HOPF ALGEBRAS

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Contents

Chapter 1

Algebras and Coalgebras

1.1. *Let k be a commutative ring. If not stated otherwise, by the word k-module we mean a symmetric k-module. Whenever k is a field, the word vector space substitutes the word k-module. A k-homomorphism between k-modules will be also called a k-linear map.* $\mathrm{Hom}_k(M,N)$ *or even* $\mathrm{Hom}(M,N)$ *the group of k-linear maps.*

1.2. *The tensor product over k will be denoted by $\otimes_k$ or even by $\otimes$ if there is no risk of confusion. For a k-module M we denote by M^n the n-th tensor power of M and for a morphism $f : M \to N$ of k-modules, we will denote by f^n the n-th tensor power of f. Also, for any k-module W, $f \otimes W$ will denote the morphism $f \otimes \mathrm{Id}_W$. a similar convention holds for $W \otimes f$.*

1.3. *Given a k-module M, we denote by l_M the obvious isomorphism $l_M : k \otimes_k M \to M$*

$$l_M(t \otimes x) = t \cdot x \text{ for every } t \in k, x \in M.$$

The morphism $r_M : M \otimes k \to M$ is similarly defined. The identity on M will be denoted by I_M or even more simply by I or M. Observe that both l_M and r_M give rise to functorial isomorphisms. In fact if $f : M \to N$ is a k-linear map we have

$$f \circ l_M = l_N \circ (k \otimes f) \quad \text{and} \quad f \circ r_M = r_N \circ (f \otimes k). \tag{1.1}$$

Moreover

$$l_{M\otimes N} = l_M \otimes N \quad r_{M\otimes N} = M \otimes r_N \quad \text{and} \quad M \otimes l_N = r_M \otimes N \tag{1.2}$$

We will also denote by $\tau_{M,N} : M \otimes N \to N \otimes M$ the usual flip. Note that if $f : M \to M'$ and $g : N \to N'$ are k-linear maps, then

$$\tau_{M',N'} \circ (f \otimes g) = (g \otimes f) \circ \tau_{M,N} \tag{1.3}$$

Notation 1.4. *Let R be a ring and let X be a non empty set. For each $x \in X$ let e_x be the element of $R^{(X)}$ defined by*

$$e_x(x) = 1_R \qquad \text{and} \qquad e_x(y) = 0_R \qquad \text{for every } y \in X, y \neq x.$$

Then every element $\alpha \in R^{(X)}$ can be uniquely written, using the left R-module structure of $R^{(X)}$, as

$$\alpha = \sum_{x \in Supp(\alpha)} \alpha(x)\, e_x.$$

From now on, for every $x \in X$, we will write x instead of e_x.

Definition 1.5. *Let k be a commutative ring. A k-*algebra *is a couple (A, u) where*

- *A is a ring*
- *$u : k \to A$ is a morphism of rings such that*

  $$\operatorname{Im}(u) \subseteq Z(A)$$

  where $Z(A)$ denotes the center of A.

Definition 1.6. *Let k be a commutative ring. A k-*algebra *is a triple (A, m, u) where*

- *A is k-module*
- *$m : A \otimes_k A \to A$ is a morphism of k-modules*
- *$u : k \to A$ is a morphism of k-modules*

such that the following diagrams are commutative:

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{A \otimes m} & A \otimes A \\ \downarrow{\scriptstyle m \otimes A} & & \downarrow{\scriptstyle m} \\ A \otimes A & \xrightarrow{m} & m \end{array} \qquad \begin{array}{ccccc} k \otimes A & \xrightarrow{l_A} & A & \xleftarrow{r_A} & A \otimes k \\ & {\scriptstyle u \otimes A} \searrow & \uparrow{\scriptstyle m} & \swarrow {\scriptstyle A \otimes u} & \\ & & A \otimes A & & \end{array}$$

Exercise 1.7. *Proof that Definition 1.5 and Definition 1.6 are equivalent.*

Definition 1.8. *Let (A, m_A, u_A) and (B, m_B, u_B) be k-algebras. A k-linear map $f : A \to B$ is called a* morphism of algebras *if it is a morphism of rings i.e.*

$$f \circ m_A = m_B \circ (f \otimes f) \quad \text{and} \quad f \circ u_B = u_A$$

.

Example 1.9. *Let R be a ring and let $(M, \cdot, 1_M)$ be a monoid. On the abelian group $R^{(M)} = \{\alpha : M \to R \mid Sup(\alpha) \text{ is finite}\}$ we define a multiplication by setting, for every $\alpha, \beta \in R^{(M)}$ and for every $x \in M$:*

$$(\alpha \cdot \beta)(x) = \sum_{\substack{z, w \in M \\ zw = x}} \alpha(z)\, \beta(w).$$

In this way $R^{(M)}$ *becomes a ring which is usually denoted by* RM *or by* $R[M]$ *and is called the* ***monoid ring*** *of* M ***over the ring*** R. *Using the notations introduced in 1.4, this product is uniquely defined by setting*

$$x \cdot y = xy$$

for every $x, y \in M$. *In particular the identity* 1_{RM} *of* RM *is*

$$1_{RM} = 1_M.$$

Let S *be a non empty set and let* $M = \left(\mathbb{N}^{(S)}, +, 0\right)$. *Then* RM *is the* ***ring of polynomials in*** S ***over*** R.

Whenever $R = k$ *is a commutative ring, the monoid ring* kM *of* M *over* k *is a* k*-algebra. The ring homomorphism* $u : k \to kM$ *is defined by setting:*

$$u(a) = a1_M \qquad \text{for every } a \in k.$$

Definition 1.10. *Let* k *be a commutative ring. A* k-coalgebra *is a triple* (C, Δ, ε) *where*

- C *is a* k*-module*
- $\Delta : C \to C \otimes_k C$ *is a morphism of* k*-modules*
- $\varepsilon : C \to k$ *is a morphism of* k*-modules*

such that the following diagrams are commutative:

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle C\otimes\Delta} \\ C \otimes C & \xrightarrow{\Delta\otimes C} & C \otimes C \otimes C \end{array} \qquad \begin{array}{ccccc} k \otimes C & \xleftarrow{l_C^{-1}} & C & \xrightarrow{r_C^{-1}} & C \otimes k \\ & {\scriptstyle\varepsilon\otimes C}\nwarrow & \downarrow{\scriptstyle\Delta} & \nearrow{\scriptstyle C\otimes\varepsilon} & \\ & & C \otimes C & & \end{array}$$

i.e. the following equalities hold:

$$(\Delta \otimes C) \circ \Delta = (C \otimes \Delta) \circ \Delta \quad \textit{(coassociativity)} \tag{1.4}$$

$$l_C \circ (\varepsilon \otimes C) \circ \Delta = I = r_C \circ (C \otimes \varepsilon) \circ \Delta \quad \textit{(counitarity).} \tag{1.5}$$

Exercise 1.11. *Let* (C, Δ, ε) *be a coalgebra. Prove that the map* Δ *is injective while the map* ε *is surjective whenever* k *is a field .*

Example 1.12. *Let* S *be a semigroup with* ***zero element*** z, *i.e.:*$s \cdot z = z = z \cdot s$ *for every* $s \in S$. *We denote* $S \setminus \{z\}$ *by* S^* *Assume also that* S *has* ***local identities*** *i.e.* S *contains a subset* E *of nonzero orthogonal idempotents such that for each* $s \in S^*$ *there exists* e_s *and* e'_s *in* E *with* $e_s s = s = s e'_s$. *Moreover assume that* S *is* ***locally finite*** *i.e., for every* $s \in S^*$ *the set*

$$\{(x,y) \in S^* \times S^* \mid x \cdot y = s\}$$

is finite.

Let k be a commutative ring and let $C(S,k)$ be the k-module $k^{(S^*)}$ endowed with the coalgebra structure defined by setting

$$\Delta(s) = \sum_{\substack{(t,v)\in S^*\times S^* \\ tv=s}} t \otimes_k v \qquad \text{for every } s \in S^*$$

and

$$\begin{aligned} \varepsilon(s) &= 0 \qquad \text{for every } s \notin E \\ \varepsilon(s) &= 1 \qquad \text{for every } s \in E \end{aligned}$$

Note that

$$\sum_{\substack{(t,v)\in S^*\times S^* \\ tv=s}} \varepsilon(t)\, v = \sum_{\substack{(e,v)\in E\times S^* \\ ev=s}} \varepsilon(e)\, v = \varepsilon(e_s)\, s = s.$$

The symmetrical equality is proved similarly. We call this the ***semigroup coalgebra*** *of S* ***with coefficients in*** *k.*

Let us consider now some particular cases.

