $g : B \longrightarrow C$ *be morphisms in a category* $\mathcal{C}$ *. Then*

- *if both* f *and* g *are monomorphisms, also* $g \circ f$ *is a monomorphism;*

- *if both* f *and* g *are epimorphisms, also* $g \circ f$ *is an epimorphism.*

Remark 7.10. *The converse of Proposition 7.8 doesn't hold in general, such as in the case of the inclusion* $\mathbb{Z} \to \mathbb{Q}$ *in the category of rings. In fact, let* $\mathcal{C}$ *be the category of rings, let*

$$i : \mathbb{Z} \longrightarrow \mathbb{Q}$$

be the canonical inclusion and let $h_1, h_2 : \mathbb{Q} \longrightarrow A$ *be such that*

$$\mathbb{Z} \xrightarrow{\ i\ } \mathbb{Q} \underset{h_2}{\overset{h_1}{\rightrightarrows}} A$$

$h_1 \circ i = h_2 \circ i$. We will prove that $h_1 = h_2$. Let $m \in \mathbb{Z}$ and let $n \in \mathbb{N}$, $n \neq 0$. Since h_j is a morphism of rings for $j = 1, 2$, we have that

$$\begin{aligned} 1_A &= h_j(1) = h_j\left(\frac{n}{n}\right) = h_j(n)\, h_j\left(\frac{1}{n}\right) \text{ and also} \\ 1_A &= h_j(1) = h_j\left(\frac{n}{n}\right) = h_j\left(\frac{1}{n}\right)\ h_j(n) \end{aligned}$$

so that

$$h_j\left(\frac{1}{n}\right) = h_j(n)^{-1}.$$

Moreover we have

$$h_j(n) = n h_j(1) = n 1_A.$$

Therefore we get

$$h_1\left(\frac{m}{n}\right) = m h_1\left(\frac{1}{n}\right) = m h_1(n)^{-1} = m h_2(n)^{-1} = m h_2\left(\frac{1}{n}\right) = h_2\left(\frac{m}{n}\right)$$

that is $h_1 = h_2$ so that i is an epimorphism. Now, let $g_1, g_2 : R \longrightarrow \mathbb{Z}$

$$R \underset{g_2}{\overset{g_1}{\rightrightarrows}} \mathbb{Z} \xrightarrow{\ i\ } \mathbb{Q}$$

be such that $i \circ g_1 = i \circ g_2$. Then $g_1 = g_2$ i.e. i is also a monomorphism. Note that i is not an isomorphism: a non-zero group morphism

$$f : \mathbb{Q} \longrightarrow \mathbb{Z}$$

does not exists since $\mathbb{Q}$ is divisible but $\mathbb{Z}$ is not. In fact, assume there exists a group morphism

$$f : D \longrightarrow \mathbb{Z}$$

where D is divisible. By definition of divisible group, for every $n \in \mathbb{N}$, $nD = D$. Since f is a group morphism, $f(D) \subseteq \mathbb{Z}$ and thus $f(D) = t\mathbb{Z}$ for some $t \in \mathbb{N} \setminus \{0\}$. Since f is a group morphism and D is divisible we have that

$$nf(D) = f(nD) = f(D) = t\mathbb{Z}$$

and therefore

$$nt\mathbb{Z} = t\mathbb{Z}.$$

In particular, for every $n \in \mathbb{N}$, there exists $y_n \in \mathbb{Z}$ such that

$$t = nty_n.$$

For $n = 2$ we get

$$t = 2ty_2$$

and thus

$$1 = 2y_2$$

contradiction since 2 is not invertible in $\mathbb{Z}$.

Proposition 7.11. *Let A be a ring and let $f : L \to M$ be a morphism in Mod-A. Then*

1) *f is injective $\Leftrightarrow$ f is a monomorphism in Mod-A.*

2) *f is surjective $\Leftrightarrow$ f is an epimorphism in Mod-A.*

3) *f is an isomorphism $\Leftrightarrow$ f is an isomorphism in Mod-A. $\Leftrightarrow$ f is both a monomorphism and an epimorphism in Mod-A.*

Proof. 1) $\Rightarrow$. It is trivial.

1) $\Leftarrow$. Let $x \in L$ such that $x \neq 0$. Let us consider the morphism in Mod-A (Proposition 2.2)

$$h_x : A_A \to L_A \text{ defined by setting } h_x(a) = xa \text{ for every } a \in A.$$

Then

$$h_x(1) = x \neq 0 = \mathbf{0}(x)$$

where $\mathbf{0}$ denotes the zero morphism from A to M. Since f is a monomorphism in Mod-A, we get

$$f \circ h_x \neq f \circ \mathbf{0}.$$

It is easy to see that this implies

$$(f \circ h_x)(1) \neq 0.$$

Since $(f \circ h_x)(1) = f(x)$ we conclude.

2) $\Rightarrow$. It is trivial.

2) $\Leftarrow$. Let $p : M \to M/\mathrm{Im}(f)$ be the canonical projection. We have to prove that $M = \mathrm{Im}(f)$ i.e. that $p = \mathbf{0}$ where $\mathbf{0} : M \to M/\mathrm{Im}(f)$ is the zero morphism.

Since $p \circ f = \mathbf{0} \circ f$ and since f is an epimorphism in Mod-A, we get that $p = \mathbf{0}$.

3) It follows easily from 1) and 2). □

Notations 7.12. *Let A be a ring. In view of the foregoing, from now on*

- *an injective homomorphism f of right (left) A-modules will also be called a monomorphism. We will also say that f is mono, for short.*
- *a surjective homomorphism of right (left) A-modules will also be called an epimorphism. We will also say that f is mono, for short.*
- *a bijective homomorphism of right (left) A-modules will also be called an isomorphism. We will also say that f is iso, for short.*

Definition 7.13. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories.* A covariant functor $F : \mathcal{C} \to \mathcal{D}$ *between $\mathcal{C}$ and $\mathcal{D}$ consists of*

1) *a collection of objects of* $\mathcal{D}$
$$(F(C))_{C\in\mathcal{C}}$$

2) *a collection of morphisms in* $\mathcal{D}$
$$(F(f) : F(C_1) \longrightarrow F(C_2))_{f\in\mathrm{Hom}_{\mathcal{C}}(C_1,C_2)} \text{ for every } C_1, C_2 \in \mathcal{C}$$
such that
$$F(\mathrm{Id}_C) = \mathrm{Id}_{F(C)} \text{ and } F(g\circ f) = F(g)\circ F(f)$$
for every morphism $f \in \mathrm{Hom}_{\mathcal{C}}(C_1, C_2)$ *and* $g \in \mathrm{Hom}_{\mathcal{C}}(C_2, C_3)$.

Definition 7.14. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be categories. A* contravariant functor $F : \mathcal{C} \to \mathcal{D}$ *between* $\mathcal{C}$ *and* $\mathcal{D}$ *consists of*

1) *a collection of objects of* $\mathcal{D}$ $(F(C))_{C\in\mathcal{C}}$

2) *a collection of morphisms in* $\mathcal{D}$
$$(F(f) : F(C_2) \longrightarrow F(C_1))_{f\in\mathrm{Hom}_{\mathcal{C}}(C_1,C_2)} \text{ for every } C_1, C_2 \in \mathcal{C}$$
such that
$$F(\mathrm{Id}_C) = \mathrm{Id}_{F(C)} \text{ and } F(g\circ f) = F(f)\circ F(g).$$
for every morphism $f \in \mathrm{Hom}_{\mathcal{C}}(C_1, C_2)$ *and* $g \in \mathrm{Hom}_{\mathcal{C}}(C_2, C_3)$..

Examples 7.15.

Let ${}_AL_R$ *be an A-R-bimodule. Then we can consider the following functors.*

1) *The covariant functor* $\mathrm{Hom}_R({}_AL_R, -) : Mod\text{-}R \to Mod\text{-}A$ *defined by setting*
$$\mathrm{Hom}_R({}_AL_R, -)(M_R) = \mathrm{Hom}_R({}_AL_R, M_R) \text{ and } \mathrm{Hom}_R({}_AL_R, -)(f) = \mathrm{Hom}_R({}_AL_R, f)$$
for every $M_R \in Mod\text{-}R$ *and* f *morphism in* $Mod\text{-}R$.

2) *The covariant functor* $-\otimes_A {}_AL_R : Mod\text{-}A \to Mod\text{-}R$ *defined by setting*
$$(-\otimes_A {}_AL_R)(M_A) = M_A \otimes_A {}_AL_R \text{ and } (-\otimes_A {}_AL_R)(f) = f \otimes_A {}_AL_R$$
for every $M_A \in Mod\text{-}A$ *and* f *morphism in* $Mod\text{-}A$.

3) *The contravariantvariant functor* $\mathrm{Hom}_R(-, {}_AL_R) : Mod\text{-}R \to A\text{-}Mod$ *defined by setting*
$$\mathrm{Hom}_R(-, {}_AL_R)(M_R) = \mathrm{Hom}_R(M_R, {}_AL_R) \text{ and } \mathrm{Hom}_R(-, {}_AL_R)(f) = \mathrm{Hom}_R(f, {}_AL_R)$$
for every $M_R \in Mod\text{-}R$ *and* f *morphism in* $Mod\text{-}R$.

Lemma 7.16. *Let $F : \mathcal{C}_1 \to \mathcal{C}_2$ and $G : \mathcal{C}_2 \to \mathcal{C}_3$, be functors. For every $C \in \mathcal{C}_1$ we set*

$$GF(C) = G(F(C))$$

and for every morphism $f : C_1 \to C_2$ we set

$$GF(f) = G(F(f)).$$

This gives rise to a functor $GF = G \circ F : \mathcal{C}_1 \to \mathcal{C}_3$ which is

1) *covariant whenever both F and G are covariant,*

2) *covariant whenever both F and G are contravariant,*

3) *contravariant whenever F is covariant and G is contravariant,*

4) *contravariant whenever F is contravariant and G is covariant.*

Proof. Exercise. □

Definitions 7.17. *Given two covariant functors $\mathcal{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{D}$, a* functorial morphism *(or natural transformation) $\alpha : F \to G$ is a collection of morphims in $\mathcal{D}$, for every $C \in \mathcal{C}$, by a morphism $\left(F(C) \xrightarrow{\alpha_C} G(C)\right)_{C \in \mathcal{C}}$ such that, for every $C_1 \xrightarrow{f} C_2$,*

$$\alpha_{C_2} \circ F(f) = G(f) \circ \alpha_{C_1}$$

i.e. the following diagram

$$\begin{array}{ccc} F(C_1) & \xrightarrow{\phi_{C_1}} & G(C_1) \\ {\scriptstyle F(f)}\downarrow & \circlearrowleft & \downarrow{\scriptstyle G(f)} \\ F(C_2) & \xrightarrow[\phi_{C_2}]{} & G(C_2) \end{array}$$

is commutative. α is called a functorial isomorphism *(or* natural equivalence*) if, for every $C \in \mathcal{C}$, α_C is an isomorphism in $\mathcal{D}$. In this case the functors are called* isomorphic *and we write $F \cong G$.*

Exercise 7.18. *Let $\alpha : F \to G$ be a functorial isomorphism. Show that the collection $\beta = \left((\alpha_C)^{-1}\right)_{C \in \mathcal{C}}$ is a functorial isomorphism from G to F.*

Notation 7.19. *Let $\alpha : F \to G$ be a functorial isomorphism. Then the functorial isomorphism β in Exercise 7.18 will be denoted by α^{-1}.*

Examples 7.20. *Let ${}_AM_R$ and ${}_AM'_R$ be A-R-bimodules and let $f : {}_AM_R \to {}_AM'_R$ be a morphism of A-R-bimodules i.e. f is both a left A-modules and also a right R-modules homomorphism. Then*

$$\mathrm{Hom}_R(f, \text{-}) : \mathrm{Hom}_R({}_AM'_R, \text{-}) \longrightarrow \mathrm{Hom}_R({}_AM_R, \text{-})$$

and

$$\text{-} \otimes_A f : \text{-} \otimes_A M \longrightarrow \text{-} \otimes_A M'$$

are functorial morphisms (Exercise).

Definitions 7.21. *Let $F : \mathcal{C} \to \mathcal{D}$ We say that*

- *F is an* equivalence *of categories if there is a functor $G : \mathcal{D} \to \mathcal{C}$ such that $FG \cong \mathrm{Id}_{\mathcal{D}}$ and $GF \cong \mathrm{Id}_{\mathcal{C}}$. In this case we also say that (F, G) is an* equivalence of categories.
- *F is an* isomorphism *of categories if there is a functor $G : \mathcal{D} \to \mathcal{C}$ such that $FG = \mathrm{Id}_{\mathcal{D}}$ and $GF = \mathrm{Id}_{\mathcal{C}}$. In this case we also say that (F, G) is an* isomorphism of categories .

Definitions 7.22. *Two categories $\mathcal{C}$ and $\mathcal{D}$ are called*

- equivalent *if there exist fuctors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that (F, G) is an equivalence of categories.*
- isomorphic *if there exist fuctors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that (F, G) is an isomorphism of categories.*

7.2 Snake Lemma

Lemma 7.23 (Snake Lemma). *Let A be a ring and let*

$$\begin{CD}
@. L @>{\epsilon}>> M @>{\pi}>> N @>>> 0 \\
@. @V{\alpha}VV @V{\beta}VV @V{\gamma}VV \\
0 @>>> L' @>>{\epsilon'}> M' @>>{\pi'}> N'
\end{CD}$$

be a commutative diagram in Mod-A with exact rows.

1) *Then there exist right A-module homomorphisms $\epsilon_\star$, $\pi_\star$, $\epsilon'_\star$, $\pi'_\star$ such that the diagram*

$$\begin{CD}
@. 0 @. 0 @. 0 \\
@. @VVV @VVV @VVV \\
@. \ker\alpha @>{\epsilon_\star}>> \ker\beta @>{\pi_\star}>> \ker\gamma \\
@. @V{i_\alpha}VV @V{i_\beta}VV @V{i_\gamma}VV \\
@. L @>{\epsilon}>> M @>{\pi}>> N @>>> 0 \\
@. @V{\alpha}VV @V{\beta}VV @V{\gamma}VV \\
0 @>>> L' @>{\epsilon'}>> M' @>{\pi'}>> N' \\
@. @V{p_\alpha}VV @V{p_\beta}VV @V{p_\gamma}VV \\
@. \operatorname{coker}\alpha @>{\epsilon'_\star}>> \operatorname{coker}\beta @>{\pi'_\star}>> \operatorname{coker}\gamma \\
@. @VVV @VVV @VVV \\
@. 0 @. 0 @. 0
\end{CD} \quad .$$

is commutative and exact. They are uniquely defined by the following equalities:

$$i_\beta \circ \epsilon_* = \epsilon_{|\mathrm{Ker}(\alpha)} = \epsilon \circ i_\alpha \tag{7.1}$$

$$i_\gamma \circ \pi_* = \pi \circ i_\beta \tag{7.2}$$

$$\epsilon'_* \circ p_\alpha = p_\beta \circ \varepsilon' \tag{7.3}$$

$$\pi'_* \circ p_\beta = p_\gamma \circ \pi'. \tag{7.4}$$

2) *There exists a right A-module homomorphism* $\omega : \mathrm{Ker}(\gamma) \longrightarrow C\mathrm{oker}(\alpha)$ *such that the sequence*

$$\mathrm{Ker}(\alpha) \xrightarrow{\epsilon_*} \mathrm{Ker}(\beta) \xrightarrow{\pi_*} \mathrm{Ker}(\gamma) \xrightarrow{\omega} \mathrm{Coker}(\alpha) \xrightarrow{\epsilon'_*} \mathrm{Coker}(\beta) \xrightarrow{\pi'_*} \mathrm{Coker}(\gamma)$$

is exact.

3) *If* ϵ *is mono,* $\epsilon_\star$ *is also mono and if* π' *is epi, so is* $\pi'_\star$.

Proof. **1) Construction of the homomorphisms** $\epsilon_\star$, $\pi_\star$, $\epsilon'_\star$, $\pi'_\star$.

ϵ_*) Let $x \in \mathrm{Ker}(\alpha)$. Then $\alpha(x) = 0$ and hence $0 = \epsilon'\alpha(x) = \beta\epsilon(x)$ so that $\epsilon(x) \in \mathrm{Ker}(\beta)$. Therefore we get $\epsilon(\mathrm{Ker}(\alpha)) \subseteq \mathrm{Ker}(\beta)$ and we can set

$$\epsilon_* = \left(\epsilon_{|\mathrm{Ker}(\alpha)}\right)^{|\mathrm{Ker}(\beta)}.$$

It follows that

$$i_\beta \circ \epsilon_* = i_\beta \circ \left(\epsilon_{|\mathrm{Ker}(\alpha)}\right)^{|\mathrm{Ker}(\beta)} = \epsilon_{|\mathrm{Ker}(\alpha)} = \epsilon \circ i_\alpha.$$

π_*) Let $m \in \mathrm{Ker}(\beta)$. Then $0 = \beta(m)$ and hence $0 = \pi'\beta(m) = \gamma\pi(m)$ so that $\pi(m) \in \mathrm{Ker}\gamma$. Therefore we get $\pi(\mathrm{Ker}(\beta)) \subseteq \mathrm{Ker}(\gamma)$ and we can set

$$\pi_* = \left(\pi_{|\mathrm{Ker}(\beta)}\right)^{|\mathrm{Ker}(\gamma)}.$$

It follows that

$$i_\gamma \circ \pi_* = i_\gamma \circ \left(\pi_{|\mathrm{Ker}(\beta)}\right)^{|\mathrm{Ker}(\gamma)} = \pi_{|\mathrm{Ker}(\beta)} = \pi \circ i_\beta.$$

ϵ'_*) We have $p_\beta \circ \varepsilon' \circ \alpha = p_\beta \circ \beta \circ \varepsilon = 0$ so that $(p_\beta \circ \varepsilon')(\mathrm{Im}(\alpha)) = 0$. Hence, by the Fundamental Theorem for Quotient Modules 1.20, there exists a unique homomorphism $\epsilon'_* : C\mathrm{oker}(\alpha) = \frac{L'}{\mathrm{Im}(\alpha)} \to C\mathrm{oker}(\beta) = \frac{M'}{\mathrm{Im}(\beta)}$ such that

$$\begin{aligned} \epsilon'_* \circ p_\alpha &= p_\beta \circ \varepsilon' \\ \text{i.e. } \epsilon'_*(x' + \mathrm{Im}(\alpha)) &= \epsilon'(x') + \mathrm{Im}(\beta) \text{ for every } x' \in L'. \end{aligned}$$

π'_*) We have $p_\gamma \circ \pi' \circ \beta = p_\gamma \circ \gamma \circ \pi = 0$ so that $(p_\gamma \circ \pi')(\mathrm{Im}(\beta)) = 0$. Hence, by the Fundamental Theorem for Quotient Modules 1.20, there exists a unique homomorphism $\pi'_* : C\mathrm{oker}(\beta) = \frac{M'}{\mathrm{Im}(\beta)} \longrightarrow C\mathrm{oker}(\gamma) = \frac{N'}{\mathrm{Im}(\gamma)}$ such that

$$\begin{aligned} \pi'_* \circ p_\beta &= p_\gamma \circ \pi' \\ \text{i.e. } \pi'_*(m' + \mathrm{Im}(\beta)) &= \pi'(m') + \mathrm{Im}(\gamma) \text{ for every } m' \in M'. \end{aligned}$$

2) The diagram is commutative. It follows from $(7.1), (7.2), (7.3)$ and (7.4).

3) The diagram is exact.

3a) $\mathrm{Im}(\epsilon_\star) \subseteq \mathrm{Ker}(\pi_*)$. We have $i_\gamma \circ \pi_* \circ \epsilon_\star = \pi \circ \epsilon \circ i_\alpha = 0 \circ i_\alpha = 0$. Since i_γ is mono we get that $\pi_* \circ \epsilon_\star = 0$.

3b) $\mathrm{Ker}(\pi_*) \subseteq \mathrm{Im}(\epsilon_\star)$. Let $m \in \mathrm{Ker}(\pi_*)$. Then $m \in \mathrm{Ker}(\beta)$ and $0 = i_\gamma \pi_*(m) = \pi i_\beta(m)$. Thus $i_\beta(m) \in \mathrm{Ker}(\pi) = \mathrm{Im}(\epsilon)$ and hence there is an $x \in L$ such that $i_\beta(m) = \epsilon(x)$. Then

$$0 = \beta(i_\beta(m)) = \beta(\epsilon(x)) = \epsilon'(\alpha(x)).$$

Since ϵ' is *mono* we deduce that $\alpha(x) = 0$ i.e. $x \in \mathrm{Ker}(\alpha)$ and hence $x = i_\alpha(x)$. Thus $i_\beta(m) = \epsilon(i_\alpha(x)) = i_\beta(\epsilon_*(x))$. Since i_β is mono, we deduce that $m = \epsilon_*(x)$.

3c) $\mathrm{Im}(\epsilon'_\star) \subseteq \mathrm{Ker}(\pi'_*)$.

$$\mathrm{Im}(\epsilon'_\star) = \mathrm{Im}(\epsilon'_\star \circ p_\alpha) = \mathrm{Im}(p_\beta \circ \varepsilon').$$

Since

$$\pi'_* \circ p_\beta \circ \varepsilon' = p_\gamma \circ \pi' \circ \varepsilon' = p_\gamma \circ 0 = 0$$

we get

$$\mathrm{Im}(\epsilon'_\star) = \mathrm{Im}(p_\beta \circ \varepsilon') \subseteq \mathrm{Ker}(\pi'_*).$$

3d) $\mathrm{Ker}(\pi'_*) \subseteq \mathrm{Im}(\epsilon'_\star)$. Let $m' + \mathrm{Im}(\beta) = p_\beta(m') \in \mathrm{Ker}(\pi'_*)$, i.e. $m' + \mathrm{Im}(\beta) \in M'/\mathrm{Im}(\beta)$ and $0 + \mathrm{Im}(\gamma) = \pi'_*(m' + \mathrm{Im}(\beta)) = \pi'_* p_\beta(m') = p_\gamma \pi'(m')$ i.e. $\pi'(m') \in \mathrm{Im}(\gamma)$ so that there exists a $y \in N$ such that

$$\pi'(m') = \gamma(y).$$

Moreover, since π is *epi*, there exists $m \in M$ such that

$$y = \pi(m)$$

so that

$$\pi'(m') = \gamma(y) = \gamma(\pi(m)) = \pi'(\beta(m)).$$

Hence $m' - \beta(m) \in \mathrm{Ker}(\pi') = \mathrm{Im}(\epsilon')$ and hence there exists $x' \in L'$ such that

$$m' - \beta(m) = \epsilon'(x').$$

Thus we have

$$p_\beta(m') = p_\beta(\epsilon'(x')) = \epsilon'_*(p_\alpha(x')) \in \mathrm{Im}(\epsilon'_*).$$

4) 4a) Construction of ω. Let $y \in \mathrm{Ker}(\gamma)$. Since π is *epi*, there exists an $m \in M$ such that $i_\gamma(y) = \pi(m)$. We have $0 = \gamma(i_\gamma(y)) = \gamma(\pi(m)) = \pi'(\beta(m))$, i.e. $\beta(m) \in \mathrm{Ker}(\pi') = \mathrm{Im}(\epsilon')$. Hence there exists an element $x' \in L'$ such that $\epsilon'(x') = \beta(m)$. We set

$$\omega(y) = x' + \mathrm{Im}(\alpha)$$

4b) ω is well-defined. Let $\overline{m} \in M$ such that $\pi(\overline{m}) = i_\gamma(y)$ and let $\overline{x}' \in L'$ such that $\epsilon'(\overline{x}') = \beta(\overline{m})$. Then we have

$$\pi(m) = \pi(\overline{m}) \text{ i.e. } m - \overline{m} \in \mathrm{Ker}(\pi) = \mathrm{Im}(\epsilon).$$

Thus there exists an $x \in L$ such that

$$\epsilon(x) = m - \overline{m}. \tag{7.5}$$

On the other hand we have

$$\epsilon'(x' - \overline{x}') = \epsilon'(x) - \epsilon'(\overline{x}') = \beta(m) - \beta(\overline{m}) = \beta(m - \overline{m}). \tag{7.6}$$

Thus from (7.5) and (7.6) we deduce that

$$\epsilon'(x' - \overline{x}') = \beta(m - \overline{m}) = \beta(\epsilon(x)) = \epsilon'(\alpha(x)).$$

Since ϵ' is *mono* we get $x' - \overline{x}' = \alpha(x)$ so that

$$x' + \mathrm{Im}(\alpha) = \overline{x}' + \mathrm{Im}(\alpha).$$

4c) ω is a homomomorphism. Let $y_1, y_2 \in \mathrm{Ker}(\gamma)$. Since π is *epi*, there exist $m_1, m_2 \in M$ such that $i_\gamma(y_1) = \pi(m_1)$and $i_\gamma(y_2) = \pi(m_2)$. By 4a) there exist $x_1', x_2' \in L'$ such that $\beta(m_1) = \epsilon'(x_1')$ and $\beta(m_2) = \epsilon'(x_2')$. Since π and β and ϵ' are homomorphisms we have that

$$\begin{aligned} \pi(m_1 + m_2) &= \pi(m_1) + \pi(m_2) = y_1 + y_2 \text{ and} \\ \beta(m_1 + m_2) &= \beta(m_1) + \beta(m_2) = \epsilon'(x_1') + \epsilon'(x_2') = \epsilon'(x_1' + x_2'). \end{aligned}$$

Therefore, by definition of ω, we have

$$\begin{aligned} \omega(y_1 + y_2) &= (x_1' + x_2') + \mathrm{Im}(\alpha) \\ &= (x_1' + \mathrm{Im}(\alpha)) + (x_2' + \mathrm{Im}(\alpha)) \\ &= \omega(y_1) + \omega(y_2). \end{aligned}$$

Let now $y \in \mathrm{Ker}(\gamma)$ and $a \in A$. Since π is epi, there exists an $m \in M$ such that $i_\gamma(y) = \pi(m)$. By 4a) there exists an element $x' \in L'$ such that $\beta(m) = \epsilon'(x')$. Since π and β and ϵ' are homomorphisms we have that

$$\pi(m \cdot a) = \pi(m) \cdot a = y \cdot a \text{ and } \beta(m \cdot a) = \beta(m) \cdot a = \epsilon'(x') \cdot a = \epsilon'(x' \cdot a).$$

Therefore, by definition of ω, we have

$$\begin{aligned}\omega(y\cdot a) &= x'\cdot a+\mathrm{Im}(\alpha)\\ &= (x'+\mathrm{Im}(\alpha))\cdot a\\ &= \omega(y)\cdot a.\end{aligned}$$

5) The sequence is exact. In view of 3) we need to prove the following.

5a) $\mathrm{Im}(\pi_\star)\subseteq \mathrm{Ker}(\omega)$. Let $y\in \mathrm{Im}(\pi_\star)$. Then there exists an $m\in \mathrm{Ker}(\beta)$ such that $y=\pi_\star(m)$ and hence $i_\gamma(y)=i_\gamma\pi_\star(m)=\pi(i_\beta(m))$. Then, by 4), there exists an $x'\in L'$ with $\epsilon'(x')=\beta(i_\beta(m))=0$. We deduce that $x'\in \mathrm{Ker}(\epsilon')$. Since ϵ' is *mono*, we get $x'=0$ so that

$$\omega(y)=x'+\mathrm{Im}(\alpha)=0+\mathrm{Im}(\alpha)$$

and hence $y\in \mathrm{Ker}(\omega)$.

5b) $\mathrm{Ker}(\omega)\subseteq \mathrm{Im}(\pi_\star)$. Let $y\in \mathrm{Ker}(\omega)$. Then $y\in \mathrm{Ker}(\gamma)$ and hence, by 4), there is an $m\in M$ such that $\pi(m)=i_\gamma(y)$, and an $x'\in L'$ such that $\beta(m)=\epsilon'(x')$ and we have

$$0+\mathrm{Im}(\alpha)=\omega(y)=x'+\mathrm{Im}(\alpha)$$

i.e. $x'\in \mathrm{Im}(\alpha)$. Hence there exists $x\in L$ such that $x'=\alpha(x)$. Then we have

$$\beta(m)=\epsilon'(x')=\epsilon'(\alpha(x))=\beta(\epsilon(x))$$

that is $m-\epsilon(x)\in \mathrm{Ker}(\beta)$. Since $\pi\epsilon=0$ we get $i_\gamma(y)=\pi(m)=\pi(m-\epsilon(x))=\pi(i_\beta(m-\epsilon(x)))=i_\gamma\pi_*(m-\epsilon(x))$, and hence $y=\pi_*(m-\epsilon(x))\in \mathrm{Im}(\pi_\star)$.

5c) $\mathrm{Im}(\omega)\subseteq \mathrm{Ker}(\epsilon'_\star)$. Let $w\in \mathrm{Im}(\omega)$. Then there exists $y\in \mathrm{Ker}(\gamma)$ with $\omega(y)=w$. By 4) there is an $m\in M$ such that $\pi(m)=i_\gamma(y)$ and there is an $x'\in L'$ with $\epsilon'(x')=\beta(m)$ and

$$w=\omega(y)=x'+\mathrm{Im}(\alpha)=\pi_\alpha(x').$$

Hence

$$\epsilon'_*(w)=\epsilon'_*(\pi_\alpha(x'))=p_\beta(\epsilon'(x'))=p_\beta(\beta(m))=0+\mathrm{Im}(\beta),$$

i.e. $x'+\mathrm{Im}(\alpha)\in \mathrm{Ker}(\epsilon'_\star)$.

5d) $\mathrm{Ker}(\epsilon'_\star)\subseteq \mathrm{Im}(\omega)$. Let $z\in \mathrm{Ker}(\epsilon'_\star)$. Then there is an $x'\in L'$ such that $z=x'+\mathrm{Im}(\alpha)=p_\alpha(x')$ and

$$0+\mathrm{Im}(\beta)=\epsilon'_\star(z)=\epsilon'_\star(p_\alpha(x'))=p_\beta(\epsilon'(x')).$$

Therefore $\epsilon'(x')\in \mathrm{Im}(\beta)$ so that there exists an $m\in M$ such that $\beta(m)=\epsilon'(x')$. Let $y=\pi(m)$. Then, we have

$$\gamma(y)=\gamma(\pi(m))=\pi'(\beta(m))=\pi'(\epsilon'(x))=0.$$

Therefore $y\in \mathrm{Ker}(\gamma)$ and,by definition ω, we have

$$\omega(y)=x'+\mathrm{Im}(\alpha)$$

so that $z=x'+\mathrm{Im}(\alpha)=\omega(y)\in \mathrm{Im}(\omega)$.

6) If ϵ is mono then ϵ_* is also mono. It follows from $i_\beta\circ\epsilon_*=\epsilon\circ i_\alpha$.

7) If π' is epi then π'_* is also epi. It follows from $\pi'_*\circ p_\beta=p_\gamma\circ\pi'$. □

7.3 Chain Complexes

Definitions 7.24. *A* chain complex *of right A-modules is a a pair* $\left(C_\bullet, d_\bullet^{C_\bullet}\right) = \left((C_n)_{n\in\mathbb{Z}}, \left(d_n^{C_\bullet}\right)_{n\in\mathbb{Z}}\right)$ *where each C_n is a right A-module, $d_n^{C_\bullet}: C_n \to C_{n-1}$ is a right A-modules homomorphism and $d_n^{C_\bullet} \circ d_{n+1}^{C_\bullet} = 0$ for every $n \in \mathbb{Z}$. For each $n \in \mathbb{Z}$*

- $d_\bullet^{C_\bullet} = \left(d_n^{C_\bullet}\right)_{n\in\mathbb{Z}}$ *is called the* differential operator *of the chain complex,*
- $Z_n(C_\bullet) := \mathrm{Ker}\left(d_n^{C_\bullet}\right)$ *is called the n-*th cycle *of the chain complex ,*
- $B_n(C_\bullet) := \mathrm{Im}\left(d_{n+1}^{C_\bullet}\right)$ *is called the n-*th boundary *of the chain complex ,*
- $B_n(C_\bullet) \subseteq Z_n(C_\bullet)$ *and* $H_n(C_\bullet) := \frac{\mathrm{Ker}\left(d_n^{C_\bullet}\right)}{\mathrm{Im}\left(d_{n+1}^{C_\bullet}\right)} = \frac{Z_n(C_\bullet)}{B_n(C_\bullet)}$ *is called the n-*th homology module *of the chain complex.*

We will denote by

- $i_{Z_n}: Z_n(C_\bullet) \to C_n$ *the canonical inclusion and by* $p_{Z_n}: C_n \to C_n/Z_n(C_\bullet)$ *the canonical projection;*
- $i_{B_n}: B_n(C_\bullet) \to C_n$ *the canonical inclusion and by* $p_{B_n}: C_n \to C_n/B_n(C_\bullet)$ *the canonical projection;*
- $j_{B_n}: B_n(C_\bullet) \to Z_n(C_\bullet)$ *the canonical inclusion and by* $q_{B_n}: Z_n(C_\bullet) \to Z_n(C_\bullet)/B_n(C_\bullet) = H_n(C_\bullet)$ *the canonical projection.*
- $j_{H_n}: H_n(C_\bullet) \to C_n/B_n(C_\bullet)$ *the canonical inclusion.*

Clearly we have

$$j_{H_n} \circ q_{B_n} = p_{B_n} \circ i_{Z_n}. \tag{7.7}$$

Whenever needed, we will write $Z_n(C_\bullet)$ and $B_n(C_\bullet)$ in the above subscripts.

Definition 7.25. *Given chain complexes* $\left(C_\bullet, d_\bullet^{C_\bullet}\right)$ *and* $\left(D_\bullet, d_\bullet^{D_\bullet}\right)$*, a* morphism of chain complexes of right A-modules $\varphi_\bullet = (\varphi_n)_{n\in\mathbb{Z}} : \left(C_\bullet, d_\bullet^{C_\bullet}\right) = \left((C_n)_{n\in\mathbb{Z}}, \left(d_n^{C_\bullet}\right)_{n\in\mathbb{Z}}\right) \longrightarrow \left(D_\bullet, d_\bullet^{D_\bullet}\right) = \left((D_n)_{n\in\mathbb{Z}}, \left(d_n^{D_\bullet}\right)_{n\in\mathbb{Z}}\right)$ *consists of a family of right A-modules homomorphisms* $(\varphi_n : C_n \longrightarrow D_n)_{n\in\mathbb{Z}}$ *such that the following diagram is commutative*

$$\begin{array}{ccc} C_{n+1} & \xrightarrow{\varphi_{n+1}} & D_{n+1} \\ {\scriptstyle d_{n+1}^{C_\bullet}}\downarrow & & \downarrow{\scriptstyle d_{n+1}^{D_\bullet}} \\ C_n & \xrightarrow[\varphi_n]{} & D_n. \end{array}$$

i.e. $d_{n+1}^{D_\bullet} \circ \varphi_{n+1} = \varphi_n \circ d_{n+1}^{C_\bullet}$*, for every $n \in \mathbb{Z}$.*

We will simply write φ instead of $\varphi_\bullet$ whenever no risk of confusion will arise.

Notation 7.26. *We will denote by $Ch\,(Mod\text{-}A)$ the category of chain complexes. Obviously the objects are chain complexes of right A-modules and morphisms are just morphism of chain complexes of right A-modules.*

Lemma 7.27. *Let $\left(C_\bullet, d_\bullet^{C_\bullet}\right)$ be a chain complex and let $n \in \mathbb{Z}$. Then the map*

$$\begin{array}{rccc} \widehat{d_n^{C_\bullet}}: & \mathrm{Coker}\left(d_{n+1}^{C_\bullet}\right) = \frac{C_n}{\mathrm{Im}\left(d_{n+1}^{C_\bullet}\right)} & \longrightarrow & \mathrm{Ker}\left(d_{n-1}^{C_\bullet}\right) \\ & x_n + \mathrm{Im}\left(d_{n+1}^{C_\bullet}\right) & \longmapsto & d_n\left(x_n\right) \end{array}.$$

is well-defined and is a right A-modules homomorphism. It is uniquely defined by

$$i_{Z_{n-1}(C_\bullet)} \circ \widehat{d_n^{C_\bullet}} \circ p_{B_n(C_\bullet)} = d_n^{C_\bullet} \tag{7.8}$$

Moreover we have

$$\mathrm{Ker}\left(\widehat{d_n^{C_\bullet}}\right) = H_n\left(C_\bullet\right) \text{ and } \mathrm{Coker}\left(\widehat{d_n^{C_\bullet}}\right) = H_{n-1}\left(C_\bullet\right).$$

Proof. Since $d_n^{C_\bullet} \circ d_{n+1}^{C_\bullet} = 0$,we have that $\mathrm{Im}\left(d_{n+1}^{C_\bullet}\right) \subseteq \mathrm{Ker}\left(d_n^{C_\bullet}\right)$. Then, by the Fundamental Theorem for Quotient Modules 1.20, there exists a unique homomorphism

$$\left(\overline{d_n^{C_\bullet}}\right): \frac{C_n}{\mathrm{Im}\left(d_{n+1}^{C_\bullet}\right)} \to C_{n-1}$$

such that

$$\overline{d_n^{C_\bullet}} \circ p_{B_n(C_\bullet)} = d_n^{C_\bullet} \text{ i.e.}$$

$$\overline{d_n^{C_\bullet}}\left(x_n + \mathrm{Im}\left(d_{n+1}^{C_\bullet}\right)\right) = d_n^{C_\bullet}\left(x_n\right) \text{ for every } x_n \in C_n.$$

Since $d_{n-1}^{C_\bullet} \circ d_n^{C_\bullet} = 0$ we have that $\mathrm{Im}\left(\overline{d_n^{C_\bullet}}\right) = \mathrm{Im}\left(d_n^{C_\bullet}\right) \subseteq \mathrm{Ker}\left(d_{n-1}^{C_\bullet}\right)$ and we can set

$$\widehat{d_n^{C_\bullet}} = \left(\overline{d_n^{C_\bullet}}\right)^{|\mathrm{Ker}\left(d_{n-1}^{C_\bullet}\right)}: \frac{C_n}{\mathrm{Im}\left(d_{n+1}^{C_\bullet}\right)} \to \mathrm{Ker}\left(d_{n-1}^{C_\bullet}\right) \text{ i.e.}$$

$$i_{Z_{n-1}(C_\bullet)} \circ \widehat{d_n^{C_\bullet}} = \overline{d_n^{C_\bullet}}$$

so that

$$i_{Z_{n-1}(C_\bullet)} \circ \widehat{d_n^{C_\bullet}} \circ p_{B_n(C_\bullet)} = d_n^{C_\bullet}.$$

We have

$$\begin{aligned} \mathrm{Ker}\left(\widehat{d_n^{C_\bullet}}\right) &= \left\{c_n + B_n\left(C_\bullet\right) \mid d_n^{C_\bullet}\left(c_n\right) = 0\right\} \\ &= \left\{c_n + B_n\left(C_\bullet\right) \mid c_n \in \mathrm{Ker}\left(d_n^{C_\bullet}\right) = Z_n\left(C_\bullet\right)\right\} \\ &= \frac{Z_n\left(C_\bullet\right)}{B_n\left(C_\bullet\right)} = H_n\left(C_\bullet\right) \end{aligned}$$

and

$$\mathrm{Coker}\left(\widehat{d_n^{C_\bullet}}\right) = \frac{\mathrm{Ker}\left(d_{n-1}^{C_\bullet}\right)}{\mathrm{Im}\left(d_n^{C_\bullet}\left(C_\bullet\right)\right)} = \frac{Z_{n-1}\left(C_\bullet\right)}{B_{n-1}\left(C_\bullet\right)} = H_{n-1}\left(C_\bullet\right).$$

□

7.28. *Let* $\varphi_\bullet : \left(C_\bullet, d_\bullet^{C_\bullet}\right) \longrightarrow \left(D_\bullet, d_\bullet^{D_\bullet}\right)$ *be a morphism of complexes We can consider the following morphisms*

1) ***A morphism between kernels of the differential operators= cycles****.*

Since $d_{n-1}^{D_\bullet} \circ \varphi_{n-1} = \varphi_{n-2} \circ d_{n-1}^{C_\bullet}$*, we have that*

$$\left(d_{n-1}^{D_\bullet} \circ \varphi_{n-1}\right)\left(\operatorname{Ker}\left(d_{n-1}^{C_\bullet}\right)\right) = \left(\varphi_{n-2} \circ d_{n-1}^{C_\bullet}\right)\left(\operatorname{Ker}\left(d_{n-1}^{C_\bullet}\right)\right) = 0$$

so that

$$\varphi_{n-1}\left(\operatorname{Ker}\left(d_{n-1}^{C_\bullet}\right)\right) \subseteq \operatorname{Ker}\left(d_{n-1}^{D_\bullet}\right) \tag{7.9}$$

and we can consider

$$\Lambda_n(\varphi) = \left(\left(\varphi_{n-1}\right)_{|Z_{n-1}(C_\bullet)}\right)^{|Z_{n-1}(D_\bullet)} : \begin{array}{ccc} Z_{n-1}(C_\bullet) = \operatorname{Ker}\left(d_{n-1}^{C_\bullet}\right) & \longrightarrow & \operatorname{Ker}\left(d_{n-1}^{D_\bullet}\right) = Z_{n-1}(D_\bullet) \\ c_{n-1} & \longmapsto & \varphi_{n-1}(c_{n-1}). \end{array}$$

so that

$$i_{Z_{n-1}(D_\bullet)} \circ \Lambda_n(\varphi) = \varphi_{n-1} \circ i_{Z_{n-1}(C_\bullet)} \tag{7.10}$$

2) ***A morphism between cokernels of the differential operators****.*

Since $d_{n+1}^{D_\bullet} \circ \varphi_{n+1} = \varphi_n \circ d_{n+1}^{C_\bullet}$*, we have that*

$$\begin{aligned} \varphi_n\left(B_n(C_\bullet)\right) &= \varphi_n\left(\operatorname{Im}\left(d_{n+1}^{C_\bullet}\right)\right) = \varphi_n d_{n+1}^{C_\bullet}(C_{n+1}) \\ &= \left(d_{n+1}^{D_\bullet} \circ \varphi_{n+1}\right)(C_{n+1}) \subseteq \operatorname{Im}\left(d_{n+1}^{D_\bullet}\right) = B_n(D_\bullet) \end{aligned}$$

so that

$$p_{B_n(D_\bullet)}\left(\varphi_n\left(B_n(C_\bullet)\right)\right) = 0$$

Then, by the Fundamental Theorem for Quotient Modules 1.20, there exists a unique homomorphism

$$\Gamma_n(\varphi) : \frac{C_n}{B_n(C_\bullet)} = \operatorname{Coker}\left(d_{n+1}^{C_\bullet}\right) \longrightarrow \operatorname{Coker}\left(d_{n+1}^{D_\bullet}\right) = \frac{D_n}{B_n(D_\bullet)}$$

such that

$$\Gamma_n(\varphi) \circ p_{B_n(C_\bullet)} = p_{B_n(D_\bullet)} \circ \varphi_n \tag{7.11}$$

i.e.

$$\Gamma_n(\varphi) : \begin{array}{ccc} \frac{C_n}{B_n(C_\bullet)} = \operatorname{Coker}\left(d_{n+1}^{C_\bullet}\right) & \longrightarrow & \operatorname{Coker}\left(d_{n+1}^{D_\bullet}\right) = \frac{D_n}{B_n(D_\bullet)} \\ c_n + B_n(C_\bullet) & \longmapsto & \varphi_n(c_n) + B_n(D_\bullet). \end{array}$$

3) ***A morphism between the homology modules****.*

We have that

$$\begin{aligned}\Gamma_n(\varphi)\left(\frac{Z_n(C_\bullet)}{B_n(C_\bullet)}\right) &= \left(\Gamma_n(\varphi)\circ p_{B_n(C_\bullet)}\right)(Z_n(C_\bullet)) = \left(p_{B_n(D_\bullet)}\circ\varphi_n\right)(Z_n(C_\bullet)) = \\ &= p_{B_n(D_\bullet)}(\varphi_n(Z_n(C_\bullet))) \overset{(7.9)}{\subseteq} p_{B_n(D_\bullet)}(Z_n(D_\bullet)) \\ &= \frac{Z_n(D_\bullet)}{B_n(D_\bullet)} = H_n(D_\bullet).\end{aligned}$$

Therefore we can consider

$$H_n(\varphi) = \left((\Gamma_n(\varphi))_{|\frac{Z_n(C_\bullet)}{B_n(C_\bullet)}}\right)^{|\frac{Z_n(D_\bullet)}{B_n(D_\bullet)}}$$

i.e.

$$\begin{array}{rccc} H_n(\varphi): & \frac{Z_n(C_\bullet)}{B_n(C_\bullet)} = H_n(C_\bullet) & \longrightarrow & H_n(D_\bullet) = \frac{Z_n(D_\bullet)}{B_n(D_\bullet)} \\ & z_n + B_n(C_\bullet) & \longmapsto & \varphi_n(z_n) + B_n(D_\bullet). \end{array}$$

We have

$$j_{H_n(D_\bullet)}\circ H_n(\varphi) = \Gamma_n(\varphi)\circ j_{H_n}(C_\bullet) \tag{7.12}$$

and

$$\begin{aligned} & j_{H_n(D_\bullet)}\circ H_n(\varphi)\circ q_{B_n(C_\bullet)} \overset{(7.12)}{=} \\ &= \Gamma_n(\varphi)\circ j_{H_n(C_\bullet)}\circ q_{B_n(C_\bullet)} = \\ &\overset{(7.7)}{=} \Gamma_n(\varphi)\circ p_{B_n(C_\bullet)}\circ i_{Z_n(C_\bullet)} \\ &\overset{(7.11)}{=} p_{B_n(D_\bullet)}\circ\varphi_n\circ i_{Z_n(C_\bullet)} \end{aligned}$$

so that we get

$$j_{H_n(D_\bullet)}\circ H_n(\varphi)\circ q_{B_n(C_\bullet)} = p_{B_n(D_\bullet)}\circ\varphi_n\circ i_{Z_n(C_\bullet)} \tag{7.13}$$

Moreover we have

$$\begin{aligned} & j_{H_{n-1}(D_\bullet)}\circ H_{n-1}(\varphi)\circ q_{B_{n-1}(C_\bullet)} \overset{(7.13)}{=} p_{B_{n-1}(D_\bullet)}\circ\varphi_{n-1}\circ i_{Z_{n-1}(C_\bullet)} \overset{(7.10)}{=} \\ &= p_{B_{n-1}(D_\bullet)}\circ i_{Z_{n-1}(D_\bullet)}\circ\Lambda_n(\varphi) \overset{(7.7)}{=} j_{H_{n-1}(D_\bullet)}\circ q_{B_{n-1}(D_\bullet)}\circ\Lambda_n(\varphi). \end{aligned}$$

Since $j_{H_{n-1}(D_\bullet)}$ *is mono, we deduce that*

$$H_{n-1}(\varphi)\circ q_{B_{n-1}(C_\bullet)} = q_{B_{n-1}(D_\bullet)}\circ\Lambda_n(\varphi). \tag{7.14}$$

Proposition 7.29. *In the notations of 7.28, we have that*

$$\Lambda_n(\varphi)\circ\widehat{d_n^{C_\bullet}} = \widehat{d_n^{D_\bullet}}\circ\Gamma_n(\varphi) \tag{7.15}$$

i.e. the following diagram is commutative:

$$\begin{array}{ccc} \frac{C_n}{B_n(C_\bullet)} & \xrightarrow{\Gamma_n(\varphi)} & \frac{D_n}{B_n(D_\bullet)} \\ {\scriptstyle \widehat{d_n^{C_\bullet}}}\downarrow & & \downarrow{\scriptstyle \widehat{d_n^{D_\bullet}}} \\ Z_{n-1}\left(C_\bullet\right) & \xrightarrow[\Lambda_n(\varphi)]{} & Z_{n-1}\left(D_\bullet\right), \end{array}$$

We have also the commutative diagram

$$\begin{array}{ccccccc} C_n & & & \xrightarrow{\varphi_n} & & & D_n \\ & \searrow^{p_{B_n(C_\bullet)}} & & & & {}^{p_{B_n(D_\bullet)}}\swarrow & \\ & & \operatorname{Coker}\left(d_{n+1}^{C_\bullet}\right) = \frac{C_n}{B_n(C_\bullet)} & \xrightarrow{\Gamma_n(\varphi)} & \frac{D_n}{B_n(D_\bullet)} = \operatorname{Coker}\left(d_{n+1}^{D_\bullet}\right) & & \\ {\scriptstyle d_n^{C_\bullet}}\downarrow & & {\scriptstyle \widehat{d_n^{C_\bullet}}}\downarrow & & \downarrow{\scriptstyle \widehat{d_n^{D_\bullet}}} & & \downarrow{\scriptstyle d_n^{D_\bullet}} \\ & & \operatorname{Ker}\left(d_{n-1}^{C_\bullet}\right) = Z_{n-1}\left(C_\bullet\right) & \xrightarrow[\Lambda_n(\phi)]{} & Z_{n-1}\left(D_\bullet\right) = \operatorname{Ker}\left(d_{n-1}^{D_\bullet}\right) & & \\ & \swarrow_{i_{Z_{n-1}(C_\bullet)}} & & & & {}_{i_{Z_{n-1}(D_\bullet)}}\searrow & \\ C_{n-1} & & & \xrightarrow[\varphi_{n-1}]{} & & & D_{n-1}. \end{array}$$

.

Proof. We have

$$\begin{aligned} & i_{Z_{n-1}(D_\bullet)} \circ \Lambda_n\left(\varphi\right) \circ \widehat{d_n^{C_\bullet}} \circ p_{B_n(C_\bullet)} \overset{(7.10)}{=} \varphi_{n-1} \circ i_{Z_{n-1}(C_\bullet)} \circ \widehat{d_n^{C_\bullet}} \circ p_{B_n(C_\bullet)} \overset{(7.8)}{=} \varphi_{n-1} \circ d_n^{C_\bullet} \\ = \ & d_n^{D_\bullet} \circ \varphi_n \overset{(7.8)}{=} i_{Z_{n-1}(D_\bullet)} \circ \widehat{d_n^{D_\bullet}} \circ p_{B_n(D_\bullet)} \circ \varphi_n \overset{(7.11)}{=} i_{Z_{n-1}(D_\bullet)} \circ \widehat{d_n^{D_\bullet}} \circ \Gamma_n\left(\varphi\right) \circ p_{B_n(C_\bullet)}. \end{aligned}$$

Since $i_{Z_{n-1}(D_\bullet)}$ is mono and $p_{B_n(C_\bullet)}$ is epi, we get

$$\Lambda_n\left(\varphi\right) \circ \widehat{d_n^{C_\bullet}} = \widehat{d_n^{D_\bullet}} \circ \Gamma_n\left(\varphi\right).$$

$$\begin{aligned} d_n^{D_\bullet}\Gamma_n\left(\varphi\right) \circ p_{B_n(C_\bullet)} &= p_{B_n(D_\bullet)} \circ \varphi_n = d_n^{D_\bullet}\left(\varphi_n\left(c_n\right) + B_n\left(D_\bullet\right)\right) \\ &= d_n^{D_\bullet}\left(\varphi_n\left(c_n\right)\right). \end{aligned}$$

The last diagram is commutative in view of (7.8)

$$i_{Z_{n-1}(C_\bullet)} \circ \widehat{d_n^{C_\bullet}} \circ p_{B_n(C_\bullet)} = d_n^{C_\bullet},$$

(7.11)

$$\Gamma_n\left(\varphi\right) \circ p_{B_n(C_\bullet)} = p_{B_n(D_\bullet)} \circ \varphi_n,$$

(7.10)

$${}_{Z_{n-1}(D_\bullet)} \circ \Lambda_n\left(\varphi\right) = \varphi_{n-1} \circ i_{Z_{n-1}(C_\bullet)}$$

and (7.15)

$$\Lambda_n\left(\varphi\right) \circ \widehat{d_n^{C_\bullet}} = \widehat{d_n^{D_\bullet}} \circ \Gamma_n\left(\varphi\right).$$

□

Lemma 7.30. *Let* $C_\bullet \xrightarrow{\varphi} D_\bullet \xrightarrow{\psi} E_\bullet$ *be morphisms of complexes, then* $\psi\circ\varphi$*, defined by setting* $(\psi\circ\varphi)_n = \psi_n \circ \varphi_n$ *for every* $n\in\mathbb{Z}$*, is also a morphism of complexes and for every* $n\in\mathbb{Z}$ *the following equalities hold.*

$$\Lambda_n(\psi\circ\varphi) = \Lambda_n(\psi)\circ\Lambda_n(\varphi). \tag{7.16}$$

$$\Gamma_n(\psi\circ\varphi) = \Gamma_n(\psi)\circ\Gamma_n(\varphi). \tag{7.17}$$

$$H_n(\psi\circ\varphi) = H_n(\psi)\circ H_n(\varphi). \tag{7.18}$$

so that we get obviously defined functors

$$H_n : Ch(Mod\text{-}A) \to Mod\text{-}A.$$

Proof. For every $n\in\mathbb{Z}$ we have

$$d_n^{E_\bullet}\circ(\psi\circ\varphi)_n = d_n^{E_\bullet}\circ\psi_n\circ\varphi_n = \psi_{n-1}\circ d_n^{D_\bullet}\circ\varphi_n = \psi_{n-1}\circ\varphi_{n-1\circ d_n^{D_\bullet}}\circ d_n^{C_\bullet} = (\psi\circ\varphi)_{n-1}\circ d_n^{C_\bullet}$$

and hence we deduce that $\psi\circ\varphi$ is a morphism of complexes.

1) Let us prove (7.16).

We compute

$$i_{Z_{n-1}(E_\bullet)}\circ\Lambda_n(\psi\circ\varphi) \overset{(7.10)}{=} (\psi_{n-1}\circ\varphi_{n-1})\circ i_{Z_{n-1}(C_\bullet)} \overset{(7.10)}{=} \psi_{n-1}\circ i_{Z_{n-1}(D_\bullet)}\circ\Lambda_n(\varphi) = \overset{(7.10)}{=} i_{Z_{n-1}(E_\bullet)}\circ\Lambda_n(\psi)\circ\Lambda_n(\varphi).$$

Since $i_{Z_{n-1}(E_\bullet)}$ is mono, we obtain(7.16).

2) Let us prove (7.17).

We compute

$$\Gamma_n(\psi\circ\varphi)\circ p_{B_n(C_\bullet)} \overset{(7.11)}{=} p_{B_n(E_\bullet)}\circ(\psi_n\circ\varphi_n) \overset{(7.11)}{=} \Gamma_n(\psi)\circ p_{B_n(D_\bullet)}\circ\varphi_n = \overset{(7.11)}{=} \Gamma_n(\psi)\circ\Gamma_n(\varphi)\circ p_{B_n(C_\bullet)}.$$

Since $p_{B_n(C_\bullet)}$ is epi, we obtain (7.17).

3) Let us prove (7.18).

$$j_{H_n(E_\bullet)}\circ H_n(\psi)\circ H_n(\varphi)\circ q_{B_n(C_\bullet)} \overset{(7.12)}{=} \Gamma_n(\psi)\circ j_{H_n(D_\bullet)}\circ H_n(\varphi)\circ q_{B_n(C_\bullet)} \overset{(7.12)}{=}$$
$$= \Gamma_n(\psi)\circ\Gamma_n(\varphi)\circ j_{H_n(C_\bullet)}\circ q_{B_n(C_\bullet)} \overset{(7.7)}{=} \Gamma_n(\psi)\circ\Gamma_n(\varphi)\circ p_{B_n(C_\bullet)}\circ i_{Z_n(C_\bullet)} =$$
$$\overset{(7.11)}{=} \Gamma_n(\psi)\circ p_{B_n(D_\bullet)}\circ\varphi_n\circ i_{Z_n(C_\bullet)} \overset{(7.11)}{=} p_{n(E_\bullet)}\circ\psi_n\circ\varphi_n\circ i_{Z_n(C_\bullet)} =$$
$$= p_{n(E_\bullet)}\circ(\psi_n\circ\varphi_n)\circ i_{Z_n(C_\bullet)} = p_{n(E_\bullet)}\circ(\psi\circ\varphi)_n\circ i_{Z_n(C_\bullet)} \overset{(7.13)}{=}$$
$$= j_{H_n}(E_\bullet)\circ H_n(\psi\circ\varphi)\circ q_{B_n(C_\bullet)}.$$

Since $j_{H_n}(E_\bullet)$ is mono and $q_{B_n(C_\bullet)}$ is epi, we get $H_n(\psi\circ\varphi) = H_n(\psi)\circ H_n(\varphi)$.

□

Definition 7.31. *Let* $\varphi_\bullet = (\varphi_n)_{n\in\mathbb{Z}} : \left(C_\bullet, d_\bullet^{C_\bullet}\right) \longrightarrow \left(D_\bullet, d_\bullet^{D_\bullet}\right)$ *and* $\psi_\bullet = (\psi_n)_{n\in\mathbb{Z}} : \left(D_\bullet, d_\bullet^{D_\bullet}\right) \longrightarrow \left(E_\bullet, d_\bullet^{E_\bullet}\right)$ *morphisms of complexes. We say that*

$$0 \to C_\bullet \xrightarrow{\varphi_\bullet} D_\bullet \xrightarrow{\psi_\bullet} E_\bullet \to 0$$

is an exact sequence of complexes if, for every $n \in \mathbb{Z}$*, the sequence*

$$0 \to C_n \xrightarrow{\varphi_n} D_n \xrightarrow{\psi_n} E_n \to 0 \text{ is exact.}$$

Theorem 7.32. *Let* $0 \longrightarrow C_\bullet \xrightarrow{\varphi_\bullet} D_\bullet \xrightarrow{\psi_\bullet} E_\bullet \longrightarrow 0$ *be an exact sequence of complexes of right* A*-modules. Then, for every* $n \in \mathbb{Z}$*, there exists a morphism* $H_n(E_\bullet) \xrightarrow{\omega_n} H_{n-1}(C_\bullet)$ *such that the sequence*

$$\ldots . \to H_n(C_\bullet) \xrightarrow{H_n(\varphi_\bullet)} H_n(D_\bullet) \xrightarrow{H_n(\psi_\bullet)} H_n(E_\bullet) \xrightarrow{\omega_n} H_{n-1}(C_\bullet) \xrightarrow{H_{n-1}(\varphi_\bullet)} H_{n-1}(D_\bullet) \xrightarrow{H_{n-1}(\psi_\bullet)} H_{n-1}(E_\bullet)$$

is exact.

Proof. Let $n \in \mathbb{Z}$ and let us consider the following diagram:

$$\begin{array}{ccccccccc}
 & & \frac{C_n}{B_n(C_\bullet)} & \xrightarrow{\Gamma_n(\varphi)} & \frac{D_n}{B_n(D_\bullet)} & \xrightarrow{\Gamma_n(\psi)} & \frac{E_n}{B_n(E_\bullet)} & \longrightarrow & 0 . \\
 & & \downarrow{\scriptstyle \widehat{d_n^{C_\bullet}}} & & \downarrow{\scriptstyle \widehat{d_n^{D_\bullet}}} & & \downarrow{\scriptstyle \widehat{d_n^{E_\bullet}}} & & \\
0 & \longrightarrow & Z_{n-1}(C_\bullet) & \xrightarrow[\Lambda_n(\varphi)]{} & Z_{n-1}(D_\bullet) & \xrightarrow[\Lambda_n(\psi)]{} & Z_{n-1}(E_\bullet) & &
\end{array}$$

In view of (7.15), this diagram is commutative. Let us prove that the rows are exact.

1) $\Gamma_n(\psi)$ **is epi**. By (7.11) we have

$$\Gamma_n(\psi) \circ p_{B_n(D_\bullet)} = p_{B_n(E_\bullet)} \circ \psi_n.$$

Since ψ_n and $p_{B_n(D_\bullet)}$ are epi, so is $\Gamma_n(\psi)$.

2) $\Lambda_n(\varphi)$ **is mono**. By (7.10) we have

$$i_{Z_{n-1}(D_\bullet)} \circ \Lambda_n(\varphi_\bullet) = \varphi_{n-1} \circ i_{Z_{n-1}(C_\bullet)}$$

Since φ_{n-1} and $i_{Z_{n-1}(C_\bullet)}$ are mono, so is $\Lambda_n(\varphi)$.

3) $\mathrm{Im}(\Gamma_n(\varphi)) \subseteq \mathrm{Ker}(\Gamma_n(\psi))$. We have

$$\Gamma_n(\psi_\bullet) \circ \Gamma_n(\varphi_\bullet) \circ p_{B_n(C_\bullet)} \overset{(7.17)}{=} \Gamma_n\left((\psi \circ \varphi)_\bullet\right) \circ p_{B_n(C_\bullet)} \overset{(7.11)}{=} p_{B_n(E_\bullet)} \circ (\psi \circ \varphi)_n = \\ = p_{B_n(E_\bullet)} \circ \psi_n \circ \varphi_n = 0.$$

Since $p_{B_n(C_\bullet)}$ is epiwe get that $\Gamma_n(\psi) \circ \Gamma_n(\varphi) = 0$.

4) $\mathrm{Ker}(\Gamma_n(\psi)) \subseteq \mathrm{Im}(\Gamma_n(\varphi))$.

Let $x_n + B_n(D_\bullet) \in \mathrm{Ker}(\Gamma_n(\psi))$, then

$$0 = \Gamma_n(\psi)(x_n + B_n(D_\bullet)) \overset{(7.11)}{=} \left(\Gamma_n(\psi) \circ p_{B_n(D_\bullet)}\right)(x_n) = \left(p_{B_n(E_\bullet)} \circ \psi_n\right)(x_n) = \psi_n(x_n) + B_n(E_\bullet),$$

i.e. $\psi_n(x_n) \in B_n(E_\bullet) = \mathrm{Im}\left(d_{n+1}^{E_\bullet}\right)$. Thus there exists $e_{n+1} \in E_{n+1}$ such that

$$\psi_n(x_n) = d_{n+1}^{E_\bullet}(e_{n+1}).$$

Since ψ_{n+1} **is epi**, there exists $y_{n+1} \in D_{n+1}$ such that $\psi_{n+1}(y_{n+1}) = e_{n+1}$; we have

$$\psi_n(x_n) = d_{n+1}^{E_\bullet}(e_{n+1}) = d_{n+1}^{E_\bullet}(\psi_{n+1}(y_{n+1})) = \psi_n\left(d_{n+1}^{D_\bullet}(y_{n+1})\right),$$

i.e. $x_n - d_{n+1}^{D_\bullet}(y_{n+1}) \in \mathrm{Ker}(\psi_n) \subseteq \mathrm{Im}(\varphi_n)$. Hence there exists $c_n \in C_n$ such that $\varphi_n(c_n) = x_n - d_{n+1}^{D_\bullet}(y_{n+1})$ so that

$$\mathrm{Im}(\Gamma_n(\varphi)) \ni \Gamma_n(\varphi)(c_n) = \varphi_n(c_n) + B_n(D_\bullet) = x_n - d_{n+1}^{D_\bullet}(y_{n+1}) + B_n(D_\bullet) = x_n + B_n(D_\bullet).$$

5) $\mathrm{Im}(\Lambda_n(\varphi)) \subseteq \mathrm{Ker}(\Lambda_n(\psi))$ .We have

$$i_{Z_{n-1}(E_\bullet)} \circ \Lambda_n(\psi) \circ \Lambda_n(\varphi) \overset{(7.16)}{=} i_{Z_{n-1}(E_\bullet)} \circ \Lambda_n(\psi \circ \varphi) \overset{(7.10)}{=} (\psi_{n-1} \circ \varphi_{n-1}) \circ i_{Z_{n-1}(C_\bullet)} = 0$$

Since $i_{Z_{n-1}(E_\bullet)}$ is mono, we deduce that $\Lambda_n(\psi) \circ \Lambda_n(\varphi) = 0$.

6) $\mathrm{Ker}(\Lambda_n(\psi)) \subseteq \mathrm{Im}(\Lambda_n(\varphi))$. Let $x_{n-1} \in \mathrm{Ker}(\Lambda_n(\psi))$, then

$$0 = \left(i_{Z_{n-1}(E_\bullet)} \circ \Lambda_n(\psi)\right)(x_{n-1}) \overset{(7.10)}{=} \left(\psi_{n-1} \circ i_{Z_{n-1}(D_\bullet)}\right)(x_{n-1}),$$

i.e. $i_{Z_{n-1}(D_\bullet)}(x_{n-1}) \in \mathrm{Ker}(\psi_{n-1}) = \mathrm{Im}(\varphi_{n-1})$. Then there exists $c_{n-1} \in C_{n-1}$ such that $i_{Z_{n-1}(D_\bullet)}(x_{n-1}) = \varphi_{n-1}(c_{n-1})$. Now we have prove that $c_{n-1} \in Z_{n-1}(C_\bullet)$. We have

$$\varphi_{n-2}\left(d_{n-1}^{C_\bullet}(c_{n-1})\right) = d_{n-1}^{D_\bullet}(\varphi_{n-1}(c_{n-1})) = d_{n-1}^{D_\bullet}(x_{n-1}) \overset{x_{n-1} \in Z_{n-1}(D_\bullet)}{=} 0$$

As φ_{n-2} is mono, we deduce that $d_{n-1}^{C_\bullet}(c_{n-1}) = 0$ so that $c_{n-1} \in Z_{n-1}(C_\bullet)$. Hence we can write

$$i_{Z_{n-1}(D_\bullet)}(x_{n-1}) = \varphi_{n-1}(c_{n-1}) = \varphi_{n-1}\left(i_{Z_{n-1}(C_\bullet)}(c_{n-1})\right) \overset{(7.10)}{=} \left(i_{Z_{n-1}(D_\bullet)} \circ \Lambda_n(\varphi)\right)(c_{n-1}) =$$
$$= i_{Z_{n-1}(D_\bullet)}(\Lambda_n(\varphi)(c_{n-1})).$$

Since $i_{Z_{n-1}(D_\bullet)}$ is mono, we deduce that

$$x_{n-1} = \Lambda_n(\varphi)(c_{n-1}) \in \Lambda_n(\varphi_\bullet).$$

Since the diagram is commutative and exact, it satisfies conditions of Snake Lemma 7.23. Now recall that, by Lemma 7.27, we have

$$\mathrm{Ker}\left(\widehat{d_n^{C_\bullet}}\right) = H_n(C_\bullet) \text{ and } \mathrm{Coker}\left(\widehat{d_n^{C_\bullet}}\right) = H_{n-1}(C_\bullet).$$

Recall also that, by formula (7.12) we have that

$$j_{H_n(D_\bullet)} \circ H_n(\varphi) = \Gamma_n(\varphi) \circ j_{H_n}(C_\bullet)$$

and by formula (7.14) we have that

$$H_{n-1}(\varphi_\bullet) \circ q_{B_{n-1}(C_\bullet)} = q_{B_{n-1}(D_\bullet)} \circ \Lambda_n(\varphi_\bullet).$$

Hence, in view of the uniqueness of the homomorphisms involved in the statement of Snake Lemma 7.23, we have the following commutative and exact diagram.

$$\begin{array}{ccccccccc}
 & & H_n(C_\bullet) & \xrightarrow{H_n(\varphi)} & H_n(D_\bullet) & \xrightarrow{H_n(\psi)} & H_n(E_\bullet) & & \\
 & & \downarrow{\scriptstyle j_{H_n(C_\bullet)}} & & \downarrow{\scriptstyle j_{H_n(D_\bullet)}} & & \downarrow{\scriptstyle j_{H_n(E_\bullet)}} & & \\
 & & \frac{C_n}{B_n(C_\bullet)} & \xrightarrow{\Gamma_n(\varphi)} & \frac{D_n}{B_n(D_\bullet)} & \xrightarrow{\Gamma_n(\psi)} & \frac{E_n}{B_n(E_\bullet)} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \widehat{d_n^{C_\bullet}}} & & \downarrow{\scriptstyle \widehat{d_n^{D_\bullet}}} & & \downarrow{\scriptstyle \widehat{d_n^{E_\bullet}}} & & \\
0 & \longrightarrow & Z_{n-1}(C_\bullet) & \xrightarrow{\Lambda_n(\varphi)} & Z_{n-1}(D_\bullet) & \xrightarrow{\Lambda_n(\psi)} & Z_{n-1}(E_\bullet) & & \\
 & & \downarrow{\scriptstyle q_{H_{n-1}(C_\bullet)}} & & \downarrow{\scriptstyle q_{H_{n-1}(D_\bullet)}} & & \downarrow{\scriptstyle q_{H_{n-1}(E_\bullet)}} & & \\
 & & H_{n-1}(C_\bullet) & \xrightarrow{H_{n-1}(\varphi)} & H_{n-1}(D_\bullet) & \xrightarrow{H_{n-1}(\psi)} & H_{n-1}(E_\bullet) & &
\end{array}$$

Moreover there exists an homomorphism $\omega_n : \operatorname{Ker}\left(\widehat{d_n^{E_\bullet}}\right) = H_n(E_\bullet) \to \operatorname{Coker}\left(\widehat{d_n^{C_\bullet}}\right) = H_{n-1}(C_\bullet)$ such that the sequence

$$\ldots . \to H_n(C_\bullet) \overset{H_n(\varphi)}{\longrightarrow} H_n(D_\bullet) \overset{H_n(\psi)}{\longrightarrow} H_n(E_\bullet) \overset{\omega_n}{\longrightarrow} H_{n-1}(C_\bullet) \overset{H_{n-1}(\varphi)}{\longrightarrow} H_{n-1}(D_\bullet) \overset{H_{n-1}(\psi)}{\longrightarrow} H_{n-1}(E_\bullet) \ldots$$

is exact. □

Remark 7.33. *Note that*

$$\omega_n(e_n + B_n(E_\bullet)) \overset{def}{=} c_{n-1} + \operatorname{Im}\left(\widehat{d_n^{C_\bullet}}\right) = c_{n-1} + B_{n-1}(C_\bullet).$$

where $e_n + B_n(E_\bullet) = \Gamma_n(\psi)(x_n + B_n(D_\bullet))$ *and* $\widehat{d_n^{D_\bullet}}(x_n + B_n(D_\bullet)) = \Lambda_n(\varphi)(c_{n-1})$.

and hence

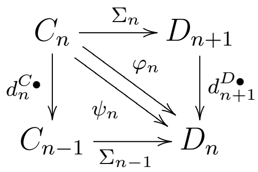
$$\omega_n(e_n + B_n(E_\bullet)) = c_{n-1} + B_{n-1}(C_\bullet).$$

where $e_n = \psi_n(x_n)$ *and* $d_n^{D_\bullet}(x_n) = \varphi_n(c_{n-1})$ *with* $c_{n-1} \in Z_n(C_\bullet)$.

7.4 Homotopies

Definition 7.34. *Let* $\varphi_\bullet, \psi_\bullet : \left(C_\bullet, d_\bullet^{C_\bullet}\right) \longrightarrow \left(D_\bullet, d_\bullet^{D_\bullet}\right)$ *be morphisms of complexes.*
A homotopy Σ *between* φ *and* ψ *consists of a family of homomorphisms* $(\Sigma_n : C_n \longrightarrow D_{n+1})_{n\in\mathbb{Z}}$ *such that*

$$\varphi_n - \psi_n = d_{n+1}^{D_\bullet} \circ \Sigma_n + \Sigma_{n-1} \circ d_n^{C_\bullet}.$$



If there is a homotopy between $\varphi_\bullet$ and $\psi_\bullet$ we say that $\varphi_\bullet$ is homotopic to *$\psi_\bullet$ and we write $\varphi_\bullet \simeq \psi_\bullet$.*

Theorem 7.35. *If $\varphi_\bullet, \psi_\bullet : C_\bullet \longrightarrow D_\bullet$ are homotopic, then $H_n(\varphi_\bullet) = H_n(\psi_\bullet)$.*

Proof. Let $\Sigma : \varphi \longrightarrow \psi$ be the homotopy between φ and ψ. Then, for every $n \in \mathbb{Z}$, we compute:

$$\begin{aligned}
j_{H_n(D_\bullet)} \circ H_n(\varphi) \circ q_{B_n(C_\bullet)} &\overset{(7.13)}{=} p_{B_n(D_\bullet)} \circ \varphi_n \circ i_{Z_n(C_\bullet)} \\
&= p_{B_n(D_\bullet)} \circ \left(\psi_n + d^{D_\bullet}_{n+1} \circ \Sigma_n + \Sigma_{n-1} \circ d^{C_\bullet}_n\right) \circ i_{Z_n(C_\bullet)} = \\
= \left(p_{B_n(D_\bullet)} \circ \psi_n + p_{B_n(D_\bullet)} \circ d^{D_\bullet}_{n+1} \circ \Sigma_n + p_{B_n(D_\bullet)} \circ \Sigma_{n-1} \circ d^{C_\bullet}_n\right) \circ i_{Z_n(C_\bullet)} &= \\
\overset{p_{B_n(D_\bullet)} \circ d^{D_\bullet}_{n+1} = 0}{=} p_{B_n(D_\bullet)} \circ \psi_n \circ i_{Z_n(C_\bullet)} + p_{B_n(D_\bullet)} \circ \Sigma_{n-1} \circ d^{C_\bullet}_n \circ i_{Z_n(C_\bullet)} &\overset{d^{C_\bullet}_n \circ i_{Z_n(C_\bullet)} = 0}{=} \\
= p_{B_n(D_\bullet)} \circ \psi_n \circ i_{Z_n(C_\bullet)} &\overset{(7.13)}{=} j_{H_n(D_\bullet)} \circ H_n(\psi) \circ q_{B_n(C_\bullet)}
\end{aligned}$$

Since $j_{H_n(D_\bullet)}$ is mono and $q_{B_n(C_\bullet)}$ is epi, we get $H_n(\varphi_\bullet) = H_n(\psi_\bullet)$. □

Proposition 7.36. *The homotopy relation $\simeq$ is an equivalence relation.*

Proof. Clearly the relation is reflexive (with $\Sigma_n = 0$) and symmetric (with $\Sigma'_n = -\Sigma_n$). Now we prove that it is also transitive: let $\varphi \overset{\Sigma}{\longrightarrow} \psi \overset{\Theta}{\longrightarrow} \chi$ be two homotopies. Then $\varphi_n - \psi_n = d^{D_\bullet}_{n+1} \circ \Sigma_n + \Sigma_{n-1} \circ d^{C_\bullet}_n$ and $\psi_n - \chi_n = d^{D_\bullet}_{n+1} \circ \Theta_n + \Theta_{n-1} \circ d^{C_\bullet}_n$. Then we have

$$\varphi_n - \chi_n = (\varphi_n - \psi_n) + (\psi_n - \chi_n) = d^{D_\bullet}_{n+1} \circ (\Sigma_n + \Theta_n) + (\Sigma_{n-1} + \Theta_{n-1}) \circ d^{C_\bullet}_n.$$

Thus $\Sigma + \Theta$ is a homotopy between φ and χ where $(\Sigma + \Theta)_n = \Sigma_n + \Theta_n$. □

Lemma 7.37. *Let $C_\bullet \overset{\varphi,\psi}{\longrightarrow} D_\bullet \overset{\varphi',\psi'}{\longrightarrow} E_\bullet$ be morphisms of complexes.*

1) If $\varphi \simeq \psi$ then $\varphi' \circ \varphi \simeq \varphi' \circ \psi$.

2) If $\varphi' \simeq \psi'$ then $\varphi' \circ \psi \simeq \psi' \circ \psi$.

3) If $\varphi \simeq \psi$ and $\varphi' \simeq \psi'$ then $\varphi' \circ \varphi \simeq \psi' \circ \psi$.

Proof. 1) Let us denote by Σ the homotopy between φ and ψ. Then we have

$$\begin{aligned}
\varphi'_n \circ \varphi_n - \varphi'_n \circ \psi_n &= \varphi'_n \circ (\varphi_n - \psi_n) \\
&= \varphi'_n \circ d^{D_\bullet}_{n+1} \circ \Sigma_n + \varphi'_n \circ \Sigma_{n-1} \circ d^{C_\bullet}_n \\
&= d^{E_\bullet}_{n+1} \circ \varphi'_{n+1} \circ \Sigma_n + \varphi'_n \circ \Sigma_{n-1} \circ d^{C_\bullet}_n
\end{aligned}$$

since $\varphi'_\bullet$ is a morphism of complexes; then $\varphi'_\bullet \circ \Sigma$ determines a homotopy between $(\varphi' \circ \varphi)_\bullet$ and $(\varphi' \circ \psi)_\bullet$, where $(\varphi' \circ \Sigma)_n = \varphi'_{n+1} \circ \Sigma_n$.

2) Let Θ be the homotopy between φ' and ψ'. Then we have

$$\begin{aligned} \varphi'_n \circ \psi_n - \psi'_n \circ \psi_n &= (\varphi'_n - \psi'_n) \circ \psi_n \\ &= d^{E_\bullet}_{n+1} \circ \Theta_n \circ \psi_n + \Theta_{n-1} \circ d^{D_\bullet}_n \circ \psi_n \\ &= d^{E_\bullet}_{n+1} \circ \Theta_n \circ \psi_n + \Theta_{n-1} \circ \psi_{n-1} \circ d^{C_\bullet}_n \end{aligned}$$

since ψ is a morphism of complexes. Thus $\Theta \circ \psi$ determines a homotopy between $\varphi' \circ \psi$ and $\psi' \circ \psi$, where $(\Theta \circ \psi)_n = \Theta_n \circ \psi_n$.

3) If $\varphi \simeq \psi$ and $\varphi' \simeq \psi'$, by 1) and 2) we have $\varphi' \circ \varphi \simeq \varphi' \circ \psi$ and $\varphi' \circ \psi \simeq \psi' \circ \psi$. Since the homotopy relation is an equivalence relation, by transitivity we get $\varphi' \circ \varphi \simeq \psi' \circ \psi$. □

Definition 7.38. *Let A and B be rings. Any functor $F : Mod\text{-}A \longrightarrow Mod\text{-}B$ is called* additive *if it satisfies*

$$F(f+g) = F(f) + F(g)$$

for every $f, g : M \to M'$.

Exercise 7.39. *Let $F : Mod\text{-}A \longrightarrow Mod\text{-}B$ be an additive functor and let $0_{M,M'} : M \to M'$ the zero homomorphism. Show that $F(0_{M,M'}) = 0_{F(M),F(M')}$ if F is covariant while $F(0_{M,M'}) = 0_{F(M'),F(M)}$ if F is contravariant.*

Exercise 7.40. *Prove that all examples in 7.15 are additive.*

Lemma 7.41. *Let $F : Mod\text{-}A \longrightarrow Mod\text{-}B$ be an additive covariant functor and let $\left(C_\bullet, d^{C_\bullet}_\bullet\right)$ be a chain complex in $Mod\text{-}A$. For every $n \in \mathbb{Z}$, set*

$$(F(C_\bullet))_n = F(C_n) \quad \text{and} \quad d^{F(C_\bullet)}_n = F\left(d^{C_\bullet}_n\right) \quad \text{for every } n \in \mathbb{Z}.$$

Then $\left(F(C_\bullet), d^{F(C_\bullet)}_\bullet\right)$ is a chain complex in $Mod\text{-}B$. Moreover if $\varphi_\bullet : \left(C_\bullet, d^{C_\bullet}_\bullet\right) \to \left(D_\bullet, d^{D_\bullet}_\bullet\right)$ is a morphism of chain complexes in $Mod\text{-}A$, for every $n \in \mathbb{Z}$, set

$$F(\varphi_\bullet)_n = F(\varphi_n).$$

Then $F(\varphi_\bullet) : \left(F(C_\bullet), d^{F(C_\bullet)}_\bullet\right) \to \left(F(D_\bullet), d^{F(D_\bullet)}_\bullet\right)$ is a morphism of chain complexes.

Proof. For every $n \in \mathbb{Z}$, we have

$$F\left(d^{C_\bullet}_{n-1}\right) \circ F\left(d^{C_\bullet}_n\right) = F\left(d^{C_\bullet}_{n-1} \circ d^{C_\bullet}_n\right) = F(0) = 0$$

and also

$$\begin{aligned} d^{F(D_\bullet)}_{n+1} \circ F(\varphi_{n+1}) &= F\left(d^{D_\bullet}_{n+1}\right) \circ F(\varphi_{n+1}) = F\left(d^{D_\bullet}_{n+1} \circ \varphi_{n+1}\right) = F\left(\varphi_n \circ d^{C_\bullet}_{n+1}\right) = \\ &= F(\varphi_n) \circ F\left(d^{C_\bullet}_{n+1}\right) = F(\varphi_n) \circ d^{F(C_\bullet)}_{n+1}. \end{aligned}$$

□

Exercise 7.42. *In the notations of Lemma 7.41, assume that also* $\psi_\bullet : \left(D_\bullet, d_\bullet^{D_\bullet}\right) \to \left(E_\bullet, d_\bullet^{E_\bullet}\right)$ *is a morphism of chain complexes in Mod-A. Show that*

$$F\left(\psi_\bullet \circ \varphi_\bullet\right) = F\left(\psi_\bullet\right) \circ F\left(\varphi_\bullet\right). \tag{7.19}$$

Lemma 7.43. *Let* $F : Mod\text{-}A \longrightarrow Mod\text{-}B$ *be an additive covariant functor and let* $\varphi_\bullet \simeq \psi_\bullet$ *be homotopic chain complex morphisms. Then* $F\left(\varphi_\bullet\right) \simeq F\left(\psi_\bullet\right)$. *In particular* $H_n\left(F\left(\varphi_\bullet\right)\right) = H_n\left(F\left(\psi_\bullet\right)\right)$.

Proof. Let $\varphi_\bullet, \psi_\bullet : C_\bullet \longrightarrow D_\bullet$ be the morphisms of chain complexes and let $\Sigma : \varphi_\bullet \longrightarrow \psi_\bullet$ be an homotopy between $\varphi_\bullet$ and $\psi_\bullet$. Thus $\varphi_n - \psi_n = d_{n+1}^{D_\bullet} \circ \Sigma_n + \Sigma_{n-1} \circ d_n^{C_\bullet}$. By applying F to this relation we get

$$\begin{aligned} F\left(\varphi_n\right) - F\left(\psi_n\right) &= F\left(d_{n+1}^{D_\bullet}\right) \circ F\left(\Sigma_n\right) + F\left(\Sigma_{n-1}\right) \circ F\left(d_n^{C_\bullet}\right) \\ &= d_{n+1}^{F(D_\bullet)} \circ F\left(\Sigma_n\right) + F\left(\Sigma_{n-1}\right) \circ d_n^{F(C_\bullet)}. \end{aligned}$$

Hence $F(\varphi) \simeq F(\psi)$ via the homotopy $F(\Sigma) : F(\varphi) \longrightarrow F(\psi)$ where $\left(F(\Sigma)\right)_n = F\left(\Sigma_n\right)$ for every $n \in \mathbb{Z}$.

The last assertion follows in view of Theorem 7.35. □

Example 7.44. *In general* $H_n\left(\varphi_\bullet\right) = H_n\left(\psi_\bullet\right)$ *does not imply* $\varphi_\bullet \simeq \psi_\bullet$. *For instance, consider two complexes* $C_\bullet$ *and* $D_\bullet$ *and the morphism* $\varphi_\bullet$ *between them:*

$$\begin{array}{ccccccccc} \dots \longrightarrow & 0 & \longrightarrow & C_1 = \mathbb{Z} & \xrightarrow{2\bullet} & C_0 = \mathbb{Z} & \longrightarrow & 0 & \longrightarrow \dots \\ & \downarrow & & \downarrow {\scriptstyle \varphi_1 = Id_{\mathbb{Z}}} & & \downarrow & & \downarrow & \\ \dots \longrightarrow & 0 & \longrightarrow & D_1 = \mathbb{Z} & \longrightarrow & D_0 = 0 & \longrightarrow & 0 & \longrightarrow \dots . \end{array}$$

Since all the compositions $\varphi_{n-1} \circ d_n^{C_\bullet}$ *and* $d_n^{D_\bullet} \circ \varphi_n$ *are zero,* $\varphi_\bullet$ *is a morphism of complexes. We have* $H_n\left(D_\bullet\right) = 0$ *for every* $n \neq 1$ *and* $H_1\left(C_\bullet\right) = 0$, *thus* $H_n(\varphi) = 0$ *for every* n, *that is* $H_n\left(\varphi_\bullet\right) = H_n(0)$, *but* $\varphi_\bullet \not\simeq 0$. *In fact assume* $\varphi_\bullet \simeq 0$. *Then, for any additive functor* F, *we get* $F\left(\varphi_\bullet\right) \simeq F(0) = 0$. *Let* F *be the functor* $- \otimes_{\mathbb{Z}} \frac{\mathbb{Z}}{2\mathbb{Z}}$. *By applying* F *and considering that* $\mathbb{Z} \otimes \frac{\mathbb{Z}}{2\mathbb{Z}} \cong \frac{\mathbb{Z}}{2\mathbb{Z}}$, *the diagram becomes*

$$\begin{array}{ccccccccc} \dots \longrightarrow & 0 & \longrightarrow & F\left(C_1\right) = \frac{\mathbb{Z}}{2\mathbb{Z}} & \xrightarrow{0} & F\left(C_0\right) = \frac{\mathbb{Z}}{2\mathbb{Z}} & \longrightarrow & 0 & \longrightarrow \dots . \\ & \downarrow & & \downarrow {\scriptstyle F(\varphi_1) = Id_{\frac{\mathbb{Z}}{2\mathbb{Z}}}} & & \downarrow & & \downarrow & \\ \dots \longrightarrow & 0 & \longrightarrow & F\left(D_1\right) = \frac{\mathbb{Z}}{2\mathbb{Z}} & \longrightarrow & F\left(D_0\right) = 0 & \longrightarrow & 0 & \longrightarrow \dots \end{array}$$

In particular $H_1\left(F\left(C_\bullet\right)\right) = \frac{\mathbb{Z}}{2\mathbb{Z}} = H_1\left(F\left(D_\bullet\right)\right)$ *and* $H_1\left(F(\varphi)\right) = \mathrm{Id}_{\frac{\mathbb{Z}}{2\mathbb{Z}}}$, *from which we deduce* $F\left(\varphi_\bullet\right) \not\simeq 0$ *and thus* $\varphi_\bullet \not\simeq 0$.

7.5 Projective resolutions

Definitions 7.45. *A* chain complex $\left(C_\bullet, d_\bullet^{C_\bullet}\right)$ *is called*

- positive *if* $C_n = 0$ *for every* $n \leq -1$,
- acyclic positive *if if* $C_n = 0$ *for every* $n \leq -1$ *and* $H_n(C_\bullet) = 0$ *i.e.* $\mathrm{Im}\left(d_{n+1}^{C_\bullet}\right) = \mathrm{Ker}\left(d_n^{C_\bullet}\right)$ *for every* $n \geq 1$,
- projective *if* C_n *is projective for every* $n \in \mathbb{Z}$.