1) Let $S = (\mathbb{N},+) \cup \{z\}$. Then $C(S,k) = \bigoplus_{n\in\mathbb{N}} kn$ and $\Delta(n) = \sum_{i+j=n} i \otimes j$. Moreover we have $\varepsilon(n) = 0$ if $n \neq 0$ and $\varepsilon(0) = 1$. This coalgebra is called the ***divided power coalgebra****.*

2) Let $\leq$ be a reflexive and transitive binary relation on a non empty set X. Assume that $(X,\leq)$ is locally finite i.e. that the set

$$\{t \mid x \leq t \leq y\}$$

is finite, for every $x, y \in X$ and set

$$X^{\leq} = \{(x,y) \in X \times X \mid x \leq y\} \cup \{z\} \text{ where } z \notin X \times X.$$

Then $X^{\leq}$ is a semigroup with zero element z whenever we define

$$(x,y)\cdot(x',y') = z \qquad \text{whenever } y \neq x' \qquad \text{and} \qquad (x,y)\cdot(y,y') = (x,y').$$

Here $E = \{(x,x) \mid x \in X\}$ is a set of local identities and we have

$$\Delta((x,y)) = \sum_{x\leq t\leq y} (x,t)\otimes(t,y) \qquad \text{and} \qquad \varepsilon((x,y)) = \delta_{x,y}.$$

This is called the ***incidence coalgebra of*** *$(X,\leq)$.*

2a) Consider the particular case when $\leq$ coincides with $=$. Then $X^{\leq} = E \cup \{z\}$ and we have

$$\Delta((x,x)) = (x,x)\otimes(x,x) \qquad \text{and} \qquad \varepsilon((x,x)) = 1.$$

By identifying E with X we obtain the ***grouplike coalgebra*** *over the set X.*

2b) Another particular case is when the set $X = \{1, \ldots, n\}$is finite and $\leq$ is the usual order on X

$$X^{\leq} = \{(i, j) \in X \times X \mid i \leq j\} \cup \{z\}$$

and we have

$$\Delta((i, j)) = \sum_{i \leq t \leq j} (i, t) \otimes (t, j) \quad \text{and} \quad \varepsilon((i, j)) = \delta_{i,j}.$$

2c) Finally consider the case when the set $X = \{1, \ldots, n\}$is finite and $\leq$ is the the trivial order i.e.

$$X^{\leq} = (X \times X) \cup \{z\}$$

and we have

$$\Delta((i, j)) = \sum_{t=1}^{n} (i, t) \otimes (t, j) \quad \text{and} \quad \varepsilon((i, j)) = \delta_{i,j}.$$

This coalgebra is usually denoted by $M^C(n, k)$ and is called the matrix coalgebra.

3) Let now $\Gamma = (V(\Gamma), A(\Gamma), s, t)$ be an oriented graph. *This means that $V(\Gamma)$ and $A(\Gamma)$ are nonempty sets and $s, t : A(\Gamma) \to V(\Gamma)$ are maps. The elements of $V(\Gamma)$ are usually called* vertices *and the elements of $A(\Gamma)$ are called* arrows *of Γ For a given arrow $a \in A(\Gamma)$ the vertex $s(a)$ is called the* source *of a while the vertix $t(a)$ is called the* target *of a. The picture*

$$s(a) \overset{a}{\to} t(a)$$

means that a is an arrow with source $s(a)$ and target $t(a)$. Let $n \in \mathbb{N}$, $n \geq 1$. A path of length n *in Γ is an n-tuple $\alpha = (a_1, \ldots, a_n)$.where each $a_i \in A(\Gamma)$ and $t(a_i) = s(a_{i+1})$ for every $i = 1, \ldots, n-1$. In this case we set $s(\alpha) = s(a_1)$ and $t(\alpha) = t(a_n)$. Let $D_n(\Gamma)$ be the set of paths of Γ of length n. For $n = 0$ set $D_0(\Gamma) = V(\Gamma)$ where, for each $x \in V(\Gamma)$, we set $s(x) = t(x) = x$.We call the elements of $D_0(\Gamma)$* paths of length 0. *Let*

$$D(\Gamma) = \bigcup_{n \in \mathbb{N}} D_n(\Gamma)$$

and set

$$S(\Gamma) = D(\Gamma) \cup \{z\} \quad \text{where } z \notin D(\Gamma).$$

$S(\Gamma)$ becomes a semigroup with zero element z by setting, for given $\alpha = (a_1, \ldots, a_n)$,$\beta = (b_1, \ldots, b_m)$ and $v \in D_0(\Gamma) = V(\Gamma)$

$$\alpha \cdot \beta = (a_1, \ldots, a_n, b_1, \ldots, b_m) \quad \text{whenever } t(a_n) = s(b_1) \quad \text{and } \alpha \cdot \beta = z \quad \text{otherwise}$$

and

$$\begin{aligned} v \cdot \alpha &= \alpha \quad \text{whenever } v = s(a) \quad \text{and } v \cdot \alpha = z \quad \text{otherwise;} \\ \alpha \cdot v &= \alpha \quad \text{whenever } t(a) = v \quad \text{and } \alpha \cdot v = z \quad \text{otherwise.} \end{aligned}$$

The set of local identities is clearly $D_0(\Gamma) = V(\Gamma)$*. Note that* $S(\Gamma)$ *is locally finite i.e. that*

$$\{(\beta,\gamma) \in D(\Gamma) \times D(\Gamma) \mid \beta \cdot \gamma = \alpha\}$$

is a finite set, for every $\alpha \in D(\Gamma)$*. Given* $\alpha \in D(\Gamma) = (S(\Gamma))^*$ *we get*

$$\begin{aligned}\Delta(\alpha) &= \sum_{\substack{\beta,\gamma\in D(\Gamma)\\ \beta\cdot\gamma=\alpha}} \beta \otimes \gamma \qquad \text{and} \\ \varepsilon(\gamma) &= 1 \text{ if } \gamma \text{ has length } 0 \qquad \text{and} \qquad \varepsilon(\gamma) = 0 \text{ otherwise.}\end{aligned}$$

This particular coalgebra is called path coalgebra *of the oriented graph* Γ.

Definition 1.13. *Let* (C,Δ,ε) *be a coalgebra. We define, by recursion, a sequence* $(\Delta_n)_{n\geq 1}$ *by setting*

$$\Delta_1 = \Delta \quad \text{and} \quad \Delta_n = \left(\Delta \otimes C^{n-1}\right) \circ \Delta_{n-1} \quad \text{for every } n \in \mathbb{N}, n \geq 2$$

Notation 1.14. *For any* k*-module* M *and any* k*-linear map* $f : L \to N$ *we set*

$$M^0 \otimes f = f = f \otimes M^0.$$

Lemma 1.15. *Let* (C,Δ,ε) *be a coalgebra. Then*

$$\Delta_n = \left(C^t \otimes \Delta \otimes C^{n-1-t}\right) \circ \Delta_{n-1} \quad \text{for every } n,t \in \mathbb{N}, n \geq 2 \quad \text{and } 0 \leq t \leq n-1.$$

Proof. We proceed by induction on n. For $n = 2$ we have to prove that $\Delta_2 = (\Delta \otimes C) \circ \Delta$ which holds in view of the given definition and that $\Delta_2 = (C \otimes \Delta) \circ \Delta$ which holds in view of the coassociativity of Δ. Let us assume that the statement holds true for some $n \in \mathbb{N}, n \geq 2$ and let us prove it for $n+1$. We proceed by induction on t. For $t = 0$ we have to prove that $\Delta_{n+1} = (\Delta \otimes C^n) \circ \Delta_n$ which holds in view of the given definition. Let $t \in \mathbb{N}, 1 \leq t \leq n$ and let us assume that the equality hold for $t-1$. Then

$$\begin{aligned}
&\left(C^t \otimes \Delta \otimes C^{n+1-1-t}\right) \circ \Delta_n \\
&= \left(C^t \otimes \Delta \otimes C^{n-t}\right) \circ \Delta_n \\
&\overset{\text{induct. on } n \text{and} t'=t-1}{=} \left(C^t \otimes \Delta \otimes C^{n-t}\right) \circ \left(C^{t-1} \otimes \Delta \otimes C^{n-1-(t-1)}\right) \circ \Delta_{n-1} \\
&= \left(C^{t-1} \otimes C \otimes \Delta \otimes C^{n-t}\right) \circ \left(C^{t-1} \otimes \Delta \otimes C^{n-t}\right) \circ \Delta_{n-1} \\
&= \left(C^{t-1} \otimes [(C \otimes \Delta) \circ \Delta] \otimes C^{n-t}\right) \circ \Delta_{n-1} \\
&= \left(C^{t-1} \otimes [(\Delta \otimes C) \circ \Delta] \otimes C^{n-t}\right) \circ \Delta_{n-1} \\
&= \left(C^{t-1} \otimes \Delta \otimes C \otimes C^{n-t}\right) \circ \left(C^{t-1} \otimes \Delta \otimes C^{n-t}\right) \circ \Delta_{n-1} \\
&\overset{\text{induct. on } n \text{and} t'=t-1}{=} \left(C^{t-1} \otimes \Delta \otimes C^{n+1-t}\right) \circ \Delta_n \\
&= \left(C^{t-1} \otimes \Delta \otimes C^{n+1-1-(t-1)}\right) \circ \Delta_n \\
&\overset{\text{induct. on } t}{=} \Delta_{n+1}
\end{aligned}$$

□

Lemma 1.16. *Let (C, Δ, ε) be a coalgebra. Then*

$$\Delta_n = (\Delta_{n-1} \otimes C) \circ \Delta \quad \textit{for every } n \geq 2$$

Proof. We proceed by induction on n. For $n = 2$ we have to prove that $\Delta_2 = (\Delta_1 \otimes C) \circ \Delta$ which holds in view of the given definition. Let us assume the statement holds for some $n \in \mathbb{N}, n \geq 2$ and let us prove it for $n + 1$.