Remark 7.46. *An acyclic positive chain complex is a chain complex of the form*

$$\ldots \longrightarrow C_2 \xrightarrow{d_2^{C_\bullet}} C_1 \xrightarrow{d_1^{C_\bullet}} C_0 \longrightarrow 0,$$

with $Z_n(C_\bullet) = B_n(C_\bullet)$ *for every* $n \geq 1$. *The sequence is not exact since* $d_1^{C_\bullet}$ *is not epi, but we can consider the following exact sequence*

$$\ldots \longrightarrow C_2 \xrightarrow{d_2^{C_\bullet}} C_1 \xrightarrow{d_1^{C_\bullet}} C_0 \xrightarrow{\pi} \frac{C_0}{B_0(C_\bullet)} = H_0(C_\bullet) \longrightarrow 0.$$

Definition 7.47. *Let* M *be a right* A*-module and let* $\left(C_\bullet, d_\bullet^{C_\bullet}\right)$ *be an acyclic positive projective chain complex with* $\frac{C_o}{B_0(C_\bullet)} \cong M$. *Then* $\left(C_\bullet, d_\bullet^{C_\bullet}\right)$ *is called a* projective resolution *of* M *and we have*

$$\ldots \longrightarrow C_2 \xrightarrow{d_2^{C_\bullet}} C_1 \xrightarrow{d_1^{C_\bullet}} C_0 \xrightarrow{\pi} M \longrightarrow 0.$$

Lemma 7.48. *Every module is epimorphic image of a projective module.*

Proof. It follow by Proposition 2.2 and Proposition 2.16. □

Proposition 7.49. *Every right* A*-module admits a projective resolution.*

Proof. Let M be a right A-module. By Lemma 7.48, every module is an epimorphic image of a projective module, i.e. there exists an epimorphism $\varphi_0 : P_0 \longrightarrow M$ with P_0 projective. We construct the complex recursively. Let us consider $\mathrm{Ker}(\varphi_0)$ and let $i_0 : \mathrm{Ker}(\varphi_0) \to P_0$ be the canonical inclusion. By Lemma 7.48 there is a projective module P_1 and an epimorphism $\varphi_1 : P_1 \to \mathrm{Ker}(\varphi_0)$. Let us set

$$d_1^{P_\bullet} = i_0 \circ \varphi_1.$$

Then

$$\mathrm{Im}\left(d_1^{P_\bullet}\right) = \mathrm{Ker}(\varphi_0).$$

Let us consider $\mathrm{Ker}\left(d_1^{P_\bullet}\right)$ and let $i_1 : \mathrm{Ker}\left(d_1^{P_\bullet}\right) \to P_1$ be the canonical inclusion. By Lemma 7.48 there is a projective module P_2 and an epimorphism $\varphi_2 : P_2 \to \mathrm{Ker}\left(d_1^{P_\bullet}\right)$. Let us set

$$d_2^{P_\bullet} = i_1 \circ \varphi_2.$$

Then

$$\mathrm{Im}\left(d_2^{P_\bullet}\right) = \mathrm{Ker}\left(d_1^{P_\bullet}\right).$$

Assume that, for some $n \in \mathbb{N}$, $n \geq 2$ we have $P_0, \ldots, P_n$ projective modules and $d_1^{P_\bullet}, \ldots, d_n^{P_\bullet}$ such that

$$\operatorname{Im}\left(d_i^{P_\bullet}\right) = \operatorname{Ker}\left(d_{i-1}^{P_\bullet}\right) \text{ for every } i = 2, \ldots, n$$

By Lemma 7.48, there exists a projective module P_{n+1} and an epimorphism $\varphi_{n+1} : P_{n+1} \longrightarrow \operatorname{Ker}\left(d_n^{P_\bullet}\right)$. Set

$$d_{n+1}^{P_\bullet} = \varphi_{n+1} \circ i_n$$

where i_n is the canonical inclusion of $\operatorname{Ker}\left(d_n^{P_\bullet}\right)$ in P_n. Then

$$\operatorname{Im}\left(d_{n+1}^{P_\bullet}\right) = \operatorname{Ker}\left(d_n^{P_\bullet}\right).$$

$$\begin{array}{ccccc} P_{n+1} & \xrightarrow{\quad d_{n+1}^{P_\bullet} \quad} & P_n & \xrightarrow{d_n^{P_\bullet}} & P_{n-1} \\ & \searrow^{\varphi_{n+1}} \quad \nearrow_{i_n} & & & \\ & \operatorname{Ker}\left(d_n^{P_\bullet}\right) & & & \end{array}$$

Thus, in this way we construct an acyclic, positive and projective complex. Moreover $\frac{P_0}{B_0(P_\bullet)} = \frac{P_0}{\operatorname{Im}(\varphi_1)} = \frac{P_0}{\operatorname{Ker}(\varphi_0)} = M$. □

Theorem 7.50 (Lifting Theorem for Chain Complexes). *Let $\left(P_\bullet, d_\bullet^{P_\bullet}\right)$ be a positive projective chain complex, let $\left(D_\bullet, d_\bullet^{D_\bullet}\right)$ be an acyclic positive complex and let $\varphi : H_0\left(P_\bullet\right) \longrightarrow H_0\left(D_\bullet\right)$ be a morphism in Mod-A. Then there exists a morphism of chain complexes $\varphi_\bullet : \left(P_\bullet, d_\bullet^{P_\bullet}\right) \longrightarrow \left(D_\bullet, d_\bullet^{D_\bullet}\right)$ such that $H_0\left(\varphi_\bullet\right) = \varphi$. Moreover, if $\psi_\bullet : \left(P_\bullet, d_\bullet^{P_\bullet}\right) \longrightarrow \left(D_\bullet, d_\bullet^{D_\bullet}\right)$ also satisfies $H_0\left(\psi_\bullet\right) = \varphi$, we have $\varphi_\bullet \simeq \psi_\bullet$. In particular $H_n\left(\varphi_\bullet\right)$ only depends on φ.*

Proof. We have the following situation where $\pi_P = p_{B_0(P_\bullet)}$ and $\pi_D = p_{B_0(D_\bullet)}$

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & P_2 & \xrightarrow{d_2^{P_\bullet}} & P_1 & \xrightarrow{d_1^{P_\bullet}} & P_0 & \xrightarrow{\pi_P} & H_0\left(P_\bullet\right) & \longrightarrow & 0 \\ & & & & & & & & \downarrow{\scriptstyle \varphi} & & \\ \ldots & \longrightarrow & D_2 & \xrightarrow{d_2^{D_\bullet}} & D_1 & \xrightarrow{d_1^{D_\bullet}} & D_0 & \xrightarrow{\pi_D} & H_0\left(D_\bullet\right) & \longrightarrow & 0. \end{array}$$

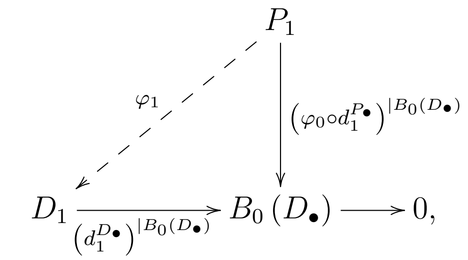
Existence of $\varphi_\bullet$. Since P_0 is projective, there exists $\varphi_0 : P_0 \longrightarrow D_0$ such that

$$\pi_D \circ \varphi_0 = \varphi \circ \pi_P. \tag{7.20}$$

Since $\operatorname{Im}\left(d_1^{P_\bullet}\right) = \operatorname{Ker}\left(\pi_P\right)$, by composing to the right with $d_1^{P_\bullet}$, we get

$$\pi_D \circ \varphi_0 \circ d_1^{P_\bullet} = \varphi \circ \pi_P \circ d_1^{P_\bullet} = 0$$

hence we have that $\mathrm{Im}\left(\varphi_0 \circ d_1^{P_\bullet}\right) \subseteq \mathrm{Ker}\left(\pi_D\right) = \mathrm{Im}\left(d_1^{D_\bullet}\right) = B_0\left(D_\bullet\right)$. Since P_1 is projective there is a morphism $\varphi_1 : P_1 \to D_1$



such that

$$\left(d_1^{D_\bullet}\right)^{|B_0(D_\bullet)} \circ \varphi_1 = \left(\varphi_0 \circ d_1^{P_\bullet}\right)^{|B_0(D_\bullet)}$$

so that

$$d_1^{D_\bullet} \circ \varphi_1 = i_{B_0(D_\bullet)} \circ \left(d_1^{D_\bullet}\right)^{|B_0(D_\bullet)} \circ \varphi_1 = i_{B_0(D_\bullet)} \circ \left(\varphi_0 \circ d_1^{P_\bullet}\right)^{|B_0(D_\bullet)} = \varphi_0 \circ d_1^{P_\bullet}.$$

Proceeding recursively we construct $\varphi_\bullet$ using the acyclicity of $D_\bullet$ which allows us to reiterate the process. Namely assume that for some $n \in \mathbb{N}, n \geq 1$

$$\varphi_n : P_n \to D_n$$

is constructed so that

$$d_n^{D_\bullet} \circ \varphi_n = \varphi_{n-1} \circ d_n^{P_\bullet}.$$

Then we have

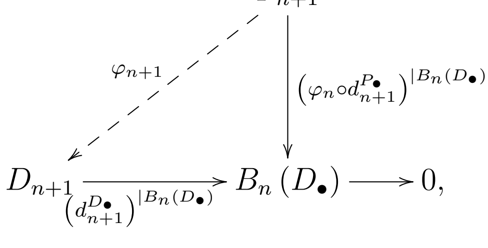
$$d_n^{D_\bullet} \circ \varphi_n \circ d_{n+1}^{P_\bullet} = \varphi_{n-1} \circ d_n^{P_\bullet} \circ d_{n+1}^{P_\bullet} = 0$$

so that $\mathrm{Im}\left(\varphi_n \circ d_{n+1}^{P_\bullet}\right) \subseteq \mathrm{Ker}\left(d_n^{D_\bullet}\right) = \mathrm{Im}\left(d_{n+1}^{D_\bullet}\right) = B_n\left(D_\bullet\right)$. Since P_{n+1} is projective, there exists a morphism $\varphi_{n+1} : P_{n+1} \to D_{n+1}$ such that

$$\left(d_{n+1}^{D_\bullet}\right)^{|B_n(D_\bullet)} \circ \varphi_{n+1} = \left(\varphi_n \circ d_{n+1}^{P_\bullet}\right)^{|B_n(D_\bullet)}$$

so that

$$d_{n+1}^{D_\bullet} \circ \varphi_{n+1} = i_{B_n(D_\bullet)} \circ \left(d_{n+1}^{D_\bullet}\right)^{|B_n(D_\bullet)} \circ \varphi_{n+1} = i_{B_n(D_\bullet)} \circ \left(\varphi_n \circ d_{n+1}^{P_\bullet}\right)^{|B_n(D_\bullet)} =$$
$$= \varphi_n \circ d_{n+1}^{P_\bullet}.$$



Now we prove that $H_0\left(\varphi_\bullet\right) = \varphi$. Note that, since $d_0^{D_\bullet} = 0$ and $d_0^{P_\bullet} = 0$ we have that $Z_0\left(D_\bullet\right) = \mathrm{Ker}\left(d_0^{D_\bullet}\right) = D_0$ and $Z_0\left(P_\bullet\right) = \mathrm{Ker}\left(d_0^{P_\bullet}\right) = P_0$. Thus $i_{Z_0(P_\bullet)} = \mathrm{Id}_{P_0}$

, $q_{B_0(P_\bullet)} = \pi_P : P_0 = Z_0(P_\bullet) \to Z_0(P_\bullet)/B_0(P_\bullet) = H_0(P_\bullet)$ and $j_{H_0(D_\bullet)} = \mathrm{Id}_{H_0(D_\bullet)} : H_0(D_\bullet) \to D_0/B_0(D_\bullet)$. Therefore we have

$$H_0(\varphi_\bullet) \circ \pi_P = j_{H_0(D_\bullet)} \circ H_0(\varphi_\bullet) \circ q_{B_0(P_\bullet)} \overset{(7.13)}{=} p_{B_0(D_\bullet)} \circ \varphi_0 \circ i_{Z_0(P_\bullet)} =$$
$$= \pi_D \circ \varphi_0 \circ i_{Z_0(P_\bullet)} \overset{(7.20)}{=} \varphi \circ \pi_P \circ i_{Z_0(P_\bullet)} = \varphi \circ \pi_P$$

so that

$$H_0(\varphi_\bullet) \circ \pi_P = \varphi \circ \pi_P$$

and since π_P is epi we get

$$H_0(\varphi_\bullet) = \varphi.$$

Uniqueness up to homotopies. Let $\psi_\bullet$ be another lifting of φ i.e. $\psi_\bullet : \left(P_\bullet, d_\bullet^{P_\bullet}\right) \longrightarrow \left(D_\bullet, d_\bullet^{D_\bullet}\right)$ is a chain complex morphism such that $H_0(\psi_\bullet) = \psi$. We look for a homotopy $\Sigma : \psi \longrightarrow \varphi$. Now for every $n \leq -1$ we have $P_n = 0$ and hence $\varphi_n = 0, \psi_n = 0$ and $\Sigma_n = 0$ for every $n \leq -1$. Thus $\Sigma_0 : P_0 \longrightarrow D_1$ must satisfy

$$\psi_0 - \varphi_0 = d_1^{D_\bullet} \circ \Sigma_0 + \Sigma_{-1} \circ d_0^{P_\bullet} = d_1^{D_\bullet} \circ \Sigma_0.$$

On the other hand we have

$$\varphi \circ \pi_P = H_0(\psi_\bullet) \circ \pi_P = j_{H_0(D_\bullet)} \circ H_0(\psi_\bullet) \circ q_{B_0(P_\bullet)} \overset{(7.13)}{=} p_{B_0(D_\bullet)} \circ \psi_0 \circ i_{Z_0(P_\bullet)} = \pi_D \circ \psi_0$$

so that we get

$$\varphi \circ \pi_P = \pi_D \circ \psi_0. \tag{7.21}$$

We compute

$$\pi_D \circ (\psi_0 - \varphi_0) = \pi_D \circ \psi_0 - \pi_D \circ \varphi_0 \overset{(7.21)(7.20)}{=} \varphi \circ \pi_P - \varphi \circ \pi_P = 0.$$

Thus we deduce that $\mathrm{Im}(\psi_0 - \varphi_0) \subseteq \mathrm{Ker}(\pi_D) = \mathrm{Im}\left(d_1^{D_\bullet}\right) = B_0(D_\bullet)$ and since P_0 is projective there exists

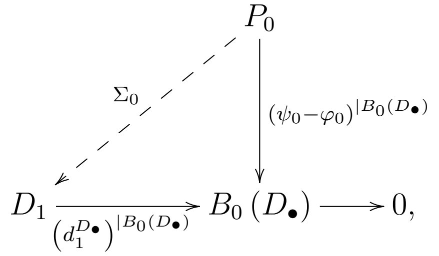
$$\Sigma_0 : P_0 \to D_1$$

such that

$$\left(d_1^{D_\bullet}\right)^{|B_0(D_\bullet)} \circ \Sigma_0 = (\psi_0 - \varphi_0)^{|B_0(D_\bullet)}$$

so that

$$d_1^{D_\bullet} \circ \Sigma_0 = i_{B_0(D_\bullet)} \circ \left(d_1^{D_\bullet}\right)^{|B_0(D_\bullet)} \circ \Sigma_0 = i_{B_0(D_\bullet)} \circ (\psi_0 - \varphi_0)^{|B_0(D_\bullet)} = \psi_0 - \varphi_0,$$



Recursively assume that, for some $n \in \mathbb{N}$, there exists $\Sigma_{n-1} : P_{n-1} \to D_n$ and $\Sigma_n : P_n \to D_{n+1}$such that

$$\psi_n - \varphi_n = d_{n+1}^{D_\bullet} \circ \Sigma_n + \Sigma_{n-1} \circ d_n^{P_\bullet}.$$

We look for a $\Sigma_{n+1} : P_{n+1} \to D_{n+2}$ such that

$$\psi_{n+1} - \varphi_{n+1} = d_{n+2}^{D_\bullet} \circ \Sigma_{n+1} + \Sigma_n \circ d_{n+1}^{P_\bullet}.$$

We have

$$\begin{aligned} d_{n+1}^{D_\bullet} \circ \left(\psi_{n+1} - \varphi_{n+1} - \Sigma_n \circ d_{n+1}^{P_\bullet}\right) &= d_{n+1}^{D_\bullet} \circ \psi_{n+1} - d_{n+1}^{D_\bullet} \circ \varphi_{n+1} - d_{n+1}^{D_\bullet} \circ \Sigma_n \circ d_{n+1}^{P_\bullet} = \\ &= \psi_n \circ d_{n+1}^{P_\bullet} - \varphi_n \circ d_{n+1}^{P_\bullet} - \left[\psi_n - \varphi_n - \Sigma_{n-1} \circ d_n^{P_\bullet}\right] \circ d_{n+1}^{P_\bullet} = \\ &= 0 \end{aligned}$$

Then we get

$$\operatorname{Im}\left(\psi_{n+1} - \varphi_{n+1} - \Sigma_n \circ d_{n+1}^{P_\bullet}\right) \subseteq \operatorname{Ker}\left(d_{n+1}^{D_\bullet}\right) = \operatorname{Im}\left(d_{n+2}^{D_\bullet}\right) = B_{n+1}(D_\bullet)$$

Thus, since P_{n+1} is projective, there exists $\Sigma_{n+1} : P_{n+1} \to D_{n+2}$ such that

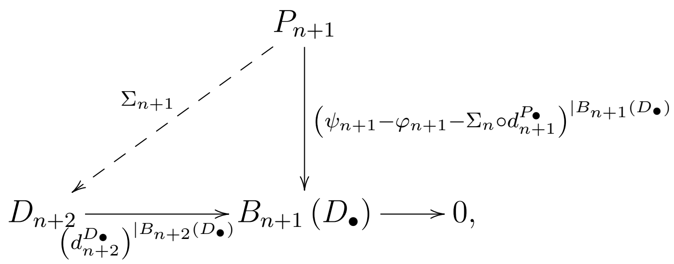
$$\left(d_{n+2}^{D_\bullet}\right)^{|B_{n+2}(D_\bullet)} \circ \Sigma_{n+1} = \left(\psi_{n+1} - \varphi_{n+1} - \Sigma_n \circ d_{n+1}^{P_\bullet}\right)^{|B_{n+1}(D_\bullet)}$$

so that

$$\begin{aligned} \left(d_{n+2}^{D_\bullet}\right) \circ \Sigma_{n+1} &= i_{B_{n+2}(D_\bullet)} \circ \left(d_{n+2}^{D_\bullet}\right)^{|B_{n+2}(D_\bullet)} \circ \Sigma_{n+1} = \\ &= i_{B_{n+2}(D_\bullet)} \circ \left(\psi_{n+1} - \varphi_{n+1} - \Sigma_n \circ d_{n+1}^{P_\bullet}\right)^{|B_{n+1}(D_\bullet)} = \psi_{n+1} - \varphi_{n+1} - \Sigma_n \circ d_{n+1}^{P_\bullet} \end{aligned}$$

i.e.

$$\psi_{n+1} - \varphi_{n+1} = \left(d_{n+2}^{D_\bullet}\right) \circ \Sigma_{n+1} + \Sigma_n \circ d_{n+1}^{P_\bullet}.$$



□

Definition 7.51. *In the notations and assumptions of Theorem 7.50, any morphism of chain complexes* $\varphi_\bullet : \left(P_\bullet, d_\bullet^{P_\bullet}\right) \longrightarrow \left(D_\bullet, d_\bullet^{D_\bullet}\right)$ *such that* $H_0(\varphi_\bullet) = \varphi$ *will be called a* lifting *of* φ.

Lemma 7.52. *Let* $M \xrightarrow{\varphi} M' \xrightarrow{\varphi'} M''$ *be morphisms in* Mod-A *and let* $\left(P_\bullet, d_\bullet^{P_\bullet}\right)$ *be a projective resolution of* M, $\left(P'_\bullet, d_\bullet^{P'_\bullet}\right)$ *a projective resolution of* M' *and* $\left(P''_\bullet, d_\bullet^{P''_\bullet}\right)$ *a projective resolution of* M''. *If* $\varphi_\bullet : \left(P_\bullet, d_\bullet^{P_\bullet}\right) \longrightarrow \left(P'_\bullet, d_\bullet^{P'_\bullet}\right)$ *is a lifting of* φ *and* $\varphi'_\bullet : \left(P'_\bullet, d_\bullet^{P'_\bullet}\right) \longrightarrow \left(P''_\bullet, d_\bullet^{P''_\bullet}\right)$ *is a lifting of* φ', *then*

$$\varphi'_\bullet \circ \varphi_\bullet : \left(P_\bullet, d_\bullet^{P_\bullet}\right) \longrightarrow \left(P''_\bullet, d_\bullet^{P''_\bullet}\right)$$

is a lifting of $\varphi' \circ \varphi$.

Proof. By Lemma 7.30 we know that $\varphi'_\bullet \circ \varphi_\bullet : \left(P_\bullet, d_\bullet^{P_\bullet}\right) \longrightarrow \left(P''_\bullet, d_\bullet^{P''_\bullet}\right)$ is a morphism of chain complexes. Moreover, for every $n \in \mathbb{Z}$, we have

$$H_n\left(\varphi'_\bullet \circ \varphi_\bullet\right) \overset{(7.18)}{=} H_n\left(\varphi'_\bullet\right) \circ H_n\left(\varphi_\bullet\right).$$

In particular, for $n = 0$, we get

$$H_0\left(\varphi'_\bullet \circ \varphi_\bullet\right) = H_0\left(\varphi'_\bullet\right) \circ H_0\left(\varphi_\bullet\right) = \varphi' \circ \varphi.$$

□

Theorem 7.53. *Let* $P_\bullet$ *and* $Q_\bullet$ *be projective resolution of a right* A*-module* M. *In view of Theorem 7.50, we can consider the liftings* $\varphi_\bullet : P_\bullet \longrightarrow Q_\bullet$ *and* $\psi_\bullet : Q_\bullet \longrightarrow P_\bullet$ *of* Id_M. *Then*

1) $\varphi_\bullet \circ \psi_\bullet \simeq \mathrm{Id}_{Q_\bullet}$ *and* $\psi_\bullet \circ \varphi_\bullet \simeq \mathrm{Id}_{P_\bullet}$.

2) $H_n\left(\varphi_\bullet\right) : H_n\left(P_\bullet\right) \to H_n\left(Q_\bullet\right)$ *is an isomorphism with inverse* $H_n\left(\psi_\bullet\right)$, *for every* $n \in \mathbb{N}$.

Proof. 1) In view of Lemma 7.52, $\varphi_\bullet \circ \psi_\bullet : Q_\bullet \longrightarrow Q_\bullet$ is a lifting of $\mathrm{Id}_M \circ \mathrm{Id}_M = \mathrm{Id}_M$. Since also $\mathrm{Id}_{Q_\bullet}$ is a lifting of Id_M, we deduce, in view of Theorem 7.50, that

$$\varphi_\bullet \circ \psi_\bullet \simeq \mathrm{Id}_{Q_\bullet}. \tag{7.22}$$

In a similar way we get also that

$$\psi_\bullet \circ \varphi_\bullet \simeq \mathrm{Id}_{P_\bullet}. \tag{7.23}$$

2) For every $n \in \mathbb{N}$, we have

$$H_n\left(\varphi_\bullet\right) \circ H_n\left(\psi_\bullet\right) \overset{(7.18)}{=} H_n\left(\varphi_\bullet \circ \psi_\bullet\right) \overset{(7.22)}{=} H_n\left(\mathrm{Id}_{Q_\bullet}\right) = \mathrm{Id}_{H_n(Q_\bullet)}$$

and

$$H_n\left(\psi_\bullet\right) \circ H_n\left(\varphi_\bullet\right) \overset{(7.18)}{=} H_n\left(\psi_\bullet \circ \varphi_\bullet\right) \overset{(7.23)}{=} H_n\left(\mathrm{Id}_{P_\bullet}\right) = \mathrm{Id}_{H_n(P_\bullet)}.$$

□

7.6 Left Derived functors

Remark 7.54. *Let A and R be rings, and let $T : Mod\text{-}A \to Mod\text{-}R$ be an additive covariant functor, e.g. $T = - \otimes {}_A L_R$ where ${}_A L_R$ is an A-R-bimodule. Let M be a right A-module and let $P_\bullet \longrightarrow M \longrightarrow 0$ be a projective resolution of M in $Mod\text{-}A$. By applying T we get, in view of By Lemma 7.41, a chain complex with $\left(T(P_\bullet), d_\bullet^{T(P_\bullet)}\right)$, which, in general, is no longer acyclic i.e. $H_n(T(P_\bullet))$ is not necessarily zero for every $n \geq 1$.*

Notations 7.55. *Let A and R be rings, and let $T : Mod\text{-}A \to Mod\text{-}R$ be an additive covariant functor. Let $n \in \mathbb{N}$. Let $M \in Mod\text{-}A$ and let $(P_\bullet, d_\bullet^{P_\bullet})$ be a projective resolution of M in $Mod\text{-}A$. We set*

$$\left(L^{P_\bullet}T\right)_n(M) = H_n(T(P_\bullet)).$$

Let $\varphi : M \to M'$ be a morphism in $Mod\text{-}A$ and let $\left(P'_\bullet, d_\bullet^{P'_\bullet}\right)$ be a projective resolution of M'. Let $\varphi_\bullet : (P_\bullet, d_\bullet^{P_\bullet}) \to \left(P'_\bullet, d_\bullet^{P'_\bullet}\right)$ be a lifting of φ (see Theorem 7.50). We set

$$\left(L^{P_\bullet P'_\bullet}T\right)_n(\varphi) = H_n(T(\varphi_\bullet))$$

Proposition 7.56. *In the assumptions and notations of 7.55, for every $n \in \mathbb{N}$, we have that*

1) $\left(L^{P_\bullet P'_\bullet}T\right)_n(\varphi)$ *is well-defined i.e. does not depend on the lifting $\varphi_\bullet$ of φ,*

2) *If $M \xrightarrow{\varphi} M' \xrightarrow{\varphi'} M''$ are morphisms in $Mod\text{-}A$ and $\left(P''_\bullet, d_\bullet^{P''_\bullet}\right)$ is a projective resolution of M'', then*

$$\left(L^{P_\bullet P''_\bullet}T\right)_n(\varphi' \circ \varphi) = \left[\left(L^{P'_\bullet P''_\bullet}T\right)_n(\varphi')\right] \circ \left[\left(L^{P_\bullet P'_\bullet}T\right)_n(\varphi)\right],$$

3) $\left(L^{P_\bullet P_\bullet}T\right)_n(\mathrm{Id}_M) = \mathrm{Id}_{L_n^{P_\bullet}T(M)}$

Proof. 1) Let $\psi_\bullet$ be another lifting of φ. Then, by Theorem 7.50 $\varphi_\bullet \simeq \psi_\bullet$. Then, by Lemma 7.43, $T(\varphi_\bullet) \simeq T(\psi_\bullet)$ and hence $H_n(T(\varphi_\bullet)) = H_n(T(\psi_\bullet))$.

2) Let $\varphi_\bullet : (P_\bullet, d_\bullet^{P_\bullet}) \to \left(P'_\bullet, d_\bullet^{P'_\bullet}\right)$ be a lifting of φ and let $\varphi'_\bullet : \left(P'_\bullet, d_\bullet^{P'_\bullet}\right) \to \left(P''_\bullet, d_\bullet^{P''_\bullet}\right)$ be a lifting of φ'. Thus we get

$$\left[\left(L^{P'_\bullet P''_\bullet}T\right)_n(\varphi')\right] \circ \left[\left(L^{P_\bullet P'_\bullet}T\right)_n(\varphi)\right] = H_n(T(\varphi'_\bullet)) \circ H_n(T(\psi'_\bullet)) =$$
$$\overset{(7.18)}{=} H_n(T(\varphi'_\bullet) \circ T(\varphi'_\bullet)) \overset{(7.19)}{=} H_n(T(\varphi'_\bullet \circ \varphi_\bullet)) =$$
$$\overset{\text{Lemma7.52}}{=} H_n(T((\varphi' \circ \varphi)_\bullet)) = \left(L^{P_\bullet P''_\bullet}T\right)_n(\varphi' \circ \varphi)$$

3) Since $\mathrm{Id}_{P_\bullet}$ is a lifting of Id_M, we have

$$\left(L^{P_\bullet P_\bullet}T\right)_n(\mathrm{Id}_M) = H_n\left(T\left(\mathrm{Id}_{P_\bullet}\right)\right) = H_n\left(\mathrm{Id}_{T(P_\bullet)}\right).$$

Since $H_n\left(\mathrm{Id}_{T(P_\bullet)}\right) = \mathrm{Id}_{H_n(T(P_\bullet))}$ (exercise) we obtain that $\left(L^{P_\bullet P_\bullet}T\right)_n(\mathrm{Id}_M) = \mathrm{Id}_{H_n(T(P_\bullet))} = \mathrm{Id}_{(L^{P_\bullet}T)_n(M)}$. □

Lemma 7.57. *Let A and R be rings, and let $T : Mod\text{-}A \to Mod\text{-}R$ be an additive covariant functor. Let $\left(P_\bullet, d_\bullet^{P_\bullet}\right)$ and $\left(Q_\bullet, d_\bullet^{Q_\bullet}\right)$ be projective resolutions of M in $Mod\text{-}A$. Let $\alpha_{P_\bullet Q_\bullet} : \left(P_\bullet, d_\bullet^{P_\bullet}\right) \to \left(Q_\bullet, d_\bullet^{Q_\bullet}\right)$ be a lifting of Id_M and let $\alpha_{Q_\bullet P_\bullet} : \left(Q_\bullet, d_\bullet^{Q_\bullet}\right) \to \left(P_\bullet, d_\bullet^{P_\bullet}\right)$ be a lifting of Id_M (see Theorem 7.50). Then*

$$H_n\left(T\left(\alpha_{P_\bullet Q_\bullet}\right)\right) = \left(L^{P_\bullet Q_\bullet}T\right)_n(\mathrm{Id}_M) \text{ and } H_n\left(T\left(\alpha_{Q_\bullet P_\bullet}\right)\right) = \left(L^{Q_\bullet P_\bullet}T\right)_n(\mathrm{Id}_M)$$

are mutual inverse and hence they determine an isomorphism between $H_n(T(P_\bullet)) = \left(L^{P_\bullet}T\right)_n(M)$ and $H_n(T(Q_\bullet)) = \left(L^{Q_\bullet}T\right)_n(M)$.

Proof. By Theorem 7.53, we have that $\alpha_{Q_\bullet P_\bullet} \circ \alpha_{P_\bullet Q_\bullet} \simeq \mathrm{Id}_{P_\bullet}$ and thus

$$T\left(\alpha_{Q_\bullet P_\bullet}\right) \circ T\left(\alpha_{P_\bullet Q_\bullet}\right) \overset{\text{Exercise7.42}}{=} T\left(\alpha_{Q_\bullet P_\bullet} \circ \alpha_{P_\bullet Q_\bullet}\right) \overset{\text{Lemma7.43}}{\simeq} T\left(\mathrm{Id}_{P_\bullet}\right) = \mathrm{Id}_{T(P_\bullet)}$$

Then we get

$$\begin{aligned}
\mathrm{Id}_{H_n(T(P_\bullet))} &= H_n\left(\mathrm{Id}_{T(P_\bullet)}\right) = H_n\left(T\left(\alpha_{Q_\bullet P_\bullet}\right) \circ T\left(\alpha_{P_\bullet Q_\bullet}\right)\right) \\
&\overset{(7.18)}{=} H_n\left(T\left(\alpha_{Q_\bullet P_\bullet}\right)\right) \circ H_n\left(T\left(\alpha_{P_\bullet Q_\bullet}\right)\right) \\
&= \left(L^{Q_\bullet P_\bullet}T\right)_n(\mathrm{Id}_M) \circ \left(L^{P_\bullet Q_\bullet}T\right)_n(\mathrm{Id}_M).
\end{aligned}$$

Similarly we also have

$$T\left(\alpha_{P_\bullet Q_\bullet}\right) \circ T\left(\alpha_{Q_\bullet P_\bullet}\right) = \mathrm{Id}_{T(Q_\bullet)}$$

and

$$\mathrm{Id}_{H_n(T(Q_\bullet))} = \left(L^{P_\bullet Q_\bullet}T\right)_n(\mathrm{Id}_M) \circ \left(L^{Q_\bullet P_\bullet}T\right)_n(\mathrm{Id}_M)$$

so that $H_n\left(T\left(\alpha_{Q_\bullet P_\bullet}\right)\right)$ and $H_n\left(T\left(\alpha_{P_\bullet Q_\bullet}\right)\right)$ determine an isomorphism between $H_n(T(P_\bullet)) = \left(L_n^{P_\bullet}T\right)(M)$ and $H_n(T(Q_\bullet)) = \left(L_n^{Q_\bullet}T\right)(M)$. □

Lemma 7.58. *Let A and R be rings, and let $T : Mod\text{-}A \to Mod\text{-}R$ be an additive covariant functor. Let $\varphi : M \longrightarrow M'$ be a morphism in $Mod\text{-}A$, let $\left(P_\bullet, d_\bullet^{P_\bullet}\right)$ and $\left(Q_\bullet, d_\bullet^{Q_\bullet}\right)$ be projective resolutions of M and let $\left(P'_\bullet, d_\bullet^{P'_\bullet}\right)$ and $\left(Q'_\bullet, d_\bullet^{Q'_\bullet}\right)$ be projective resolutions of M'. Then we have*

$$\left[\left(L^{Q_\bullet Q'_\bullet}T\right)_n(\varphi)\right] \circ \left[\left(L^{P_\bullet Q_\bullet}T\right)_n(\mathrm{Id}_M)\right] = \left[\left(L^{P'_\bullet Q'_\bullet}T\right)_n(\mathrm{Id}_{M'})\right] \circ \left[\left(L^{P_\bullet P'_\bullet}T\right)_n(\varphi)\right]$$

Proof. Let $\varphi_\bullet : P_\bullet \to P'_\bullet$ be a lifting of φ. Then, in the notation of Lemma 7.57 and in view of Lemma 7.52, we have that

$$\alpha_{P'_\bullet Q'_\bullet} \circ \varphi_\bullet \circ \alpha_{Q_\bullet P_\bullet} : \left(Q_\bullet, d_\bullet^{Q_\bullet}\right) \to \left(Q'_\bullet, d_\bullet^{Q'_\bullet}\right)$$

is also a lifting of φ. Therefore, for every $n \in \mathbb{N}$, we have

$$\left(L^{Q_\bullet Q'_\bullet} T\right)_n (\varphi) = H_n \left(T \left(\alpha_{P'_\bullet Q'_\bullet} \circ \varphi_\bullet \circ \alpha_{Q_\bullet P_\bullet}\right)\right)$$

Now, by Exercise7.42, we have

$$T \left(\alpha_{P'_\bullet Q'_\bullet} \circ \varphi_\bullet \circ \alpha_{Q_\bullet P_\bullet}\right) = T \left(\alpha_{P'_\bullet Q'_\bullet}\right) \circ T \left(\varphi_\bullet\right) \circ T \left(\alpha_{Q_\bullet P_\bullet}\right)$$

and, by Lemma 7.30, we know that

$$H_n \left(T \left(\alpha_{P'_\bullet Q'_\bullet}\right) \circ T \left(\varphi_\bullet\right) \circ T \left(\alpha_{Q_\bullet P_\bullet}\right)\right) = H_n \left(T \left(\alpha_{P'_\bullet Q'_\bullet}\right)\right) \circ H_n \left(T \left(\varphi_\bullet\right)\right) \circ H_n \left(T \left(\alpha_{Q_\bullet P_\bullet}\right)\right).$$

Thus we deduce that

$$\left(L^{Q_\bullet Q'_\bullet} T\right)_n (\varphi) = H_n \left(T \left(\alpha_{P'_\bullet Q'_\bullet} \circ \varphi_\bullet \circ \alpha_{Q_\bullet P_\bullet}\right)\right) = H_n \left(T \left(\alpha_{P'_\bullet Q'_\bullet}\right)\right) \circ H_n \left(T \left(\varphi_\bullet\right)\right) \circ H_n \left(T \left(\alpha_{Q_\bullet P_\bullet}\right)\right).$$

Thus we obtain

$$\left(L^{Q_\bullet Q'_\bullet} T\right)_n (\varphi) = \left[\left(L^{P'_\bullet Q'_\bullet} T\right)_n (\mathrm{Id}_{M'})\right] \circ \left[\left(L^{P_\bullet P'_\bullet} T\right)_n (\varphi)\right] \circ \left[\left(L^{Q_\bullet P_\bullet} T\right)_n (\mathrm{Id}_M)\right]$$

By Lemma 7.57 we know that $\left(L^{P_\bullet Q_\bullet} T\right)_n (\mathrm{Id}_M)$ is the two-sided inverse of $\left(L^{Q_\bullet P_\bullet} T\right)_n (\mathrm{Id}_M)$, so that we get

$$\left[\left(L^{Q_\bullet Q'_\bullet} T\right)_n (\varphi)\right] \circ \left[\left(L^{P_\bullet Q_\bullet} T\right)_n (\mathrm{Id}_M)\right] = \left[\left(L^{P'_\bullet Q'_\bullet} T\right)_n (\mathrm{Id}_M)\right] \circ \left[\left(L^{P_\bullet P'_\bullet} T\right)_n (\varphi)\right]$$

□

Notations 7.59. *Let A and R be rings, and let $T : Mod\text{-}A \to Mod\text{-}R$ be an additive covariant functor. By Lemma 7.57 and Lemma 7.58 we can omit the projective resolutions and set*

$$L_n T (M) = \left(L^{P_\bullet} T\right)_n (M) = H_n \left(T \left(P_\bullet\right)\right).$$

for every $M \in Mod\text{-}A$ and

$$L_n T (\varphi) = \left(L^{P_\bullet P'_\bullet} T\right)_n (\varphi) = H_n \left(T \left(\varphi_\bullet\right)\right)$$

for every left R-module homomorphism $\varphi : M \to M'$.

Remark 7.60. *Clearly $L_n T (M)$ and $L_n T (\varphi)$ are defined only up to "well-behaved" isomorphisms.*

Proposition 7.61. *In the notations of 7.59, the assignment* $M \mapsto L_nT(M)$ *and* $\varphi \mapsto L_nT(\varphi)$ *gives rise to a covariant functor* $L_nT : Mod\text{-}A \to Mod\text{-}R$.

Proof. By Proposition 7.56, we have

$$\left(L^{P_\bullet P"_\bullet}T\right)_n (\varphi' \circ \varphi) = \left[\left(L^{P'_\bullet P''_\bullet}T\right)_n (\varphi')\right] \circ \left[\left(L^{P_\bullet P'_\bullet}T\right)_n (\varphi)\right],$$

and

$$\left(L_n^{P_\bullet P_\bullet}T\right)(\mathrm{Id}_M) = \mathrm{Id}_{L_n^{P_\bullet}T(M)}.$$

□

Definition 7.62. *The functor* L_nT *in Proposition 7.61 is called* n-th left derived functor of T.

Lemma 7.63. *Let us consider the following diagram with exact rows, where* P' *and* P'' *are projective modules,* $i : P' \to P' \oplus P"$ *is the canonical injection and* $p : P' \oplus P"$ $\to P"$ *is the canonical projection,*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P' & \xrightarrow{i} & P' \oplus P'' & \xrightarrow{p} & P'' & \longrightarrow & 0\,. \\
 & & \downarrow{\scriptstyle \pi'} & & & & \downarrow{\scriptstyle \pi''} & & \\
0 & \longrightarrow & M' & \xrightarrow[\alpha']{} & M & \xrightarrow[\alpha'']{} & M'' & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & &
\end{array}$$

Then there is an epimorphism $P' \oplus P'' \xrightarrow{\pi} M$ *such that the diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P'_0 & \xrightarrow{i} & P'_0 \oplus P''_0 & \xrightarrow{p} & P''_0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi''} & & \\
0 & \longrightarrow & M' & \xrightarrow[\alpha']{} & M & \xrightarrow[\alpha'']{} & M'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & ,
\end{array}$$

is commutative.

Proof. Since P'' is projective and α'' is epi, there exists $\beta : P'' \to M$ such that

$$\alpha \circ \beta = \pi''$$

$$\begin{array}{ccc}
 & & P'' \\
 & \swarrow{\scriptstyle \beta} & \downarrow{\scriptstyle \pi''} \\
M & \xrightarrow[\alpha'']{} & M'';
\end{array}$$

Let us set

$$\pi = \nabla\left(\alpha' \circ \pi', \beta\right)$$

i.e.

$$\pi\left(\left(y', y''\right)\right) := \alpha'\left(\pi'\left(y'\right)\right) + \beta\left(y''\right) \text{ for all } y' \in P' \text{ and } y'' \in P''.$$

Then we have

$$\pi \circ i = \alpha' \circ \pi'$$

so that the right-hand square is commutative. In the left-hand one we have

$$\begin{aligned} \alpha''\left(\pi\left(\left(y', y''\right)\right)\right) &= \alpha''\left(\alpha'\left(\pi'\left(y'\right)\right) + \beta\left(y''\right)\right) = \alpha''\left(\alpha'\left(\pi'\left(y'\right)\right)\right) + \alpha''\left(\beta\left(y''\right)\right) \\ &= \alpha''\left(\beta\left(y''\right)\right) = \pi''\left(y''\right) = \pi''\left(\pi\left(\left(y', y''\right)\right)\right) \text{ for all } y' \in P' \text{ and } y'' \in P''. \end{aligned}$$

Let us prove that π is surjective. Let $x \in M$, then $\alpha''(x) \in M''$ and since π'' is surjective there exists $y'' \in P''$ such that $\alpha''(x) = \pi''(y'') = \alpha''(\beta(y''))$. Then $x - \beta(y'') \in \operatorname{Ker}(\alpha'') = \operatorname{Im}(\alpha')$ so that there exists $x' \in M'$ with $\alpha'(x') = x - \beta(y'')$. Since π' is surjective there exists $y' \in P'$ such that $\pi'(y') = x'$. We get $\pi((y', y'')) = \alpha'(\pi'(y')) + \beta(y'') = \alpha'(x') + \beta(y'') = x$.

Theorem 7.64 (Horseshoe Lemma). *Let A be a ring, let*

$$0 \longrightarrow M' \xrightarrow{\alpha'} M \xrightarrow{\alpha''} M'' \longrightarrow 0$$

be an exact sequence in Mod-A. Let $\left(P'_\bullet, d_\bullet^{P'_\bullet}\right)$ be a projective resolution of M'_0 and let $\left(P''_\bullet, d_\bullet^{P''_\bullet}\right)$ be a proiective resolution of $M''_.$ For every $n \in \mathbb{Z}$ set

$$P_n = P'_n \oplus P''_n.$$

Then

1) *the modules P_n give rise to a projective resolution $\left(P_\bullet, d_\bullet^{P_\bullet}\right)$ of M;*

2) *for every $n \in \mathbb{Z}$, let $i_{n:} : P'_n \to P'_n \oplus P''_n$ be the canonical inclusion and let $p_n : P'_n \oplus P''_n \to P''_n$ be the canonical projection. Then*

$$i_\bullet = (i_n)_{n \in \mathbb{N}} : \left(P'_\bullet, d_\bullet^{P'_\bullet}\right) \to \left(P_\bullet, d_\bullet^{P_\bullet}\right)$$

and

$$p_\bullet = (p_n)_{n \in \mathbb{N}} : \left(P_\bullet, d_\bullet^{P_\bullet}\right) \to \left(P''_\bullet, d_\bullet^{P''_\bullet}\right)$$

are morphism of chain complexes;

3) *$i_\bullet$ is a lifting of α' and $p_\bullet$ is a lifting of α'';*

4) *the sequence*

$$0 \longrightarrow \left(P'_\bullet, d_\bullet^{P'_\bullet}\right) \xrightarrow{i_\bullet} \left(P_\bullet, d_\bullet^{P_\bullet}\right) \xrightarrow{p_\bullet} \left(P''_\bullet, d_\bullet^{P''_\bullet}\right) \longrightarrow 0$$

is exact.

□

Proof. Since $\left(P'_\bullet, d^{P'_\bullet}_\bullet\right)$ is a projective resolution of M'_0 and $\left(P''_\bullet, d^{P''_\bullet}_\bullet\right)$ is a proiective resolution of $M''_.$ we have epimorhisms

$$\pi'_0 : P'_0 \to M'_0,\ \pi''_0 : P''_0 \to M''.$$

such that the sequences

$$P'_1 \xrightarrow{d_1^{P'_\bullet}} P'_0 \xrightarrow{\pi'_0} M'_0 \to 0$$
$$P''_1 \xrightarrow{d_1^{P''_\bullet}} P''_0 \xrightarrow{\pi''_0} M'' \to 0$$

are exact. Then, by Lemma 7.63, there exists an epimorphism $P' \oplus P'' \xrightarrow{\pi_0} M$ such that the diagram

$$\begin{array}{ccccccccc}
0 & \to & P'_0 & \xrightarrow{i_0} & P'_0 \oplus P''_0 & \xrightarrow{p_0} & P''_0 & \to & 0 \\
 & & \pi'_0 \downarrow & & \pi_0 \downarrow & & \pi''_0 \downarrow & & \\
0 & \to & M' & \xrightarrow{\alpha'} & M & \xrightarrow{\alpha''} & M'' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

is commutative and exact. Then assumptions of Snake Lemma 7.23 are fulfilled so that the sequence

$$0 \longrightarrow \operatorname{Ker}(\pi'_0) \xrightarrow{\alpha'_0} \operatorname{Ker}(\pi_0) \xrightarrow{\alpha''_0} \operatorname{Ker}(\pi''_0) \longrightarrow \operatorname{Coker}(\pi'_0) = \{0\}$$

is exact. Let

$$\begin{array}{rcl}
j'_0 & : & \operatorname{Ker}(\pi'_0) \to P'_0 \\
j_0 & : & \operatorname{Ker}(\pi_0) \to P_0 \\
j''_0 & : & \operatorname{Ker}(\pi''_0) \to P''_0
\end{array}$$

be the canonical inclusions. Recall that α'_0 and α''_0 are uniquely defined by

$$j_0 \circ \alpha'_0 = i_0 \circ j'_0 \tag{7.24}$$
$$j''_0 \circ \alpha''_0 = p_0 \circ j_0. \tag{7.25}$$

Therefore we get the commutative and exact diagram:

$$\begin{array}{ccccccccc}
0 & \to & \operatorname{Ker}(\pi'_0) & \xrightarrow{\alpha'_0} & \operatorname{Ker}(\pi_0) & \xrightarrow{\alpha''_0} & \operatorname{Ker}(\pi''_0) & \to & 0 \\
 & & j'_0 \downarrow & & j_0 \downarrow & & j''_0 \downarrow & & \\
0 & \to & P'_0 & \xrightarrow{i_0} & P'_0 \oplus P''_0 & \xrightarrow{p_0} & P''_0 & \to & 0 \\
 & & \pi'_0 \downarrow & & \pi_0 \downarrow & & \pi''_0 \downarrow & & \\
0 & \to & M' & \xrightarrow{\alpha'} & M & \xrightarrow{\alpha''} & M'' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}.$$

Since $\left(P'_\bullet, d_\bullet^{P'_\bullet}\right)$ is a projective resolution of M'_0 and $\left(P''_\bullet, d_\bullet^{P''_\bullet}\right)$ is a proiective resolution of M'' we have that $\operatorname{Ker}(\pi'_0) = \operatorname{Im}\left(d_1^{P'_\bullet}\right)$ and $\operatorname{Ker}(\pi''_0) = \operatorname{Im}\left(d_1^{P''_\bullet}\right)$. Let

$$\pi'_1 = \left(d_1^{P'_\bullet}\right)^{|\operatorname{Im}\left(d_1^{P'_\bullet}\right)} \text{ and } \pi''_1 = \left(d_1^{P''_\bullet}\right)^{\operatorname{Im}\left(d_1^{P''_\bullet}\right)}$$

Then, by Lemma 7.63, there exists an epimorphism $P'_1 \oplus P''_1 \xrightarrow{\pi_1} \operatorname{Ker}(\pi_0)$ such that the diagram

$$\begin{array}{ccccccccc}
0 & \to & P'_1 & \xrightarrow{i_1} & P'_1 \oplus P''_1 & \xrightarrow{p_1} & P''_1 & \to & 0 \\
 & & \pi'_1 \downarrow & & \pi_1 \downarrow & & \pi''_1 \downarrow & & \\
0 & \to & \operatorname{Ker}(\pi'_0) & \xrightarrow{\alpha'_0} & \operatorname{Ker}(\pi_0) & \xrightarrow{\alpha''_0} & \operatorname{Ker}(\pi''_0) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

is commutative and exact. By Snake Lemma 7.23, we get the exact sequence

$$0 \longrightarrow \operatorname{Ker}(\pi'_1) = \operatorname{Ker}\left(d_1^{P'_\bullet}\right) \xrightarrow{\alpha'_1} \operatorname{Ker}(\pi_1) \xrightarrow{\alpha''_1} \operatorname{Ker}(\pi''_1) = \operatorname{Ker}\left(d_1^{P''_\bullet}\right) \longrightarrow \operatorname{Coker}(\pi'_1) = 0.$$

Let

$$\begin{array}{rcl}
j'_1 & : & \operatorname{Ker}(\pi'_1) \to P'_1 \\
j_1 & : & \operatorname{Ker}(\pi_1) \to P_1 \\
j''_1 & : & \operatorname{Ker}(\pi''_1) \to P''_1
\end{array}$$

be the canonical inclusions. Recall that α'_1 and α''_1 are uniquely defined by

$$j_1 \circ \alpha'_1 = i_1 \circ j'_1 \tag{7.26}$$

$$j''_1 \circ \alpha''_1 = p_1 \circ j_1. \tag{7.27}$$

Now we get

$$i_0 \circ d_1^{P'_\bullet} = i_0 \circ j'_0 \circ \pi'_1 \overset{(7.24)}{=} j_0 \circ \alpha'_0 \circ \pi'_1 = j_0 \circ \pi_1 \circ i_1$$

$$p_0 \circ j_0 \circ \pi_1 \overset{(7.25)}{=} j''_0 \circ \alpha''_0 \circ \pi_1 = j''_0 \circ \pi''_1 \circ p_1$$

and hence the exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & P'_1 & \xrightarrow{i_1} & P'_1 \oplus P''_1 & \xrightarrow{p_1} & P''_1 & \to & 0 \\
 & & d_1^{P'_\bullet} = j'_0 \circ \pi'_1 \downarrow & & j_0 \circ \pi_1 \downarrow & & j''_0 \circ \pi''_1 = d_1^{P''_\bullet} \downarrow & & \\
0 & \to & P'_0 & \xrightarrow{i_0} & P'_0 \oplus P''_0 & \xrightarrow{p_0} & P''_0 & \to & 0 \\
 & & \pi'_0 \downarrow & & \pi_0 \downarrow & & \pi''_0 \downarrow & & \\
0 & \to & M' & \xrightarrow{\alpha'} & M & \xrightarrow{\alpha''} & M'' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}.$$

We set

$$d_1^{P_\bullet} = j_0 \circ \pi_1.$$

Since j_0 is mono, note that,

$$\operatorname{Ker}\left(d_1^{P_\bullet}\right) = \operatorname{Ker}(\pi_1)$$

so that we have the exact sequence

$$0 \longrightarrow \operatorname{Ker}(\pi_1') = \operatorname{Ker}\left(d_1^{P'_\bullet}\right) \xrightarrow{\alpha_1'} \operatorname{Ker}(\pi_1) = \operatorname{Ker}\left(d_1^{P_\bullet}\right) \xrightarrow{\alpha_1''} \operatorname{Ker}(\pi_1'') = \operatorname{Ker}\left(d_1^{P''_\bullet}\right) \longrightarrow 0.$$

and we can consider the diagram

$$\begin{array}{ccccccccc}
0 & \to & P_2' & \xrightarrow{i_2} & P_2' \oplus P_2'' & \xrightarrow{p_2} & P_2'' & \to & 0 \\
 & & \pi_2' \downarrow & & & & \pi_2'' \downarrow & & \\
0 & \to & \operatorname{Im}\left(d_2^{P'_\bullet}\right) = \operatorname{Ker}\left(d_1^{P'_\bullet}\right) & \xrightarrow{\alpha_1'} & \operatorname{Ker}\left(d_1^{P_\bullet}\right) & \xrightarrow{\alpha_1''} & \operatorname{Im}\left(d_2^{P''_\bullet}\right) = \operatorname{Ker}\left(d_1^{P''_\bullet}\right) & \longrightarrow & 0 \cdot \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & &
\end{array}$$

Then, by Lemma 7.63, there exists an epimorphism $\pi_2 : P_2 = P_2' \oplus P_2'' \to \operatorname{Ker}\left(d_1^{P_\bullet}\right)$ such that the diagram

$$\begin{array}{ccccccccc}
0 & \to & P_2' & \xrightarrow{i_2} & P_2' \oplus P_2'' & \xrightarrow{p_2} & P_2'' & \to & 0 \\
 & & \pi_2' \downarrow & & \pi_2 \downarrow & & \pi_2'' \downarrow & & \\
0 & \to & \operatorname{Im}\left(d_2^{P'_\bullet}\right) = \operatorname{Ker}\left(d_1^{P'_\bullet}\right) & \xrightarrow{\alpha_1'} & \operatorname{Ker}\left(d_1^{P_\bullet}\right) & \xrightarrow{\alpha_1''} & \operatorname{Im}\left(d_2^{P''_\bullet}\right) = \operatorname{Ker}\left(d_1^{P''_\bullet}\right) & \longrightarrow & 0 \cdot \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

is commutative and exact. Then by Snake Lemma 7.23, we get the exact sequence

$$0 \longrightarrow \operatorname{Ker}(\pi_2') = \operatorname{Ker}\left(d_2^{P'_\bullet}\right) \xrightarrow{\alpha_2'} \operatorname{Ker}(\pi_2) \xrightarrow{\alpha_2''} \operatorname{Ker}(\pi_2'') = \operatorname{Ker}\left(d_2^{P''_\bullet}\right) \longrightarrow \operatorname{Coker}(\pi_2') = 0.$$

Let

$$\begin{aligned}
j_2' &: \operatorname{Ker}(\pi_2') \to P_2' \\
j_2 &: \operatorname{Ker}(\pi_2) \to P_2 \\
j_2'' &: \operatorname{Ker}(\pi_2'') \to P_2''
\end{aligned}$$

be the canonical inclusions. Recall that α_2' and α_2'' are uniquely defined by

$$j_2 \circ \alpha_2' = i_2 \circ j_2' \tag{7.28}$$

$$j_2'' \circ \alpha_2'' = p_2 \circ j_2. \tag{7.29}$$

Now we get

$$i_1 \circ d_2^{P'\bullet} = i_1 \circ j_1' \circ \pi_2' \overset{(7.26)}{=} j_1 \circ \alpha_1' \circ \pi_2' = j_1 \circ \pi_2 \circ i_2$$
$$p_1 \circ j_1 \circ \pi_2 \overset{(7.27)}{=} j_1'' \circ \alpha_1'' \circ \pi_2 = j_1'' \circ \pi_2'' \circ p_2$$

and hence the exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & P_2' & \xrightarrow{i_2} & P_2' \oplus P_2'' & \xrightarrow{p_2} & P_2'' & \to & 0 \\
 & & d_2^{P'\bullet} = j_1' \circ \pi_2' \downarrow & & j_1 \circ \pi_2 \downarrow & & j_1'' \circ \pi_2'' = d_2^{P''\bullet} \downarrow & & \\
0 & \to & P_1' & \xrightarrow{i_1} & P_1' \oplus P_1'' & \xrightarrow{p_1} & P_1'' & \to & 0 \\
 & & \pi_1' \downarrow & & \pi_1 \downarrow & & \pi_1'' \downarrow & & \\
0 & \to & \mathrm{Im}\left(d_2^{P'\bullet}\right) = \mathrm{Ker}\left(d_1^{P'\bullet}\right) & \xrightarrow{\alpha_1'} & \mathrm{Ker}\left(d_1^{P\bullet}\right) & \xrightarrow{\alpha_1''} & \mathrm{Im}\left(d_2^{P''\bullet}\right) = \mathrm{Ker}\left(d_1^{P''\bullet}\right) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}.$$

We set

$$d_2^{P\bullet} = j_1 \circ \pi_2.$$

By induction assume that, for some $n \geq 2$ we have for all $t = 1, \dots, n$

$$d_t^{P\bullet} : P_t = P_t' \oplus P_t'' \to P_{t-1} = P_{t-1}' \oplus P_{t-1}''$$

such that the diagrams

$$\begin{array}{ccccccccc}
0 & \to & P_t' & \xrightarrow{i_t} & P_t' \oplus P_t'' & \xrightarrow{p_t} & P_t'' & \to & 0 \\
 & & d_t^{P'\bullet} \downarrow & & d_t^{P\bullet} \downarrow & & d_t^{P''\bullet} \downarrow & & \\
0 & \to & P_{t-1}' & \xrightarrow{i_{t-1}} & P_{t-1}' \oplus P_{t-1}'' & \xrightarrow{p_{t-1}} & P_{t-1}'' & \to & 0 \\
 & & d_{t-1}^{P'\bullet} \downarrow & & d_{t-1}^{P\bullet} \downarrow & & d_{t-2}^{P''\bullet} \downarrow & & \\
0 & \to & P_{t-2}' & \xrightarrow{i_{t-2}} & P_{t-2}' \oplus P_{t-2}'' & \xrightarrow{p_{t-1}} & P_{t-2}'' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}$$

are commutative and exact. For every t, let

$$\begin{array}{rcl}
j_{t-1}' & : & \mathrm{Ker}\left(d_{t-1}^{P'\bullet}\right) \to P_{t-1}' \\
j_{t-1} & : & \mathrm{Ker}\left(d_{t-1}^{P\bullet}\right) \to P_{t-1} \\
j_{t-1}'' & : & \mathrm{Ker}\left(d_{t-1}^{P''\bullet}\right) \to P_{t-1}''
\end{array}$$

denote the canonical inclusions. Let

$$\pi_t' = \left(d_t^{P'\bullet}\right)^{|\mathrm{Im}\left(d_t^{P'\bullet}\right)}, \pi_t = \left(d_t^{P\bullet}\right)^{|\mathrm{Im}\left(d_t^{P\bullet}\right)} \text{ and } \pi_t'' = \left(d_t^{P''\bullet}\right)^{|\mathrm{Im}\left(d_t^{P''\bullet}\right)}.$$

For every $t \geq 2$, we have that

$$\mathrm{Im}\left(d_t^{P'_\bullet}\right) = \mathrm{Ker}\left(d_{t-1}^{P'_\bullet}\right), \mathrm{Im}\left(d_t^{P_\bullet}\right) = \mathrm{Ker}\left(d_{t-1}^{P_\bullet}\right), \mathrm{Im}\left(d_t^{P''_\bullet}\right) = \mathrm{Ker}\left(d_{t-1}^{P''_\bullet}\right)$$

and hence

$$j'_{t-1} \circ \pi'_t = d_t^{P'_\bullet}, j_{t-1} \circ \pi_t = d_t^{P_\bullet} \text{ and } j''_{t-1} \circ \pi''_t = d_t^{P''_\bullet} \tag{7.30}$$

Now, by applying Snake Lemma 7.23 to the commutative and exact diagram

$$\begin{array}{ccccccccc} 0 & \to & P'_{n-1} & \xrightarrow{i_{n-1}} & P'_{n-1} \oplus P''_{n-1} & \xrightarrow{p_{n-1}} & P''_{n-1} & \to & 0 \\ & & d_{n-1}^{P'_\bullet} \downarrow & & d_{n-1}^{P_\bullet} \downarrow & & d_{n-1}^{P''_\bullet} \downarrow & & \\ 0 & \to & P'_{n-2} & \xrightarrow{i_{n-2}} & P'_{n-2} \oplus P''_{n-2} & \xrightarrow{p_{n-2}} & P''_{n-2} & \to & 0 \end{array}.$$

we get the exact sequence

$$0 \to \mathrm{Ker}\left(d_{n-1}^{P'_\bullet}\right) \overset{\alpha'_{n-1}}{\to} \mathrm{Ker}\left(d_{n-1}^{P_\bullet}\right) \overset{\alpha''_{n-1}}{\to} \mathrm{Ker}\left(d_{n-1}^{P''_\bullet}\right)$$

where α'_{n-1} and α''_{n-1} are canonically defined by

$$j_{n-1} \circ \alpha'_{n-1} = i_{n-1} \circ j'_{n-1} \tag{7.31}$$

$$j''_{n-1} \circ \alpha''_{n-1} = p_{n-1} \circ j_{n-1}. \tag{7.32}$$

Since $n \geq 2$

$$\mathrm{Im}\left(d_n^{P'_\bullet}\right) = \mathrm{Ker}\left(d_{n-1}^{P'_\bullet}\right), \mathrm{Im}\left(d_n^{P_\bullet}\right) = \mathrm{Ker}\left(d_{n-1}^{P_\bullet}\right), \mathrm{Im}\left(d_n^{P''_\bullet}\right) = \mathrm{Ker}\left(d_{n-1}^{P''_\bullet}\right)$$

we can consider the diagram

$$\begin{array}{ccccccccc} 0 & \to & P'_n & \xrightarrow{i_n} & P'_n \oplus P''_n & \xrightarrow{p_n} & P''_n & \to & 0 \\ & & \pi'_n \downarrow & & \pi_n \downarrow & & \pi''_n \downarrow & & \\ 0 & \to & \mathrm{Im}\left(d_n^{P'_\bullet}\right) & \xrightarrow{\alpha'_{n-1}} & \mathrm{Im}\left(d_n^{P_\bullet}\right) & \xrightarrow{\alpha''_{n-1}} & \mathrm{Im}\left(d_n^{P''_\bullet}\right) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Note that this diagram is commutative. In fact we have

$$j_{n-1} \circ \alpha'_{n-1} \circ \pi'_n \overset{(7.31)}{=} i_{n-1} \circ j'_{n-1} \circ \pi'_n = i_{n-1} \circ d_n^{P'_\bullet} = d_n^{P_\bullet} \circ i_n = j_{n-1} \circ \pi_n \circ i_n$$

so that, since j_{n-1} is mono, we get

$$\alpha'_{n-1} \circ \pi'_n = \pi_n \circ i_n.$$

We also have

$$j''_{n-1} \circ \alpha''_{n-1} \circ \pi_n \overset{(7.32)}{=} p_{n-1} \circ j_{n-1} \circ \pi_n = p_{n-1} \circ d_n^{P_\bullet} = d_n^{P''_\bullet} \circ p_n = j''_{n-1} \circ \pi''_n \circ p_n$$

so that, since j''_{n-1} is mono, we get

$$\alpha''_{n-1} \circ \pi_n = \pi''_n \circ p_n.$$

Note that this implies that α''_{n-1} is epi so that we have the commutative and exact diagram

$$\begin{array}{ccccccccc} 0 & \to & P'_n & \stackrel{i_n}{\longrightarrow} & P'_n \oplus P''_n & \stackrel{p_n}{\longrightarrow} & P''_n & \to & 0 \\ & & \pi'_n \downarrow & & \pi_n \downarrow & & \pi''_n \downarrow & & \\ 0 & \to & \operatorname{Im}\left(d_n^{P'_\bullet}\right) & \stackrel{\alpha'_{n-1}}{\longrightarrow} & \operatorname{Im}\left(d_n^{P_\bullet}\right) & \stackrel{\alpha''_{n-1}}{\longrightarrow} & \operatorname{Im}\left(d_n^{P''_\bullet}\right) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \cdot \tag{7.33}$$

By applying Snake Lemma 7.23 to the diagram (7.33), we get the exact sequence

$$0 \to \operatorname{Ker}(\pi'_n) = \operatorname{Ker}\left(d_n^{P'_\bullet}\right) \stackrel{\alpha'_n}{\to} \operatorname{Ker}(\pi_n) = \operatorname{Ker}\left(d_n^{P_\bullet}\right) \stackrel{\alpha''_n}{\to} \operatorname{Ker}(\pi''_n) = \operatorname{Ker}\left(d_n^{P''_\bullet}\right) \to 0$$

where α'_n and α''_n are uniquely defined by

$$j_n \circ \alpha'_n = i_n \circ j'_n \tag{7.34}$$

$$j''_n \circ \alpha''_n = p_n \circ j_n. \tag{7.35}$$

Now we can consider the diagram

$$\begin{array}{ccccccccc} 0 & \to & P'_{n+1} & \stackrel{i_{n+1}}{\longrightarrow} & P'_{n+1} \oplus P''_{n+1} & \stackrel{p_{n+1}}{\longrightarrow} & P''_{n+1} & \to & 0 \\ & & \pi'_{n+1} \downarrow & & & & \pi''_{n+1} \downarrow & & \\ 0 & \to & \operatorname{Im}\left(d_{n+1}^{P'_\bullet}\right) = \operatorname{Ker}\left(d_n^{P'_\bullet}\right) & \stackrel{\alpha'_n}{\longrightarrow} & \operatorname{Ker}\left(d_n^{P_\bullet}\right) & \stackrel{\alpha''_n}{\longrightarrow} & \operatorname{Im}\left(d_{n+1}^{P''_\bullet}\right) = \operatorname{Ker}\left(d_n^{P''_\bullet}\right) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Then, by Lemma 7.63, there exists an epimorphism $\pi_{n+1} : P_{n+1} = P'_{n+1} \oplus P''_{n+1} \to \operatorname{Ker}\left(d_n^{P_\bullet}\right)$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \to & P'_{n+1} & \stackrel{i_{n+1}}{\longrightarrow} & P'_{n+1} \oplus P''_{n+1} & \stackrel{p_{n+1}}{\longrightarrow} & P''_{n+1} & \to & 0 \\ & & \pi'_{n+1} \downarrow & & \pi_{n+1} \downarrow & & \pi''_{n+1} \downarrow & & \\ 0 & \to & \operatorname{Im}\left(d_{n+1}^{P'_\bullet}\right) = \operatorname{Ker}\left(d_n^{P'_\bullet}\right) & \stackrel{\alpha'_n}{\longrightarrow} & \operatorname{Ker}\left(d_n^{P_\bullet}\right) & \stackrel{\alpha''_n}{\longrightarrow} & \operatorname{Im}\left(d_{n+1}^{P''_\bullet}\right) = \operatorname{Ker}\left(d_n^{P''_\bullet}\right) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

is commutative and exact.

Now we get

$$i_n \circ d^{P'_\bullet}_{n+1} = i_n \circ j'_n \circ \pi'_{n+1} \overset{(7.34)}{=} j_n \circ \alpha'_n \circ \pi'_{n+1} = j_n \circ \pi_{n+1} \circ i_{n+1}$$
$$p_n \circ j_n \circ \pi_{n+1} \overset{(7.35)}{=} j''_n \circ \alpha''_n \circ \pi_{n+1} = j''_n \circ \pi''_{n+1} \circ p_{n+1}$$

and hence the commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & P'_{n+1} & \overset{i_{n+1}}{\longrightarrow} & P'_{n+1} \oplus P''_{n+1} & \overset{p_{n+1}}{\longrightarrow} & P''_{n+1} & \to & 0 \\
 & & d^{P'_\bullet}_{n+1} = j'_n \circ \pi'_{n+1} \downarrow & & j_n \circ \pi_{n+1} \downarrow & & j''_n \circ \pi''_{n+1} = d^{P''_\bullet}_{n+1} \downarrow & & \\
0 & \to & P'_n & \overset{i_n}{\longrightarrow} & P'_n \oplus P''_n & \overset{p_n}{\longrightarrow} & P''_n & \to & 0 \\
 & & d^{P'_\bullet}_n \downarrow & & d^{P_\bullet}_n \downarrow & & d^{P''_\bullet}_n \downarrow & & \\
0 & \to & P'_{n-1} & \overset{i_{n-1}}{\longrightarrow} & P'_{n-1} \oplus P''_{n-1} & \longrightarrow & P''_{n-1} & \to & 0
\end{array}$$

We set

$$d^{P_\bullet}_{n+1} = j_n \circ \pi_{n+1.}$$

Note that, since $\pi_{n+1.}$ is epi,

$$\mathrm{Im}\left(d^{P_\bullet}_{n+1}\right) = \mathrm{Im}\left(j_n\right) = \mathrm{Ker}\left(d^{P_\bullet}_n\right).$$

□

Remark 7.65. *The previous Theorem is called "Horseshoe Lemma" because we have to complete the horseshoe-shaped diagram*

$$\begin{array}{ccccccccccc}
 & & & & & & & & 0 & & \\
 & & & & & & & & \downarrow & & \\
\cdots \to & P'_2 & \to & P'_1 & \to & P'_0 & \to & & M' & \to & 0 \\
 & & & & & & & & \downarrow & & \\
 & & & & & & & & M & & \\
 & & & & & & & & \downarrow & & \\
\cdots \to & P''_2 & \to & P''_1 & \to & P''_0 & \to & & M'' & \to & 0 \\
 & & & & & & & & \downarrow & & \\
 & & & & & & & & 0 & &
\end{array}.$$

Lemma 7.66. *Let A and R be rings, let $T : Mod\text{-}A \to Mod\text{-}R$ be an additive covariant functor and let $0 \to L \overset{f}{\longrightarrow} M \overset{g}{\longrightarrow} N \to 0$ be a split exact sequence in $Mod\text{-}A$. Then the sequence*

$$0 \to T(L) \overset{T(f)}{\longrightarrow} T(M) \overset{T(g)}{\longrightarrow} T(N) \to 0$$

is split exact.

Proof. By Theorem 1.84, there exists an R-module homomorphism $p : M \to L$ and an R-module homomorphism $s : N \to M$ such that

$$p \circ f = \mathrm{Id}_L, g \circ s = \mathrm{Id}_N \text{ and } \mathrm{Id}_M = f \circ p + s \circ g.$$

By applying T we get

$$T(p) \circ T(f) = \mathrm{Id}_{T(L)}, T(g) \circ T(s) = \mathrm{Id}_{T(N)} \text{ and } \mathrm{Id}_{T(M)} = T(f) \circ T(p) + T(s) \circ T(g).$$

By applying Theorem 1.84 once more, we get that the sequence

$$0 \to T(L) \xrightarrow{T(f)} T(M) \xrightarrow{T(g)} T(N) \to 0$$

is split exact. □

Theorem 7.67. *Let A and R be rings, and let $T : Mod\text{-}A \to Mod\text{-}R$ be an additive covariant functor. Let*

$$0 \longrightarrow M' \xrightarrow{\alpha'} M \xrightarrow{\alpha''} M'' \longrightarrow 0$$

be an exact sequence in $Mod\text{-}A$. For every $n \geq 1$ there exists a (connection) morphism $L_n T(M'') \xrightarrow{\omega_n} L_{n-1} T(M')$ in $Mod\text{-}R$ such that the sequence in $Mod\text{-}R$

$$\begin{array}{l} \ldots \longrightarrow L_{n+1} T(M'') \xrightarrow{\omega_{n+1}} L_n T(M') \xrightarrow{L_n T(\alpha')} L_n T(M) \xrightarrow{L_n T(\alpha'')} L_n T(M'') \longrightarrow \ldots \\ \ldots \longrightarrow L_1 T(M'') \xrightarrow{\omega_1} L_0 T(M') \xrightarrow{L_0 T(\alpha')} L_0 T(M) \xrightarrow{L_0 T(\alpha'')} L_0 T(M'') \longrightarrow 0 \end{array}$$

is exact.

Proof. By Theorem 7.64 there are projective resolutions $P'_\bullet$, $P_\bullet := P'_\bullet \oplus P''_\bullet$ and $P''_\bullet$ respectively of M', M and M'', and morphism of chain complexes

$$i_\bullet = (i_n)_{n \in \mathbb{N}} : \left(P'_\bullet, d_\bullet^{P'_\bullet}\right) \to \left(P_\bullet, d_\bullet^{P_\bullet}\right)$$

and

$$p_\bullet = (p_n)_{n \in \mathbb{N}} : \left(P_\bullet, d_\bullet^{P_\bullet}\right) \to \left(P''_\bullet, d_\bullet^{P''_\bullet}\right)$$

such that $i_\bullet$ is a lifting of α' and $p_\bullet$ is a lifting of α'' and the sequence

$$0 \longrightarrow \left(P'_\bullet, d_\bullet^{P'_\bullet}\right) \xrightarrow{i_\bullet} \left(P_\bullet, d_\bullet^{P_\bullet}\right) \xrightarrow{p_\bullet} \left(P''_\bullet, d_\bullet^{P''_\bullet}\right) \longrightarrow 0$$

is split exact. Then, By Lemma 7.66, the sequence

$$0 \longrightarrow T(P'_\bullet) \xrightarrow{T(i_\bullet)} T(P_\bullet) \xrightarrow{T(p_\bullet)} T(P''_\bullet) \longrightarrow 0$$

is split exact. Then we can apply Theorem 7.32 and get that for every $n \in \mathbb{Z}$, there exists a morphism $H_n(T(P''_\bullet)) \xrightarrow{\omega_n} H_{n-1}(T(P'_\bullet))$ such that the sequence

$$\ldots . \to H_n T(P'_\bullet) \xrightarrow{H_n(T(i_\bullet))} H_n(T(P_\bullet)) \xrightarrow{H_n(T(p_\bullet))} H_n(T(P''_\bullet)) \xrightarrow{\omega_n} H_{n-1} T(P'_\bullet) \xrightarrow{H_{n-1}(T(i_\bullet))} H_{n-1}(T(P_\bullet$$

is exact. Then we have

1) Since $P'_\bullet$, $P_\bullet := P'_\bullet \oplus P''_\bullet$ and $P''_\bullet$ are projective resolutions of M', M and M respectively, then

$$H_nT\left(P'_\bullet\right) = L_nT\left(M'\right), H_nT\left(P_\bullet\right) = L_nT\left(M\right), H_n\left(T\left(P''_\bullet\right)\right) = L_nT\left(M''\right).$$

2) Since $i_\bullet$ is a lifting of α' and $p_\bullet$ is a lifting of α'', then

$$L_nT\left(\alpha'\right) = H_n\left(T\left(i_\bullet\right)\right) \text{ and } L_nT\left(\alpha''\right) = H_n\left(T\left(p_\bullet\right)\right).$$

□

Proposition 7.68. *Let A and R be rings, let $T : Mod\text{-}A \to Mod\text{-}R$ be an additive covariant functor and let P be a projective module. Then $L_nT\left(P\right) = 0$ for every $n > 0$ and $L_0T\left(P\right) = T\left(P\right)$.*

Proof. Clearly, a projective resolution of P is given by $P_0 = P$ and $P_n = 0$ for every $n \neq 0$. In fact $\frac{\mathrm{Ker}\left(d_0^{P_\bullet}\right)}{\mathrm{Im}\left(d_1^{P_\bullet}\right)} = \frac{P_0}{\{0\}} = P = H_0\left(P_\bullet\right)$. By applying T to this resolution we get that $L_nT\left(P\right) = H_n\left(T\left(P_\bullet\right)\right)$ is always 0 whenever $n \neq 0$ and equal to $\frac{\mathrm{Ker}T\left(d_0^{P_\bullet}\right)}{\mathrm{Im}T\left(d_1^{P_\bullet}\right)} = T\left(P\right)$ for $n = 0$. □

Definition 7.69. *Let A and R be rings, and let $T : Mod\text{-}A \to Mod\text{-}R$ be an additive covariant functor. T is said to be* right exact *if, for every exact sequence $M' \xrightarrow{\alpha'} M \xrightarrow{\alpha''} M'' \longrightarrow 0$, the sequence $T\left(M'\right) \xrightarrow{T(\alpha')} T\left(M\right) \xrightarrow{T(\alpha'')} T\left(M''\right) \longrightarrow 0$ is also exact.*

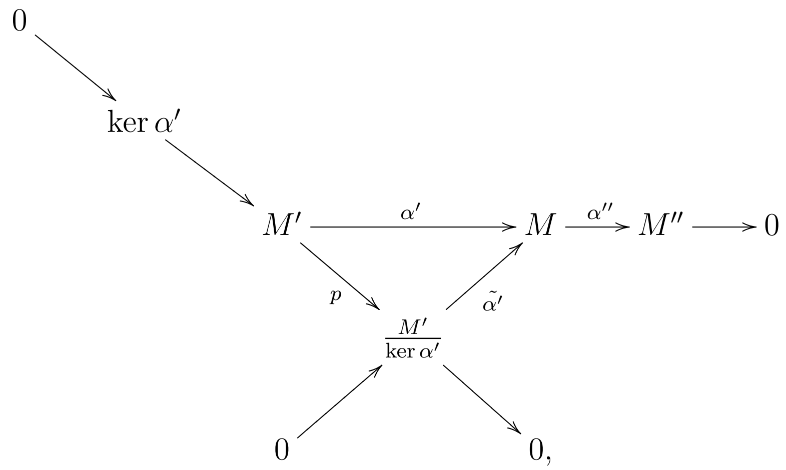
Proposition 7.70. *Let A and R be rings, let $T : Mod\text{-}A \to Mod\text{-}R$ be an additive covariant functor. Then the following statements are equivalent:*

(a) T is right exact.

(b) For every exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$, the sequence $T\left(M'\right) \longrightarrow T\left(M\right) \longrightarrow T\left(M''\right) \longrightarrow 0$ is exact.

Proof. $(a) \Rightarrow (b)$. It is trivial.

$(b) \Rightarrow (a)$. Let $M' \xrightarrow{\alpha'} M \xrightarrow{\alpha''} M'' \longrightarrow 0$ be an exact sequence and consider the commutative and exact diagram



Then

$$0 \longrightarrow \mathrm{Ker}\,(\alpha') \xrightarrow{i} M' \xrightarrow{p} \frac{M'}{\mathrm{Ker}\,(\alpha')} \longrightarrow 0$$

and

$$0 \longrightarrow \frac{M'}{\mathrm{Ker}\,(\alpha')} \xrightarrow{\overline{\alpha'}} M \xrightarrow{\alpha''} M'' \longrightarrow 0$$

are exact. Hence we get the following exact sequences:

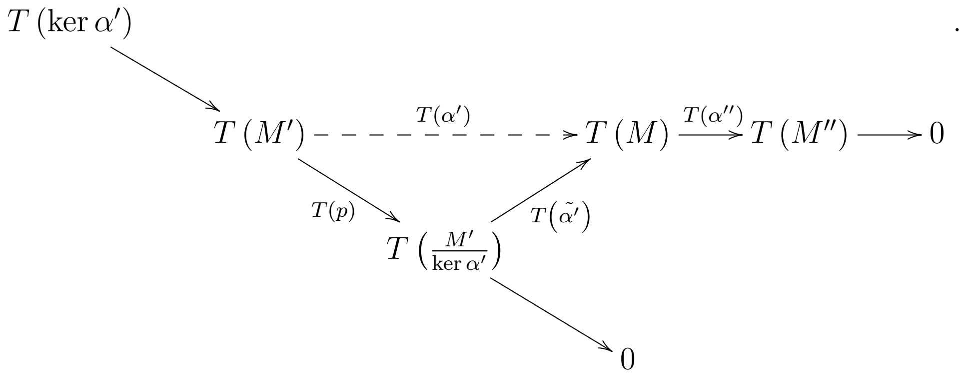
$$T\,(\mathrm{Ker}\,(\alpha')) \xrightarrow{T(i)} T\,(M') \xrightarrow{T(p)} T\left(\frac{M'}{\mathrm{Ker}\,(\alpha')}\right) \longrightarrow 0,$$

$$T\left(\frac{M'}{\mathrm{Ker}\,(\alpha')}\right) \xrightarrow{T\left(\overline{\alpha'}\right)} T\,(M) \xrightarrow{T(\alpha'')} T\,(M'') \longrightarrow 0.$$

Since T is a functor, we have that $T\left(\overline{\alpha'}\right) \circ T\,(p) = T\left(\overline{\alpha'} \circ p\right) = T\,(\alpha')$. Moreover we have

$$\mathrm{Im}\,(T\,(\alpha')) = \mathrm{Im}\left(T\left(\overline{\alpha'}\right) \circ T\,(p)\right) \overset{T(p)\text{isepi}}{=} \mathrm{Im}\left(T\left(\overline{\alpha'}\right)\right) = \mathrm{Ker}\,(T\,(\alpha'')).$$

Thus we obtain the following commutative and exact diagram



□

Proposition 7.71. *Let A and R be rings, and let T : Mod-$A \to$ Mod-R be an additive right exact covariant functor. Then L_0T and T are isomorphic.*

Proof. Let $M \in$ *Mod*-A and let $\left(P_\bullet, d_\bullet^{P_\bullet}\right)$ be a projective resolution of M. Then from the exact sequence

$$P_1 \xrightarrow{d_1^{P_\bullet}} P_0 \xrightarrow{\pi} M \longrightarrow 0$$

we deduce that

$$T\,(P_1) \xrightarrow{T\left(d_1^{P_\bullet}\right)} T\,(P_0) \xrightarrow{T(\pi)} T\,(M) \longrightarrow 0$$

is exact. In particular $\mathrm{Im}\left(T\left(d_1^{P_\bullet}\right)\right) = \mathrm{Ker}\,(T\,(\pi))$. Note that $P_0 \xrightarrow{d_0^{P_\bullet}} 0$, so that $T\,(P_0) \xrightarrow{T\left(d_0^{P_\bullet}\right)} 0$ and hence $\mathrm{Ker}\left(T\left(d_0^{P_\bullet}\right)\right) = T\,(P_0)$. Thus we get

$$L_0T\,(M) = H_0\,(T\,(P_\bullet)) = \frac{\mathrm{Ker}\left(T\left(d_0^{P_\bullet}\right)\right)}{\mathrm{Im}\left(T\left(d_1^{P_\bullet}\right)\right)} = \frac{T\,(P_0)}{\mathrm{Ker}\,(T\,(\pi))}$$

and therefore

$$L_0T(M) = \frac{T(P_0)}{\mathrm{Ker}(T(\pi))} \simeq T(M)$$

that is $L_0T \simeq T$. □

Corollary 7.72. *Let A and R be rings, and let $T : Mod\text{-}A \to Mod\text{-}R$ be an additive right exact covariant functor. Then the sequence*

$$\begin{array}{l} \ldots \longrightarrow L_{n+1}T(M'') \overset{\omega_{n+1}}{\longrightarrow} L_nT(M') \overset{L_nT(\alpha')}{\longrightarrow} L_nT(M) \overset{L_nT(\alpha'')}{\longrightarrow} L_nT(M'') \longrightarrow \ldots \\ \ldots \longrightarrow L_1T(M'') \overset{\omega_1}{\longrightarrow} T(M') \overset{T(\alpha')}{\longrightarrow} T(M) \overset{T(\alpha'')}{\longrightarrow} T(M'') \longrightarrow 0 \end{array}$$

is exact.

Proof. Apply Theorem 7.67 and Proposition 7.71. □

7.73. *Let ${}_AN_R$ be an A-R-bimodule and let $T = - \otimes_A N : Mod - A \to Mod - R$. Then by Proposition 6.14 and Exercise 7.40,T is an additive right exact functor. For every $n \in \mathbb{N}$, we set*

$$Tor_n^A(-, N) = L_nT.$$

Then, by Corollary 7.72, we have the exact sequence

$$\cdots \longrightarrow Tor_2^A(M'', N) \overset{\omega_2}{\longrightarrow} Tor_1^A(M', N) \overset{Tor_1^A(\alpha', N)}{\longrightarrow} Tor_1^A(M, N) \overset{Tor_1^A(\alpha'', N)}{\longrightarrow} Tor_1^A(M'', N) \overset{\omega_1}{\longrightarrow}$$
$$\overset{\omega_1}{\longrightarrow} M' \otimes_A N \overset{\alpha' \otimes_A N}{\longrightarrow} M \otimes_A N \overset{\alpha'' \otimes_A N}{\longrightarrow} M'' \otimes_A N \longrightarrow 0$$

Proposition 7.74. *Let $T : Mod - A \to Mod - R$ be a right exact additive covariant functor and let $\left(P_\bullet, d_\bullet^{P_\bullet}\right)$ be a projective resolution of M in Mod-A. Let $n \in \mathbb{N}, n \geq 2$, let $K_n = \mathrm{Ker}\left(d_{n-1}^{P_\bullet}\right)$ and let $\mu : K_n \to P_{n-1}$ be the canonical injection. Then $L_nT(M) \cong \mathrm{Ker}(T(\mu))$.*

Proof. Let

$$\varphi_n = \left(d_n^{P_\bullet}\right)^{|\mathrm{Im}\left(d_n^{P_\bullet}\right)}.$$

Since $n \geq 2$ we have that

$$\mathrm{Im}\left(d_n^{P_\bullet}\right) = \mathrm{Ker}\left(d_{n-1}^{P_\bullet}\right) = K_n$$

so that

$$\varphi_n \circ \mu = d_n^{P_\bullet}$$

In particular we have that the sequence

$$\ldots \longrightarrow P_{n+1} \overset{d_{n+1}^{P_\bullet}}{\longrightarrow} P_n \overset{\varphi_n}{\longrightarrow} K_n \longrightarrow 0$$

is exact. Then, by applying the right exact functor T, we get the following exact diagram

$$\begin{array}{ccccccccc}
 & & T\left(P_{n+1}\right) & \overset{T\left(d_{n+1}^{P_\bullet}\right)}{\longrightarrow} & T\left(P_{n}\right) & \overset{T(\varphi_n)}{\longrightarrow} & T\left(K_{n}\right) & \rightarrow & 0\\
 & & \downarrow & & T\left(d_{n}^{P_\bullet}\right)\downarrow & & T\left(\mu\right)\downarrow & & \\
0 & \rightarrow & 0 & \longrightarrow & T\left(P_{n-1}\right) & \overset{T\left(\mathrm{Id}_{P_{n-1}}\right)=\mathrm{Id}_{T\left(P_{n-1}\right)}}{\longrightarrow} & T\left(P_{n-1}\right) & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

Then we can apply the Snake Lemma 7.23, from which we deduce that the following sequence is exact

$$T\left(P_{n+1}\right) \overset{f}{\longrightarrow} \mathrm{Ker}\left(T\left(d_{n}^{P_\bullet}\right)\right) \overset{g}{\longrightarrow} \mathrm{Ker}\left(T\left(\mu\right)\right) \longrightarrow \mathrm{Coker}\left(0\right) = 0.$$

Here

$$f = \left(T\left(d_{n+1}^{P_\bullet}\right)\right)^{|\mathrm{Ker}\left(T\left(d_{n}^{P_\bullet}\right)\right)} \text{ and } g = \left(T\left(\varphi_{n}\right)_{|\mathrm{Ker}\left(T\left(d_{n}^{P_\bullet}\right)\right)}\right)^{|\mathrm{Ker}(T(\mu))}$$

so that

$$\mathrm{Ker}\left(g\right) = \mathrm{Im}\left(f\right) = \mathrm{Im}\left(T\left(d_{n+1}^{P_\bullet}\right)\right)$$

Thus we get

$$\mathrm{Ker}\left(T\left(\mu\right)\right) \cong \frac{\mathrm{Ker}\left(T\left(d_{n}^{P_\bullet}\right)\right)}{\mathrm{Im}\left(T\left(d_{n+1}^{P_\bullet}\right)\right)} = H_{n}\left(T\left(P_\bullet\right)\right) = L_{n}T\left(M\right).$$

□

7.7 Cochain Complexes and Right Derived Functors

Definitions 7.75. *A* cochain complex *of right A-modules is a a pair* $\left(C^\bullet, d_{C^\bullet}^\bullet\right) = \left(\left(C^n\right)_{n\in\mathbb{Z}}, \left(d_{C^\bullet}^n\right)_{n\in\mathbb{Z}}\right)$ *where each* C^n *is a right A-module,* $d_{C^\bullet}^n : C^n \to C^{n+1}$ *is a right A-modules homomorphism and* $d_{C^\bullet}^{n+1} \circ d_{C^\bullet}^n = 0$ *for every* $n \in \mathbb{Z}$. *For each* $n \in \mathbb{Z}$

- $\left(d_{C^\bullet}^\bullet\right) = \left(d_{C^\bullet}^n\right)_{n\in\mathbb{Z}}$ *is called the* differential operator *of the cochain complex,*
- $Z^n\left(C^\bullet\right) := \mathrm{Ker}\left(d_{C^\bullet}^n\right)$ *is called the n-*th cococycle *of the cochain complex ,*
- $B^n\left(C^\bullet\right) := \mathrm{Im}\left(d_{C^\bullet}^{n-1}\right)$ *is called the n-*th coboundary *of the cochain complex ,*
- $B^n\left(C^\bullet\right) \subseteq Z^n\left(C^\bullet\right)$ *and* $H^n\left(C^\bullet\right) := \frac{\mathrm{Ker}\left(d_{C^\bullet}^n\right)}{\mathrm{Im}\left(d_{C^\bullet}^n\right)} = \frac{Z^n(C^\bullet)}{B^n(C^\bullet)}$ *is called the n-*th cohomology module *of the cochain complex.*

Definition 7.76. *Given cochain complexes* $(C^\bullet, d^\bullet_{C^\bullet})$ *and* $(D^\bullet, d^\bullet_{D^\bullet})$, *a* morphism of cochain complexes of right A-modules $\varphi^\bullet = (\varphi^n)_{n\in\mathbb{Z}} : (C^\bullet, d^\bullet_{C^\bullet}) = \left((C^n)_{n\in\mathbb{Z}}, (d^n_{C^\bullet})_{n\in\mathbb{Z}}\right) \longrightarrow (D^\bullet, d^\bullet_{D^\bullet}) = \left((D^n)_{n\in\mathbb{Z}}, (d^n_{D^\bullet})_{n\in\mathbb{Z}}\right)$ *consists of a family of right A-modules homomorphisms* $(\varphi^n : C^n \longrightarrow D^n)_{n\in\mathbb{Z}}$ *such that* $d^n_{D^\bullet} \circ \varphi^n = \varphi^{n+1} \circ d^n_{C^\bullet}$, *for every* $n \in \mathbb{Z}$.