We have

$$\begin{aligned} & \Delta_{n+1} \overset{\text{def.}}{=} (\Delta \otimes C^n) \circ \Delta_n \overset{\text{induct. assumpt.}}{=} (\Delta \otimes C^n) \circ (\Delta_{n-1} \otimes C) \circ \Delta \\ = \ & \left(\Delta \otimes C^{n-1} \otimes C\right) \circ (\Delta_{n-1} \otimes C) \circ \Delta = \left(\left[\left(\Delta \otimes C^{n-1}\right) \circ \Delta_{n-1}\right] \otimes C\right) \circ \Delta \overset{\text{def}}{=} \\ = \ & (\Delta_n \otimes C) \circ \Delta. \end{aligned}$$

□

Theorem 1.17. *Let (C, Δ, ε) be a coalgebra. Then*

$$\Delta_n = \left(C^m \otimes \Delta_i \otimes C^{n-i-m}\right) \circ \Delta_{n-i} \quad \textit{for every } n, i, m \in \mathbb{N}, n \geq 2, 1 \leq i \leq n-1 \textit{ and } 0 \leq m \leq n-i.$$

Proof. Let us fix an $n \in \mathbb{N}, n \geq 2$ and let us prove the statement by induction on i where $1 \leq i \leq n-1$. For $i = 1$ we have to prove that

$$\Delta_n = \left(C^m \otimes \Delta_1 \otimes C^{n-1-m}\right) \circ \Delta_{n-1} \text{ for every } 0 \leq m \leq n-1$$

which holds true in view of Lemma 1.15. Let us assume that the statement holds for some $i, 1 \leq i \leq n-2$ and let us prove it for $i+1$. We have, for every $0 \leq m \leq n-(i+1) < n-i$

$$\begin{aligned} \Delta_n & \overset{\text{induct on } i}{=} \left(C^m \otimes \Delta_i \otimes C^{n-i-m}\right) \circ \Delta_{n-i} \\ & \overset{\text{Lem1.15}}{=} \left(C^m \otimes \Delta_i \otimes C^{n-i-m}\right) \circ \left(C^m \otimes \Delta \otimes C^{n-i-1-m}\right) \circ \Delta_{n-i-1} \\ & = \left(C^m \otimes \Delta_i \otimes C \otimes C^{n-i-1-m}\right) \circ \left(C^m \otimes \Delta \otimes C^{n-i-1-m}\right) \circ \Delta_{n-i-1} \\ & = \left(C^m \otimes [(\Delta_i \otimes C) \circ \Delta] \otimes C^{n-i-1-m}\right) \circ \Delta_{n-i-1} = \\ & \overset{\text{Lem1.16}}{=} \left(C^m \otimes \Delta_{i+1} \otimes C^{n-(i+1)-m}\right) \circ \Delta_{n-(i+1)}. \end{aligned}$$

Note that, by induction assumption, actually all the first equality holds for every $0 \leq m \leq n-i$ while, in the second one we have to restrict to $0 \leq m \leq n-(i+1)$ in order to apply 1.15 for $n-i$ which forces $0 \leq m \leq n-i-1$. □

Notation 1.18. *(Sweedler's Sigma Notation) Let (C, Δ, ε) be a coalgebra. For a given $c \in C$ we have*

$$\Delta(c) = \sum_{i=1}^{n_c} c_{1i} \otimes c_{2i} \quad \textit{where} \quad n_c \in \mathbb{N}, n_c \geq 1, c_{1i}, c_{2i} \in C \quad \textit{for every } i = 1, \ldots n_c.$$

We adopt the notation

$$\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$$

or even

$$\Delta(c) = \sum c_1 \otimes c_2$$

where the index i is suppressed.

Note that, using this notation, equalities in 1.4 and in 1.5 become respectively

$$\sum (c_1)_1 \otimes (c_1)_2 \otimes c_2 = \sum c_1 \otimes (c_2)_1 \otimes (c_2)_2 \tag{1.6}$$

and

$$\sum \varepsilon(c_1) c_2 = c = \sum c_1 \varepsilon(c_2). \tag{1.7}$$

Notation 1.19. *More generally, for any $n \in \mathbb{N}$, $n \geq 1$ we write*

$$\Delta_n(c) = \sum c_1 \otimes \ldots \otimes c_{n+1}.$$

Using this notation, equality 1.6 gives rise to

$$\sum c_1 \otimes c_2 \otimes c_3 = \sum (c_1)_1 \otimes (c_1)_2 \otimes c_2 = \sum c_1 \otimes (c_2)_1 \otimes (c_2)_2$$

Since, from Theorem 1.17, we have that $\Delta_{m+n} = (C^a \otimes \Delta_m \otimes C^{n-a}) \circ \Delta_n$, for every $a, m, n \in \mathbb{N}, m, n \geq 1$, and $0 \leq a \leq n$, we obtain that

$$\sum c_1 \otimes \ldots \otimes c_{m+n+1} =$$
$$= \sum c_1 \otimes \ldots \otimes c_a \otimes (c_{a+1})_1 \otimes \ldots (c_{a+1})_{m+1} \otimes c_{a+2} \ldots \otimes c_{n+1}$$
$$\text{for every } a \in \mathbb{N}, 1 \leq a \leq n-1$$

and

$$\sum c_1 \otimes \ldots \otimes c_{m+n+1} = (c_1)_1 \otimes \ldots (c_1)_{m+1} \otimes c_2 \ldots \otimes c_{n+1}$$

$$\sum c_1 \otimes \ldots \otimes c_{m+n} = c_1 \otimes \ldots \otimes c_n \otimes (c_{n+1})_1 \otimes \ldots (c_{n+1})_{m+1}$$

Proposition 1.20. *Let (C, Δ, ε) be a coalgebra, let $n, i \in \mathbb{N}$, $i \geq 1, n \geq i$. Let $f : C^{i+1} \to C$ and $g : C^{n+1} \to C$ be k-homomorphisms. Then for every $t \in \mathbb{N}$, $2 \leq t \leq n+1$ we have*

$$\begin{aligned} & \sum g\left(c_1 \otimes \cdots \otimes c_{t-1} \otimes f\left(c_t \otimes \cdots \otimes c_{t+i}\right) \otimes c_{t+i+1} \cdots \otimes c_{n+i+1}\right) \\ = \; & \sum g\left(c_1 \otimes \cdots \otimes c_{t-1} \otimes f\left((c_t)_1 \otimes \cdots \otimes (c_t)_{i+1}\right) \otimes c_{t+1} \cdots \otimes c_{n+1}\right). \end{aligned}$$

Proof. Set

$$\overline{f} = f \circ \Delta_i.$$

Since $t-1 \leq (n+i)-i = n$, we can apply Theorem 1.17 to the case when "n" $= n+i$ and "i" $= i$ and "m" $= t-1$ to get $\Delta_{n+i} = \left(C^{t-1} \otimes \Delta_i \otimes C^{(n+i)-i-(t-1)}\right) \circ \Delta_{(n+i)-i} = (C^{t-1} \otimes \Delta_i \otimes C^{n-t+1}) \circ \Delta_n$ so that

$$
\begin{aligned}
& \sum g\left(c_1 \otimes \cdots \otimes c_{t-1} \otimes \overline{f}\left(c_t\right) \otimes c_{t+1} \cdots \otimes c_{n+1}\right) \\
= \; & g\left(\sum \left(c_1 \otimes \cdots \otimes c_{t-1} \otimes \overline{f}\left(c_t\right) \otimes c_{t+1} \cdots \otimes c_{n+1}\right)\right) \\
= \; & \left[g \circ \left(C^{t-1} \otimes \overline{f} \otimes C^{n-t+1}\right) \circ \Delta_n\right](c) \\
= \; & \left[g \circ \left(C^{t-1} \otimes f \otimes C^{n-t+1}\right) \circ \left(C^{t-1} \otimes \Delta_i \otimes C^{n-t+1}\right) \circ \Delta_n\right](c) \\
= \; & \left[g \circ \left(C^{t-1} \otimes f \otimes C^{n-t+1}\right) \circ \Delta_{n+i}\right](c)
\end{aligned}
$$