Definition 7.77. *Let* $\varphi^\bullet, \psi^\cdot : (C^\bullet, d^\cdot_{C^\bullet}) \longrightarrow (D^\bullet, d^\cdot_{D^\bullet})$ *be morphisms of cocomplexes. A* homotopy Δ *between* $\varphi^\cdot$ *and* $\psi^\cdot$ *consists of a family of homomorphisms* $(\Delta^n : C^n \longrightarrow D^{n-1})_{n\in\mathbb{Z}}$ *such that*

$$\varphi^n - \psi^n = d^{n-1}_{D^\cdot} \circ \Delta^n + \Delta^{n+1} \circ d^n_{C^\cdot}.$$

If there is a homotopy between $\varphi^\cdot$ *and* $\psi^\cdot$ *we say that* $\varphi^\cdot$ is homotopic to $\psi^\cdot$ *and we write* $\varphi^\cdot \simeq \psi^\cdot$.

Notation 7.78. *We will denote by* $Coch\,(Mod\text{-}A)$ *the category of cochain complexes. Obviously the objects are cochain complexes of right A-modules and morphisms are just morphism of cochain complexes of right A-modules.*

Theorem 7.79. *The assignments*

$$\begin{aligned} \left((C_n)_{n\in\mathbb{Z}}, \left(d^{C_\bullet}_n\right)_{n\in\mathbb{Z}}\right) &\mapsto \left((C_{-n})_{n\in\mathbb{Z}}, \left(d^{C_\bullet}_{-n}\right)_{n\in\mathbb{Z}}\right) \\ (\varphi_n)_{n\in\mathbb{Z}} &\mapsto (\varphi_{-n})_{n\in\mathbb{Z}} \end{aligned}$$

define a covariant functor $F : Ch\,(Mod\text{-}A) \to Coch\,(Mod\text{-}A)$ *which is an isomorphism of categories. The inverse of F is the functor* $G : Coch\,(Mod\text{-}A) \to Ch\,(Mod\text{-}A)$ *defined by setting*

$$\begin{aligned} G\left((C^n)_{n\in\mathbb{Z}}, (d^n_{C^\bullet})_{n\in\mathbb{Z}}\right) &= \left(C^{-n}\right)_{n\in\mathbb{Z}}, \left(d^{-n}_{C^\bullet}\right)_{n\in\mathbb{Z}} \text{ for every } \left((C^n)_{n\in\mathbb{Z}}, (d^n_{C^\bullet})_{n\in\mathbb{Z}}\right) \in Coch\,(Mod\text{-}A) \\ G\left((\varphi^n)_{n\in\mathbb{Z}}\right) &= \left(\varphi^{-n}\right)_{n\in\mathbb{Z}} \text{ for every morphism } (\varphi^n)_{n\in\mathbb{Z}} \text{ in } Coch\,(Mod\text{-}A). \end{aligned}$$

Moreover, for every $n \in \mathbb{Z}$, *we have*

$$\begin{aligned} H^n \circ F &= H_{-n} \\ H_n \circ G &= H^{-n}. \end{aligned}$$

Proof. We have

$$H^n\left(F\left((C_\bullet)\right)\right) = \frac{\operatorname{Ker}\left(d^n_{F(C_\bullet)}\right)}{\operatorname{Im}\left(d^{n-1}_{F(C_\bullet)}\right)} = \frac{\operatorname{Ker}\left(d^{C_\bullet}_{-n}\right)}{\operatorname{Im}\left(d^{C_\bullet}_{-n+1}\right)} = H_{-n}\left(C_\bullet\right).$$

□

Theorem 7.80. *Let* $0 \longrightarrow C^\bullet \xrightarrow{\varphi^\bullet} D^\bullet \xrightarrow{\psi^\bullet} E^\bullet \longrightarrow 0$ *be an exact sequence of cochain complexes of right A-modules. Then, for every* $n \in \mathbb{Z}$, *there exists a morphism* $H^n(E^\bullet) \xrightarrow{\omega^n} H^{n+1}(C^\bullet)$ *such that the sequence*

$$\ldots. \to H^n(C^\bullet) \xrightarrow{H^n(\varphi^\bullet)} H^n(D^\bullet) \xrightarrow{H_n(\psi^\bullet)} H^n(E^\bullet) \xrightarrow{\omega^n} H^{n+1}(C^\bullet) \xrightarrow{H^{n+1}(\varphi^\bullet)} H^{n+1}(D^\bullet) \xrightarrow{H^{n+1}(\psi^\bullet)} H^{n+1}(E^\bullet)\ldots$$

is exact.

Definitions 7.81. *A cochain complex* $(C^\bullet, d^\bullet_{C^\bullet})$ *is called*

1) positive *if* $C^n = 0$ *for every* $n \leq -1$.

2) acyclic positive *if it is positive and* $H^n(C^\bullet) = 0$ *for every* $n \geq 1$.

3) injective *if* C^n *is injective for every* n.

Definition 7.82. *An* injective resolution *of a right A-module M is an acyclic positive and injective cochain complex* $(E^\bullet, d^\bullet_{E^\bullet})$ *such that* $H^0(E^\bullet) = \frac{\mathrm{Ker}\left(d^0_{E^\bullet}\right)}{\mathrm{Im}\left(d^{-1}_{E^\bullet}\right)} = \mathrm{Ker}\left(d^0_{E^\bullet}\right) \cong M$ *so that the sequence*

$$0 \longrightarrow M \longrightarrow E^0 \xrightarrow{d^0_{E^\bullet}} E^1 \xrightarrow{d^1_{E^\bullet}} E^2 \xrightarrow{d^2_{E^\bullet}} \cdots$$

is exact.

Proposition 7.83. *Every right A-module admits an injective resolution.*

Proof. Follow the same pattern of Proposition 7.49, using Theorem 3.28. □

Theorem 7.84 (Lifting Theorem for Cochain Complexes)**.** *Let* $(C^\bullet, d^\bullet_{C^\bullet})$ *be an acyclic positive cochain complex, let* $(E^\bullet, d^\bullet_{E^\bullet})$ *be an injective positive cochain complex and let* $\varphi : H^0(C^\bullet) \longrightarrow H^0(E^\bullet)$ *be a morphism in Mod-A. Then there exists a morphism of cochain complexes* $\varphi^\bullet : (C^\bullet, d^\bullet_{C^\bullet}) \longrightarrow (E^\bullet, d^\bullet_{E^\bullet})$ *such that* $H^0(\varphi^\bullet) = \varphi$. *Moreover, if* $\psi^\bullet : (C^\bullet, d^\bullet_{C^\bullet}) \longrightarrow (E^\bullet, d^\bullet_{E^\bullet})$ *also satisfies* $H_0(\psi^\bullet) = \varphi$, *we have* $\varphi^\bullet \simeq \psi^\bullet$. *In particular* $H^n(\varphi^\bullet)$ *only depends on* φ.

Definition 7.85. *In the notations and assumptions of Theorem 7.84, any morphism of cochain complexes* $\varphi^\bullet : (C^\bullet, d^\bullet_{C^\bullet}) \longrightarrow (E^\bullet, d^\bullet_{E^\bullet})$ *such that* $H^0(\varphi^\bullet) = \varphi$ *will be called a* lifting of φ.

Theorem 7.86. *Let* $(E^\bullet, d^\bullet_{E^\bullet})$ *and* $(G^\bullet, d^\bullet_{G^\bullet})$ *be injective resolution of a right A-module M. In view of Theorem 7.84, we can consider the liftings* $\varphi^\bullet : E^\bullet \longrightarrow G$ *and* $\psi^\bullet : G^\bullet \longrightarrow E^\bullet$ *of* Id_M. *Then*

1) $\varphi^\bullet \circ \psi^\bullet \simeq \mathrm{Id}_{G^\cdot}$ *and* $\psi^\bullet \circ \varphi^\bullet \simeq \mathrm{Id}_{E^\bullet}$.

2) $H^n(\varphi^\bullet) : H^n(E^\bullet) \to H^n(G^\bullet)$ *is an isomorphism with inverse* $H^n(\psi^\bullet)$, *for every* $n \in \mathbb{N}$.

7.87. *Let A and R be rings, and let* $T : Mod\text{-}A \to Mod\text{-}R$ *be an additive covariant functor. By applying T to an acyclic positive injective resolution* $(E^\bullet, d^\bullet_{E^\bullet})$ *of* $M \in Mod\text{-}A$ *, we set*

$$(R_{E^\bullet}T)^n(M) = H^n(T(E^\bullet)).$$

Let $\varphi : M \to \overline{M}$ *be a morphism in Mod-A and let* $\left(\overline{E}^\bullet, d^\bullet_{\overline{E}^\bullet}\right)$ *be a projective resolution of* $\overline{M}$. *Let* $\varphi^\bullet : (E^\bullet, d^\bullet_{E^\bullet}) \to \left(\overline{E}^\bullet, d^\bullet_{\overline{E}^\bullet}\right)$ *be a lifting of* φ *(see Theorem 7.84). We set*

$$(R_{E^\bullet \overline{E}^\bullet}T)^n(\varphi) = H^n(T(\varphi^\bullet))$$

One can prove a suitable version of Lemma 7.57:

Lemma 7.88. *Let A and R be rings, and let $T : Mod\text{-}A \to Mod\text{-}R$ be an additive covariant functor. Let $(E^\bullet, d^\bullet_{E^\bullet})$ and $(F^\bullet, d^\bullet_{F^\bullet})$ be injective resolutions of M in Mod-A. Let $\alpha_{E^\bullet F^\bullet} : (E^\bullet, d^\bullet_{E^\bullet}) \to (F^\bullet, d^\bullet_{F^\bullet})$ be a lifting of Id_M and let $\alpha_{F^\bullet E^\bullet} : (F^\bullet, d^\bullet_{F^\bullet}) \to (E^\bullet, d^\bullet_{E^\bullet})$ be a lifting of Id_M (see Theorem 7.50). Then*

$$H^n\left(T\left(\alpha_{E^\bullet F^\bullet}\right)\right) = \left(R_{E^\bullet F^\bullet} T\right)^n \left(\mathrm{Id}_M\right) \text{ and } H^n\left(T\left(\alpha_{F^\bullet E^\bullet}\right)\right) = \left(R_{F^\bullet E^\bullet} T\right)^n \left(\mathrm{Id}_M\right)$$

determine an isomorphism between $H^n(T(E^\bullet)) = (R_{E^\bullet}T)^n(M)$ and $H^n(T(F^\bullet)) = (R_{F^\bullet}T)^n(M)$.

and a suitable version of Lemma 7.58:

Lemma 7.89. *Let A and R be rings, and let $T : Mod\text{-}A \to Mod\text{-}R$ be an additive covariant functor. Let $\varphi : M \longrightarrow \overline{M}$ be a morphism in Mod-A, let $(E^\bullet, d^\bullet_{E^\bullet})$ and $(F^\bullet, d^\bullet_{F^\bullet})$ be injective resolutions of M and let $\left(\overline{E}^\bullet, d^\bullet_{\overline{E}^\bullet}\right)$ and $\left(\overline{F}^\bullet, d^\bullet_{\overline{F}^\bullet}\right)$ be injective resolutions of $\overline{M}$. Then we have*

$$\left[\left(R_{F^\bullet \overline{F}^\bullet} T\right)^n (\varphi)\right] \circ \left[\left(R_{E^\bullet F^\bullet} T\right)^n \left(\mathrm{Id}_M\right)\right] = \left[\left(R_{\overline{E}^\bullet \overline{F}^\bullet} T\right)^n \left(\mathrm{Id}_{M'}\right)\right] \circ \left[\left(R_{E^\bullet \overline{E}^\bullet} T\right)^n (\varphi)\right].$$

Notations 7.90. *Let A and R be rings, and let $T : Mod\text{-}A \to Mod\text{-}R$ be an additive covariant functor. By Lemma 7.88 and Lemma 7.89 we can omit the injective resolutions and set*

$$R^n T(M) = \left(R_{E^\bullet} T\right)^n (M) = H^n\left(T\left(E^\bullet\right)\right).$$

for every $M \in Mod\text{-}A$

$$R^n T(\varphi) = \left(R_{E^\bullet \overline{E}^\bullet} T\right)^n (\varphi) = H^n\left(T\left(\varphi^\bullet\right)\right).$$

In this way we get a functor $R^nT : Mod\text{-}A \to Mod\text{-}R$.

Definition 7.91. *The functor $R^nT : Mod\text{-}A \to Mod\text{-}R$ is called* n-th right derived functor of T.

Theorem 7.92. *Let A and R be rings, and let $T : Mod\text{-}A \to Mod\text{-}R$ be an additive covariant functor. Let*

$$0 \longrightarrow M' \xrightarrow{\alpha'} M \xrightarrow{\alpha''} M'' \longrightarrow 0$$

be an exact sequence in Mod-A. For every $n \geq 0$ there exists a (connection) morphism $R^{n-1}T(M'') \xrightarrow{\omega^n} R^nT(M')$ in Mod-R such that the sequence in Mod-R

$$\begin{array}{lllllll} 0 & \longrightarrow & R^0T(M') \xrightarrow{R^0T(\alpha')} R^0T(M) \xrightarrow{R^0T(\alpha'')} R^0T(M'') \xrightarrow{\omega^1} R^1T(M') \longrightarrow \dots \\ \dots & \longrightarrow & R^nT(M') \xrightarrow{R^nT(\alpha')} R^nT(M) \xrightarrow{R^nT(\alpha'')} R^nT(M'') \xrightarrow{\omega^n} R^{n+1}T(M') \longrightarrow \dots \end{array}$$

is exact.

Proposition 7.93. *Let A and R be rings, let $T : Mod\text{-}A \to Mod\text{-}R$ be an additive covariant functor and let E be an injective right A-module. Then $R^nT(E) = 0$ for every $n > 0$ and $R^0T(E) = T(E)$.*

Definition 7.94. *Let A and R be rings, and let $T : Mod\text{-}A \to Mod\text{-}R$ be an additive covariant functor. T is said to be* left exact *if, for every exact sequence $0 \to M' \xrightarrow{\alpha'} M \xrightarrow{\alpha''} M''$, the sequence $0 \to T(M') \xrightarrow{T(\alpha')} T(M) \xrightarrow{T(\alpha'')} T(M'')$ is also exact.*

Proposition 7.95. *Let A and R be rings, and let $T : Mod\text{-}A \to Mod\text{-}R$ be an additive left exact covariant functor. Then R^0T and T are isomorphic.*

Corollary 7.96. *Let A and R be rings, and let $T : Mod\text{-}A \to Mod\text{-}R$ be an additive left exact covariant functor. Then the sequence*

$$\begin{array}{rcl} 0 & \longrightarrow & T(M') \xrightarrow{T(\alpha')} T(M) \xrightarrow{T(\alpha'')} T(M'') \xrightarrow{\omega^1} R^1T(M') \longrightarrow \dots \\ \dots & \longrightarrow & R^nT(M') \xrightarrow{R^nT(\alpha')} R^nT(M) \xrightarrow{R^nT(\alpha'')} R^nT(M'') \xrightarrow{\omega^n} R^{n+1}T(M') \longrightarrow \dots \end{array}$$

is exact.

7.97. *Let ${}_AN_R$ be an A-R-bimodule and let $T = \mathrm{Hom}_A({}_AN_R, -) : Mod\text{-}A \to Mod\text{-}R$. Then by Proposition 1.91 and Exercise 7.40,T is an additive left exact functor. For every $n \in \mathbb{N}$, we set*

$$Ext^n_A(N, -) = R^nT.$$

Then, by Corollary 7.96, we have the exact sequence

$$\begin{gathered} 0 \longrightarrow \mathrm{Hom}_A({}_AN_R, M') \xrightarrow{\mathrm{Hom}_A({}_AN_R,\alpha')} \mathrm{Hom}_A({}_AN_R, M) \xrightarrow{\mathrm{Hom}_A({}_AN_R,\alpha'')} \\ \mathrm{Hom}_A({}_AN_R, M'') \xrightarrow{\omega^1} Ext^1_A({}_AN_R, M') \longrightarrow \dots \\ \dots \longrightarrow Ext^n_A({}_AN_R, M') \xrightarrow{\mathrm{Ext}_A({}_AN_R,\alpha')} Ext^n_A({}_AN_R, M) \xrightarrow{\mathrm{Ext}_A({}_AN_R,\alpha'')} \\ Ext^n_A({}_AN_R, M'') \xrightarrow{\omega^n} Ext^{n+1}_A({}_AN_R, M') \longrightarrow \dots \end{gathered}$$

7.98. *Let us consider an additive contravariant functor $W : Mod\text{-}A \to Mod\text{-}R$. The right derived functors R^nW are obtained as right derived functors of the covariant functor $W' : (Mod - A)^{opp} \to Mod\text{-}R$. In order to compute $R^nW(M)$ we consider a projective resolution $\left(P_\bullet, d^{P_\bullet}_\bullet\right)$ of M in $Mod\text{-}A$, form the cochain complex $\left(WP_\bullet, d^{WP_\bullet}_\bullet\right)$ and take the cohomology*

$$R^nW(M) = H^n(WP_\bullet)$$

for every $n \in \mathbb{N}$.

Analogously we obtain the left derived functors of contravariant functors via injective resolutions.

Definition 7.99. *Let A and R be rings, and let $W : Mod\text{-}A \to Mod\text{-}R$ be an additive contravariant functor. W is said to be* left exact *if, for every exact sequence $M' \xrightarrow{\alpha'} M \xrightarrow{\alpha''} M'' \longrightarrow 0$, the sequence $0 \longrightarrow W(M'') \xrightarrow{W(\alpha'')} W(M) \xrightarrow{W(\alpha')} W(M')$is also exact.*

Example 7.100. *Let ${}_AN_R$ be an A-R-bimodule and let $W = \mathrm{Hom}_R(-, {}_AN_R) : Mod\text{-}R \to Mod-A$. Then by Proposition 1.91 and Exercise 7.40, W is a left exact additive contravariant functor. The right derived functor of W are denoted by $Ext^n_R(-, N_R)$.*

7.101. *Analogously one defines right-exactness. Results similar to Proposition 7.68, Proposition 7.71 and Proposition 7.74 may be proved.*

Chapter 8

Semisimple modules and Jacobson radical

8.1. *Throught this chapter R will denote a ring.*

Definition 8.2. *Let M_R be a right R-module. M_R is said to be semisimple if there is a family $(S_\lambda)_{\lambda\in\Lambda}$ of right simple R-submodules such that*

$$M = \dot{\bigoplus_{\lambda\in\Lambda}} S_\lambda.$$

Exercise 8.3. *Let $(S_\lambda)_{\lambda\in\Lambda}$ be a family of right simple R-modules and assume that $M_R \cong \bigoplus_{\lambda\in\Lambda} S_\lambda$. Prove that M_R is semisimple.*

Lemma 8.4. *Let M_R be a right R-module and let $(S_\lambda)_{\lambda\in\Lambda}$ be a family of right simple R-submodules such that*

$$M = \sum_{\lambda\in\Lambda} S_\lambda.$$

Then for each submodule L of M, there exists a subset $\Gamma \subseteq \Lambda$ such that

$$M = L \oplus \dot{\bigoplus_{\gamma\in\Gamma}} S_\gamma.$$

In particular, M is semisimple.

Proof. Let us assume that $L \subsetneqq M$. Let

$$\mathcal{E} = \left\{ \Gamma \subseteq \Lambda \mid \sum_{\gamma\in\Gamma} S_\gamma = \dot{\bigoplus_{\gamma\in\Gamma}} S_\gamma \text{ and } L \cap \sum_{\gamma\in\Gamma} S_\gamma = \{0\} \right\}.$$

Then $\mathcal{E} \neq \varnothing$. In fact, since $L \subsetneqq M$ there is at least a $\gamma \in \Lambda$ such that $S_\gamma \nsubseteq L$ so that $L \cap S_\gamma = \{0\}$. Let us prove that $(\mathcal{E}, \subseteq)$ is inductive. Let $(\Gamma_i)_{i\in I}$ be a chain in $\mathcal{E}$ and let

$$\Gamma = \bigcup_{i\in I} \Gamma_i.$$

We want to prove that $\Gamma \in \mathcal{E}$. First of all, let us prove that $\sum_{\gamma\in\Gamma} S_\gamma = \bigoplus_{\gamma\in\Gamma} S_\gamma$. Assume that $\sum_{\gamma\in\Gamma} S_\gamma \neq \bigoplus_{\gamma\in\Gamma} S_\gamma$. Then there is a $\gamma_0 \in \Gamma$ such that

$$S_{\gamma_0} \cap \sum_{\gamma\in\Gamma\smallsetminus\{\gamma_0\}} S_\gamma \neq \{0\}$$

i.e.

$$S_{\gamma_0} \subseteq \sum_{\gamma\in\Gamma\smallsetminus\{\gamma_0\}} S_\gamma. \tag{8.1}$$

Since $\Gamma = \bigcup_{i\in I} \Gamma_i$, there is an $i_0 \in I$ such that $\gamma_0 \in \Gamma_{i_0}$ and for every $i \in I$ we have either $\Gamma_{i_0} \subseteq \Gamma_i$ or $\Gamma_i \subseteq \Gamma_{i_0}$. Therefore

$$\Gamma = \bigcup_{\substack{i\in I\\ \Gamma_{i_0}\subseteq\Gamma_i}} \Gamma_i \cup \bigcup_{\substack{i\in I\\ \Gamma_i\subseteq\Gamma_{i_0}}} \Gamma_i = \bigcup_{\substack{i\in I\\ \Gamma_{i_0}\subseteq\Gamma_i}} \Gamma_i \cup \Gamma_{i_0} = \bigcup_{\substack{i\in I\\ \Gamma_{i_0}\subseteq\Gamma_i}} \Gamma_i$$

where, in the last equality we have used that

$$\Gamma_{i_0} \subseteq \bigcup_{\substack{i\in I\\ \Gamma_{i_0}\subseteq\Gamma_i}} \Gamma_i.$$

Moreover

$$\Gamma \smallsetminus \{\gamma_0\} = \bigcup_{\substack{i\in I\\ \Gamma_{i_0}\subseteq\Gamma_i}} (\Gamma_i \smallsetminus \{\gamma_0\}).$$

Let $0 \neq x_{\gamma_0} \in S_{\gamma_0}$. Then $x_{\gamma_0} \in S_{\gamma_0} \overset{(8.1)}{\subseteq} \sum_{\gamma\in\Gamma\smallsetminus\{\gamma_0\}} S_\gamma$. Hence there is an $n \in \mathbb{N}, n \geq 1$, elements $\gamma_1, \ldots, \gamma_n \in \Gamma \smallsetminus \{\gamma_0\}$ and elements $x_{\gamma_1} \in S_{\gamma_1}, \ldots, x_{\gamma_n} \in S_{\gamma_n}$ such that

$$x_{\gamma_0} = x_{\gamma_1} + \ldots + x_{\gamma_n}.$$

Since $\gamma_1, \ldots, \gamma_n \in \Gamma\smallsetminus\{\gamma_0\}$, for every $t = 1, \ldots, n$ there is a set Γ_{i_t} such that $\Gamma_{i_0} \subseteq \Gamma_{i_t}$ and $\gamma_t \in \Gamma_{i_t} \smallsetminus \{\gamma_0\}$. Let $1 \leq u \leq n$ be such that $\Gamma_{i_t} \subseteq \Gamma_{i_u}$ for every $t = 1, \ldots, n$. Then $\gamma_0 \in \Gamma_{i_0} \subseteq \Gamma_{i_u}$ and $\gamma_1, \ldots, \gamma_n \in \Gamma_{i_u} \smallsetminus \{\gamma_0\}$ so that

$$0 \neq x_{\gamma_0} = x_{\gamma_1} + \ldots + x_{\gamma_n} \in \sum_{\gamma\in\Gamma_{i_u}\smallsetminus\{\gamma_0\}} S_\gamma. \tag{8.2}$$

Since $\Gamma_{i_u} \in \mathcal{E}$ we know that

$$\sum_{\gamma\in\Gamma_{i_u}} S_\gamma = \dot{\bigoplus_{\gamma\in\Gamma_{i_u}}} S_\gamma$$

and since $\gamma_0 \in \Gamma_{i_u}$ we deduce that

$$S_{\gamma_0} \cap \sum_{\gamma\in\Gamma_{i_u}\smallsetminus\{\gamma_0\}} S_\gamma = \{0\}$$

which contradicts (8.2).

Let us prove that $L\cap\sum_{\gamma\in\Gamma}S_\gamma=\{0\}$. Assume that $0\neq x\in L\cap\sum_{\gamma\in\Gamma}S_\gamma$. Then there is an $n\in\mathbb{N}, n\geq 1$, elements $\gamma_1,\dots,\gamma_n\in\Gamma$ and elements $x_{\gamma_1}\in S_{\gamma_1},\dots,x_{\gamma_n}\in S_{\gamma_n}$ such that

$$x=x_{\gamma_1}+\ldots+x_{\gamma_n}.$$

Since $\gamma_1,\dots,\gamma_n\in\Gamma$, for every $t=1,\dots,n$ there is a set Γ_{i_t} such that $\Gamma_{i_0}\subseteq\Gamma_{i_t}$ and $\gamma_t\in\Gamma_{i_t}$. Let $1\leq u\leq n$ be such that $\Gamma_{i_t}\subseteq\Gamma_{i_u}$ for every $t=1,\dots,n$. Then $\gamma_1,\dots,\gamma_n\in\Gamma_{i_u}$ so that

$$0\neq x=x_{\gamma_1}+\ldots+x_{\gamma_n}\in\sum_{\gamma\in\Gamma_{i_u}}S_\gamma.$$

and we deduce that

$$L\cap\sum_{\gamma\in\Gamma_{i_u}}S_\gamma\neq\{0\}$$

which contradicts the fact that $\Gamma_{i_u}\in\mathcal{E}$.

Therefore $\Gamma\in\mathcal{E}$ and clearly Γ is an upper bound of the chain $(\Gamma_i)_{i\in I}$. Thus $(\mathcal{E},\subseteq)$ is inductive. By Zorn's Lemma, there is a maximal element $\Gamma_0\in\mathcal{E}$. Then

$$\sum_{\gamma\in\Gamma_0}S_\gamma=\dot{\bigoplus_{\gamma\in\Gamma_0}}S_\gamma \text{ and } L\cap\sum_{\gamma\in\Gamma_0}S_\gamma=\{0\}\,.$$

Let us prove that $M=L+\sum_{\gamma\in\Gamma_0}S_\gamma$. Let $\lambda\in\Lambda$ such that

$$S_\lambda\nsubseteq L+\sum_{\gamma\in\Gamma_0}S_\gamma.$$

Then $S_\lambda\nsubseteq\sum_{\gamma\in\Gamma_0}S_\gamma$ i.e.

$$S_\lambda\cap\sum_{\gamma\in\Gamma_0}S_\gamma=\{0\}\,. \tag{8.3}$$

Let $\Xi=\Gamma_0\cup\{\lambda\}$ and let us prove that $\Xi\in\mathcal{E}$.

First of all, let us prove that

$$\sum_{\gamma\in\Xi}S_\gamma=\dot{\bigoplus_{\gamma\in\Xi}}S_\gamma$$

i.e. that, for every $\xi\in\Xi$,

$$S_\xi\cap\sum_{\gamma\in\Xi\setminus\{\xi\}}S_\gamma=\{0\}$$

We already know this for $\xi=\lambda$ in view of (8.3). Assume that $\xi\in\Gamma_0$ and let

$$x\in S_\xi\cap\left(\sum_{\gamma\in\Gamma_0\setminus\{\xi\}}S_\gamma+S_\lambda\right).$$

Then there is an $n \in \mathbb{N}, n \geq 1$, elements $\gamma_1, \ldots, \gamma_n \in \Gamma_0 \backslash \{\xi\}$ and elements $x_{\gamma_1} \in S_{\gamma_1}, \ldots, x_{\gamma_n} \in S_{\gamma_n}$ and an element $x_\lambda \in S_\lambda$ such that

$$x = x_{\gamma_1} + \ldots + x_{\gamma_n} + x_\lambda.$$

Then

$$x - (x_{\gamma_1} + \ldots + x_{\gamma_n}) = x_\lambda \in \sum_{\gamma \in \Gamma_0} S_\gamma \cap S_\lambda \overset{(8.3)}{=} \{0\}$$

and we deduce that

$$x = x_{\gamma_1} + \ldots + x_{\gamma_n} \in S_\xi \cap \sum_{\gamma \in \Gamma_0 \backslash \{\xi\}} S_\gamma = \{0\} \text{ as } \Gamma_0 \in \mathcal{E}.$$

Let us prove that

$$L \cap \sum_{\gamma \in \Xi} S_\gamma = \{0\}.$$

If

$$L \cap \sum_{\gamma \in \Xi} S_\gamma \neq \{0\}$$

then there is an element

$$0 \neq x \in L \cap \sum_{\gamma \in \Xi} S_\gamma.$$

Write

$$x = x_{\Gamma_0} + x_\lambda \text{ where } x_{\Gamma_0} \in \sum_{\gamma \in \Gamma_0} S_\gamma \text{ and } x_\lambda \in S_\lambda.$$

Then

$$x_\lambda = x - x_{\Gamma_0} \in L + \sum_{\gamma \in \Gamma_0} S_\gamma$$

and from $S_\lambda \nsubseteq L + \sum_{\gamma \in \Gamma_0} S_\gamma$ we deduce that $x_\lambda = 0$. Hence

$$x = x_{\Gamma_0} \in L \cap \sum_{\gamma \in \Gamma_0} S_\gamma = \{0\}.$$

Therefore $x = x_{\Gamma_0} + x_\lambda = 0$. Contradiction.

We conclude that $\Xi \in \mathcal{E}$ and $\Gamma_0 < \Xi$ which contradicts the maximality of Γ_0. $\square$

Corollary 8.5. *Let M_R be semisimple right R-module and let $(S_\lambda)_{\lambda \in \Lambda}$ be a family of right simple R-submodules such that*

$$M_R = \dot{\bigoplus_{\lambda \in \Lambda}} S_\lambda.$$

Let L be a submodule of M_R. Then there is a subset Ξ of Λ such that

$$L \cong \bigoplus_{\xi \in \Xi} S_\xi \text{ and } M/L \cong \bigoplus_{\lambda \in \Lambda \backslash \Xi} S_\lambda.$$

In particular L and M/L are semisimple.

Proof. By Lemma 8.4, there exists a subset $\Gamma \subseteq \Lambda$ such that

$$M = L \oplus \dot{\bigoplus_{\gamma \in \Gamma}} S_\gamma.$$

Then

$$L \cong M / \dot{\bigoplus_{\gamma \in \Gamma}} S_\gamma \text{ and } M/L \cong \dot{\bigoplus_{\gamma \in \Gamma}} S_\gamma$$

Since

$$M = \dot{\bigoplus_{\gamma \in \Gamma}} S_\gamma \oplus \dot{\bigoplus_{\lambda \in \Lambda \setminus \Gamma}} S_\lambda$$

we get

$$L \cong M / \dot{\bigoplus_{\gamma \in \Gamma}} S_\gamma \cong \dot{\bigoplus_{\lambda \in \Lambda \setminus \Gamma}} S_\lambda.$$

□

Theorem 8.6. *Let M_R be a right R-module. Then the following statements are equivalent;*

(a) *M is semisimple.*

(b) *M is a sum of a family of simple submodules.*

(c) *M is the sum of all its simple submodules.*

(d) *Every submodule of M is a direct summand of M.*

(e) *Every short exact sequence*

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

splits.

Proof. $(a) \Rightarrow (b) \Rightarrow (c)$ is trivial.

$(c) \Rightarrow (a)$ and $(c) \Rightarrow (d)$. They follow by Lemma 8.4.

$(d) \Leftrightarrow (e)$ It follows by Theorem 1.84.

$(d) \Rightarrow (c)$.

Let L be a submodule of a left R-module M. First of all let us prove that every submodule H of L is a direct summand of L. In fact, by assumption, there is a submodule K of M such that

$$M = H \oplus K$$

so that (exercise)

$$L = H \oplus (K \cap L).$$

Let us prove that every non-zero submodule L of M_R contains a simple submodule. Since $L \neq \{0\}$, there is an $x \in L$, $x \neq 0$. Let $V \leq L_R$ be a submodule maximal with respect to the property $x \notin V$. Let U/V be a non-zero submodule of $R(x+V)$. Since $V \subsetneqq U$ we get that $x \in V$ so that

$$U/V = R(x+V).$$

Therefore $R(x+V)$ is simple. By the foregoing, there is a submodule W of L such that

$$L = V \oplus W.$$

Since $W \cong V/L$, we deduce that W has a simple submodule. □

Definition 8.7. *A ring R is said to be* right semisimple *if the right R-module R_R is semisimple.*

Theorem 8.8. *Let R be a ring. The following statements are equivalent.*

(a) *Every right R-module is semisimple.*

(b) *Every short exact sequence*

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

splits.

(c) *Every right R-module is projective.*

(d) *Every right R-module is injective.*

(e) *R is right semisimple i.e. R_R is semisimple.*

(f) *R_R is a sum of a family of simple right ideals.*

(g) *R_R is a sum of a finite family of simple right ideals.*

(h) *R_R is a direct sum of a finite family of simple right ideals.*

Proof. $(a) \iff (b)$. It follows by Theorem 8.6.

$(b) \iff (c)$. It follows by Proposition 2.17.

$(b) \iff (d)$. It follows by Proposition 3.29.

$(e) \iff (f)$. It follows by Theorem 8.6.

$(f) \Rightarrow (g)$. Let $(S_\lambda)_{\lambda \in \Lambda}$ be a family of right simple R-modules such that $R_R = \sum_{\lambda \in \Lambda} S_\lambda$. Then there is a finite subset $F \subseteq \Lambda$ and elements $x_\lambda \in S_\lambda, \lambda \in F$, such that

$$1 = \sum_{\lambda \in F} x_\lambda.$$

Then

$$R_R = 1 \cdot R \subseteq \sum_{\lambda \in F} S_\lambda.$$

$(g) \Rightarrow (f)$. It is trivial.
$(g) \Rightarrow (h)$. It follows by Proposition 2.17.
$(h) \Rightarrow (g)$. It is trivial
$(a) \Rightarrow (e)$. It is trivial.
$(e) \Rightarrow (a)$. Let M_R be a right R-module. By Proposition 2.2, there is an epimorphism

$$h : {R_R}^{(M)} \to M.$$

Since R_R is semisimple, ${R_R}^{(M)}$ is semisimple so that, by Corollary 8.5, M is semisimple too. □

Theorem 8.9. *Let D be a division ring and let $n \in \mathbb{N}, n \geq 1$. Let $R = M_n(D)$. Then*

1) *There is, up to isomorphism, only one simple right R-module V_R and $R_R \cong (V_R)^n$.*

2) *R is right semisimple.*

1') *There is, up to isomorphism, only one simple left R-module ${}_RW$ and ${}_RR \cong ({}_RW)^n$.*

2') *R is left semisimple.*

Proof. 1) Let e_{ij} be the matrix with all zero entries except for (i, j) where the entry is 1_D. For any matrix $A \in M_n(D)$ let A_i denote its i-th row and A^i its i-th column. Set

$$S_i = e_{ii}R.$$

Since

$$e_{ii}e_{hk} = \delta_{ih}e_{ik}$$

we have that

$$S_i = \sum_{k=1}^{n} e_{ik}D = \{A \in M_n(D) \mid A_t = 0 \text{ for every } t \neq i\}.$$

Then S_i is a right R-module and

$$R = S_1 \dot{\oplus} \ldots \dot{\oplus} S_n.$$

Let us check that S_i is a simple right R-module. Let $x \neq 0, x \in S_i$. Then there exist element $d_k, k = 1, \ldots, n$ such that

$$x = \sum_{k=1}^{n} e_{ik}d_k$$

and since $x \neq 0$ there is at least a $w \in \{1, \ldots, n\}$ such that $d_w \neq 0$. For every $t \in \{1, \ldots, n\}$, let

$$r_t = d_w^{-1} e_{wt}.$$

Then

$$xr_t = \sum_{k=1}^{n} e_{ik} d_k d_w^{-1} e_{wt} = e_{it}.$$

Therefore $xR \supseteq S_i$ and hence S_i is simple.

Let us check that, for every $j \in \{1, \ldots, n\}$, $S_i \cong S_j$. Let us consider the homomorphism

$$\mu_{ij} : R_R \to R_R$$

defined by setting

$$\mu_{ij}(r) = e_{ji} \cdot r.$$

Then

$$\mu_{ij}(S_i) = \mu_{ij}\left(\sum_{k=1}^{n} e_{ik} D\right) = e_{ji} \cdot \left(\sum_{k=1}^{n} e_{ik} D\right) = \sum_{k=1}^{n} e_{jk} D = S_j.$$

Since S_i is simpe, this implies that $S_i \cong S_j$.

Let now S be a simple right R-module and let $x \in S, x \neq 0$. Then the epimorphism

$$\begin{array}{rccc} h_x : & R & \longrightarrow & xR = S \\ & r & \longmapsto & xr \end{array}$$

is non-zero. Since $R = S_1 \oplus \ldots \oplus S_n$, this implies tht there is a $j, 1 \leq j \leq n$, such that $h_x(S_j) \neq \{0\}$. Since S_j and S are both simple, this implies that $h_{x|S_j} : S_j :\to S$ is an isomorphism.

2) It is now trivial.

1') It can be proved in an analogous way working on the left instead of the right side. □

Lemma 8.10. *Let R be a ring and let M and M' be simple right R-module. Let $f : M \to M'$ be a left R-module homomorphism and assume that $f \neq 0$. Then*

1) *If M is simple, f is a is monomorphism.*

2) *If M' is simple, then f is an epimorphism.*

3) *If both M and M' are simple, then f is an isomorphism.*

Proof. Since $f \neq 0$, then $\text{Ker}(f) \subsetneqq M$ and $\{0\} \subsetneqq Im(f) \subseteq M'$. Thus M simple implies $\text{Ker}(f) = \{0\}$ while m' simple implies $Im(f) = M'$. □

Lemma 8.11. ***(Schur's Lemma)*** *Let R be a ring and let S_R be a simple right R-module. Then $D = End(S_R)$ is a division ring.*

Proof. Let $f \in D$, $f \neq 0$. Then, by Lemma 8.10, f is an isomorphism. □

Lemma 8.12. *Let R be a ring, let S_R be a simple right R-module and let $n \in \mathbb{N}, n \geq 1$. Then*

$$\mathrm{End}_R\left(S_R^n\right) \cong M_n\left(D\right)$$

where $D = End(S_R)$.

Proof. For every $1 \leq h, k \leq n$ let $i_h : S \to S^n$ be the h-th canonical injection and let $p_k : S^n \to S$ be the k-th canonical projection. Let

$$\varphi : \mathrm{End}_R\left(S_R^n\right) \to M_n\left(D\right)$$

be the map defined by setting

$$\varphi\left(f\right) = \sum_{h,k=1}^{m} \left(p_h \circ f \circ i_k\right) e_{hk} \text{ for every } f \in \mathrm{End}_R\left(S_R^n\right)$$

Let us check that φ is a ring homomorphism. Let $f, g \in \mathrm{End}_R\left(S_R^n\right)$. Then

$$\varphi\left(f \circ g\right) = \varphi\left(f\right) \cdot \varphi\left(g\right).$$

$$\varphi\left(f \circ g\right) = \sum_{h,k=1}^{m} \left(p_h \circ f \circ g \circ i_k\right) e_{hk} = \left[\sum_{h,k=1}^{m} p_h \circ f \circ \left(\sum_{v=1}^{n} i_v \circ p_v\right) \circ g \circ i_k\right] e_{hk} =$$
$$= \left(\sum_{h,k,v=1}^{m} p_h \circ f \circ i_v \circ p_v \circ g \circ i_k\right) e_{hk} = \left[\sum_{h,v=1}^{m} \left(p_h \circ f \circ i_v\right) e_{hv}\right]\left[\sum_{k,t=1}^{m} \left(p_t \circ g \circ i_k\right) e_{tk}\right]$$
$$= \varphi\left(f\right) \cdot \varphi\left(g\right).$$

The other checkings are straightforward. It is an easy exercise to prove that φ is bijective. □

Theorem 8.13. *Let R be a right semisimple ring. Then there exists a $k \in \mathbb{N}, k \geq 1$ and $n_1, \ldots, n_k \in \mathbb{N}, n_1, \ldots, n_k \geq 1$ and division rings $D_1, \ldots, D_k$ such that*

$$R \cong M_{n_1}\left(D_1\right) \times \ldots \times M_{n_k}\left(D_k\right) \text{ as rings.}$$

Proof. By Theorem 8.8, there is a finite set F such that

$$R_R = \dot{\bigoplus_{i \in F}} S_i$$

where each S_i is simple. For each $i \in F$ let

$$F_i = \left\{j \in F \mid S_j \cong S_i\right\}.$$

Note that

$$S_i \cong S_j \Longleftrightarrow F_i = F_j$$

Let

$$m = |\{F_i \mid i \in F\}|$$

and let $F_{i_1}, \ldots, F_{i_m}$ be such that

$$\{F_i \mid i \in F\} = \{F_{i_1}, \ldots, F_{i_m}\}.$$

Then

$$F = \bigcup_{i \in I} F_i = F_{i_1} \cup \ldots \cup F_{i_m}.$$

Note that

$$j \in F_{i_t} \iff S_j \cong S_{i_t} \iff F_j = F_{i_t}$$

For every $t \in \{1, \ldots, , m\}$ let $n_t = |F_{i_t}|$ and let

$$\Sigma_t = \bigoplus_{j \in F_{i_t}} S_j \cong (S_{i_t})^{n_t}.$$

Note that, $t, u \in \{1, \ldots, , m\}$ and $t \neq u$ implies that for each $j \in F_{i_t}$ and for every $h \in F_{i_u}$, $S_j \ncong S_h$. Infact $j \in F_{i_t}$ implies that $F_j = F_{i_t}$ and $h \in F_{i_u}$ implies that $F_h = F_{i_u}$. Now $S_j \cong S_h$ implies $F_j = F_h$ so that we get $F_{i_t} = F_j = F_h = F_{i_u}$ which implies that $t = u$. Hence, by Lemma 8.10, we have that

$$\mathrm{Hom}_R(S_j, S_h) = \{0\}.$$

This implies that

$$\mathrm{Hom}_R(\Sigma_t, \Sigma_u) = \{0\}$$

and hence

$$R \cong \mathrm{End}(R_R) \cong \mathrm{Hom}_R\left(\bigoplus_{t=1}^{m} \Sigma_t, \bigoplus_{t=1}^{m} \Sigma_t\right) \cong \mathrm{End}_R(\Sigma_1) \times \ldots \times \mathrm{End}_R(\Sigma_t).$$

In view of Lemma 8.12, we conclude. □

Exercise 8.14. *Let R_1 and R_2 be right semisimple rings. Then $R_1 \times R_2$ is right semisimple.*

Theorem 8.15. *Let R be a ring. Then R is right semisimple if and only if R is left semisimple.*

Proof. Assume that R is right semisimple. By Theorem 8.13, there exists a $k \in \mathbb{N}, k \geq 1$ and $n_1, \ldots, n_k \in \mathbb{N}, n_1, \ldots, n_k \geq 1$ and division rings $D_1, \ldots, D_k$ such that

$$R \cong M_{n_1}(D_1) \times \ldots \times M_{n_k}(D_k) \text{ as rings.}$$

By Theorem 8.9, each $M_{n_i}(D_i)$ is a left semisimple ring. □

Lemma 8.16. *Let $(L_i)_{i\in I}$ be a chain of submodules of a right R-module M. Then*

$$L = \bigcup_{i\in I} L_i$$

is a submodule of M.