□

Notation 1.21. *Let (C, Δ, ε) be a coalgebra. In the sequel, for any $c \in C$ and $i, j \in \mathbb{N}, i, j \geq 1$, we will write c_{i_j} instead of $(c_i)_j$ e.g. c_{1_2} instead of $(c_1)_2$.*

Exercise 1.22. *Let (C, Δ, ε) be a coalgebra. Prove that, for any $c \in C$, we have*

$$\sum \varepsilon\left(c_1\right) \varepsilon\left(c_2\right) c_3 = c.$$

Definition 1.23. *Let (C, Δ, ε) be a coalgebra and let $\tau : C \otimes C \to C \otimes C$ be the usual flip. We say that the coalgebra C is* cocommutative *if $\tau \circ \Delta = \Delta$ i.e. if*

$$\sum c_1 \otimes c_2 = \sum c_2 \otimes c_1 \quad \text{for every } c \in C.$$

Examples 1.24. *The coalgebra in example 2a) is always cocommutative, while the coalgebra in example 2b) is, in general, not cocommutative. A typical example of not cocommutative coalgebra is the path coalgebra of the oriented graph*

$$e_0 \xrightarrow{d_o} e_1 \xrightarrow{d_1} e_2 \xrightarrow{d_2} e_3 \cdots e_n \xrightarrow{d_n} e_{n+1} \cdots$$

In fact we have

$$\Delta\left(d_i\right) = e_i \otimes d_i + d_i \otimes e_{i+1} \quad \text{for every } i \in \mathbb{N}.$$

Definition 1.25. *Let $(C, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$ be coalgebras. A k-linear map $\varphi : C \to D$ will be called a morphism of coalgebras if the following diagrams are commutative:*

i.e. if

$$(\varphi \otimes \varphi) \circ \Delta_C = \Delta_D \circ \varphi \quad \text{and} \quad \varepsilon_D \circ \varphi = \varepsilon_C$$

Which can be rewritten as

$$\sum \varphi\left(c_1\right) \otimes \varphi\left(c_2\right) = \sum \varphi\left(c\right)_1 \otimes \varphi\left(c\right)_2 \quad \text{and} \quad \varepsilon_D\left(\varphi\left(c\right)\right) = \varepsilon_C\left(c\right) \quad \text{for every } c \in C.$$

1.26. *We will denote by* ***Coalg***$_k$ *the category of coalgebras over the ring* k*. Note that* k *can be equipped by the structure of a coalgebra by setting*

$$\Delta_k = r_k^{-1} = l_k^{-1} : k \to k \otimes k \quad i.e. \quad \Delta_k(a) = a \otimes 1 = 1 \otimes a \quad \text{for every } a \in k$$
$$\text{and } \varepsilon_k = \mathrm{Id}_k : k \to k \quad i.e. \quad \varepsilon_k(a) = a \quad \text{for every } a \in k.$$

Note that, given any coalgebra $(C, \Delta_C, \varepsilon_C)$*,* $\varepsilon_C : C \to k$ *is a coalgebra morphism. In fact we have*

$$(\varepsilon_C \otimes \varepsilon_C) \circ \Delta_C = r_k^{-1} \circ \varepsilon_C \quad \text{and} \quad \varepsilon_k \circ \varepsilon_C = \varepsilon_C.$$

Moreover ε_C *is unique with respect to this property: given a coalgebra morphism* $\alpha : C \to k$ *we get that* $\alpha = \varepsilon_k \circ \alpha = \varepsilon_C$*. Hence we can claim that* $(k, \Delta_k, \varepsilon_k)$ *is a final object for the category* ***Coalg***$_k$*.*

Theorem 1.27. *Let* $(C, \Delta_C, \varepsilon_C)$ *and* $(D, \Delta_D, \varepsilon_D)$ *be coalgebras.*
Then $(C \otimes D, \Delta_{C\otimes D}, \varepsilon_{C\otimes D})$ *is a coalgebra where*

$$\Delta_{C\otimes D} = (C \otimes \tau_{C,D} \otimes D) \circ (\Delta_C \otimes \Delta_D) \quad \text{and} \quad \varepsilon_{C\otimes D} = l_k \circ (\varepsilon_C \otimes \varepsilon_D). \tag{1.8}$$

Here $\tau_{C,D} : C \otimes D \to D \otimes C$ *denotes the usual flip. Moreover the map*

$$p_C : C \otimes D \to C \quad \text{defined by setting} \quad p_C(c \otimes d) = c\varepsilon_D(d)$$

is a morphism of coalgebras.

Proof. We compute

$$\begin{aligned}
& [((C \otimes D) \otimes \Delta_{C\otimes D}) \circ \Delta_{C\otimes D}](c \otimes d) \\
= \; & [((C \otimes D) \otimes C \otimes \tau_{C,D} \otimes D) \circ ((C \otimes D) \otimes (\Delta_C \otimes \Delta_D))] \sum (c_1 \otimes d_1 \otimes c_2 \otimes d_2) \\
= \; & \sum c_1 \otimes d_1 \otimes c_{2_1} \otimes d_{2_1} \otimes c_{2_2} \otimes d_{2_2} = \sum c_{1_1} \otimes d_{1_1} \otimes c_{1_2} \otimes d_{1_2} \otimes c_2 \otimes d_2 \\
= \; & [(\Delta_{C\otimes D} \otimes C \otimes D)] \left(\sum c_1 \otimes d_1 \otimes c_2 \otimes d_2\right) \\
= \; & [(\Delta_{C\otimes D} \otimes C \otimes D) \circ \Delta_{C\otimes D}](c \otimes d)
\end{aligned}$$

and

$$\begin{gathered}
[l_{C\otimes D} \circ (\varepsilon_{C\otimes D} \otimes C \otimes D) \circ \Delta_{C\otimes D}](c \otimes d) = \\
= l_{C\otimes D} \left[l_k \left[\sum (\varepsilon_C(c_1) \otimes \varepsilon_D(d_1)) \right] \otimes c_2 \otimes d_2 \right] \\
= \sum \varepsilon_C(c_1) c_2 \otimes \varepsilon_D(d_1) d_2 = c \otimes d \\
= \sum c_1 \varepsilon_C(c_2) \otimes d_1 \varepsilon_D(d_2) \\
= r_{C\otimes D} \circ \left[\sum c_1 \otimes d_1 \otimes r_k \circ (\varepsilon_C(c_2) \otimes \varepsilon_D(d_2)) \right] \\
[r_{C\otimes D} \circ (C \otimes D \otimes \varepsilon_{C\otimes D}) \circ \Delta_{C\otimes D}](c \otimes d).
\end{gathered}$$

The last statement is left as an exercise to the reader. □

Proposition 1.28. *Let C and D be cocommutative coalgebras. Then the tensor product $C \otimes D$ is the product of C and D in the full subcategory* $\mathrm{CoCoalg}_k$ *of cocommutative coalgebras.*

Proof. Let $\varphi : L \to C$ and $\psi : L \to D$ be coalgebra morphisms where L is a cocommutative coalgebra. Set $\zeta = (\varphi \otimes \psi) \circ \Delta_L$. Then ζ is a coalgebra morphism. In fact, for any $x \in L$ we have

$$\begin{aligned} \sum x_{1_1} \otimes x_{1_2} \otimes x_{2_1} \otimes x_{2_2} &= \sum x_1 \otimes x_2 \otimes x_3 \otimes x_4 = \sum x_1 \otimes x_{2_1} \otimes x_{2_2} \otimes x_3 \\ \overset{L\text{cocomm}}{=} \sum x_1 \otimes x_{2_2} \otimes x_{2_1} \otimes x_3 &= \sum x_1 \otimes x_3 \otimes x_2 \otimes x_4 = \sum x_{1_1} \otimes x_{2_1} \otimes x_{1_2} \otimes x_{2_2} \end{aligned}$$

so that we obtain

$$\sum x_{1_1} \otimes x_{2_1} \otimes x_{1_2} \otimes x_{2_2} = \sum x_{1_1} \otimes x_{1_2} \otimes x_{2_1} \otimes x_{2_2} \tag{1.9}$$

and hence

$$\begin{aligned} \Delta_{C\otimes D}(\zeta)(x) &= \sum \varphi(x_1)_1 \otimes \psi(x_2)_1 \otimes \varphi(x_1)_2 \otimes \psi(x_2)_2 \\ &= \sum \varphi(x_{1_1}) \otimes \psi(x_{2_1}) \otimes \varphi(x_{1_2}) \otimes \psi(x_{2_2}) \\ &\overset{(1.9)}{=} \sum \varphi(x_{1_1}) \otimes \psi(x_{1_2}) \otimes \varphi(x_{2_1}) \otimes \psi(x_{2_2}) \\ &= \sum \zeta(x_1) \otimes \zeta(x_2). \end{aligned}$$

Moreover we have

$$\begin{aligned} \varepsilon_{C\otimes D}(\zeta(x)) &= \sum \varepsilon_C(\varphi(x_1)) \cdot \varepsilon_D(\psi(x_2)) = \sum \varepsilon_C(\varphi(x_1)) \cdot \varepsilon_D(\psi(x_2)) \\ &= \sum \varepsilon_L(x_1)\, \varepsilon_L(x_2) = \varepsilon_L\left(\sum x_1 \varepsilon_L(x_2)\right) = \varepsilon_L(x). \end{aligned}$$

We compute

$$\begin{aligned} p_C(\zeta(x)) &= p_C\left(\sum \varphi(x_1) \otimes \psi(x_2)\right) = \sum \varphi(x_1)\, \varepsilon_D(\psi(x_2)) = \sum \varphi(x_1)\, \varepsilon_L(x_2) \\ &= \varphi\left(\sum x_1 \varepsilon_L(x_2)\right) = \varphi(x). \end{aligned}$$

In a similar way, one gets $p_D(\zeta(x)) = \psi(x)$.

Now we have to prove that ζ is unique with respect to this property. Thus let $\chi : L \to C \otimes D$ be a morphism of coalgebras such that $p_C \circ \chi = \varphi$ and $p_D \circ \chi = \psi$. Note that, given $c \in C$ and $d \in D$, we have

$$\begin{aligned} c \otimes d &= \sum (c_1 \otimes d_1)\, \varepsilon_{C\otimes D}(c_2 \otimes d_2) = \sum (c_1 \varepsilon_D(d_2) \otimes d_1 \varepsilon_C(c_2)) \\ &\overset{D \text{ cocomm}}{=} \sum (c_1 \varepsilon_D(d_1) \otimes d_2 \varepsilon_C(c_2)) = \sum p_C(c_1 \otimes d_1) \otimes p_D(c_2 \otimes d_2) \\ &= (p_C \otimes p_D)(\Delta_{C\otimes D}(c \otimes d)) \end{aligned}$$

and hence we get that $(p_C \otimes p_D) \circ \Delta_{C\otimes D} = I_{C\otimes D}$. From this we obtain

$$\begin{aligned} \chi &= I_{C\otimes D} \circ \chi = (p_C \otimes p_D) \circ \Delta_{C\otimes D} \circ \chi = (p_C \otimes p_D) \circ (\chi \otimes \chi) \circ \Delta_L = \\ &= (p_C \circ \chi \otimes p_D \circ \chi) \circ \Delta_L = (\varphi \otimes \psi) \circ \Delta_L = \zeta. \end{aligned}$$

□

1.29. *Let $(C, \Delta_C, \varepsilon_C)$ be a coalgebra. We denote by C^{cop} the coalgebra defined by setting:*

$$\Rightarrow \Delta_{C^{cop}} = \tau \circ \Delta_C \quad \text{and} \quad \varepsilon_{C^{cop}} = \varepsilon_C.$$

Clearly C is cocommutative if and only if $C = C^{cop}$.

Exercise 1.30. *Check that $(C^{cop}, \Delta_{C^{cop}}, \varepsilon_{C^{cop}})$ is indeed a coalgebra.*

Assumption 1.31. ***From now on we will assume that** k **is a field**. This will imply, in particular, that, given a subspace W_j of a k-vector space $V_j, j = 1, 2$, we can identify $W_1 \otimes W_2$ with a subspace of $V_1 \otimes V_2$.*

Definition 1.32. *Let $(C, \Delta_C, \varepsilon_C)$ be a coalgebra and let D be a k-subspace of C. D is called a* subcoalgebra *of C if $\Delta_C(D) \subseteq D \otimes D$. Note that D becomes a coalgebra by setting $\Delta_D = \left(\Delta_{|D}\right)^{|D\otimes D}$ and $\varepsilon_D = \varepsilon_{C|D}$. Moreover the inclusion map $i_D : D \to C$ becomes a morphism of coalgebras.*

Definitions 1.33. *Let $(C, \Delta_C, \varepsilon_C)$ be a coalgebra and let I be a k-subspace of C. I is called*

- a *right coideal* of C if $\Delta(I) \subseteq I \otimes C$,
- a *left coideal* of C if $\Delta(I) \subseteq C \otimes I$,
- a *(two-sided) coideal of C* if $\Delta(I) \subseteq I \otimes C + C \otimes I$ and $\varepsilon_C(I) = \{0\}$.

Exercise 1.34. *Let $f : C \to D$ be a coalgebra morphism. Then* $\mathrm{Im}(f)$ *is a subcoalgebra of D and $Ker(f)$ is a coideal of C. (Use Lemma 15.1).*

Theorem 1.35. ***(The Fundamental Theorem of the Quotient Coalgebra)** Let $(C, \Delta_C, \varepsilon_C)$ be a coalgebra, let I be a coideal of C.and let $p = p_I : C \to C/I$ be the canonical projection. Then C/I can be endowed by a unique coalgebra structure (*called quotient coalgebra*) such that p becomes a coalgebra morphism. Moreover given any coalgebra morphism $f : C \to D$ such that $I \subseteq$* Ker(f)*, there exists a unique coalgebra morphism $\overline{f} : C/I \to D$ such that $f = \overline{f} \circ p$.*

Proof. Since $\Delta_C(I) \subseteq I \otimes C + C \otimes I \overset{\text{Lemma}(15.1)}{=} Ker(p \otimes p)$ we deduce that $I \subseteq Ker((p \otimes p) \circ \Delta_C)$, so that there exists a unique linear map $\overline{\Delta} : C/I \to (C/I) \otimes (C/I)$ such that

$\overline{\Delta} \circ p = (p \otimes p) \circ \Delta_C$ and we have

$$\begin{aligned}
\left(\overline{\Delta} \otimes C/I\right) \circ \overline{\Delta} \circ p &= \left(\overline{\Delta} \otimes C/I\right) \circ [(p \otimes p) \circ \Delta_C] \\
&= \left(\left[\overline{\Delta} \circ p\right] \otimes p\right) \circ \Delta_C = ([(p \otimes p) \circ \Delta_C] \otimes p) \circ \Delta_C \\
&= (p \otimes p \otimes p) \circ (\Delta_C \otimes C) \circ \Delta_C = (p \otimes p \otimes p) \circ (C \otimes \Delta_C) \circ \Delta_C \\
&= (p \otimes [(p \otimes p) \circ \Delta_C]) \circ \Delta_C = \left(p \otimes \left[\overline{\Delta} \circ p\right]\right) \circ \Delta_C \\
&= \left(C/I \otimes \overline{\Delta}\right) \circ [(p \otimes p) \circ \Delta_C] = \left(C/I \otimes \overline{\Delta}\right) \circ \overline{\Delta} \circ p.
\end{aligned}$$

Since p is surjective, we get that $\left(\overline{\Delta} \otimes I_{C/I}\right) \circ \overline{\Delta} = \left(C/I \otimes \overline{\Delta}\right) \circ \overline{\Delta}$. Analogously, since $\varepsilon_C(I) = 0$, there exists a unique map $\overline{\varepsilon} : C/I \to k$ such that $\overline{\varepsilon} \circ p = \varepsilon_C$ and we have

$$\begin{aligned}
l_{C/I} \circ (\overline{\varepsilon} \otimes C/I) \circ \overline{\Delta} \circ p &= l_{C/I} \circ (\overline{\varepsilon} \otimes C/I) \circ (p \otimes p) \circ \Delta_C \\
&= l_{C/I} \circ (\varepsilon_C \otimes p) \circ \Delta_C = l_{C/I} \circ (k \otimes p) \circ (\varepsilon_C \otimes C) \circ \Delta_C \\
&= p \circ l_C \circ (\varepsilon_C \otimes C) \circ \Delta_C = p.
\end{aligned}$$

Since p is surjective, we get $l_{C/I} \circ (\overline{\varepsilon} \otimes C/I) \circ \overline{\Delta} = C/I$. In a similar way one proves that $r_{C/I} \circ (C/I \otimes \overline{\varepsilon}) \circ \overline{\Delta} = C/I$. Therefore $\left(C/I, \overline{\Delta}, \overline{\varepsilon}\right)$ is a coalgebra. Note that p becomes automatically a coalgebra morphism.