Proof. Let $x, y \in L$ and let $r \in R$. Then there are $i, j \in I$ such that $x \in L_i$ and $y \in L_j$. Since $(L_i)_{i\in I}$ is a chain, we have that $L_i \cup L_j = L_h$ where $h \in \{i, j\}$ and hence $x - y \in L_h \subseteq L$. On the other hand $rx \in L_i \subseteq L$. □

Lemma 8.17 (Generalized Krull's Lemma)**.** *Every non-zero finitely generated right R-module M has a maximal submodule. In particular any proper right ideal I of R is contained in a maximal right ideal of R.*

Proof. Let M be a non-zero finitely generated right R-module. We set

$$\mathcal{E} = \{L \mid L \subsetneqq M\}.$$

Since $\{0\} \subsetneqq M$ we have that $\{0\} \in \mathcal{E}$ and hence $\mathcal{E} \neq \varnothing$. Let us prove that $(\mathcal{E}, \subseteq)$ is inductive. Let $(L_i)_{i\in I}$ be a chain of elements of $\mathcal{E}$ and let

$$L = \bigcup_{i\in I} L_i.$$

By Lemma 8.16, L is a submodule of M.

Now we claim that $L \subsetneqq M$. In fact, assume that $M = L$ and let $\{x_1, \ldots, x_n\}$ be a set of generators of M. Then, for any $i \in \{1, \ldots, n\}$, there is a $j_i \in I$ such that $x_i \in L_{i_j}$. Since $(L_i)_{i\in I}$ is a chain, there is a $t \in \{1, \ldots, n\}$ such that

$$L_{i_1} \cup \ldots \cup L_{i_n} = L_{i_t}.$$

We deduce that

$$M = x_1R + \ldots + x_nR \subseteq L_{i_t}$$

and hence we get that $M = L_{i_t}$. Since $L_{i_t} \in \mathcal{E}$, this is a contradiction. Thus $L \in \mathcal{E}$ and L is an upper bound for the chain $(L_i)_{i\in I}$. We deduce that $(\mathcal{E}, \subseteq)$ is inductive so that, by Zorn's Lemma, it has a maximal element. If L_0 is a maximal element of $\mathcal{E}$ then L_0 is not properly contained in any proper submodule of M i.e. L_0 is a maximal submodule of M.

If I is a proper right ideal of R, then the right R-module R/I is finitely generated and nozero. Hence it has a maximal submodule L/I. Then L is a maximal right ideal of R which contains I. □

Notations 8.18. *Let R be a ring. We set*

$\Omega_l = \Omega_l(R) = \{L \mid L$ *is a maximal left ideal of* $R\}$

$\Omega_r = \Omega_r(R) = \{M \mid M$ *is a maximal right ideal of* $R\}$

${}_R\mathcal{S} = \{S \in {}_R\mathcal{M} \mid S$ *is a simple left* R*-module*$\}$

$\mathcal{S}_R = \{S \in \mathcal{M}_R \mid S$ *is a simple right* R*-module*$\}$

Definition 8.19. *A left ideal I of R is called left primitive if there is a simple left R-module S such that $I = \mathrm{Ann}_R(S)$.*

Exercise 8.20. *Every left primitive ideal of R is a two-sided ideal of R.*

Notation 8.21. *Let $\mathcal{P}_l$=$\{I \leq_R R \mid I$ is left primitive$\}$*

Lemma 8.22. *Let R be a ring. Then*

$$\bigcap_{L\in\Omega_l} L = \bigcap_{I\in\mathcal{P}_l} I.$$

In particular $\bigcap_{L\in\Omega_l} L$ is a two-sided ideal of R.

Proof. Let $I \in \mathcal{P}_l$ and let S be a simple left R-module such that

$$I = \mathrm{Ann}_R(S) = \bigcap_{\substack{x\in S\\ x\neq 0}} \mathrm{Ann}_R(x).$$

By Proposition 4.10, for every $x \in S$, $x \neq 0$ we have that $Rx = S$ and by Proposition 4.11 we get that $\mathrm{Ann}_R(x)$ is a left maximal ideal of R. This implies that

$$I \supseteq \bigcap_{L\in\Omega_l} L$$

so that

$$\bigcap_{I\in\mathcal{P}_l} I \supseteq \bigcap_{L\in\Omega_l} L.$$

On the other hand, if $L \in \Omega_l$, then

$$R(1+L) = R/L$$

is a simple left R-module and

$$L = \mathrm{Ann}_R(1+L) \supseteq \mathrm{Ann}_R(R(1+L)) \in \mathcal{P}_l$$

so that

$$L \supseteq \bigcap_{I\in\mathcal{P}_l} I$$

and hence

$$\bigcap_{L\in\Omega_l} L \supseteq \bigcap_{I\in\mathcal{P}_l} I.$$

□

Theorem 8.23. *Let R be a ring. Then*

$$\bigcap_{L\in\Omega_l} L = \bigcap_{M\in\Omega_r} M$$

Proof. Let $H = \bigcap_{M \in \Omega_r} M$ and let us prove that $\bigcap_{L \in \Omega_l} L \subseteq H$. Thus let $r \in \bigcap_{L \in \Omega_l} L$ and let $M \in \Omega_r$. Let us assume that $r \notin M$. Then

$$M + rR = R$$

and hence there is an $x \in M$ and an $s \in R$ such that

$$1 = x + rs.$$

Since, by Lemma 8.22, $\bigcap_{L \in \Omega_l} L$ is a two-sided ideal of R, we get that $rs \in \bigcap_{L \in \Omega_l} L$. Hence $1 - rs \notin L$ for every $L \in \Omega_l$ and hence, by Krull's Lemma 8.17, $R(1 - rs) = R$. Then there is an element $t \in R$ such that

$$t \cdot (1 - rs) = 1. \tag{8.4}$$

Then we get

$$t = 1 + trs.$$

Since $\bigcap_{L \in \Omega_l} L$ is a two-sided ideal of R, we know that $trs \in \bigcap_{L \in \Omega_l} L$. Hence $1 + trs \notin L$ for every $L \in \Omega_l$ so that, by Krull's Lemma, $R(1 + trs) = R$. Thus there is a $v \in R$ such that

$$v(1 + trs) = 1.$$

Then

$$v \cdot t = v(1 + trs) = 1 \tag{8.5}$$

so that

$$v = v \cdot 1 \overset{(8.4)}{=} v \cdot t \cdot (1 - rs) \overset{(8.5)}{=} 1 - rs$$

and hence

$$v = 1 - rs. \tag{8.6}$$

Therefore we get

$$(1 - rs) \cdot t \overset{(8.6)}{=} v \cdot t \overset{(8.5)}{=} 1.$$

Thus we deduce that

$$(1 - rs) \cdot t = 1. \tag{8.7}$$

Thus from (8.4) and from (8.7), we obtain that $(1 - rs)$ is invertible in R. Since

$$1 - rs = x \in M$$

this is a contradiction. □

Definition 8.24. *Let R be a ring. We set*

$$J(R) = \bigcap_{L \in \Omega_l} L \overset{Theo8.23}{=} \bigcap_{M \in \Omega_r} M$$

$J(R)$ is called the Jacobson radical *of R.*

Theorem 8.25 (Nakayama's Lemma)**.** *Let I be a right ideal of a ring R. The following statements are equivalent.*

(*a*) $I \subseteq J(R)$.

(*b*) *For every finitely generated right R-module M, $M = MI$ implies that $M = \{0\}$.*

(*c*) *For any submodule L of a right R-module M, if M/L is finitely generated and $L + MI = M$, then $L = M$.*

Proof. $(a) \Rightarrow (b)$. Assume that $M \neq \{0\}$ is a finitely generated. By Krull's Lemma 8.17, M contains a maximal submodule L. Thus we get that $S = M/L$ is a simple right R-module. By Proposition 4.10, for every $x \in S$, $x \neq 0$ we have that $xR = S$ and by Proposition 4.11 we get that

$$\mathrm{Ann}_R(x) = \{r \in R \mid x \cdot r = 0\}$$

is a right maximal ideal of R so that

$$I \subseteq J(R) \subseteq \mathrm{Ann}_R(x)$$

and hence

$$xI = \{0\} \text{ for every } x \in S, x \neq 0.$$

Thus we deduce that $SI = \{0\}$ i.e.

$$\frac{MI + L}{L} = \frac{M}{L} \cdot I = \{0\}$$

which means that

$$MI + L = L$$

i.e. $M = MI \subseteq L$ which contradicts the maximality of L.

$(b) \Rightarrow (c)$. Since M/L is finitely generated and $L + MI = M$ implies that

$$\frac{M}{L} \cdot I = \frac{MI + L}{L} = \frac{M}{L}$$

we deduce that $M/L = \{0\}$ i.e. $M = L$.

$(c) \Rightarrow (a)$. Assume that $I \nsubseteq J(R)$. Then there is an $x \in I$ and a right maximal ideal L of R such that $x \notin L$. This implies that $L + xR = R$ and hence $L + I = R$. Therefore we get that R/L is finitely generated and $L + RI = R$. By (b) we deduce that $L = R$, a contradiction. □

Proposition 8.26. *Let R be a ring. The following statements are equivalent*

(a) *R has a unique maximal right ideal.*

(b) *$J(R)$ is a maximal right ideal.*

(c) *$R/J(R)$ is a division ring.*

(a') *R has a unique maximal left ideal.*

(b') *$J(R)$ is a maximal left ideal.*

Proof. $(a) \Rightarrow (b)$. It is trivial.

$(b) \Rightarrow (c)$. Since $J(R)$ is a maximal right ideal the right R-module $R/J(R)$ is simple. Let $\overline{R} = R/J(R)$. Then $\overline{R}_{\overline{R}}$ is simple and

$$\overline{R} = \mathrm{End}_{\overline{R}}\left(\overline{R}_{\overline{R}}\right).$$

By Schur's Lemma 8.11, $\overline{R}$ is a division ring.

$(c) \Rightarrow (a)$ Let L be a right maximal ideal of R. Then

$$\frac{L}{J(R)} = \left\{0_{\frac{R}{J(R)}}\right\} = \frac{J(R)}{J(R)}$$

so that $L = J(R)$. Hence R has a unique maximal right ideal.

The equivalences $(a') \Leftrightarrow (b') \Leftrightarrow (c)$ follow by simmetry. □

Definition 8.27. *A ring R is satisfying the equivalent conditions of Proposition 8.26 is called a* local ring.

Proposition 8.28. *Let R be a local ring and let $J = J(R)$. Let M be a right R-module and assume that the elements*

$$x_1 + MJ, \dots, x_n + MJ$$

are a set of generators of M/MJ as a vector space over R/J. Then $x_1, \dots, x_n$ generate M.

Proof. Let $N = x_1R + \ldots + x_nR$. Then

$$\frac{M}{MJ} = \frac{N + MJ}{MJ}$$

so that $M = N + MJ$. Since M/MJ is finitely generated, by Nakayama's Lemma 8.25 we deduce that $M = N$. □

Chapter 9

Chain Conditions.

9.1. *Throught this chapter R will denote a ring.*

Definitions 9.2. *Let M be a right R-module.*
We say that

- *M satisfies the* Ascending Chain Condition *(*A.C.C.*)* on submodules *if for every ascending chain*

$$M_0 \leq M_1 \leq \cdots \leq M_n \leq \cdots$$

  of submodules of M there is an $n \in \mathbb{N}$ tale che $M_i = M_n$ for every $i \geq n$.

- *M satisfies the* Maximum Condition on submodules, *if every nonempty set of submodules of M has a maximal element.*

Definitions 9.3. *Let M be a right R-module.*
We say that

- *M satisfies the* Descending Chain Condition *(*D.C.C.*)* on submodules *if for every descending chain*

$$\cdots \leq M_n \leq \cdots \leq M_1 \leq M_0$$

  of submodules of M there is an $n \in \mathbb{N}$ tale che $M_i = M_n$ for every $i \geq n$.

- *M satisfies the* Minimum Condition on submodules, *if every nonempty set of submodules of M has a minimal element.*

Theorem 9.4. *Let M be a right R-module. The following statements are equivalent.*

(*a*) *M satisfies the Ascending Chain Condition on submodules.*

(*b*) *M satisfies the Maximum Condition on submodules..*

(*c*) *Every submodule of M is finitely generated.*

Proof. $(a) \Rightarrow (b)$ Let $\mathcal{F}$ be a nonempty set of submodules of M. Since $\mathcal{F}$ is nonempty, then there is a submodule $M_0 \in \mathcal{F}$. Assume that $\mathcal{F}$ has no maximal element. Then, for each element $L \in \mathcal{F}$ there is at least an element $L' \in \mathcal{F}$ such that $L \subsetneqq L'$. For each $L \in \mathcal{F}$ we can choose, by the Axiom of Choice, one such L'. Let

$$\begin{array}{rccc} f: & \mathcal{F} & \longrightarrow & \mathcal{F} \\ & L & \longmapsto & L'. \end{array}$$

By the Recursion Theorem, there is a map $f_{M_0} : \mathbb{N} \to \mathcal{F}$ such that

$$f_{M_0}(0) = M_0 \text{ and } f_{M_0}(n+1) = f(f_{M_0}(n)) = (f_{M_0}(n))'.$$

This implies that

$$f_{M_0}(n) \subsetneqq (f_{M_0}(n))' \text{ for every } n \in \mathbb{N}.$$

For every $n \in \mathbb{N}$, let us set

$$M_n = f_{M_0}(n).$$

Then, for every $n \in \mathbb{N}$, we get

$$M_n \subsetneqq M_{n+1}$$

and hence a strictly ascending chain

$$M_0 \subsetneqq M_1 \subsetneqq \cdots \subsetneqq M_n \subsetneqq M_{n+1} \subsetneqq \cdots \subsetneqq$$

which contradicts A.C.C..

$(b) \Rightarrow (c)$ Let L be an R-submodule of M and set

$$\mathcal{F} = \{N_R \leq L_R \mid N_R \text{ is finitely generated}\}.$$

Since $\{0\} = 0R \in \mathcal{F}$, we have that $\mathcal{F} \neq \varnothing$ so that $\mathcal{F}$ has a maximal element N. Let us show that $L = N$. Let $x \in L$. Then

$$N + xR \leq L \qquad \text{and} \qquad N + xR \text{ is finitely generated.}$$

Hence $L \in \mathcal{F}$. Since $N \leq L$, by the maximality property of N we deduce that

$$N = N + xR$$

so that $x \in N$.
$(c) \Rightarrow (a)$ Let

$$M_0 \leq M_1 \leq \cdots$$

be a chain of submodules of M. By Lemma 8.16, $L = \cup_{i \in \mathbb{N}} M_i$ is a submodule of M. Hence there is an $n \in \mathbb{N}, n \geq 1$ and elements $x_1, \ldots, x_n \in L$ such that

$$L = x_1 R + \ldots + x_n R.$$

For every $i \in \{1, \ldots, n\}$ there is a $j_i \in \mathbb{N}$ such that $x_i \in L_{j_i}$. Let $t = \max\{j_1, \ldots, j_n\}$. Then

$$L = x_1 R + \ldots + x_n R \subseteq M_{i_1} \cup \ldots \cup M_{j_n} \subseteq M_t$$

so that, for every $i \in \mathbb{N}$

$$M_i \subseteq L \subseteq M_t.$$

This implies that, for every $i \geq t$, we have $M_i = M_t$. □

Definition 9.5. *Let M_R be a right R-module. We say that M_R is* noetherian if *M satisfies one of the equivalent conditions of Theorem 9.4.*

Definition 9.6. *Let M_R be a right R-module. We say that M is* finitely cogenerated *if, for every set $\mathcal{L}$ of submodules of M tale che*

$$\bigcap_{L\in\mathcal{L}} L = \{0\}$$

there is a finite subset F of $\mathcal{L}$ such that

$$\bigcap_{L\in F} L = \{0\}.$$

Definition 9.7. *Let M_R be a right R-module. We say that M is* finitely embedded *if its socle is essential and finitely generated.*

Lemma 9.8. *Let H_R be a semisimple right R-module. H_R is finitely cogenerated $\Leftrightarrow$ $H = \bigoplus_{\lambda\in F} S_\lambda$ where F is a finite set and each $S_\lambda \in \mathcal{S}_r$.*

Proof. ($\Rightarrow$). Let $(S_\lambda)_{\lambda\in\Lambda}$ be a family of right R-modules such that

$$H = \bigoplus_{\lambda\in\Lambda} S_\lambda.$$

For each $\gamma \in \Lambda$ set

$$H_\gamma = \bigoplus_{\lambda\in\Lambda\setminus\{\gamma\}} S_\lambda.$$

Let $x \in \bigcap_{\gamma\in\Lambda} H_\gamma$. Then

$$Supp\,(x) \subseteq \bigcap_{\gamma\in\Lambda} (\Lambda\setminus\{\gamma\}) = \varnothing$$

so that $x = 0$. Thus we get that

$$\bigcap_{\gamma\in\Lambda} H_\gamma = \{0\}.$$

Since H is finitely cogenerated, there is a finite subset $F \subseteq \Lambda$ such that

$$\bigcap_{\gamma\in F} H_\gamma = \{0\}.$$

Then we have

$$\{0\} = \bigcap_{\gamma\in F} H_\gamma = \bigoplus_{\lambda\in\Lambda\setminus F} S_\lambda$$

i.e.

$$H = \bigoplus_{\lambda\in F} S_\lambda.$$

($\Leftarrow$) Assume that $H = \bigoplus_{\lambda \in F} S_\lambda$ where F is a finite set and each $S_\lambda \in \mathcal{S}_r$. Let $\mathcal{F}$ be a set of submodules of H such that

$$\bigcap_{L \in A} L \neq \{0\}$$

for each finite subset A of $\mathcal{F}$ and let us show that

$$\bigcap_{L \in \mathcal{F}} L \neq \{0\}$$

Let us proceed by induction on $|F|$. If $|F| = 1$, then $F = \{\lambda\}$ and $H = S_\lambda$ so that there is nothing to prove. Let us assume that our statement hold true for some $n \in \mathbb{N}, n \geq 1$ and let us prove it for $n+1$. Let us fix a $\lambda_0 \in F$ and let us write

$$H = T \oplus S_{\lambda_0} \text{ where } T = \bigoplus_{\lambda \in F \setminus \{\lambda_0\}} S_\lambda$$

In the case when, or each finite subset A of $\mathcal{F}$, we have

$$\bigcap_{L \in A} (L \cap T) \neq \{0\}$$

then, by Induction hypothesis, we get that $\bigcap_{L \in \mathcal{F}} (L \cap T) \neq \{0\}$ and hence $\bigcap_{L \in \mathcal{F}} L \neq \{0\}$. Otherwise there is a finite subset A of of $\mathcal{F}$ such that

$$\bigcap_{L \in A} (L \cap T) = \{0\}.$$

Let $K = \bigcap_{L \in A} L$. Then

$$\{0\} \neq K \cong \frac{K}{K \cap T} \cong \frac{T+K}{T} \subseteq \frac{H}{T} = \frac{T \oplus S_{\lambda_0}}{T} \cong S_{\lambda_0}$$

and hence $K \cong S_{\lambda_0}$ so that K is a simple right submodule of H. We have

$$\bigcap_{L \in \mathcal{F}} L = \left(\bigcap_{L \in \mathcal{F}} L \right) \cap \left(\bigcap_{L \in A} L \right) = \left(\bigcap_{L \in \mathcal{F}} L \right) \cap K = \bigcap_{L \in \mathcal{F}} (L \cap K).$$

Since,

$$L \cap K = \bigcap_{N \in A \cup \{L\}} N \neq \{0\}$$

we deduce that $K \subseteq L$ for every $L \in \mathcal{F}$ and hence we conclude that

$$\bigcap_{L \in \mathcal{F}} L \neq \{0\}.$$

$\square$

Proposition 9.9. *Let M_R be a right R-module. The following statements are equivalent.*

(a) M_R is finitely cogenerated.

(b) M_R is finitely embedded.

Proof. $(a) \Rightarrow (b)$. Let $\{0\} \neq L$ be a submodule of M_R and let us set

$$\mathcal{E} = \{H \mid \{0\} \neq H \subseteq L\}.$$

Clearly $L \in \mathcal{E}$ so that $\mathcal{E} \neq \varnothing$. Let us consider the partially ordered set

$$(\mathcal{E}, \supseteq)$$

and let us prove it is inductive. Let $(H_i)_{i \in I}$ be a chain in $(\mathcal{E}, \supseteq)$ and let

$$H = \bigcap_{i \in I} H_i.$$

Let us show that $H \in \mathcal{E}$ i.e. that $H \neq \{0\}$. In fact assume that $H = \{0\}$. Since M_R is finitely cogenerated, there is a finite subset $F \subseteq I$ such that

$$\bigcap_{i \in F} H_i = \{0\}.$$

Since $(H_i)_{i \in I}$ is a chain in $(\mathcal{E}, \supseteq)$, there is an element $t \in F$ such that

$$H_i \supseteq H_t \text{ for every } i \in F$$

so that

$$\{0\} = \bigcap_{i \in F} H_i \supseteq H_t$$

which yields a contradiction since $H_t \in \mathcal{E}$. Thus $H \in \mathcal{E}$ and H is an upper bound for the chain $(H_i)_{i \in I}$ in $(\mathcal{E}, \supseteq)$. Hence, by Zorn's Lemma, there is at least a maximal element, say H_L in $(\mathcal{E}, \supseteq)$. Let us prove that H_L is simple. Let $0 \neq x \in H_L$. Then $\{0\} \neq x \cdot R \subseteq H_L \subseteq L$ so that $x \cdot R \in \mathcal{E}$ and hence, by the maximality property of H_L in $(\mathcal{E}, \supseteq)$. we get that $x \cdot R = H_L$. Therefore H_L is simple.

Hence every nonzero submodule L of M_R contains a simple right R-module which implies that $\mathrm{Soc}\,(M)$ is essential in M. Since $\mathrm{Soc}\,(M)$ is a submodule of M_R and M_R is finitely cogenerated, also $\mathrm{Soc}\,(M)$ is finitely cogenerated. By Lemma 9.8, we deduce that $\mathrm{Soc}\,(M) = \bigoplus_{\lambda \in F} S_\lambda$ where F is a finite set and each $S_\lambda \in \mathcal{S}_r$.

$(b) \Rightarrow (a)$. Assume that $\mathrm{Soc}\,(M) = \bigoplus_{\lambda \in F} S_\lambda$ where F is a finite set and each $S_\lambda \in \mathcal{S}_r$. Let $\mathcal{F}$ be a set of submodules of M such that

$$\bigcap_{L \in \mathcal{F}} L = \{0\}.$$

Then we have

$$\bigcap_{L\in\mathcal{F}}[\mathrm{Soc}\,(M)\cap L]=\mathrm{Soc}\,(M)\cap\bigcap_{L\in\mathcal{F}}L=\{0\}\,.$$

By Lemma 9.8, we deduce that there is a finite subset A of $\mathcal{F}$ such that

$$\bigcap_{L\in A}[\mathrm{Soc}\,(M)\cap L]=\{0\}$$

Since $\mathrm{Soc}\,(M)=\bigoplus_{\lambda\in F}S_\lambda$ is essential in M, we get that

$$\bigcap_{L\in A}L=\{0\}\,.$$

□

Theorem 9.10. *Let M be a right R-module. The following statements are equivalent.*

(a) *M satisfies the Descending Chain Condition on submodules.*

(b) *M satisfies the Minimum Condition on submodules.*

(c) *Every quotient of M is finitely cogenerated.*

Proof. $(a)\Rightarrow(b)$. It is analogous to $(a)\Rightarrow(b)$ in Theorem 9.4.
$(b)\Rightarrow(c)$ Let L be a submodule of M_R and let $\mathcal{Q}$ be a nonempty set of submodules of M/L such that

$$\bigcap_{Q\in\mathcal{Q}}Q=\{0\}\,.$$

Now, for every $Q\in\mathcal{Q}$, there is a submodule $L_Q\leq M$ such that

$$Q=\frac{L_Q}{L}$$

Let

$$\mathcal{F}=\left\{\bigcap_{Q\in F}L_Q\mid F\subseteq\mathcal{Q}\text{ and }F\text{ is finite}\right\}.$$

Since $\mathcal{Q}$ is nonempty, there is a $Q\in\mathcal{Q}$. Then $L_Q=\bigcap_{K\in\{Q\}}L_K\in\mathcal{F}$ so that $\mathcal{F}\neq\varnothing$. Hence $\mathcal{F}$ has a minimal element N. Then there is a finite subset F of $\mathcal{Q}$ such that

$$N=\bigcap_{Q\in F}L_Q.$$

Let $K\in\mathcal{Q}$ and let $F_K=F\cup\{K\}$. Then

$$\bigcap_{H\in F_K}L_H=\left(\bigcap_{Q\in F}L_Q\right)\cap L_K\ \leq\bigcap_{Q\in F}L_Q=N$$

By the minimality of N we deduce that

$$\bigcap_{H\in F_K} L_H = \left(\bigcap_{Q\in F} L_Q\right) \cap L_K = \bigcap_{Q\in F} L_Q = N \text{ for every } K \in \mathcal{Q}.$$

and hence

$$N = \bigcap_{Q\in F} L_Q \subseteq L_K \text{ for every } K \in \mathcal{Q}.$$

Therefore

$$N = \bigcap_{Q\in F} L_Q \subseteq \bigcap_{K\in\mathcal{Q}} L_K \subseteq N = \bigcap_{Q\in F} L_Q$$

so that

$$\{0\} = \bigcap_{Q\in\mathcal{Q}} \frac{L_Q}{L} = \frac{\bigcap_{Q\in\mathcal{Q}} L_Q}{L} = \frac{\bigcap_{Q\in F} L_Q}{L} = \bigcap_{Q\in F} \frac{L_Q}{L} = \bigcap_{Q\in F} Q.$$

$(c) \Rightarrow (a)$ Let

$$\cdots \leq M_2 \leq M_1 \leq M_0$$

be a decreasing chain of submodules of M_R and let

$$L = \bigcap_{n\in\mathbb{N}} M_n.$$

Then

$$\bigcap_{n\in\mathbb{N}} \frac{M_n}{L} = \frac{\bigcap_{n\in\mathbb{N}} M_n}{L} = \{0\}\,.$$

Since M/L is finitely cogenerated, there is a finite subset $F \subseteq \mathbb{N}$ such that

$$\frac{\bigcap_{n\in F} M_n}{L} = \bigcap_{n\in F} \frac{M_n}{L} = \{0\}\,.$$

Let $t = \max F$. Then we get

$$M_t = \bigcap_{n\in F} M_n = L = \bigcap_{n\in\mathbb{N}} M_n \subseteq M_n \text{ for every } n \in \mathbb{N}$$

so that, for every $n \geq t$ we get

$$M_n \leq M_t \leq M_n$$

i.e. $M_n = M_t$. □

Definition 9.11. *Let M_R be a right R-module. We say that M_R is* artinian *if M satisfies one of the equivalent conditions of Theorem 9.10.*

Examples 9.12. *1) Every commutative principal ideal ring R is right and left noetherian.*

3) Note that

$$\mathbb{Z}(p^\infty) = \left\{ \frac{m}{p^t} + \mathbb{Z} \mid m \in \mathbb{Z}, t \in \mathbb{N} \right\} \subseteq \mathbb{Q}/\mathbb{Z}$$

is a right (and left) artinian $\mathbb{Z}$-module which is not noehterian. In fact

$$\mathcal{L}(\mathbb{Z}(p^\infty)) = \{\mathbb{Z}(p^\infty)\} \cup \left\{ \left\langle \frac{1}{p^n} + \mathbb{Z} \right\rangle \mid n \in \mathbb{N} \right\},$$

which yields the following strictly ascending chain of $\mathbb{Z}$-submodules

$$\{0\} = \left\langle \frac{1}{p^0} + \mathbb{Z} \right\rangle \lneqq \left\langle \frac{1}{p} + \mathbb{Z} \right\rangle \lneqq \left\langle \frac{1}{p^2} + \mathbb{Z} \right\rangle \lneqq \left\langle \frac{1}{p^3} + \mathbb{Z} \right\rangle \lneqq \cdots$$

Theorem 9.13. *Let*

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

be a short exact sequence of right R-modules. The following statements are equivalent.
(a) *M is right noetherian (artinian).*
(b) *Both L and N are noetherian (artinian).*

Proof. $(a) \Rightarrow (b)$ Since $L \cong f(L)$ we may assume that $L \leq M$. Then every ascending chain

$$L_0 \leq L_1 \leq \cdots \leq L_n \leq \cdots$$

of submodules of L is an ascending chain of submodules of M_R. Thus L is right noetherian

Let

$$N_0 \leq N_1 \leq \cdots \leq N_n \leq \cdots$$

be an ascending chain of submodules of N. Then

$$g^{\leftarrow}(N_0) \leq g^{\leftarrow}(N_1) \leq \cdots \leq g^{\leftarrow}(N_n) \leq \cdots$$

is an ascending chain of submodules of M. Hence there is a $t \in \mathbb{N}$ such that $g^{\leftarrow}(N_i) = g^{\leftarrow}(N_t)$ per ogni $i \geq t$. Since g is surjective, we infer that

$$N_i = g[g^{\leftarrow}(N_i)] = g[g^{\leftarrow}(N_t)] = N_t$$

for every $i \geq t$.
$(b) \Rightarrow (a)$ Let

$$M_0 \leq M_1 \leq \cdots \leq M_n \leq \cdots$$

be an ascending chain of submodules of M_R. Then

$$f^{\leftarrow}(M_0) \leq f^{\leftarrow}(M_1) \leq \cdots \leq f^{\leftarrow}(M_n) \leq \cdots$$

is an ascending chain of submodules of L and

$$g(M_0) \leq g(M_1) \leq \cdots \leq g(M_n) \leq \cdots$$

is an ascending chain of submodules of N. Hence there is a $t \in \mathbb{N}$ such that

$$f^{\leftarrow}(M_i) = f^{\leftarrow}(M_t) \qquad \text{and} \qquad g(M_i) = g(M_t) \text{ for every } i \in \mathbb{N}, i \geq t.$$

Let $i \geq t$ and let us prove that $M_i \subseteq M_t$. We have

$$\begin{aligned} M_i \cap f(L) &= f[f^{\leftarrow}(M_i)] = f[f^{\leftarrow}(M_t)] = M_t \cap f(L), \\ M_i + f(L) &= M_i + \operatorname{Ker}(g) = g^{\leftarrow}[g(M_i)] = g^{\leftarrow}[g(M_t)] = M_t + \operatorname{Ker}(g) = M_t + f(L). \end{aligned}$$

Let $x_i \in M_i$. Then

$$x_i \in M_i \subseteq M_i + f(L) = M_t + f(L)$$

and hence there are $y \in L$ and $x_t \in M_t$ such that

$$x_i = x_t + f(y)$$

so that

$$f(y) = x_i - x_t \in M_i + M_t \subseteq M_i.$$

Thus we get

$$f(y) \in M_i \cap f(L) = M_t \cap f(L)$$

and hence

$$x_i = x_t + f(y) \in M_t + (M_t \cap f(L)) \subseteq M_t.$$

The proof in the artinian case is dual. □

Corollary 9.14. *Let $M_1, M_2, \ldots, M_n$ be right R-modules. The following statements are equivalent.*

(a) $M_1 \oplus M_2 \oplus \cdots \oplus M_n$ is noetherian (artinian).

(b) For every $1 \leq i \leq n$, M_i is noetherian (artinian).

Proof. Let us consider the short exact sequence

$$0 \to M_1 \xrightarrow{i_1} M_1 \oplus M_2 \xrightarrow{p_2} M_2 \to 0$$

where i_1 is the canonical injection and p_2 is the canonical projection. Then, in view of Theorem 9.13, we deduce that $M_1 \oplus M_2$ is noetherian (artinian) if and only if both M_1 and M_2 are noetherian (artinian). □

Lemma 9.15. *Let H be a semisimple right R-module. The following statements are equivalent.*

(a) H is right noetherian.

(b) H is finitely generated.

(c) *H is right artinian.*

(d) *H is right finitely cogenerated.*

(e) *$H = \bigoplus_{\lambda \in F} S_\lambda$ where F is a finite set and each $S_\lambda \in \mathcal{S}_r$.*

Proof. Let $(S_\lambda)_{\lambda \in \Lambda}$ be a family of right R-modules such that

$$H = \bigoplus_{\lambda \in \Lambda} S_\lambda.$$

$(a) \Rightarrow (b)$. It follows by Theorem 9.4.

$(b) \Rightarrow (c)$. Let $n \in \mathbb{N}, n \geq 1$ snf let $x_1, \ldots, x_n \in H$ such that $H = x_1R + \ldots + x_nR$. For each $i \in \{1, \ldots, n\}$, there is a finite subset $F_i \subseteq \Lambda$ such that

$$x_i \in \bigoplus_{\lambda \in F_i} S_\lambda.$$

Let

$$F = \bigcup_{i=1}^{n} F_i.$$

Then we get

$$H = x_1R + \ldots + x_nR \subseteq \bigoplus_{\lambda \in F} S_\lambda$$

and hence

$$H = \bigoplus_{\lambda \in F} S_\lambda.$$

Since each S_λ is right artinian, by Corollary 9.14, also H is artinian.

$(c) \Rightarrow (d)$. It follows by Theorem 9.10.

$(d) \Rightarrow (e)$. It follows by Lemma 9.8.

$(e) \Rightarrow (a)$. We have

$$H = \bigoplus_{\lambda \in F} S_\lambda.$$

where F is a finite set and each $S_\lambda \in \mathcal{S}_r$. Since each S_λ is right noetherian, by Corollary 9.14, also H is right noetherian. □

Definition 9.16. *The ring R is called* right noetherian *if R_R is noetherian.*

Remark 9.17. *Let M be a right R-module. Then, by Theorem 9.4, every submodule of M_R is finitely generated . In particular ${}_RM$ is finitely generated. The converse is, in general, not true. In fact, if R is a ring, then R_R is always finitely generated.*

Theorem 9.18. *Let R be a ring. The following statements are equivalent.*

(a) *R is right noetherian (artinian)*

(b) *Every finitely generated right R-module is right noetherian (artinian).*

Proof. $(a) \Rightarrow (b)$ Let M_R be a finitely generated right R-module and let $\{x_1, \ldots, x_n\}$ be a finite set of generators of M. For every $i \in I = \{1, \ldots, , n\}$, let us consider the homomorphism

$$\begin{array}{cccc} h_{x_i}: & R & \longrightarrow & M \\ & r & \longmapsto & x_i r \end{array}$$

and let

$$h = \nabla (h_{x_i})_{i \in I} : R^n \longrightarrow M$$

Then h is an epimorphism. By Corollary 9.14, $(R_R)^n$ is right noetherian so that, by Theorem 9.13, M is right noetherian.

$(b) \Rightarrow (a)$. Since $R_R = R1_R$, we conclude. □

Definition 9.19. *The ring R is called* right artinian *if R_R is artinian.*

Definition 9.20. *Let R be a ring and let $J = J(R)$. R is called* semiprimary *if*

- *R/J is semisimple*
- *J is nilpotent, i.e. there is an $n \in \mathbb{N}$ such that $J^n = \{0\}$.*

Theorem 9.21. *Let R be a right artinian ring. Then R is semiprimary.*

Proof. Let

$$\mathcal{E} = \left\{ \bigcap_{i=1}^{n} L_i \mid n \in \mathbb{N}, L_i \in \Omega_r \right\}.$$

Since R is right artinian, $\mathcal{E}$ has a minimal element. Let H be a minimal element for $\mathcal{E}$. Then there exists an $h \in \mathbb{N}$ and $I_1, \ldots, I_h \in \Omega_r$ such that

$$H = \bigcap_{j=1}^{h} I_j.$$

Let $L \in \Omega_r$. Then

$$H \supseteq H \cap L = \bigcap_{j=1}^{h} I_j \cap L \in \mathcal{E}.$$

By the minimality of H we deduce that $H = H \cap L$ i.e. $H \subseteq L$. Therefore we get

$$H \subseteq \bigcap_{L \in \Omega_r} L = J \subseteq H$$

i.e.

$$H = J.$$

Hence we have an embedding

$$\frac{R}{J} = \frac{R}{\bigcap_{j=1}^{h} I_j} \hookrightarrow \prod_{j=1}^{n} \frac{R}{I_j}.$$

Since, for every $j \in \{1, \ldots, n\}$, $R/I_j \in \mathcal{S}_r$, in view of Corollary 8.5, $\frac{R}{J}$ is semisimple.
Now let us consider the descending chain of (right) ideals of R :

$$J \geq J^2 \geq \ldots \geq J^n \geq \ldots .$$

Since R is right artinian, there is an $n \in \mathbb{N}, n \geq 1$ such that $J^k = J^n$ for every $k \geq n$. Let us assume that $J^n \neq \{0\}$ and let

$$\mathcal{F} = \{L \mid L \leq R_R \text{ and } L \cdot J^n \neq \{0\}\}.$$

Then $J \in \mathcal{F}$. Therefore $\mathcal{F}$ is nonempty and hence it has a minimal element. Let I be a minimal element of $\mathcal{F}$. Then $I \cdot J^n \neq \{0\}$ so that there is an $x \in I$ such that

$$x \cdot J^n \neq \{0\}.$$

Then

$$(x \cdot J) \cdot J^n = x \cdot J^{n+1} = x \cdot J^n \neq \{0\}.$$

Since $x \cdot J \subseteq x \cdot R \subseteq I$, by the minimality of I we get $x \cdot J = I$ and hence

$$x \cdot J = x \cdot R$$

so that

$$(x \cdot R) \cdot J = x \cdot R.$$

Since $x \neq 0$ this contradicts Nakayama's Lemma 8.25. □

Proposition 9.22. *Let R be a semiprimary ring and let M be a right R-module. The following statements are equivalent.*

(*a*) *M is right noetherian.*

(*b*) *M is right artinian.*

Proof. Let $J = J(R)$. We know that there is an $n \in \mathbb{N}$ such that $J^n = \{0\}$ and R/J is right semisimple. Let us consider the finite chain of right submodules of M :

$$M = MJ^0 \geq MJ \geq \ldots \geq MJ^{n-1} \geq MJ^n = \{0\}.$$

For every $i \in \{0, \ldots, n\}$, we have that

$$\frac{MJ^{i-1}}{MJ^i} \cdot J = \{0\}$$

so that each MJ^{i-1}/MJ^i has a natural structure of right R/J-module defined by setting

$$(r + J) \cdot x = r \cdot x \text{ for every } x \in \frac{MJ^{i-1}}{MJ^i}.$$

Note that, with respect to this structure, a subset of MJ^{i-1}/MJ^i is an R/J-submodule of MJ^{i-1}/MJ^i if and only if it is an R-submodule of MJ^{i-1}/MJ^i. Since

R/J is semisimple, by Theorem 8.8, MJ^{i-1}/MJ^i is a semisimple R/J-module and hence a semisimple R-module.

By Lemma 9.15 each MJ^{i-1}/MJ^i is right noetherian if and only if it is right artinian.

$(a) \Rightarrow (b)$. Since M is right noetherian, by Theorem 9.13 each MJ^{i-1}/MJ^i is right noetherian and hence right artinian. Let us show that M is right artinian by induction on n. Assume that $n = 1$ i.e. $J = \{0\}$. Then $M = MJ^0/M\ J'$ is right artinian. Assume that the statement hold for sum $n \in \mathbb{N}$, $n \geq 1$ and let us prove it for $n + 1$. Let us set

$$M' = MJ \text{ and } R' = \frac{R}{J^{n-1}}.$$

Then

$$J' = J(R') = J\left(\frac{R}{J^{n-1}}\right) = \frac{J}{J^{n-1}}$$

so that

$$(J')^{n-1} = \{0\}.$$

On the other hand

$$M' \cdot J^{n-1} = M \cdot J^n = \{0\}$$

and hence M' has a natural structure of R'-module. Since M is right noetherian, by Theorem 9.13, also M' is right noetherian. Thus, by Induction we get that M' is right artinian as a right R'-module and hence alos as an R-mdoule. Let us consider the exact sequence

$$0 \longrightarrow MJ \longrightarrow M \longrightarrow \frac{M}{MJ} \longrightarrow 0.$$

Since both MJ and M/MJ are artinian, by Theorem 9.13 we get that also M is artinian.

$(b) \Rightarrow (a)$. It is analogous. □

Theorem 9.23 (Hopkins-Levitzki). *Let R be a ring and let $J = J(R)$. The following statements are equivalent.*

(a) *R is right artinian*

(b) *R is right noetherian and semiprimary i.e. J is nilpotent and R/J is semisimple.*

Proof. $(a) \Rightarrow (b)$. By Theorem 9.21, R is semiprimary. Then, by Proposition 9.22 we get that R is right noetherian.

$(b) \Rightarrow (a)$. It follows by Proposition 9.22. □

Examples 9.24. *Still MISSING!!!!*

Chapter 10

Progenerators and Morita Equivalence

10.1 Progenerators

10.1. *Let A and B be rings and let $_AM_B$ be an A-B-bimodule. For every $a \in A$, the map*

$$\begin{array}{rccc} {}_a^A\mu : & M & \to & M \\ & x & \mapsto & ax \end{array}$$

is a right B-module homomorphism. For every $b \in B$, the map μ_b^B is analogously defined.

Proposition 10.2. *Let A and B be rings and let $_AM_B$ be an A-B-bimodule. In the notations of 10.1, the maps*

$$\begin{array}{rccc} {}^A\mu : & A & \to & \mathrm{End}\,(M_B) \\ & a & \mapsto & {}_a^A\mu \end{array} \quad \text{and} \quad \begin{array}{rccc} \mu^B : & B & \to & \mathrm{End}\,({}_AM) \\ & b & \mapsto & \mu_b^B \end{array}$$

are ring homomorphism.