Let now $f : C \to D$ be a coalgebra morphism such that $I \subseteq Ker(f)$. Then there exists a unique k-linear map $\overline{f} : C/I \to D$ such that $\overline{f} \circ p = f$. Let us check that $\overline{f}$ is a coalgebra morphism. Indeed we have

$$\begin{aligned}
\left(\overline{f} \otimes \overline{f}\right) \circ \overline{\Delta} \circ p &= \left(\overline{f} \otimes \overline{f}\right) \circ (p \otimes p) \circ \Delta_C = (f \otimes f) \circ \Delta_C \\
&= \Delta_D \circ f = \left(\Delta_D \circ \overline{f}\right) \circ p
\end{aligned}$$

and

$$\varepsilon_D \circ \overline{f} \circ p = \varepsilon_D \circ f = \varepsilon_C = \overline{\varepsilon} \circ p$$

and since p is surjective, we conclude. □

Notation 1.36. *For every k-vector space V we will denote by V^* the dual of V i.e. $V^* = \mathrm{Hom}_k(V, k)$. We will also denote by $\omega = \omega_V : V \to V^{**}$ the canonical morphism defined by setting $\omega(x) = \widetilde{x}$ where $\widetilde{x} = ev_x : V^* \to k$ is the evaluation in x: $ev_x(f) = f(x)$ for every $f \in V^*$.*

Lemma 1.37. *For any vector space V, $\omega_V : V \to V^{**}$ is a monomorphism. Moreover, for any $\alpha \in V^{**}$ and for any finite subset $F = \{\xi_1, \ldots, \xi_n\}$ of V^*, there exists an element $x \in V$ such that*

$$\alpha(\xi_i) = \xi_i(x) = \widetilde{x}(\xi_i).$$

Proof. Let $x \in V, x \neq 0$. Then there exists a k-linear morphism $\xi : V \to k$ such that $\xi(x) \neq 0$ so that $\widetilde{x}(\xi) = \xi(x) \neq 0$. We deduce that $\omega_V(x) = \widetilde{x} \neq 0$.

Let now $\alpha \in V^{**}$ and let $F = \{\xi_1, \ldots, \xi_n\}$ be a finite subset of V^*. Set $U = \{(\xi_1(x), \ldots, \xi_n(x)) \mid x \in V\} \subseteq k^n$. Assume that

$$y = (\alpha(\xi_1), \ldots, \alpha(\xi_n)) \in k^n \setminus U.$$

Then there exists a k-linear map $\zeta : k^n \to k$ such that $\zeta(U) = \{0\}$ and $\zeta(y) \neq 0$. Let $e_1, \ldots, e_n$ be the canonical basis of k^n and let $\theta : V \to k$ be the linear map defined by

$$\theta = \sum_{i=1}^{n} \xi_i \zeta(e_i).$$

Then we have

$$\theta(x) = \sum_{i=1}^{n} \xi_i(x) \zeta(e_i) = \zeta\left(\sum_{i=1}^{n} \xi_i(x) e_i\right) = \zeta((\xi_1(x), \ldots, \xi_n(x))) = 0 \quad \text{for every } x \in V$$

and hence $\theta = 0$. Therefore we deduce that

$$\begin{aligned} 0 &= \alpha(\theta) = \alpha\left(\sum_{i=1}^{n} \xi_i \zeta(e_i)\right) = \sum_{i=1}^{n} \alpha(\xi_i) \zeta(e_i) = \zeta\left(\sum_{i=1}^{n} \alpha(\xi_i) e_i\right) \\ &= \zeta((\alpha(\xi_1), \ldots, \alpha(\xi_n))) = \zeta(y) \neq 0. \text{ Contradiction.} \end{aligned}$$

□

Proposition 1.38. *Let V and W be k-vector spaces. Then, for every $v^* \in V^*, w^* \in W^*$, the assignment $v \otimes w \mapsto v^*(v) w^*(w)$ defines a k-linear map $\Lambda_{v^*,w^*} : V \otimes W \to k$. Moreover the assignment $v^* \otimes w^* \mapsto \Lambda_{v^*,w^*}$ defines an injective k-linear map*

$$\Lambda = \Lambda_{V,W} : V^* \otimes W^* \to (V \otimes W)^*$$

which is also bijective whenever W has finite dimension.

Proof. It is easy to check that the map $\Gamma_{v^*,w^*} V \times W \to k$ defined by setting $\Gamma_{v^*,w^*}((v, w)) = v^*(v) w^*(w)$ is bilinear. Thus we can consider the map $\Gamma : V^* \times W^* \to (V \otimes W)^*$defined by setting $\Gamma((v^*, w^*)) = \Lambda_{v^*,w^*}$. Even this map is bilinear so that it gives rise to the k-linear map Λ. Let us prove that Λ is injective. Let

$n \in \mathbb{N}, n \geq 1$, and let $v_1^*, \ldots, v_n^* \in V^*$ and $w_1^*, \ldots, w_n^* \in W^*$ such that the element $\sum_{i=1}^{n} v_i^* \otimes w_i^* \neq 0$ in $V^* \otimes W^*$. We can assume w.l.o.g. that $v_1^*, \ldots, v_n^*$ are linearly independent and that $w_1^* \neq 0$. By expanding $F = \{v_1^*, \ldots, v_n^*\}$ to a basis of V, we can construct a k-linear map $\alpha : V^* \to k$ such that $\alpha(v_1^*) = 1$ and $\alpha(v_j^*) = 0$ for every $j = 2, \ldots n$. In view of Lemma 1.37 there exists a $v \in V$ such that

$$\alpha(v_i^*) = v_i^*(v) \quad \text{for every } i = 1, \ldots n.$$

Therefore we get

$$v_1^*(v) = 1 \quad \text{and} \quad v_j^*(v) = 0 \quad \text{for every } j = 2, \ldots n.$$

Since $w_1^* \neq 0$ there exists a $w \in W$ such that $w_1^*(w) \neq 0$. Thus we obtain

$$\Lambda\left(\sum_{i=1}^{n} v_i^* \otimes w_i^*\right)(v \otimes w) = \sum_{i=1}^{n} v_i^*(v)\, w_i^*(w) = v_1^*(v)\, w_1^*(w) = w_1^*(w) \neq 0$$

and hence we deduce that $\Lambda\left(\sum_{i=1}^{n} v_i^* \otimes w_i^*\right) \neq 0$.

Assume now that $\dim(W) < \infty$ and let $w_1, \ldots w_m$ be a basis of W and let $w_1^*, \ldots w_m^*$ denote the dual basis of W^*. Let $\xi \in (V \otimes W)^*$ and let $\xi_i \in V^*$ be defined by setting $\xi_i(v) = \xi(v \otimes w_i)$, for every $v \in V$. Then, for every $v \in V$ and $j = 1, \ldots m$ we have

$$\Lambda\left(\sum_{i=1}^{n} \xi_i \otimes w_i^*\right)(v \otimes w_j) = \sum_{i=1}^{n} \xi_i(v)\, w_i^*(w_j) = \xi_j(v) = \xi(v \otimes w_j)$$

and hence we deduce that $\Lambda\left(\sum_{i=1}^{n} \xi_i \otimes w_i^*\right) = \xi$. □

Proposition 1.39. *The k-linear maps $\Lambda_{V,W}$ give rise to a functorial morphism $\Lambda : (-)^* \otimes (-)^* \to (- \otimes -)^*$. Moreover for given vector spaces U, V, W, we have*

$$(\Lambda_{U,V\otimes W}) \circ (U^* \otimes \Lambda_{V,W}) = \Lambda_{U\otimes V,W} \circ (\Lambda_{U,V} \otimes W^*).$$

Proof. Let $\alpha : U \to V$ and $\beta : T \to W$ be k-linear maps. We have to prove that

$$\Lambda_{U,T} \circ (\alpha^* \otimes \beta^*) = (\alpha \otimes \beta)^* \circ \Lambda_{V,W.}$$

For given $v^* \in V^*, w^* \in W^*, u \in U$ and $t \in T$ we compute

$$\begin{gathered} ([\Lambda_{U,T} \circ (\alpha^* \otimes \beta^*)](v^* \otimes w^*))(u \otimes t) = [\Lambda_{U,T}(\alpha^*(v^*) \otimes \beta^*(w^*))](u \otimes t) \\ = [\Lambda_{U,T}((v^* \circ \alpha) \otimes (w^* \circ \beta))](u \otimes t) = [(v^* \circ \alpha)(u)][(w^* \circ \beta)(t)] \\ = v^*(\alpha(u))\, w^*(\beta(t)) = [\Lambda_{V,W}(v^* \otimes w^*)](\alpha(u) \otimes \beta(t)) \\ = ([\Lambda_{V,W}(v^* \otimes w^*)] \circ (\alpha \otimes \beta))(u \otimes t) = [(\alpha \otimes \beta)^*(\Lambda_{V,W}(v^* \otimes w^*))](u \otimes t) \\ = ([(\alpha \otimes \beta)^* \circ \Lambda_{V,W}](v^* \otimes w^*))(u \otimes t) \end{gathered}$$

Let now $u^* \in U^*, v^* \in V^*, w^* \in W^*$ and $u \in U, v \in V, w \in W$. We have

$$\begin{aligned} & \{[(\Lambda_{U,V\otimes W}) \circ (U^* \otimes \Lambda_{V,W})](u^* \otimes v^* \otimes w^*)\}(u \otimes v \otimes w) \\ = \; & [(\Lambda_{U,V\otimes W})(u^* \otimes \Lambda_{v^*,w^*})](u \otimes v \otimes w) = u^*(u)\, \Lambda_{v^*,w^*}(v \otimes w) \\ = \; & u^*(u)\, v^*(v)\, w^*(w) = \Lambda_{u^*,v^*}(u \otimes v)\, w^*(w) \\ = \; & [(\Lambda_{U\otimes V,W})(\Lambda_{u^*,v^*} \otimes w^*)](u \otimes v \otimes w) \\ = \; & \{[\Lambda_{U\otimes V,W} \circ (\Lambda_{U,V} \otimes W^*)](u^* \otimes v^* \otimes w^*)\}(u \otimes v \otimes w). \end{aligned}$$

□

Proposition 1.40. *Let* (A,m,u) *be a finite dimensional algebra. Then* $A^* = \mathrm{Hom}_k(A,k)$ *has a natural coalgebra structure defined by setting*

$$\Delta_{A^*} : A^* \xrightarrow{m^*} (A\otimes A)^* \xrightarrow{\Lambda^{-1}_{A,A}} A^*\otimes A^* \quad \text{and} \quad \varepsilon_{A^*} : A^* \xrightarrow{u^*} k^* \overset{ev_1}{\simeq} k$$

This coalgebra is called the ***dual coalgebra*** *of the algebra* A.

Proof. Let $\alpha : U \to V$ and $\beta : T \to W$ be k-linear maps between finite dimensional vector spaces. Note that, by Proposition 1.39, we have

$$(\alpha^*\otimes\beta^*)\circ\Lambda^{-1}_{V,W.} = \Lambda^{-1}_{U,T}\circ(\alpha\otimes\beta)^* \tag{1.10}$$

and

$$\left(\Lambda^{-1}_{U,V}\otimes W^*\right)\circ\Lambda^{-1}_{U\otimes V,W} = \left(U^*\otimes\Lambda^{-1}_{V,W}\right)\left(\Lambda^{-1}_{U,V\otimes W}\right) \tag{1.11}$$

We compute

$$\begin{aligned}
(\Delta_{A^*}\otimes A^*)\circ\Delta_{A^*} &= \left[\left(\Lambda^{-1}_{A,A}\circ m^*\right)\otimes A^*\right]\circ\left(\Lambda^{-1}_{A,A}\circ m^*\right)\\
&= \left(\Lambda^{-1}_{A,A}\otimes A^*\right)\circ(m^*\otimes A^*)\circ\Lambda^{-1}_{A,A}\circ m^*\\
&\overset{(1.10)}{=} \left(\Lambda^{-1}_{A,A}\otimes A^*\right)\circ\Lambda^{-1}_{A\otimes A,A}\circ(m\otimes A)^*\circ m^*\\
&= \left(\Lambda^{-1}_{A,A}\otimes A^*\right)\circ\Lambda^{-1}_{A\otimes A,A}\circ[m\circ(m\otimes A)]^*\\
&= \left(\Lambda^{-1}_{A,A}\otimes A^*\right)\circ\Lambda^{-1}_{A\otimes A,A}\circ[m\circ(A\otimes m)]^*\\
&= \left(\Lambda^{-1}_{A,A}\otimes A^*\right)\circ\Lambda^{-1}_{A\otimes A,A}\circ(A\otimes m)^*\circ m^*\\
&\overset{(1.11)}{=} \left(A^*\otimes\Lambda^{-1}_{A,A}\right)\circ\Lambda^{-1}_{A,A\otimes A}\circ(A\otimes m)^*\circ m^*\\
&\overset{(1.10)}{=} \left(A^*\otimes\Lambda^{-1}_{A,A}\right)\circ(A^*\otimes m^*)\circ\Lambda^{-1}_{A,A}\circ m^*\\
&= \left[A^*\otimes\left(\Lambda^{-1}_{A,A}\circ m^*\right)\right]\circ\Lambda^{-1}_{A,A}\circ m^* = (A^*\otimes\Delta_{A^*})\circ\Delta_{A^*}
\end{aligned}$$

and

$$\begin{aligned}
l_{A^*}\circ(\varepsilon_{A^*}\otimes A^*)\circ\Delta_{A^*} &= l_{A^*}\circ((ev_1)\circ u^*\otimes A^*)\circ\Lambda^{-1}_{A,A}\circ m^*\\
&= l_{A^*}\circ(ev_1\otimes A^*)\circ(u^*\otimes A^*)\circ\Lambda^{-1}_{A,A}\circ m^* =\\
\overset{(1.10)}{=} l_{A^*}\circ(ev_1\otimes A^*)\circ\Lambda^{-1}_{k,A}\circ(u\otimes A)^*\circ m^* &= l_{A^*}\circ(ev_1\otimes A^*)\circ\Lambda^{-1}_{k,A}\circ[m\circ(u\otimes A)]^*\\
&= l_{A^*}\circ(ev_1\otimes A^*)\circ\Lambda^{-1}_{k,A}\circ(l_A)^*
\end{aligned}$$

Now we have $\Lambda^{-1}_{k,A}(a^*\circ l_A) = \mathrm{Id}_k\otimes a^*$ in fact

$$\begin{aligned}
\Lambda_{k,A}(\mathrm{Id}_k\otimes a^*)(x\otimes a) &= x\cdot a^*(a) = a^*(xa) = (a^*\circ l_A)(x\otimes a)\\
\text{for every } x &\in k \text{ and } a\in A.
\end{aligned}$$

It follows that

$$\begin{aligned}
\left[l_{A^*}\circ(ev_1\otimes A^*)\circ\Lambda^{-1}_{k,A}\circ(l_A)^*\right](a^*) &= [l_{A^*}\circ(ev_1\otimes A^*)]\left(\Lambda^{-1}_{k,A}(a^*\circ l_A)\right)\\
&= [l_{A^*}\circ(ev_1\otimes A^*)](\mathrm{Id}_k\otimes a^*) = l_{A^*}(1\otimes a^*) = a^*.
\end{aligned}$$

A similar proof showes that $r_{A^*}\circ(A^*\otimes\varepsilon_{A^*})\circ\Delta_{A^*} = I_{A^*}$ □

1.41. *Let (A, m, u) be a finite dimensional algebra and let $f \in A^*$. Then*

$$\Delta_{A^*}(f) = \Lambda_{A,A}^{-1} \circ m^*(f) = \sum f_1 \otimes f_2$$

where $\sum f_1 \otimes f_2$ is uniquely determined by

$$\Lambda_{A,A}\left(\sum f_1 \otimes f_2\right) = m^*(f)$$

i.e. for every $a, b \in A$

$$\Lambda_{A,A}\left(\sum f_1 \otimes f_2\right)(a \otimes b) = m^*(f)(a \otimes b)$$

since

$$\Lambda_{A,A}\left(\sum f_1 \otimes f_2\right)(a \otimes b) = \sum f_1(a) f_2(b)$$

and

$$m^*(f)(a \otimes b) = f(m(a \otimes b)) = f(ab)$$

we conclude that $\sum f_1 \otimes f_2$ is uniquely determined by

$$\sum f_1(a) f_2(b) = f(ab) \quad \text{for every } a, b \in A. \tag{1.12}$$

Moreover

$$\varepsilon_{A^*}(f) = (ev_{1_k} \circ u^*)(f) = (f \circ u)(1_k) = f(1_A)$$

Exercise 1.42. *Let M be a finite monoid and let kM the monoid algebra over M. Then in $(kM)^*$ we can consider the so called "dual basis" $(x^*)_{x \in M}$ where x^* is defined by setting $x^*(y) = \delta_{x,y}$. Let $\Delta = \Delta_{(kM)^*}$ and let us compute $\Delta(x^*)$. Accordingly to (1.12) we have:*

$$\begin{aligned} \Delta(x^*) &= \sum f_1 \otimes f_2 \quad \text{such that} \\ \sum f_1(y) f_2(z) &= x^*(yz) \quad \text{for every } y, z \in M \end{aligned}$$

Since $x^*(yz) = 1$ if and only if $yz = x$ and $x^*(yz) = 0$ otherwise, and since $\sum_{\substack{s,t \in M \\ st=x}} s^* \otimes t^*$ has the property that

$$\begin{aligned} \sum_{\substack{s,t \in M \\ st=x}} s^*(y) t^*(z) &= y^*(y) z^*(z) = 1 \quad \text{if } yz = x \quad \text{and} \\ \sum_{\substack{s,t \in M \\ st=x}} s^*(y) t^*(z) &= 0 \text{ otherwise.} \end{aligned}$$

We conclude that

$$\Delta(x^*) = \sum_{\substack{s,t \in M \\ st=x}} s^* \otimes t^*.$$

A computation on ε^* reveals that $(kM)^*$ is just the coalgebra of the semigroup M as introduced in Example 1.12.