Proof. Let $a, a' \in A$. For every $x \in M$ we compute

$$\left({}_a^A\mu \circ {}_{a'}^A\mu\right)(x) = {}_a^A\mu\,(a'x) = a\,(a'x) = (aa')\,x = {}_{aa'}^A\mu\,(x)\,.$$

□

Definition 10.3. *Let A and B be rings. An A-B-bimodule $_AM_B$ is called faithfully balanced if the maps μ^A and $^B\mu$ of Proposition 10.2 are ring isomorphism.*

Lemma 10.4. *Let R be a ring, let M_R be a right R-module. For every $m \in M$ and $f \in \mathrm{Hom}_R\,(M, R)$, let $m \cdot f$ denote the map from M into M defined by setting*

$$(m \cdot f)\,(x) = m \cdot (f\,(x))\,.$$

Then $m \cdot f \in \mathrm{End}_R\,(M)$.

Proof. Let $r \in R$ and $x \in M$. We have

$$(m \cdot f)(xr) = m \cdot (f(xr)) = m \cdot [f(x) \cdot r] = [m \cdot (f(x))] \cdot r = (m \cdot f)(x) \cdot r.$$

$\square$

Notations 10.5. *Let R be a ring and let X and Y be non-empty sets. Then an $X \times Y$-*matrix over R *is simply a map $\Lambda : X \times Y \to R$. Then, for each $(x, y) \in X \times Y$ we set*

$$\Lambda_{x,y} = \Lambda((x, y))$$

and call it the (x, y) entry of Λ. We will also write

$$\Lambda = (\Lambda_{x,y})_{(x,y) \in X \times Y}.$$

Let $x \in X$ and let $y \in Y$. Then

$(\Lambda_{x,y})_{(x,y) \in \{x\} \times Y}$ *is called the x row of Λ and* $(\Lambda_{x,y})_{(x,y) \in X \times \{y\}}$ *is called the y column of Λ*

The matrix A is said to be row finite *(resp.* column finite*) in case each row (column) of A has at most finitely many non-zero entries. The set of all $X \times Y$-matrix over R will be denoted by $M_{X \times Y}(R)$ and the subsets of row finite and column finite matrices by $RFM_{X \times Y}(R)$ and $CFM_{X \times Y}(R)$ respectively.*

Consider the right R-module

$$F = R^{(X)} = \bigoplus_{x \in X} R_x \text{ where, for each } x \in X,\ R_x = R_R.$$

For every $t \in X$, let $\varepsilon_t : R_t \to \bigoplus_{x \in X} R_x$ be the canonical injection and let $e_t = \varepsilon_t(1)$. Let $\alpha \in \mathrm{Hom}_{-R}\left(R^{(Y)}, R^{(X)}\right)$ and write

$$\alpha(e_y) = (\alpha_{x,y})_{x \in X} = \sum_{x \in X} e_x \alpha_{x,y}$$

Then the assignment

$$\alpha \mapsto (\alpha_{x,y})_{(x,y) \in X \times Y}$$

defines a bijection

$$\Phi : \mathrm{Hom}_{-R}\left(R^{(Y)}, R^{(X)}\right) \to CFM_{X \times Y}(R).$$

When $Y = X$ we have

$$\Phi(\alpha \circ \beta) =$$

$$(\alpha \circ \beta)(e_y) = \alpha(\beta(e_y)) = \alpha\left(\sum_{x \in X} e_x \beta_{x,y}\right) = \sum_{x \in X} \alpha(e_x) \beta_{x,y} = \sum_{x \in X} \sum_{t \in X} e_t \alpha_{t,x} \beta_{x,y} = \sum_{t \in X} e_t \left(\sum_{x \in X} \alpha_{t,x} \beta_{x,y}\right.$$

so that

$$\Phi(\alpha \circ \beta) = \left(\sum_{x \in X} \alpha_{t,x} \beta_{x,y} \right)_{(t,y) \in X \times X}$$

Hence $CFM_{X \times Y}(R)$ inherits a ring structure by setting

$$\Lambda \cdot \Gamma = \left(\sum_{t \in X} \Lambda_{x,t} \Gamma_{t,y} \right)_{(x,y) \in X \times X}$$

Clearly, in this way, Φ becomes a ring isomorphism.

Theorem 10.6. *Let R be a ring, let M_R be a generator, let $A = \mathrm{End}(M_R)$ and $B = \mathrm{End}({}_A M)$. Then the ring homomorphism*

$$\begin{array}{rccc} \mu^R : & R & \to & \mathrm{End}({}_A M) \\ & r & \mapsto & \mu_r^R \end{array}$$

is an isomorphism i.e. the bimodule ${}_A M_R$ is faithfully balanced.

Proof. (First Proof) Since M_R is a generator, there exists an $n \in \mathbb{N}, n \geq 1$ and an epimorphism

$$\pi : M_R^n \to R_R.$$

For every $1 \leq t \leq n$ let

$$i_t : M_R \to M_R^n$$

denote the t-th canonical injection and $\pi_t = \pi \circ i_t$. Since π is surjective there exists $(x_1, \ldots, x_n) \in M_R^n$ such that

$$1_R = \pi((x_1, \ldots, x_n)) = \sum_{i=1}^n \pi_i(x_i).$$

Let $r \in \mathrm{Ker}(\mu^R)$. Then $\mu_r^R = 0$ i.e. $xr = 0$ for every $x \in M$ and hence

$$r = 1_R \cdot r = \sum_{i=1}^n \pi_i(x_i) \cdot r = \sum_{i=1}^n \pi_i(x_i \cdot r) = 0$$

Thus μ^R is injective.

Let now $b \in B = \mathrm{End}({}_A M)$. For every $x \in M$ we have

$$(x) b = (x \cdot 1_R) b = \left(x \cdot \sum_{i=1}^n \pi_i(x_i) \right) b = \left(\sum_{i=1}^n x \cdot \pi_i(x_i) \right) b = \left(\sum_{i=1}^n (x \cdot \pi_i)(x_i) \right) b.$$

By Lemma 10.4, we have $x \cdot \pi_i(x_i) = (x \cdot \pi_i)(x_i)$ and $x \cdot \pi_i \in A = \mathrm{End}_R(M_R)$. Since $b \in B = \mathrm{End}({}_A M)$ we get

$$\begin{aligned} (x) b &= \left(\sum_{i=1}^n x \cdot \pi_i(x_i) \right) b = \left(\sum_{i=1}^n (x \cdot \pi_i)(x_i) \right) b = \left(\sum_{i=1}^n (x \cdot \pi_i) \cdot x_i \right) b \\ &= (x \cdot \pi_i) \cdot \sum_{i=1}^n (x_i) b = x \cdot \left[\pi_i \left(\sum_{i=1}^n (x_i) b \right) \right]. \end{aligned}$$

Therefore we deduce that

$$b = \mu_r^R \text{ where } r = x \cdot \left[\pi_i \left(\sum_{i=1}^{n} (x_i)\, b \right) \right].$$

(Second Proof) Since M_R is a generator, there exists an $n \in \mathbb{N}, n \geq 1$ and a map

$$\pi : M_R^n \to R_R$$

which is an epimorphism of right R-modules. Since R_R is projective, there is a right R-module homomorphism

$$\sigma : R_R \to M_R^n$$

such that

$$\pi\sigma = \mathrm{Id}_R.$$

Therefore we get that

$$M^n = \mathrm{Im}(\sigma) \oplus X \text{ where } X = \mathrm{Ker}(\pi). \tag{10.1}$$

Let $y_1, \ldots, y_n \in M$ be such that

$$\sigma(1_R) = (y_1, \ldots, y_n).$$

Then, for every $r \in R$ we have

$$\sigma(r) = (y_1 r, \ldots, y_n r)$$

and

$$\mathrm{Im}(\sigma) = (y_1, \ldots, y_n) R = yR \text{ where } y = (y_1, \ldots, y_n). \tag{10.2}$$

Let $r \in \mathrm{Ker}(\mu^R)$. Then $\mu_r^R = 0$ i.e. $xr = 0$ for every $x \in M$ and hence $\sigma(r) = (y_1 r, \ldots, y_n r) = 0$. Since σ is a monomorphism, we deduce that $r = 0$ and hence μ^R is injective.

Let now $b \in B = \mathrm{End}({}_A M)$ and let us assume that

$$z = (y_1 b, \ldots, y_n b) \notin yR = \mathrm{Im}(\sigma).$$

In view of (10.1) and of (10.2), there exists an $\overline{r} \in R$ and an $\overline{x} \in X$, $\overline{x} \neq 0$ such that

$$z = y\overline{r} + \overline{x}.$$

Let

$$i_X : X \to M^n \text{ and } \pi_X : M^n \to X$$

denote respectively the canonical injection of X and the canonical projection on X with respect to the decomposition (10.1). We set

$$\alpha = i_X \pi_X : M_R^n \to M_R^n.$$

Then we have

$$\alpha(z) = i_X(\overline{x}) = \overline{x} \neq 0$$

and

$$\alpha(yr) = 0 \text{ for every } r \in R.$$

For every $1 \leq t \leq n$ let

$$i_t : M_R \to M_R^n \text{ and } p_t : M_R^n \to M_R$$

denote the t-th canonical injection and projection. Since $0 \neq \alpha(z) \in M_R^n$ there exists an $s \in \{1, \ldots, n\}$ such that

$$0 \neq p_s\alpha(z) = p_s\alpha(y_1b, \ldots, y_nb) = p_s\alpha\left(\sum_{t=1}^{n} i_tp_t[(yb)]\right) =$$
$$= p_s\alpha\left(\sum_{t=1}^{n} i_t[p_t(yb)]\right) = \sum_{t=1}^{n} p_s\alpha i_t[p_t(yb)].$$

Since

$$p_t(yb) = y_tb = (p_t(y))\,b$$

we get

$$i_t[p_t(yb)] = i_t[(p_t(y))\,b]$$

and since $p_s\alpha i_t \in \text{End}(M_R) = A$ and $b \in B = \text{End}({}_AM)$, we deduce that, for every $t \in \{1, \ldots, n\}$ so that

$$p_s\alpha i_t[p_t(yb)] = (p_s\alpha i_t)[(p_t(y))\,b] = [(p_s\alpha i_t)\,p_t(y)]\,b$$

and hence

$$0 \neq p_s\alpha(z) = \sum_{t=1}^{n}[(p_s\alpha i_t)\,p_t(y)]\,b = \left(\sum_{t=1}^{n}(p_s\alpha i_t)\,p_t(y)\right)b =$$
$$= \left(\sum_{t=1}^{n} p_s\alpha i_t(y_t)\right)b = \left[p_s\alpha\left(\sum_{t=1}^{n} i_t(y_t)\right)\right]b =$$
$$= [p_s\alpha(y)]\,b = [p_s(0)]\,b = 0$$

which is a contradiction. Therefore we infer that $z = (y_1b, \ldots, y_nb) \in yR$ and hence there exists an $\widetilde{r} \in R$ such that

$$z = (y_1b, \ldots, y_nb) = y\widetilde{r} = (y_1\widetilde{r}, \ldots, y_n\widetilde{r})$$

i.e.

$$y_ib = y_i\widetilde{r} \text{ for every } 1 \leq i \leq n. \tag{10.3}$$

For every $x \in M$ let us consider the right R-module homomorphism

$$\begin{array}{rccc} h_x: & R_R & \to & M_R \\ & r & \mapsto & xr \end{array}$$

we have

$$x = h_x(1_R) = h_x[\pi\sigma(1_R)] = h_x(\pi(y)) = h_x\left(\pi\left(\sum_{t=1}^{n}(i_t)(y_t)\right)\right) =$$
$$= \sum_{t=1}^{n}(h_x\pi i_t)(y_t) = \sum_{t=1}^{n} a_t^x(y_t)$$

where

$$a_t^x = h_x\pi i_t \in \operatorname{End}(M_R) = A \text{ for every } 1 \le t \le n.$$

Since $b \in B = \operatorname{End}({}_AM)$, we get

$$xb = \left(\sum_{t=1}^{n} a_t^x(y_t)\right)b = \sum_{t=1}^{n}[a_t^x(y_t)]\,b = \sum_{t=1}^{n} a_t^x[(y_t)\,b] \overset{(10.3)}{=} \sum_{t=1}^{n} a_t^x(y_t\widetilde{r}) =$$
$$= \sum_{t=1}^{n}(a_t^x(y_t))\,\widetilde{r} = \left(\sum_{t=1}^{n} a_t^x(y_t)\right)\widetilde{r} = x\widetilde{r} = \mu_{\widetilde{r}}^R(x).$$

Since this holds for every $x \in M$, we deduce that

$$b = \mu_{\widetilde{r}}^R = \mu^R(\widetilde{r}).$$

□

10.7. *Let P_R be a right R-module. We set*

$$P^* = \operatorname{Hom}_R(P_R, R_R).$$

By Proposition 6.28, P^* has a natural structure of left R-module defined by setting

$$(rf)(x) = rf(x) \text{ for all } r \in R, f \in P^*, x \in P.$$

Definition 10.8. *Let P_R be a right R-module. A* dual basis *for P_R is a pair $\left((x_i)_{i\in I}, (x_i^*)_{i\in I}\right)$ where $(x_i)_{i\in I}$ is a family of elements of P and $(x_i^*)_{i\in I}$ is a family of elements of P^* subject to the conditions*

P1) *For every $x \in P, x_i^*(x) = 0$ for almost every $i \in I$, i.e. there is a finite subset $F_x \subseteq I$ such that $x_i^*(x) = 0$ for every $i \notin F_x$.*

P2) *For every $x \in P$, the following equality holds:*

$$x = \sum_{i\in I} x_i \cdot x_i^*(x).$$

A dual basis is said to be finite *whenever I is a finite set.*

Theorem 10.9. *(Dual Basis Lemma) Let P_R be a right R-module. Then*

a) *P_R is projective if and only if it has a dual basis.*

b) *P_R is projective and finitely generated if and only if it has a finite dual basis.*

Proof. Let X be a system of generators of P. For every $x \in P$, let

$$h_x : R_R \to P_R \text{ defined by setting } h_x(r) = xr \text{ for every } r \in R.$$

Then h_x is a right R-module homomorphism and, by Proposition 2.2

$$h = \nabla (h_x)_{x \in X} : {}_R R^{(X)} \to P.$$

is a surjective homomorphism.

Assume that P is projective. Then, by Proposition (2.17), there exists a right R-module homomorphism $\gamma : P \to {R_R}^{(P)}$ such that

$$h \circ \gamma = \mathrm{Id}_P.$$

For every $x \in X$ let

$$\pi_x : {R_R}^{(X)} \to R_R$$

denote the xth canonical projection.

$$x = (h \circ \gamma)(x) = \sum_{y \in X} h_y (\pi_y (\gamma (x))) = \sum_{y \in X} y \left[(\pi_y \circ \gamma)(x) \right]. \tag{10.4}$$

For every $y \in X$, set

$$y^* = \pi_y \circ \gamma$$

and

$$F_x = Supp\,(\gamma (x)).$$

Then, $y^* \in P^*$ and for every $y \notin F_x$ we have

$$y^*(x) = \pi_y \circ \gamma (x) = 0.$$

Moreover, from (10.4) we get that

$$x = \sum_{y \in X} y \cdot y^* (x)$$

Conversely assume that $\left((x_i)_{i \in I}, (x_i^*)_{i \in I} \right)$ is a dual basis for P_R and let

$$\lambda = \Delta (x_i^*)_{i \in I} : P_R \to R_R^I.$$

Since, for every $x \in P, x_i^*(x) = 0$ for almost every $i \in I$, we have that $\mathrm{Im}(\lambda) \subseteq R_R^{(I)}$ so that we can consider the corestriction γ of λ to $R_R^{(I)}$. Now let

$$\chi = \nabla (h_{x_i})_{i \in I} : R_R{}^{(I)} \to P.$$

For every $y \in P$, we have

$$(\chi \circ \gamma)(y) = \sum_{i \in I} h_{x_i}(\pi_i(\gamma(x))) = \sum_{i \in I} h_{x_i}(x_i^*(y)) = \sum_{i \in I x_i} x_i \cdot x_i^*(y) = y.$$

Therefore we deduce that

$$\chi \circ \gamma = \mathrm{Id}_P$$

and hence, in view of Proposition (2.17), P_R is projective. □

Lemma 10.10. *If $((x_1, \ldots, x_n), (x_1^*, \ldots, x_n^*))$ is a finite dual basis of a finitely generated projective right R-module P_R, then for every $\xi \in \mathrm{Hom}_R(P, R)$, using the left R-module structure of $\mathrm{Hom}_R(P, R)$ enduced by ${}_R R$, we have*

$$\xi = \sum_{i=1}^{n} \xi(x_i) \cdot x_i^* \tag{10.5}$$

Thus the left R-module $\mathrm{Hom}_R(P, R)$ is projective and finitely generated with dual basis $((x_1^*, \ldots, x_n^*), (\widetilde{x}_1, \ldots, \widetilde{x}_n))$ where, for every $i = 1, \ldots, n$

$$\widetilde{x}_i(\xi) = \xi(x_i) \text{ for every } \xi \in \mathrm{Hom}_R(P, R).$$

Proof. For every $y \in P$, we compute

$$\left[\sum_{i=1}^{n} \xi(x_i) \cdot x_i^*\right](y) = \sum_{i=1}^{n} \xi(x_i) \cdot x_i^*(y) = \sum_{i=1}^{n} \xi[x_i \cdot x_i^*(y)] = \xi\left[\sum_{i=1}^{n} x_i \cdot x_i^*(y)\right] = \xi(y).$$

We have to prove that for every $i = 1, \ldots, n$,the map $\widetilde{x}_i$ is left R-linear. In fact we have

$$\widetilde{x}_i(r\xi) = (r\xi)(x_i) = r \cdot \xi(x_i) = r \cdot \widetilde{x}_i(\xi).$$

□

Proposition 10.11. *Let P_R be a right R-module. Then the map*

$$\begin{array}{rccl} \omega_P : & P & \to & \mathrm{Hom}_R(\mathrm{Hom}_R(P, R), R) \\ & y & \mapsto & \xi \mapsto \xi(y) \end{array}$$

is well defined and is a right R-module homomorphism. If P_R is a finitely generated projective, then it is an isomorphism. Namely if $((x_1, \ldots, x_n), (x_1^*, \ldots, x_n^*))$ is a finite dual basis of P_R, then the inverse ζ_P of ω_P is defined by setting

$$\zeta_P(\alpha) = \sum_{i=1}^{n} x_i \cdot (x_i^*) \alpha \text{ for every } \alpha \in \mathrm{Hom}_R(\mathrm{Hom}_R(P, R), R).$$

Proof. For every $\xi \in \mathrm{Hom}_R(P,R), \alpha \in \mathrm{Hom}_R(\mathrm{Hom}_R(P,R),R)$ and $y \in P$, we compute

$$(\xi)[(\omega_P \circ \zeta_P)(\alpha)] = (\xi)\zeta_P(\alpha) = \xi(\zeta_P(\alpha)) = \xi\left(\sum_{i=1}^n x_i \cdot (x_i^*)\alpha\right) =$$

$$= \sum_{i=1}^n \xi(x_i)\cdot(x_i^*)\alpha = \left[\sum_{i=1}^n \xi(x_i)\cdot x_i^*\right]\alpha \overset{10.5}{=} (\xi)\alpha$$

$$(\zeta_P \circ \omega_P)(y) = \sum_{i=1}^n x_i \cdot [(x_i^*)\omega_P(y)] = \sum_{i=1}^n x_i \cdot x_i^*(y) = y.$$

□

Proposition 10.12. *Let ${}_AP_R$ be an A-R-bimodule. For every $M \in Mod\text{-}R$ the map*

$$\begin{array}{llll} \alpha_M : & M \otimes_R \mathrm{Hom}_R(P,R) & \to & \mathrm{Hom}_R(P,M) \\ & m \otimes f & \mapsto & y \mapsto mf(y) \end{array}$$

is well defined and is a right A-module homomorphism. If P_R is finitely generated and projective, then α_M is an isomorphism. Namely if $((x_1,\ldots,x_n),(x_1^*,\ldots,x_n^*))$ is a finite dual basis of P_R, then the inverse β_M of α_M is defined by setting

$$\beta_M(h) = \sum_{i=1}^n h(x_i) \otimes_R x_i^* \text{ for every } h \in \mathrm{Hom}_R(P_R,M_R).$$

In particular

$$P \otimes_R P^* = {}_AP \otimes_R \mathrm{Hom}_R(P,R) \overset{\alpha_P}{\cong} \mathrm{Hom}_R(P,P)$$

is an isomorphism of A-A-bimodules.

Moreover the collection $(\alpha_M)_{M\in Mod\text{-}R}$ yields a functorial isomorphism

$$\mathrm{Hom}_R(P,-) \cong - \otimes_R \mathrm{Hom}_R(P,R).$$

Proof. Let $m \in M$ and $f \in \mathrm{Hom}_R(P,R)$. We compute

$$(\beta_M \circ \alpha_M)(m \otimes f) = \sum_{i=1}^n [\alpha_M(m\otimes f)(x_i)] \otimes_R x_i^* = \sum_{i=1}^n mf(x_i) \otimes_R x_i^* =$$

$$= m \otimes_R \sum_{i=1}^n f(x_i)x_i^* \overset{10.5}{=} m \otimes_R f.$$

Let now $h \in \mathrm{Hom}_R(P_R,M_R)$ and let us compute, for every $y \in P$

$$\alpha\left(\sum_{i=1}^n h(x_i)\otimes_R x_i^*\right)(y) = \sum_{i=1}^n h(x_i)\cdot x_i^*(y) = \sum_{i=1}^n h(x_i)\cdot x_i^*(y) =$$

$$= h\left(\sum_{i=1}^n x_i \cdot x_i^*(y)\right) = h(y).$$

We deduce that $\alpha\left(\sum_{i=1}^{n} h\left(x_i\right) \otimes_R x_i^*\right) = h$.

$$\begin{aligned}
\left[\alpha_P\left(a\left(z \otimes_R \xi\right) b\right)\right](y) &= \left[\alpha_P\left(az \otimes_R \xi b\right)\right](y) = az \cdot \left[(\xi b)(y)\right] = a \cdot z \cdot \xi\left(b \cdot y\right) \\
\left[a \cdot \alpha_P\left(z \otimes_R \xi\right) \cdot b\right](y) &= a \cdot \left[\left(\alpha_P\left(z \otimes_R \xi\right) \cdot b\right)(y)\right] = a \cdot \left[\left(\alpha_P\left(z \otimes_R \xi\right)\right)\left(b \cdot y\right)\right] \\
&= a \cdot \left[z \cdot \xi\left(b \cdot y\right)\right]
\end{aligned}$$

□

Definition 10.13. *A right R-module P_R is called a* progenerator *if it is a finitely generated projective generator.*

Lemma 10.14. *Let ${}_A P_R$ be a faithfully balanced A-R-bimodule. Then the following are equivalent*

(a) P_R is a progenerator.

(b) ${}_A P$ is a progenerator.

Proof. Assume that P_R is a progenerator. Then we have a two splitting epimorphism of right R-modules

$$R_R^n \to P_R \text{ and } P_R^m \to R_R$$

which give rise, by applying $\mathrm{Hom}_R\left(-, P_R\right)$ to two splitting monomorphism of left A-modules

$$A = \mathrm{Hom}_R\left(P, P\right) \to \mathrm{Hom}_R\left(R^n, P\right) \cong \left[\mathrm{Hom}_R\left(R, P\right)\right]^n \overset{\text{Prop6.29}}{\cong} P^n$$
$$\text{and } P \overset{\text{Prop6.29}}{\cong} \mathrm{Hom}_R\left(R, P\right) \to \mathrm{Hom}_R\left(P^m, P\right) \cong A^m.$$

□

Lemma 10.15. *Let P_R be a progenerator and let ${}_R P^* = \mathrm{Hom}_R\left(P_R, R_R\right)$. Then ${}_R P^*$ is a progenerator.*

Proof. Since P_R is a progenerator, we have a two splitting epimorphism of right R-modules

$$R_R^n \to P_R \text{ and } P_R^m \to R_R$$

which give rise, by applying $\mathrm{Hom}_R\left(-, R_R\right)$, to two splitting monomorhisms of left R-modules

$$P^* = \mathrm{Hom}_R\left(P, R\right) \to \mathrm{Hom}_R\left(R^n, R\right) \cong \left[\mathrm{Hom}_R\left(R, R\right)\right]^n \overset{\text{Prop6.29}}{\cong} R^n$$
$$\text{and } R \overset{\text{Prop6.29}}{\cong} \mathrm{Hom}_R\left(R, R\right) \to \mathrm{Hom}_R\left(P^m, R\right) \cong \left[\mathrm{Hom}_R\left(P, R\right)\right]^m = \left(P^*\right)^m.$$

□

Proposition 10.16. *Let P_R be a progenerator and let $A = \mathrm{End}(P_R)$. Then both the bimodules ${}_AP_R$ and ${}_RP^*_A$ are faithfully balanced.*

Proof. Since P_R is a generator, by Theorem 10.6, ${}_AP_R$ is faithfully balanced. Now, by Lemma 10.15, ${}_RP^*$ is a progenerator. Let $B = \mathrm{Hom}_R(P^*, P^*)$. Then, by Theorem 10.6, ${}_RP^*_B$ is faithfully balanced. Let us consider the canonical ring homomorphism

$$\begin{array}{rccl} \mu = \mu^A : & A & \to & \mathrm{Hom}_R(P^*, P^*) = B \\ & a & \mapsto & \mu^A_a : \xi \mapsto \xi \cdot a \end{array}$$

We will prove that μ is an isomorphism. First of all, note that, for every $\xi \in P^*, a \in A, y \in P$ we have

$$(\xi \cdot a)(y) = \xi(a \cdot y) = \xi(a(y)) = (\xi \circ a)(y)$$

which entails

$$\xi \cdot a = \xi \circ a \tag{10.6}$$

By Lemma 10.15, ${}_RP^*$ is a progenerator and hence, by Proposition 10.12

$$\begin{array}{rccl} \alpha_{P^*} : & \mathrm{Hom}_R(P^*, R) \otimes_R P^* & \to & \mathrm{Hom}_R(P^*, P^*) \\ & f \otimes \xi & \mapsto & \zeta \mapsto f(\zeta)\xi \end{array}.$$

is an isomorphism. By Proposition 10.11,

$$\begin{array}{rccl} \omega_P : & P & \to & \mathrm{Hom}_R(P^*, R) \\ & y & \mapsto & \xi \mapsto \xi(y) \end{array}$$

is also an isomorphism. Therefore we have the chain of isomorphisms

$$\mathrm{Hom}_R(P, P) \overset{\alpha_P^{-1}}{\cong} P \otimes_R \mathrm{Hom}_R(P, R) = P \otimes_R P^* \overset{\omega_P \otimes_R P^*}{\cong}$$
$$\cong \mathrm{Hom}_R(P^*, R) \otimes_R P^* \overset{\alpha_{P^*}}{\cong} \mathrm{Hom}_R(P^*, P^*).$$

Let us prove that $\alpha_{P^*} \circ (\omega_P \otimes_R P^*) \circ \left(\alpha_P^{-1}\right) = \lambda$. For any $a \in A$, we have

$$\begin{aligned} \left[\alpha_{P^*} \circ (\omega_P \otimes_R P^*) \circ \left(\alpha_P^{-1}\right)\right](a) &= \left[\alpha_{P^*} \circ (\omega_P \otimes_R P^*)\right]\left(\alpha_P^{-1}\right)(a) \\ &= \left[\alpha_{P^*} \circ (\omega_P \otimes_R P^*)\right]\left(\sum_{i=1}^n a(x_i) \otimes_R x_i^*\right) \\ &= \alpha_{P^*}\left[\sum_{i=1}^n \omega_P(a(x_i)) \otimes_R x_i^*\right] \end{aligned}$$

so that we get

$$\begin{aligned} \left\{\left[\alpha_{P^*} \circ (\omega_P \otimes_R P^*) \circ \left(\alpha_P^{-1}\right)\right](a)\right\}(\zeta) &= \sum_{i=1}^n \left[\omega_P(a(x_i))(\zeta)\right] \cdot x_i^* \\ &= \sum_{i=1}^n \zeta(a(x_i)) \cdot x_i^* = \sum_{i=1}^n (\zeta \circ a)(x_i) \cdot x_i^* \\ &\overset{(10.5)}{=} \zeta \circ a \overset{(10.6)}{=} \zeta.a \end{aligned}$$

Hence we deduce that μ is an isomorphism. $\square$

Corollary 10.17. *Let P_R be a progenerator, let $A = \mathrm{End}(P_R)$. Then both the bimodules ${}_AP_R$ and ${}_RP^*_A$ are faithfully balanced and each of the modules*

$$P_R, {}_AP, {}_RP^*, P^*_A$$

is a progenerator.

Proof. By Proposition 10.16, the bimodules ${}_AP_R$ and ${}_RP^*_A$ are faithfully balanced. By Lemma 10.15, ${}_RP^*$ is a progenerator. Then, by Lemma 10.14, also ${}_AP$ and P^*_A are progenerators. □

Theorem 10.18. *Let P_R be a progenerator and let $A = \mathrm{End}(P_R)$. Then the functor $\mathrm{Hom}_R({}_AP_R, -) : Mod\text{-}R \to Mod\text{-}A$ is an equivalence of categories whose inverse is the functor $- \otimes_A {}_AP_R : Mod\text{-}A \to Mod\text{-}R$.*

Proof. Let $M \in Mod\text{-}R$ and let us consider the evaluation map

$$\begin{array}{rlcl} \nu_M : & \mathrm{Hom}_R({}_AP_R, M) \otimes_A {}_AP_R & \to & M \\ & f \otimes_A y & \mapsto & f(y) \end{array}.$$

It is easy to check that ν_M is well defined and it is a right R-module homomorphism. By Proposition (4.3) we know that

$$M = \sum_{h \in \mathrm{Hom}_R(P,M)} \mathrm{Im}(h).$$

Thus given $x \in M$ there exists a finite subset $F_x \subseteq \mathrm{Hom}_R(P, M)$ such that

$$x \in \sum_{h \in F_x} \mathrm{Im}(h).$$

Thus, for every $h \in F_x$ there exists an $y_h \in P$ such that

$$x = \sum_{h \in F_x} h(y_h) = \nu_M\left(\sum_{h \in F_x} h \otimes_A y_h\right).$$

Therefore ν_M is surjective. Assume now that $m \in \mathbb{N}, m \geq 1$, and $f_1, \ldots, f_m$ are elements in $\mathrm{Hom}_R({}_AP_R, M)$ and $y_1, \ldots, y_m$ are elements in P such that

$$0 = \nu_M\left(\sum_{i=1}^m f_i \otimes_A y_i\right) = \sum_{i=1}^m f_i(y_i).$$

Let

$$f = \nabla(f_1, \ldots, f_m) : P^m \to M.$$

and for every $1 \leq i \leq m$, let $e_i : P \to P^m$ and $p_i : P^n \to P$ be the i-th canonical injection and projection respectively.Then, for every $w = (w_1, \ldots, w_m) \in P^m$ we have that

$$f(w) = f\left[\sum_{i=1}^m (e_i \circ p_i)(w)\right] = \sum_{i=1}^m (f \circ e_i \circ p_i)(w) = \left(\sum_{i=1}^m f_i \circ p_i\right)(w)$$

i.e.

$$f=\sum_{i=1}^{m} f_i\circ p_i \tag{10.7}$$

In particular for

$$y=(y_1,\dots,y_m)$$

we have

$$f(y)=\sum_{i=1}^{m} f_i(y_i)=0 \text{ so that } y=(y_1,\dots,y_m)\in \operatorname{Ker}(f).$$

Since P_R is a generator of $Mod\text{-}R$, There exists a surjective right R-module homomorphism

$$\chi : P^{(X)}\to \operatorname{Ker}(f)\subseteq P^m.$$

For every $x\in X$ let

$$\varepsilon_x : P\to P^{(X)} \text{ and } \pi_x : P^{(X)}\to P$$

be the canonical injection and projection respectively. Then

$$\chi=\nabla(\chi_x)_{x\in X} \text{ where } \chi_x=\chi\circ\varepsilon_x\in \operatorname{Hom}_R(P_R,\operatorname{Ker}(f)).$$

Since $y\in\operatorname{Ker}(f)$, there exist a $z\in P^{(X)}$ such that $\chi(z)=y$. Let $F=Supp(z)$. Then

$$z=\sum_{x\in F}\varepsilon_x(z_x)=\sum_{x\in F}(\varepsilon_x\circ\pi_x)(z)$$

and

$$y=\chi(z)=\chi\left(\sum_{x\in F}(\varepsilon_x\circ\pi_x)(z)\right)=\sum_{x\in F}(\chi\circ\varepsilon_x\circ\pi_x)(z)=$$
$$=\sum_{x\in F}(\chi\circ\varepsilon_x)(\pi_x(z))=\sum_{x\in F}\chi_x(z_x)$$

where $\chi_x=\chi\circ\varepsilon_x\in\operatorname{Hom}_R(P_R,\operatorname{Ker}(f))$. Hence we have

$$f\circ\chi_x=0 \tag{10.8}$$

and hence, since $p_i\circ\chi_x\in\operatorname{End}(P_R)=A$, we get

$$\sum_{i=1}^{m} f_i\otimes_A y_i=\sum_{i=1}^{m} f_i\otimes_A p_i(y)=\sum_{i=1}^{m} f_i\otimes_A p_i\left(\sum_{x\in F}\chi_x(z_x)\right)=$$
$$=\sum_{i=1}^{m}\sum_{x\in F} f_i\otimes_A(p_i\circ\chi_x)(z_x)=\sum_{x\in F}\sum_{i=1}^{m} f_i\otimes_A(p_i\circ\chi_x)\cdot z_x=\sum_{x\in F}\sum_{i=1}^{m} f_i\cdot(p_i\circ\chi_x)\otimes_A z_x=$$
$$=\sum_{x\in F}\left(\sum_{i=1}^{m} f_i\circ p_i\right)\circ\chi_x\otimes_A z_x\overset{(10.7)}{=}\sum_{x\in F} f\circ\chi_x\otimes_A z_x\overset{(10.8)}{=}0.$$

Let now $L \in Mod\text{-}A$ and let us prove that the natural map

$$\begin{array}{rccl} \gamma_L : & L & \to & \mathrm{Hom}_R\left({}_AP_R, L \otimes_A {}_AP_R\right) \\ & x & \mapsto & y \mapsto x \otimes_A y \end{array}.$$

is an isomorphism. Let us consider the isomorphism of Proposition 6.43

$$\begin{array}{rccc} \mu^L : & L \otimes_A A & \to & L \\ & x \otimes_A a & \longmapsto & x \cdot a \end{array}.$$

and the composition of homomorphisms

$$L \overset{\left(\mu^L\right)^{-1}}{\cong} L \otimes_A A \overset{L \otimes_A \beta_P}{\cong} L \otimes_A P \otimes_R \mathrm{Hom}_R\left(P, R\right) =$$
$$= L \otimes_A P \otimes_R P^* \overset{\alpha_{L \otimes_A P}}{\cong} \mathrm{Hom}_R\left(P, L \otimes_A {}_AP\right)$$

where β_P is as in Proposition 10.12 and $\alpha_{L \otimes_A P}$ as in Proposition 10.12. For every $x \in L$ and $y \in P$ we compute

$$\left[\left(\alpha_{L \otimes_A P} \circ \left(L \otimes_A \beta_P\right) \circ \left(\mu^L\right)^{-1}\right)(x)\right](y) = \left[\left(\alpha_{L \otimes_A P} \circ \left(L \otimes_A \beta_P\right)\right)\left(x \otimes_A 1_A\right)\right](y)$$
$$= \left[\alpha_{L \otimes_A P}\left(x \otimes_A \sum_{i=1}^{n} x_i \otimes_R x_i^*\right)\right](y) = \left[\sum_{i=1}^{n} \alpha_{L \otimes_A P}\left(x \otimes_A x_i \otimes_R x_i^*\right)\right](y)$$
$$= \sum_{i=1}^{n}\left(x \otimes_A x_i\right) x_i^*\left(y\right) = x \otimes_A \sum_{i=1}^{n} x_i x_i^*\left(y\right) = x \otimes_A y.$$

Therefore we deduce that $\gamma_L = \alpha_{L \otimes_A P} \circ \left(L \otimes_A \beta_P\right) \circ \left(\mu^L\right)^{-1}$ is an isomorphism. □

Corollary 10.19. *Let P_R be a progenerator and let $A = \mathrm{End}\left(P_R\right)$. Then the functor $\mathrm{Hom}_A\left(P, -\right) : A\text{-}Mod \to R\text{-}Mod$ is an equivalence of categories whose inverse is the functor ${}_AP_R \otimes_R - : Mod\text{-}A \to Mod\text{-}R$.*

Proof. By Corollary 10.17, ${}_AP$ is a progenerator and $R = \mathrm{End}\left({}_AP\right)$. Apply now Theorem 10.18. □

Exercise 10.20. *Let $n \in \mathbb{N}$, $n \geq 1$ and let $P_R = R_R^n$. Then $\mathrm{End}_R\left(P_R\right) \cong M_n\left(R\right)$ as rings.*

Example 10.21. *Let $n \in \mathbb{N}$, $n \geq 1$ and let $P_R = R_R^n$. Then P_R is a progenerator and $A = \mathrm{End}_R\left(P_R\right) \cong M_n\left(R\right)$. Hence, by Theorem 10.18, the functor*

$$\mathrm{Hom}_R\left({}_AP_R, -\right) : Mod - R \to Mod - A \cong Mod - M_n\left(R\right)$$

is an equivalence of categories whose inverse is the functor $- \otimes_A {}_AP_R : Mod\text{-}A \to Mod\text{-}R$.

Lemma 10.22. *Let P_R be a progenerator, let $A = \mathrm{End}(P_R)$ and let us consider the bimodule$_A\left((P^*)^*\right)_R := \mathrm{Hom}_A(P^*, A)$ where $P^* = \mathrm{Hom}_R(P, R)$. Then the map*

$$\begin{array}{rccl}\Omega : & P & \to & \mathrm{Hom}_A\left({}_RP^*, \mathrm{Hom}_R\left(P, {}_AP\right)\right) \\ & x & \mapsto & \xi \mapsto \left(y \mapsto x \cdot \xi(y)\right)\end{array}.$$

is well defined and is an isomorphism of A-R-bimodules.

Proof. By Theorem 10.18, for every $M \in Mod\text{-}R$ the evaluation map

$$\begin{array}{rccl}\nu_M : & \mathrm{Hom}_R(P, M) \otimes_A P & \to & M \\ & f \otimes_A y & \mapsto & f(y)\end{array}.$$

is well defined and it is a right R-module isomorphism. In particular, for $M = R$ we have that

$$\begin{array}{rccl}\nu_R : & \mathrm{Hom}_R\left({}_AP_R, R\right) \otimes_A {}_AP_R & \to & R \\ & f \otimes_A y & \mapsto & f(y)\end{array}$$

is a right R-module isomorphism. Now we have the following chain of isomorphisms

$$P \cong \mathrm{Hom}_R(R, P) \cong \mathrm{Hom}_R(P^* \otimes_A P, P) \cong \mathrm{Hom}_A\left(P^*, \mathrm{Hom}_R(P, P)\right) = \mathrm{Hom}_A(P^*, A)$$

where the first one is $\rho'_P : P \to \mathrm{Hom}_R(R, P)$ which is the isomorphism of Prop 6.29, the second one is $\mathrm{Hom}_R(\nu_R, P)$ and the third one is $\Lambda_P^{P^*}$ of Theorem 6.59. Let us prove that the composition of these isomorphisms is Ω. Let $x, y \in P$ and $\xi \in P^*$. We have

$$\begin{aligned}\left\{\left[\left(\Lambda_P^{P^*} \circ \mathrm{Hom}_R(\nu_R, P) \circ \rho'_P\right)(x)\right](\xi)\right\}(y) &= \left\{\left[\Lambda_P^{P^*}\left(\rho'_P(x) \circ \nu_R\right)\right](\xi)\right\}(y) \\ &= \left(\rho'_P(x) \circ \nu_R\right)(\xi \otimes_A y) = \\ &= \rho'_P(x)(\xi(y)) = x \cdot \xi(y) = [\Omega(x)(\xi)](y).\end{aligned}$$

Let us prove that Ω is a homomorphism of A-R-bimodules. Let $a \in A, r \in R, x \in P, \xi \in P^*$ and $y \in P$. We compute

$$\begin{aligned}[(a \cdot \Omega(x) \cdot r)](\xi)(y) &= \{a \cdot [\Omega(x)(r \cdot \xi)]\}(y) = a \cdot \{[\Omega(x)(r \cdot \xi)](y)\} \\ &= a \cdot (x \cdot [(r \cdot \xi)(y)]) = a \cdot (x \cdot [r \cdot \xi(y)]) \\ &= (a \cdot x \cdot r) \cdot \xi(y) = [\Omega(a \cdot x \cdot r)(\xi)](y).\end{aligned}$$

□

Theorem 10.23. *Let P_R be a progenerator, let $A = \mathrm{End}(P_R)$. By Proposition 10.16, the bimodules ${}_AP_R$ and ${}_RP^*_A = \mathrm{Hom}_R(P, R)$ are faithfully balanced. Let us consider the following functors:*

$$\begin{aligned}H &= \mathrm{Hom}_R(P, -) : Mod\text{-}R \longrightarrow Mod\text{-}A \\ T' &= - \otimes_R P^* : Mod\text{-}R \longrightarrow Mod\text{-}A \\ T &= - \otimes_A P_R : Mod\text{-}A \longrightarrow Mod\text{-}R \\ H' &= \mathrm{Hom}_A(P^*, -) : Mod\text{-}A \longrightarrow Mod\text{-}R.\end{aligned}$$

Then we have functorial isomorphisms

$$H \cong T' \text{ and } T \cong H'.$$

Proof. For every $M \in Mod\text{-}R$ let

$$\begin{array}{rlcl} \alpha_M : & M \otimes_R \mathrm{Hom}_R(P,R) & \to & \mathrm{Hom}_R(P,M) \\ & m \otimes f & \mapsto & y \mapsto mf(y) \end{array}.$$

be the isomorphism of Proposition 10.12. Then the family of maps $(\alpha_M)_{M \in Mod-R}$ gives rise to a functorial isomorphism

$$\alpha : - \otimes_R P^* \longrightarrow \mathrm{Hom}_R(P,-)$$

between the functors H and T'. Similarly consider the progenerator $P_A^* = \mathrm{Hom}_R(P,R)$ with $R \cong \mathrm{End}(P_A^*)$ and the bimodule ${}_A((P^*)^*)_R := \mathrm{Hom}_A(P^*, A)$. For every $L \in Mod\text{-}A$ let

$$\begin{array}{rlcl} \alpha'_L : & L \otimes_A (P^*)^* & \to & \mathrm{Hom}_A(P^*,L) \\ & x \otimes f & \mapsto & y \mapsto xf(y) \end{array}.$$

be the analogous of the isomorphism of Proposition 10.12 for the bimodule ${}_R P_A^*$ with P_A^* finitely generated and projective.Then the family of maps $(\alpha'_L)_{L \in Mod-A}$ gives rise to a functorial isomorphism $\alpha' : - \otimes_A (P^*)^* \longrightarrow \mathrm{Hom}_A(P^*,-) = H'$. By Lemma 10.22, the map

$$\begin{array}{rlcl} \Omega : & P & \to & \mathrm{Hom}_A(P^*, \mathrm{Hom}_R(P,P) = A) \\ & x & \mapsto & \xi \mapsto (y \mapsto x \cdot \xi(y)) \end{array}.$$

is well defined and is an isomorphism of A-R-bimodules. Hence we conclude that

$$- \otimes_A P \overset{- \otimes_A \Omega}{\cong} - \otimes_A (P^*)^*$$

is a functorial isomorphism. In conclusion we have a functorial isomorphism $T = - \otimes_A P \cong \mathrm{Hom}_A(P^*,-) = H'$. □

Proposition 10.24. *Let ${}_A W_R$ be an A-R-bimodule. By means of Proposition 6.28, for every $M \in Mod\text{-}R$ let us consider the left A-module* $\mathrm{Hom}_R(M,W)$ *and for any $L \in A\text{-}Mod$ let us consider the right R-module* $\mathrm{Hom}_R(M,W)$. *Then the map*

$$\begin{array}{rlcl} \vartheta : & \mathrm{Hom}_A({}_A L, \mathrm{Hom}_R(M,W)) & \to & \mathrm{Hom}_R(M_R, \mathrm{Hom}_A(L,W)) \\ & f & \mapsto & x \mapsto [()f](x) : L \to W \end{array}.$$

is an isomorphism natural in each variable.

Proof. Consider the map

$$\begin{array}{rlcl} \zeta : & \mathrm{Hom}_R(M, \mathrm{Hom}_A({}_A L, W)) & \to & \mathrm{Hom}_A(L, \mathrm{Hom}_R(M,W)) \\ & h & \mapsto & l \mapsto (l)h() : L \to W \end{array}.$$

Let us prove that it is a two-sided inverse of ϑ. For every $l \in L$ and $x \in M$, $f \in \mathrm{Hom}_R(M, \mathrm{Hom}_A(L,W))$ and $h \in \mathrm{Hom}_R(M, \mathrm{Hom}_A(L,W))$ we have

$$\{[(l)(\zeta \circ \vartheta)(f)]\}(x) = (l)[\vartheta(f)(x)] = [(l)f](x)$$

and

$$\{(l)[(\vartheta \circ \zeta)(h)](x)\} = [(l)\zeta(h)](x) = (l)h(x).$$

The remaining of the proof is left to the reader. □

Exercise 10.25. *The family of isomorphisms* $(\rho_M)_{M\in Mod\text{-}R}$ *, where*

$$\begin{array}{rlcl}\rho_M: & \mathrm{Hom}_R(R,M) & \to & M\\ & f & \mapsto & f(1_R)\end{array}$$

is the map of Proposition 6.29, defines a functorial isomorphism $\rho:\mathrm{Hom}_R(R,-)\to \mathrm{Id}_{Mod\text{-}R}$.

Lemma 10.26. *Let* $F: Mod\text{-}R\to Mod\text{-}A$ *be an additive functor. Assume that* (F,G) *is an equivalence of categories via the functorial isomorphisms* $\omega: G\circ F\to \mathrm{Id}_{Mod\text{-}R}$ *and* $\omega': F\circ G\to \mathrm{Id}_{Mod\text{-}A}$ *. Then, for every family* $(M_i)_{i\in I}$ *in* $Mod\text{-}R$ *we have that*

$$F\left(\bigoplus_{i\in I}M_i\right)\cong\bigoplus_{i\in I}F(M_i).$$

Proof. By..., F is full and faithful. Let $\varepsilon_i: M_i\to\bigoplus_{i\in I}M_i$ denote the i-th canonical injection. Let

$$\vartheta_i=F(\zeta_i):F(M_i)\to L=F(M)$$

be a family of morphisms in Mod-A. Then there exists a unique morphism

$$\zeta:\bigoplus_{i\in I}M_i\to M$$

such that

$$\zeta\circ\varepsilon_i=\zeta_i \text{ for every } i\in I.$$

Thus we get

$$F(\zeta)\circ F(\varepsilon_i)=F(\zeta_i)=\vartheta_i \text{ for every } i\in I.$$

Assume that

$$\chi:F\left(\bigoplus_{i\in I}M_i\right)\to F(M)$$

is another morphism such that

$$\chi\circ F(\varepsilon_i)=F(\zeta_i)=\vartheta_i \text{ for every } i\in I.$$

Then $\chi=F(\xi)$ for some $\xi:\bigoplus_{i\in I}M_i\to M$ and, since F is faithful, we get

$$\xi\circ\varepsilon_i=\zeta_i \text{ for every } i\in I.$$

By the unicity of ζ, we conclude. □

Let $F: Mod\text{-}R\to Mod\text{-}A$ be an additive functor. Assume that (F,G) is an equivalence of categories via the functorial isomorphisms $\omega: G\circ F\to\mathrm{Id}_{Mod\text{-}R}$ and $\omega': F\circ G\to\mathrm{Id}_{Mod\text{-}A}$ By..., F is full and faithful i.e. for every $M_1,M_2\in Mod$-R, the map

$$F_{M_2}^{M_1} : \mathrm{Hom}_{Mod\text{-}R}(M_1, M_2) = \mathrm{Hom}_R(M_1, M_2) \to \mathrm{Hom}_{Mod\text{-}A}(F(M_1), F(M_2)) = \mathrm{Hom}_A(F(M_1), F(M_2))$$

defined by setting

$$F_{M_2}^{M_1}(f) = F(f)$$

is a group isomorphism. In particular, for $M \in Mod\text{-}R$,

$$F_M^M : \mathrm{End}_R(M) = \mathrm{Hom}_{Mod\text{-}R}(M, M) \to \mathrm{Hom}_{Mod\text{-}A}(F(M), F(M)) = \mathrm{End}_A(F(M))$$

is a group isomorphism. Let us prove it is a ring homomorphism. Let $f, g \in \mathrm{End}_R(M)$. we have

$$F_M^M(f \circ g) = F(f \circ g) = F(f) \circ F(g) = F_M^M(f) \circ F_M^M(g).$$

Hence $F(M)$ is an $\mathrm{End}_R(M)$-A-bimodule.

Let us consider the particular case of $M = R_R$. Set $Q_A = F(R_R)$. By the foregoing we have

$$R \cong \mathrm{End}_A(Q)$$

so that Q is an R-A-bimodule.

Similar results hold for G. Let $P_R = G(A_A)$. Then $\mathrm{End}_R(P) \cong A$, P is an A-R-bimodule and, for every $M \in Mod\text{-}R$, we have the chain of isomorphisms

$$F(M) \overset{\rho_M^{-1}}{\cong} \mathrm{Hom}_A(A, F(M)) \overset{G_{F(M)}^A}{\cong} \mathrm{Hom}_R(G(A), GF(M)) \overset{\mathrm{Hom}_R(G(A), \omega_M)}{\cong} \mathrm{Hom}_R(G(A), M) = \mathrm{Hom}_R(P, M).$$

We leave it as an exercise to the reader to prove that this is an isomorphism of right A-modules. Since ρ, G_-^A and $\mathrm{Hom}_R(G(A), \omega)$ are functorial isomorphisms, we get a functorial isomorphism between the functors $F, \mathrm{Hom}_R(P, -) : Mod\text{-}R \to Mod\text{-}A$,

$$\varphi : F \to \mathrm{Hom}_R(P, -).$$

By Theorem (G, F) is an adjunction. Since also $(- \otimes_A P, \mathrm{Hom}_R(P, -))$ is an adjunction, By Theorem —, we get that $G \cong - \otimes_A P$. In particular G is a right exact functor. By interchanging the role of F and G, we get that also F is a right exact functor and since the functors $F, \mathrm{Hom}_R(P, -) : Mod\text{-}R \to Mod\text{-}A$ are isomorphic, we deduce that even $\mathrm{Hom}_R(P, -)$ is a right exact, and hence an exact, functor. Hence P_R is a projective right R-module. Let $M \in Mod\text{-}R$. Then, in $Mod\text{-}A$ we have an exact sequence of the type

$$A^{(X)} \longrightarrow F(M) \to 0$$

which, in view of Lemma 10.26 yields the exact sequence in $Mod\text{-}R$

$$P^{(X)} = (G(A))^{(X)} \cong G\left(A^{(X)}\right) \longrightarrow GF(M) \cong M \to 0.$$

Thus we deduce that P_R is also a generator. By symmetry we also get that Q_A is a generator. Hence in Mod-A we have an epimorphism of the form

$$Q_A^n \longrightarrow A_A \to 0$$

which yields the exact sequence in Mod-R

$$R^n \cong [GF(R)]^n = G(Q)^n \longrightarrow G(A) \to 0$$

so that we get that P_R is also finitely generated. Therefore we obtain the following theorem.

Theorem 10.27. *Let $F : Mod\text{-}R \to Mod\text{-}A$ be an additive functor. Assume that (F, G) is an equivalence of categories. Set $P_R = G(A_A)$. Then P_R is a progenerator and we have functorial isomorphisms*

$$F \cong \mathrm{Hom}_R(P, -) \text{ and } G \cong \otimes_A P.$$

Proposition 10.28.

$$\mathrm{Hom}_{A\text{-}}(E, G_B) \otimes_B F \cong \mathrm{Hom}_{A\text{-}}(E, G_B \otimes_B F)$$

$$\begin{array}{llll} \alpha : & \mathrm{Hom}_{A\text{-}}(E, G_B) \otimes_B F & \to & \mathrm{Hom}_{A\text{-}}(E, G_B \otimes_B F) \\ & f \otimes x & \mapsto & [y \mapsto (y) f \otimes_B x] \end{array}$$

when

- ${}_AE$ *is proj.f.g. or*
- ${}_BF$ *is proj.f.g.*

α is an isomorphism. If ${}_BF$ is proj.f.g. with dual basis $((x_1, \ldots, x_n), (x_1^*, \ldots, x_n^*))$,and $f \in \mathrm{Hom}_{A\text{-}}(E, G_B \otimes_B F)$, we have

$$\begin{aligned} \alpha^{-1}(f) &= \sum_i \sum_t [e \mapsto g_t \cdot (y_t) x_i^*] \otimes_B x_i \\ (e) f &= \sum_t g_t \otimes_B y_t \end{aligned}$$

$$F \otimes_B \mathrm{Hom}_{\text{-}A}(E, {}_BG) \cong \mathrm{Hom}_{\text{-}A}(E, F \otimes_B G)$$

$$\begin{array}{llll} \beta : & F \otimes_B \mathrm{Hom}_{\text{-}A}(E, {}_BG) & \to & \mathrm{Hom}_{\text{-}A}(E, F \otimes_B G) \\ & x \otimes f & \mapsto & [y \mapsto x \otimes_B f(y)] \end{array}$$

when

- E_A *is proj.f.g. or*

- F_B *is proj.f.g.*

β is an isomorphism. If F_B is proj.f.g. with dual basis $((x_1,\ldots,x_n),(x_1^*,\ldots,x_n^*))$ *,and* $f\in \mathrm{Hom}_{\text{-}A}(E,F\otimes_B G)$ *, we have*

$$\begin{aligned}\beta^{-1}(f) &= \sum_i x_i\otimes_B \sum_t [e\mapsto x_i^*(y_t)\,g_t]\\ f(e) &= \sum_t y_t\otimes_B g_t\end{aligned}$$

Proof. Assume F_B is projective and finitely generated. Then, by Proposition 10.11, we have that

$$\begin{array}{rccl}\omega_F: & F & \to & \mathrm{Hom}_{B\text{-}}(\mathrm{Hom}_{\text{-}B}(F,B),B)={}^*(F^*)\\ & y & \mapsto & \xi\mapsto \xi(y)\end{array}$$

is a right B-module isomorphism and by Lemma 10.10 F^* is a finitely generated and projective left B-module. Hence, by Proposition 10.12

$$F\otimes_B \mathrm{Hom}_{\text{-}A}(E,{}_BG)\cong {}^*(F^*)\otimes_B \mathrm{Hom}_{\text{-}A}(E,{}_BG)\overset{\text{prop10.12}}{\cong}\mathrm{Hom}_{B\text{-}}(F^*,\mathrm{Hom}_{\text{-}A}(E,{}_BG))\cong$$
$$\overset{\text{Prop10.24}}{\cong}\mathrm{Hom}_{\text{-}A}(E,\mathrm{Hom}_{B\text{-}}(F^*,G))\overset{\text{prop10.12}}{\cong}\mathrm{Hom}_{\text{-}A}(E,{}^*(F^*)\otimes_B G)\cong \mathrm{Hom}_{\text{-}A}(E,F\otimes_B G).$$

□

Corollary 10.29.

$$\begin{array}{rlcl}\alpha: & \mathrm{Hom}_{A\text{-}}(E,G_A)\otimes_A F & \to & \mathrm{Hom}_{A\text{-}}(E,{}_AG_A\otimes_A F)\\ & f\otimes x & \mapsto & [y\mapsto (y)\,f\otimes_A x]\end{array}$$

$$\begin{array}{rlcl}\alpha: & \mathrm{Hom}_{A\text{-}}(E,A_A)\otimes_A F & \to & \mathrm{Hom}_{A\text{-}}(E,A\otimes_A F)\cong \mathrm{Hom}_{A\text{-}}(E,F)\\ & f\otimes x & \mapsto & y\mapsto (y)\,f\otimes_A x\mapsto (y)\,f\cdot x\\ \alpha^{-1}: & \mathrm{Hom}_{A\text{-}}(E,F)\cong \mathrm{Hom}_{A\text{-}}(E,A\otimes_A F) & \to & \mathrm{Hom}_{A\text{-}}(E,A)\otimes_A F\\ & \varphi & \mapsto & \sum_i [e\mapsto 1_A\cdot((e)\,\varphi)\,x_i^*]\otimes_A x_i\end{array}$$

$$\begin{array}{rlcl}\alpha: & \mathrm{Hom}_{A\text{-}}(E,A_A)\otimes_A \mathrm{Hom}_{A\text{-}}(G_A,A) & \to & \begin{array}{l}\mathrm{Hom}_{A\text{-}}(E,A\otimes_A \mathrm{Hom}_{A\text{-}}(G_A,A))\cong \mathrm{Hom}_{A\text{-}}(E,\mathrm{Hom}_{A}\\ \cong \mathrm{Hom}_{A\text{-}}(G\otimes_A E,A)\end{array}\\ & f\otimes g & \mapsto & \begin{array}{l} y\mapsto (y)\,f\otimes_A g\mapsto (y)\,f\cdot g\\ x\otimes y\mapsto (x)\,[(y)\,f\cdot g]=[x\cdot (y)\,f]\,g\end{array}\\ \alpha^{-1}: & \mathrm{Hom}_{A\text{-}}(G\otimes_A E,A)\cong \mathrm{Hom}_{A\text{-}}(E,\mathrm{Hom}_{A\text{-}}(G_A,A))\cong \mathrm{Hom}_{A\text{-}}(E,A\otimes_A \mathrm{Hom}_{A\text{-}}(G_A,A)) & \to & \\ & \varphi & \mapsto & \end{array}$$

$$\begin{array}{llll} \alpha: & \mathrm{Hom}_{A\text{-}}(E, A_A) \otimes_A \mathrm{Hom}_{A\text{-}}(G_A, A) & \to & \mathrm{Hom}_{A\text{-}}(G \otimes_A E, A) \\ & f \otimes g & \mapsto & x \otimes y \mapsto [x \cdot (y) f] g \end{array}$$

$$\begin{array}{llll} \alpha^{-1}: & \mathrm{Hom}_{A\text{-}}(G \otimes_A E, A) & \to & \mathrm{Hom}_{A\text{-}}(E, A_A) \otimes_A \mathrm{Hom}_{A\text{-}}(G_A, A) \\ & \varphi & \mapsto & \sum_i [e \mapsto (x_i \otimes e) \varphi] \otimes_A x_i^* \end{array}$$

where $((x_1, \ldots, x_n), (x_1^*, \ldots, x_n^*))$ *is a dual basis for* G_A

Corollary 10.30. *Case* A *is commutative and we have symmetric modules*

$$\begin{array}{llll} \alpha: & \mathrm{Hom}_{A\text{-}}(E, A_A) \otimes_A \mathrm{Hom}_{A\text{-}}(G_A, A) & \to & \mathrm{Hom}_{A\text{-}}(G \otimes_A E, A) \\ & f \otimes g & \mapsto & x \otimes y \mapsto g(x) f(y) \end{array}$$

$$\begin{array}{llll} \alpha^{-1}: & \mathrm{Hom}_{A\text{-}}(G \otimes_A E, A) & \to & \mathrm{Hom}_{A\text{-}}(E, A_A) \otimes_A \mathrm{Hom}_{A\text{-}}(G_A, A) \\ & \varphi & \mapsto & \sum_i [e \mapsto \varphi(x_i \otimes e)] \otimes_A x_i^* \end{array}$$

Definition 10.31. *Let* Z *be a commutative ring. A* Z*-algebra* R *is called an Azumaya algebra over* Z *if*

1) *the map*

$$\begin{array}{llll} \varphi: & R \otimes_z R & \to & \mathrm{End}_{\text{-}Z}(R) \\ & \sum a_i \otimes_Z a_i' & \mapsto & [x \mapsto \sum a_i x a_i'] \end{array}$$

is an isomorphism

2) ${}_Z R$ *is a progenerator.*

Proposition 10.32. *Let* R *and* S *be algebras over a commutative ring* Z*. Then* R*-Mod-*$S \cong$ *Mod-*$(S \otimes_Z R^{op})$ *via*

$$x \cdot (s \otimes r) = r \cdot x \cdot s$$

Similarly R*-Mod-*$S \cong (S^{op} \otimes_Z R)$

Proof.

$$[x \cdot (s \otimes r)] \cdot (s' \otimes r') \stackrel{?}{=} x \cdot [(s \otimes r) \cdot (s' \otimes r')]$$

$$[x \cdot (s \otimes r)] \cdot (s' \otimes r') = (r \cdot x \cdot s) \cdot (r' \otimes s') = r' \cdot (r \cdot x \cdot s) \cdot s' = (r' \cdot r) \cdot x \cdot (s \cdot s')$$

$$x \cdot [(s \otimes r) \cdot (s' \otimes r')] = x \cdot (s \cdot s' \otimes r' \cdot r) = (r' \cdot r) \cdot x \cdot (s \cdot s')$$

□

Notation 10.33. *Let* R *be an algebra over a commutative ring* Z*. We set*

$$R^e = R \otimes_Z R^{op} \text{ and } {}^eR = R^{op} \otimes_Z R$$

Then, by the foregoing we have

$${}^eR\text{-}Mod \cong R\text{-}Mod\text{-}R \cong Mod\text{-}R^e.$$

Notation 10.34. *Let M be a bimodule over a ring R. We set*

$$M^R = \{x \in M \mid rx = xr \text{ for every } r \in R.\}$$

Lemma 10.35. *Let M be a bimodule over a ring R. Then the map*

$$\begin{array}{llll} \varphi_M : & \mathrm{Hom}_{R^e}(R, M) = \mathrm{Hom}_{R\text{-}R}(R, M) & \to & M^R \\ & f & \mapsto & f(1_R) \end{array}$$

is an isomorphism.

Proof. Let us consider the isomorphism

$$\begin{array}{llll} \rho_M : & \mathrm{Hom}_R(R, M) & \to & M \\ & f & \mapsto & f(1_R) \end{array}$$

of Proposition 6.29. Let $f \in \mathrm{Hom}_{\text{-}R}(R, M)$ and assume that $f \in \mathrm{Hom}_{R-R}(R, M)$. Then, for every $r \in R$ we have

$$r \cdot f(1_R) = f(r) = f(1_R) \cdot r$$

so that $f(1_R) \in M^R$. Conversely, assume that $f(1_R) \in M^R$. Then, for every $r \in R$, we have

$$f(r) = f(1_R) \cdot r = r \cdot f(1_R)$$

so that for every $x \in R$ we have

$$f(rx) = f(r) \cdot (x) = [r \cdot f(1_R)] \cdot x = r \cdot [f(1_R) \cdot x] = r \cdot f(x).$$

□

Lemma 10.36.

$$[\mathrm{Hom}_{\text{-}T}({}_SX_T, {}_SY_T)]^S = \mathrm{Hom}_{S\text{-}T}({}_SX_T, {}_SY_T)$$

Proof. Let $f \in \mathrm{Hom}_{\text{-}T}({}_SX_T, {}_SY_T)$. Then, for every $s \in S$ we have, for every $x \in X$

$$(s \cdot f)(x) = s \cdot f(x) \text{ and } (f \cdot s)(x) = f(s \cdot x)$$

so that

$$s \cdot f = f \cdot s \Longleftrightarrow s \cdot f(x) = f(s \cdot x) \text{ for every } x \in X \Longleftrightarrow f \in \mathrm{Hom}_{S\text{-}T}({}_SX_T, {}_SY_T).$$

□

Corollary 10.37.

$$[\mathrm{Hom}_{\text{-}S}(X, Y)]^S = \mathrm{Hom}_{S\text{-}S}(X, Y) = \mathrm{Hom}_{\text{-}S^e}(X, Y) = \mathrm{Hom}_{{}^eS\text{-}}(X, Y)$$

Proposition 10.38. *Let Z be a subring of a ring S which centralize S i.e. $z\cdot s = s\cdot z$ for every $\in Z$ and $s \in S$. Then M^S is a right Z-submodule of M. Let*

$$i : M^S \to M$$

be the canonical injection. Then the map

$$\mathrm{Hom}_{\text{-}Z}(W, i) : \mathrm{Hom}_{\text{-}Z}\left(W, M^S\right) \to \mathrm{Hom}_{\text{-}Z}(W, M)$$

yields an isomorphism

$$\mathrm{Hom}_{\text{-}Z}\left(W, M^S\right) \cong \left[\mathrm{Hom}_{\text{-}Z}(W, M)\right]^S.$$

Proof. For every $s \in S, z \in Z, m \in M^S$ we have

$$s\cdot(z\cdot m) = (s\cdot z)\cdot m = m\cdot(s\cdot z) = m\cdot(z\cdot s) = (m\cdot z)\cdot s.$$

$$\mathrm{Hom}_{\text{-}Z}\left(W, M^S\right) \cong \mathrm{Hom}_{\text{-}Z}\left(W, \mathrm{Hom}_{\text{-}S^e}(S, M)\right) \cong \mathrm{Hom}_{\text{-}S^e}(W\otimes_Z S, M) = \mathrm{Hom}_{S\text{-}S}(W\otimes_Z S, M) =$$
$$\overset{\text{Lemma10.36}}{=} \left[\mathrm{Hom}_{\text{-}S}(W\otimes_Z S, M)\right]^S \cong \left[\mathrm{Hom}_{\text{-}Z}(W, \mathrm{Hom}_S(S, M))\right]^S \cong \left[\mathrm{Hom}_{\text{-}Z}(W, M)\right]^S$$

□

Lemma 10.39. *Let R be an algebra over a commutative ring Z and let S be a Z-subalgebra of R. Then the map*

$$\begin{array}{rlll} \Theta : & R^S\otimes_Z R & \to & (R\otimes_Z R)^S \\ & a\otimes_Z b & \mapsto & a\otimes_Z b \end{array}$$

is well defined and it is an isomorphism of S-S-bimodules.

Proof. By Lemma 10.35 the map

$$\begin{array}{rlll} \vartheta : & \mathrm{Hom}_{{}^eS}(S, R) = \mathrm{Hom}_{S\text{-}S}(S, R) & \to & R^S \\ & f & \mapsto & f(1_S) \end{array}$$

is an isomorphism. By Lemma 10.36

$$\left[\mathrm{Hom}_{\text{-}T}({}_SX_T, {}_SY_T)\right]^S = \mathrm{Hom}_{S\text{-}T}({}_SX_T, {}_SY_T)$$

$$\left(\mathrm{Hom}_{\text{-}S}(S, R\otimes_Z R)\right)^S \overset{10.36}{=} \mathrm{Hom}_{S\text{-}S}(S, R\otimes_Z R)$$

Since ${}_ZR$ is projective and f.g., by Proposition 10.28, we have that

$$\begin{array}{rlll} \alpha : & \mathrm{Hom}_{A\text{-}}(E, G_B)\otimes_B F & \to & \mathrm{Hom}_{A\text{-}}(E, G_B\otimes_B F) \\ & f\otimes x & \mapsto & [y \mapsto (y)\,f\otimes_B x] \end{array}$$

$$\begin{array}{rlll} \alpha : & \mathrm{Hom}_{{}^eS\text{-}}(S, R)\otimes_Z R & \to & \mathrm{Hom}_{{}^eS\text{-}}(S, R\otimes_Z R) \\ & f\otimes x & \mapsto & [y \mapsto (y)\,f\otimes_B x] \end{array}$$

is an isomorfism. Therefore we deduce that

$$R^S \otimes_Z R \overset{10.35}{\cong} \mathrm{Hom}_{^eS\text{-}}(S,R) \otimes_Z R \overset{10.28}{\cong} \mathrm{Hom}_{^eS\text{-}}(S, R \otimes_Z R) = \mathrm{Hom}_{S\text{-}S}(S, R \otimes_Z R) =$$
$$\overset{10.36}{=} (\mathrm{Hom}_{\text{-}S}(S, R \otimes_Z R))^S \cong (R \otimes_Z R)^S.$$

Excplicitely let

$$\sum a_t \otimes_Z r_t \text{ where } a_t \in R^S \text{ for every } t.$$

Then we have

$$\left[\sum a_t \otimes_Z r_t\right] \mapsto \left[\sum \widehat{a_t} \otimes_Z r_t\right] \mapsto \left[s \mapsto \sum s a_t \otimes_Z r_t\right] \mapsto \left[\sum a_t \otimes_Z r_t\right].$$

□

Lemma 10.40. *Let R be an Azumaya algebra over the commutative ring Z and let S be a Z-subalgebra of R. Then the map*

$$\begin{array}{cccc} \chi : & R^S \otimes_Z R & \to & \mathrm{Hom}_{S\text{-}}(R,R) \\ & \sum a_t \otimes_Z b_t & \mapsto & [x \mapsto \sum a_t \cdot x \cdot b_t] \end{array}$$

is well defined and it is an isomorphism.

Proof. By Lemma 10.39 we have that the map

$$\begin{array}{cccc} \Theta : & R^S \otimes_Z R & \to & (R \otimes_Z R)^S \\ & a \otimes_Z b & \mapsto & a \otimes_Z b \end{array}$$

gis well defined and it is an isomorphism of S-S-bimodules. Now, by definition of Azumaya algebra we have that the map

$$\begin{array}{cccc} \varphi : & R \otimes_z R & \to & \mathrm{End}_Z(R) \\ & \sum a_i \otimes_Z a'_i & \mapsto & [x \mapsto \sum a_i x a'_i] \end{array}$$

is an isomorphism of S-bimodules. Therefore we deduce that

$$(R \otimes_Z R)^S \overset{\text{defAzumaya}}{\cong} (\mathrm{End}_{\text{-}Z}(R))^S \overset{\text{Lem10.36}}{\cong} \mathrm{Hom}_{S\text{-}Z}(R,R) \overset{Z \subseteq S + Z\text{comm}}{\cong} \mathrm{Hom}_{S\text{-}}(R,R)$$

$$[\mathrm{Hom}_{\text{-}T}({}_S X_T, {}_S Y_T)]^S = \mathrm{Hom}_{S\text{-}T}({}_S X_T, {}_S Y_T)$$

□

Lemma 10.41. *Let R be an Azumaya algebra over the commutative ring Z and let S be a Z-subalgebra of R. Assume ${}_S R$ f.g. projective. Then the map*

$$\begin{array}{cccc} \Omega : & R \otimes_S R & \to & \mathrm{Hom}_{\text{-}Z}\left(R^S, R\right) \\ & a \otimes_S b & \mapsto & [\alpha \mapsto a\alpha b] \end{array}$$

is well defined and it is an isomorphism.

Proof.

$$\mathrm{Hom}_{\text{-}Z}\left(R^S, R\right) \cong \mathrm{Hom}_{\text{-}Z}\left(R^S, \mathrm{Hom}_{\text{-}R}\left(R, R\right)\right) \cong \mathrm{Hom}_{\text{-}R}\left(R^S \otimes_Z R, R\right) \overset{\text{Lem10.40}}{\cong} \mathrm{Hom}_{\text{-}R}\left(\mathrm{Hom}_{S\text{-}}\left(R, R\right.\right.$$

$$\overset{{}_SR\text{f.g.proj+Prop10.12}}{\cong} \mathrm{Hom}_{\text{-}R}\left({}^*R \otimes_S R, R\right) \cong \mathrm{Hom}_{\text{-}S}\left({}^*R, \mathrm{Hom}_{\text{-}R}\left(R, R\right)\right) \cong \mathrm{Hom}_{\text{-}S}\left({}^*R, R\right) \cong R \otimes_S \left({}^*R\right)^*$$

$$\cong R \otimes_S R$$

∎

Lemma 10.42. *Let R be an Azumaya algebra over the commutative ring Z and let S be a Z-subalgebra of R. Assume ${}_SR$ f.g. projective. Then the map*

$$\begin{array}{llll} \Psi: & R \otimes_S R \otimes_S R & \to & \mathrm{Hom}_{\text{-}Z}\left(R^S \otimes_Z R^S, R\right) \\ & a \otimes_S b \otimes_S c & \mapsto & [\alpha \otimes_Z \beta \mapsto a\alpha b\beta c] \end{array}$$

is well defined and it is an isomorphism.

$$R \otimes_S R \otimes_S R \cong \mathrm{Hom}_{\text{-}Z}\left(R^S \otimes_Z R^S, R\right)$$

Proof. ...

$$\left(R \otimes_S R\right) \otimes_S R \overset{\text{Lem10.41}}{\cong} \mathrm{Hom}_{\text{-}Z}\left(R^S, R\right) \otimes_S R \overset{{}_SR\text{f.g.proj+Prop10.12}}{\cong}$$

$$\cong \mathrm{Hom}_{\text{-}Z}\left(R^S, R \otimes_S R\right) \overset{\text{Lem10.41}}{\cong} \mathrm{Hom}_{\text{-}Z}\left(R^S, \mathrm{Hom}_{\text{-}Z}\left(R^S, R\right)\right) \cong \mathrm{Hom}_{\text{-}Z}\left(R^S \otimes_Z R^S, R\right)$$

$$\mathrm{Hom}_{\text{-}Z}\left(R^S \otimes_Z R^S, R\right) \cong \mathrm{Hom}_{\text{-}Z}\left(R^S, \mathrm{Hom}_{\text{-}Z}\left(R^S, R\right)\right) \overset{\text{Lem10.41}}{\cong} \mathrm{Hom}_{\text{-}Z}\left(R^S, R \otimes_S R\right) \cong$$

$$\overset{{}_SR\text{f.g.proj+Prop10.12}}{\cong} \mathrm{Hom}_{\text{-}Z}\left(R^S, R\right) \otimes_S R \overset{\text{Lem10.41}}{\cong} \left(R \otimes_S R\right) \otimes_S R$$

∎

Lemma 10.43. *Let R be an Azumaya algebra over the commutative ring Z and let S be a Z-subalgebra of R. Assume ${}_SR$ f.g. projective. Then the map*

$$\begin{array}{llll} \Gamma: & \left(R \otimes_S R\right)^S & \to & \mathrm{End}_{\text{-}Z}\left(R^S\right) \\ & a \otimes_S b & \mapsto & [\alpha \mapsto a\alpha b] \end{array}$$

is well defined and it is an isomorphism.

$$\left(R \otimes_S R\right)^S \cong \mathrm{End}_{\text{-}Z}\left(R^S\right)$$

Proof.

$$\left(R \otimes_S R\right)^S \overset{\text{Lem10.41}}{\cong} \left[\mathrm{Hom}_{\text{-}Z}\left(R^S, R\right)\right]^S \overset{\text{Pro10.38}}{\cong} \mathrm{Hom}_{\text{-}Z}\left(R^S, R^S\right)$$

∎

Lemma 10.44. *Let R be an Azumaya algebra over the commutative ring Z and let S be a Z-subalgebra of R. Assume ${}_SR$ f.g. projective. Then the map*

$$\begin{array}{rlcl}\Xi: & (R\otimes_S R\otimes_S R)^S & \to & \mathrm{Hom}_{\text{-}Z}\left(R^S\otimes_Z R^S, R^S\right)\\ & a\otimes_S b\otimes_S c & \mapsto & [\alpha\otimes_Z\beta\mapsto a\alpha b\beta c]\end{array}$$

is well defined and it is an isomorphism.

$$(R\otimes_S R\otimes_S R)^S\cong\mathrm{Hom}_{\text{-}Z}\left(R^S\otimes_Z R^S, R^S\right)$$

Proof. $(R\otimes_S R\otimes_S R)^S \overset{\text{Lem10.42}}{\cong} \left[\mathrm{Hom}_{\text{-}Z}\left(R^S\otimes_Z R^S, R\right)\right]^S \overset{\text{Pro10.38}}{\cong} \mathrm{Hom}_{\text{-}Z}\left(R^S\otimes_Z R^S, R^S\right)$

$$\mathrm{Hom}_{\text{-}Z}\left(W, M^S\right)\cong\left[\mathrm{Hom}_{\text{-}Z}(W,M)\right]^S.$$

□

Notation 10.45. *Let S be a subring of a ring R. We define on $R\otimes_S R$ an R-coring structure by setting*

$$\Delta(a\otimes_S b)=(a\otimes_S 1_R)\otimes_R(1_R\otimes_S b)$$

and

$$\varepsilon(a\otimes_S b)=ab.$$

Lemma 10.46. *Let S be a subring of a ring R. Then for every $M\in R$-Mod-R, we have*

$$\begin{array}{rlcl}\Psi: & \mathrm{Hom}_{R-Mod-R}(R\otimes_S R, M) & \to & M^S\\ & f & \mapsto & f(1_R\otimes_S 1_R)\end{array}$$

is an isomorphism of S-S-bimodules.

$$\mathrm{Hom}_{R-Mod-R}(R\otimes_S R, M)\cong M^S$$

so that

$$\mathrm{End}_{R\text{-}cor}(R\otimes_S R)\cong(R\otimes_S R)^S\cap Gr(R\otimes_S R)$$

Proposition 10.47. *Let R be an Azumaya algebra over the commutative ring Z and let S be a Z-subalgebra of R. Assume ${}_SR$ f.g. projective. Then*

$$\begin{array}{rlcl}\Gamma: & \mathrm{Hom}_{R-Mod-R}(R\otimes_S R, R\otimes_S R) & \to & \mathrm{End}_{\text{-}Z}\left(R^S\right)\\ & f & \mapsto & \left[\alpha\mapsto\sum a_i\alpha b_i\right]\end{array}\quad\text{where } f(1_R\otimes_S 1_R)=\sum a_i\otimes_S b_i$$

induces an isomorphism

$$\mathrm{End}_{R\text{-}cor}(R\otimes_S R)\cong\mathrm{End}_{Z\text{-}alg}\left(R^S\right)$$

Proof. Let $f \in \mathrm{Hom}_{R-Mod-R}(R \otimes_S R, R \otimes_S R)$. Then $f \in \mathrm{End}_{R\text{-cor}}(R \otimes_S R)$ if and only if

$$\Delta \circ f = (f \otimes_S f) \circ \Delta.$$

Let

$$\begin{array}{cccc} \Lambda : & \mathrm{Hom}_{R-Mod-R}(R \otimes_S R, R \otimes_S R \otimes_R R \otimes_S R) & \to & \mathrm{Hom}_{\text{-}Z}\left(R^S \otimes_Z R\right. \\ & h & \mapsto & [\alpha \otimes_Z \beta \mapsto \sum a_i \end{array}$$

where $h(1_R \otimes_S 1_R) = \sum a_i \otimes_S b_{i,j} \otimes_R c_{i,j,k} \otimes_S d_{i,j,k}$

Then $\Delta \circ f = (f \otimes_S f) \circ \Delta$ if and only if

$$\Lambda(\Delta \circ f) = \Lambda((f \otimes_S f) \circ \Delta).$$

Let

$$f(1_R \otimes_S 1_R) = \sum a_i \otimes_S b_i.$$

Then

$$(\Delta \circ f)(1_R \otimes_S 1_R) = \sum a_i \otimes_S 1_R \otimes_R 1_R \otimes_S b_i$$

so that

$$[\Lambda(\Delta \circ f)](\alpha \otimes_Z \beta) = \sum a_i \cdot (\alpha \cdot \beta) \cdot b_i = \Gamma(f)((\alpha \cdot \beta))$$

and

$$[(f \otimes_S f) \circ \Delta](1_R \otimes_S 1_R) = f(1_R \otimes_S 1_R) \otimes_R f(1_R \otimes_S 1_R) = \sum a_i \otimes_S b_i \otimes_R \sum a_j \otimes_S b_j$$

so that

$$[\Lambda((f \otimes_S f) \circ \Delta)](\alpha \otimes_Z \beta) = \sum_i \sum_j a_i \cdot \alpha \cdot b_i \cdot a_j \cdot \beta \cdot b_j = \Gamma(f)(\alpha) \cdot \Gamma(f)(\beta)$$

Therefore $\Lambda(\Delta \circ f) = \Lambda((f \otimes_S f) \circ \Delta)$ if and only if

$$\Gamma(f)((\alpha \cdot \beta)) = \Gamma(f)(\alpha) \cdot \Gamma(f)(\beta) \text{ for every } \alpha, \beta \in R^S.$$

□

10.2 Frobenius

Lemma 10.48. *Let R be a ring. Assume that P_R is projective and finitely generated. Let $P^* = \mathrm{Hom}_R(P_R, R_R)$ and regard it has a left R-module via*

$$(r \cdot f)(x) = r \cdot f(x).$$

Let $P^{**} = \mathrm{Hom}_R({}_RP^*, {}_RR)$ which is a right R-module via

$$(f)(\alpha \cdot r) = [(f)(\alpha)] \cdot r$$

and let $\omega = \omega_P : P \to P^{**}$ the map defined by $\omega(x) = \widetilde{x}$ where

$$(f)\widetilde{x} = f(x) \text{ for every } f \in P^*.$$

Then ω is well-defined and it is an isomorphism of right R-modules.

Proof. Let $x \in P$. Then

$$(r \cdot f)\,\widetilde{x} = (r \cdot f)(x) = r \cdot f(x) = r \cdot [(f)\,\widetilde{x}]$$

which means that $\widetilde{x} \in \mathrm{Hom}_R({}_RP^*, {}_RR)$. Let us check that ω is right R-linear. Let $f \in P^*$

$$(f)[\omega(x \cdot r)] = (f)[\omega(x) \cdot r]$$

$$(f)[\omega(x \cdot r)] = f(x \cdot r) = f(x) \cdot r = (f)(\widetilde{x} \cdot r) = (f)[\omega(x) \cdot r].$$

Let $((x_1, \ldots, x_n), (x_1^*, \ldots, x_n^*))$ be a finite dual basis for P. Let us check that ω is injective. Let $0 \neq x \in P$. Then

$$x = \sum x_i \cdot x_i^*(x).$$

Hence there exists an i such that $x_i^*(x) \neq 0$. Hence $(x_i^*)\,\widetilde{x} = x_i^*(x) \neq 0$. Let us check that ω is surjective. Let $\alpha \in P^{**}$. By lemma 10.10, the left R-module $\mathrm{Hom}_R(P, R)$ is projective and finitely generated with dual basis $((x_1^*, \ldots, x_n^*), (\widetilde{x}_1, \ldots, \widetilde{x}_n))$. Hence

$$\alpha = \sum \widetilde{x}_i \cdot [(x_i^*)\,\alpha] = \left[\sum x_i \cdot (x_i^*)\,\alpha\right] \omega.$$

□

10.49. *Let R be a commutative ring and let A be an R-algebra i.e. there is a ring morphism $\eta : R \to A$ such that $Im(\eta) \subseteq Z(A)$ where $Z(A)$ denotes the center of A.* ***In this case we will write also morphism of left A-modules on the left.*** *The abelian group $\mathrm{Hom}_R(A_R, R_R)$ has a structure of right A-module defined by setting*

$$(f \cdot a)(x) = f(ax).$$

The abelian group $\mathrm{Hom}_R({}_RA, {}_RR)$ has a structure of left A-module defined by setting

$$(a \cdot f)(x) = f(xa).$$

Since A is a symmetrical R-bimodule we have that $\mathrm{Hom}_R(A_R, R_R) = \mathrm{Hom}_R({}_RA, {}_RR)$. We set $A^\vee = \mathrm{Hom}_R(A_R, R_R) = \mathrm{Hom}_R({}_RA, {}_RR)$. Then $A^\vee$ is a left and also a right A-module. Let us check that it is indeed an A-A-bimodule. In fact we have

$$[a \cdot (f \cdot b)](x) = (f \cdot b)(x \cdot a) = f(b \cdot (x \cdot a)) = f((b \cdot x) \cdot a) = (a \cdot f)(b \cdot x) = [(a \cdot f) \cdot b](x).$$

Note that the induced R-R-bimodule structure on $A^\vee$ makes it a symmetrical R-bimodule.

Corollary 10.50. *Let A be an algebra over a commutative ring R. Then, in the notations of 10.49 and Lemma 10.48, let*

$$A^{\vee\vee} = \mathrm{Hom}_R({}_RA^\vee, {}_RR)$$

endowed the left A-module structure defined by

$$(f)(a \cdot \alpha) = (f \cdot a)\,\alpha \text{ for every } a \in A, \alpha \in A^{\vee\vee}, f \in A^\vee.$$

Then $\omega = \omega_A : {}_AA \to {}_AA^{**} = A^{\vee\vee}$ is an isomorphism of left A-modules.

Proof. By Lemma 10.48 we have only to prove that ω is a morphism of left A-modules.

Let $a, x \in A, f \in A^{\vee}$. We have

$$[a \cdot \omega(x)](f) = \omega(x)(f \cdot a) = (f \cdot a)(x) = f(a \cdot x) = [\omega(a \cdot x)](f).$$

$\square$

10.51. *Let $\varphi : A_A \to A_A^{\vee}$ be an isomorphism of right A-modules. Then $\varphi : {}_R A \to {}_R A^{\vee}$ is also a left R-modules homomorphism so that we can consider*

$$\mathrm{Hom}_R(\varphi, {}_R R) : \mathrm{Hom}_R({}_R A^{\vee}, {}_R R) \to \mathrm{Hom}_R({}_R A, {}_R R)$$

which is a group isomorphism. Let us check it is a left A-modules isomorphism. For every $x, a \in A, f \in \mathrm{Hom}_R({}_R A^{\vee}, {}_R R)$, we have

$$\begin{aligned} [\mathrm{Hom}_R(\varphi, {}_R R)(a \cdot f)](x) &= [(a \cdot f) \circ \varphi](x) = (a \cdot f)(\varphi(x)) = f[\varphi(x) \cdot a] \overset{\varphi is right A-modules}{=} \\ &= f(\varphi(x \cdot a)) = [a \cdot (\varphi \circ f)](x) = (a \cdot [\mathrm{Hom}_R(\varphi, {}_R R)(f)])(x) \end{aligned}$$

so that

$$\mathrm{Hom}_R(\varphi, {}_R R) : {}_A A^{**} = A^{\vee\vee} \to \mathrm{Hom}_R({}_R A, {}_R R) = {}_A A^{\vee}$$

is an isomorphism of left A-modules. Since $\omega = \omega_A : {}_A A \to {}_A A^{**} = A^{\vee\vee}$ is also an isomorphism of left A-modules, we get that

$$\zeta = \mathrm{Hom}_R(\varphi, {}_R R) \circ \omega_A : {}_A A \to {}_A A^{\vee}$$

is an isomorphism of left A-modules. We have

$$\zeta(1) = \mathrm{Hom}_R(\varphi, {}_R R)\left(\widetilde{1}\right) = \widetilde{1} \circ \varphi$$

so that, for every $a \in A$, we get

$$[\zeta(1)](a) = \left[\widetilde{1} \circ \varphi\right](a) = \varphi(a)(1) = \varphi(1 \cdot a)(1) \overset{\varphi is right A-modules}{=} [\varphi(1) \cdot a](1) = \varphi(1)(a \cdot 1) = \varphi(1$$

Thus we deduce that

$$\zeta(1) = \varphi(1).$$