Exercise 1.43. *Prove that for* $A = M_n(k)$*, the algebra of the* $n \times n$ *matrices,* $A^* = M^c(n,k)$.

Exercise 1.44. *Prove that for an oriented finite graph* Γ*, the dual coalgebra of the path algebra of* Γ *is the path coalgebra of* Γ.

1.45. *Let* (C, Δ, ε) *be a* k*-coalgebra and let* (A, m, u) *be a* k*-algebra. Then*

$$\mathrm{Hom}_k(C, A)$$

is always an algebra, called convolution algebra*. The multiplication* $*$ *of this algebra is defined by setting, for every* $f, g \in \mathrm{Hom}_k(C, A)$ *and* $c \in C$

$$(f * g)(c) = \sum f(c_1) \cdot g(c_2) \tag{1.13}$$

Proposition 1.46. *Let* (C, Δ, ε) *be a* k*-coalgebra and let* (A, m, u) *be a* k*-algebra. Then* $\mathrm{Hom}_k(C, A)$*, with respect to the product defined in* (1.13) *becomes an algebra whose identity is* $u \circ \varepsilon$.

Proof. Let $f, g, h \in \mathrm{Hom}_k(C, A)$. For every $c \in C$, we calculate

$$\begin{aligned}((f * g) * h)(c) &= \sum (f * g)(c_1) \cdot h(c_2) = \sum (f(c_{1_1}) \cdot g(c_{1_2})) \cdot h(c_2) = \\ &= \sum f(c_1) \cdot g(c_{2_1}) \cdot h(c_{2_2}) = \sum f(c_1) \cdot (g * h)(c_2) \\ &= (f * (g * h))(c)\end{aligned}$$

and

$$(f * (u \circ \varepsilon))(c) = \sum f(c_1) \cdot (\varepsilon(c_2) u(1_k)) = f\left(\sum c_1 \varepsilon(c_2)\right) \cdot 1_A = f(c).$$

Thus we get that $f * (u \circ \varepsilon) = f$. A similar proof shows that $(u \circ \varepsilon) * f = f$. □

Proposition 1.47. *Let* $\varphi : C_2 \to C_1$ *be a morphism of* k*-coalgebras and let* $\psi : A_1 \to A_2$ *be a morphism of* k*-algebras. Then* $\mathrm{Hom}(\varphi, \psi) : \mathrm{Hom}(C_1, A_1) \to \mathrm{Hom}(C_2, A_2)$ *is an algebra morphism.*

Proof. Let $f, g \in \mathrm{Hom}(C_1, A_1)$. Then $\mathrm{Hom}(\varphi, \psi)(f * g) = \psi \circ (f * g) \circ \varphi$ and we have

$$\begin{gathered}[\mathrm{Hom}(\varphi, \psi)(f * g)](c) = [\psi \circ (f * g) \circ \varphi](c) = \psi\left[f \sum (\varphi(c)_1) g(\varphi(c)_2)\right] \\ \overset{\varphi \text{coalg.morph}}{=} \psi\left[\sum f(\varphi(c_1)) g(\varphi(c_2))\right] \overset{\psi \text{alg.morph}}{=} \sum \psi(f(\varphi(c_1))) \psi(g(\varphi(c_2))) = \\ [(\psi \circ f \circ \varphi) * (\psi \circ g \circ \varphi)](c) = [\mathrm{Hom}(\varphi, \psi)(f) * \mathrm{Hom}(\varphi, \psi)(g)](c)\end{gathered}$$

and

$$\begin{aligned}\mathrm{Hom}(\varphi, \psi)(u_{A_1} \circ \varepsilon_{C_1}) &= \psi \circ (u_{A_1} \circ \varepsilon_{C_1}) \circ \varphi = (\psi \circ u_{A_1}) \circ (\varepsilon_{C_1} \circ \varphi) = \\ &\overset{\psi \text{alg.morph}, \varphi \text{coalg.morph}}{=} u_{A_2} \circ \varepsilon_{C_2}\end{aligned}$$

□

Example 1.48. *In particular, we can consider the case when* $A = k$. *In this case* $\mathrm{Hom}_k(C, A) = C^*$ *and, in view of Proposition 1.47 the assignment* $C \mapsto C^*$ *and* $f \mapsto f^*$ *defines a covariant functor* $^* : Coalg_k \to Alg_k$.

Exercise 1.49. *Prove that for the divided power coalgebra* C *(see example 1) in Example 1.12)* C^* *is isomorphic to the formal power series ring* $k[[X]]$.

Definition 1.50. *Let* (C, Δ, ε) *be a* k*-coalgebra and let* $g \in C$. *The element* g *is called a* grouplike element *if* $g \neq 0$ *and* $\Delta(g) = g \otimes g$. *We will denote by* $G(C)$ *the set of grouplike elements of* C.

Lemma 1.51. *Let* (C, Δ, ε) *be a* k*-coalgebra and let* $g \in C$ *such that* $\Delta(g) = g \otimes g$. *Then*

$$g \neq 0 \Longleftrightarrow \varepsilon(g) = 1.$$

Proof. Since $\Delta(g) = g \otimes g$ we get that $g = \varepsilon(g) g$. From $g \neq 0$ we deduce that $\varepsilon(g) = 1$. □

Proposition 1.52. *Let* A *be a finite dimensional algebra. Then*

$$G(A^*) = Alg(A, k)$$

where $Alg(A, k)$ *is the set of algebra morphisms from* A *to* k.

Proof. Let $f \in A^*$. Then $\Delta_{A^*}(f) = \sum f_1 \otimes f_2$ is uniquely determined by

$$\sum f_1(a) f_2(b) = f(ab) \quad \text{for every } a, b \in A.$$

Hence $\Delta_{A^*}(f) = f \otimes f$ if and only if $f(a) f(b) = f(ab)$ for every $a, b \in A$. Since $\varepsilon_{A^*}(f) = f(1)$, we conclude. □

Example 1.53. *Let us consider the matrix coalgebra* $M^C(n, k)$. *Then* $M^C(n, k) = (M_n(k))^*$ *so that, by Proposition 1.52,*

$$G\left(M^C(n, k)\right) = Alg\left(M_n(k), k\right).$$

Let $\varphi : M_n(k) \to k$ *be an algebra morphism. Then* $Ker(\varphi) = \{0\}$ *which is impossible if* $n > 1$. *Hence we deduce that* $G\left(M^C(n, k)\right)$ *is empty.*

Theorem 1.54. *Let* (C, Δ, ε) *be a* k*-coalgebra and assume that* $G(C)$ *is nonempty. Then the set* $G(C)$ *is a linearly independent subset of* C.

Proof. Assume that $G(C)$ is not linearly independent. Since any grouplike element is linearly independent, there exists an $n \in \mathbb{N}$, $n \geq 1$ such that any subset of n elements in $G(C)$ is linearly independent but there is a subset $\{g_1, \ldots, g_n, g_{n+1}\}$, consisting of $n + 1$ distinct elements of $G(C)$, which is not linearly independent. Hence there exists $\lambda_1, \ldots \lambda_n \in k$ such that

$$g_{n+1} = \lambda_1 g_1 + \ldots + \lambda_n g_n.$$

By applying Δ we get

$$g_{n+1} \otimes g_{n+1} = \sum_{i=1}^{n} \lambda_i g_i \otimes g_i$$

and hence

$$\sum_{t=1}^{n} \lambda_t g_t \otimes \sum_{s=1}^{n} \lambda_s g_s = \sum_{i=1}^{n} \lambda_i g_i \otimes g_i$$

so that

$$\sum_{t,s=1}^{n} \lambda_t \lambda_s g_t \otimes g_s = \sum_{i=1}^{n} \lambda_i g_i \otimes g_i$$

Then, since the set $\{g_t \otimes g_s \mid t, s = 1, \ldots, n\}$ is linearly independent, for any t, s with $t \neq s$ we get that $\lambda_t \lambda_s = 0$. This forces, by a possible renumbering of $g_1, \ldots, g_n$, $n = 1$ and $g_{n+1} = \lambda_1 g_1$. Since $1 = \varepsilon(g_{n+1}) = \lambda_1 \varepsilon(g_1)$ we obtain that $\lambda_1 = 1$ and $g_{n+1} = g_1$, a contradiction. □

Remark 1.55. *Let (C, Δ, ε) be a k-coalgebra and assume that $G(C)$ is nonempty. Then the subspace $kG(C)$ spanned by $G(C)$ is a subcoalgebra of C.*

Chapter 2

Comodules and Rational Modules

Definitions 2.1. *Let* (C, Δ, ε) *be a* k*-coalgebra. A* right C-comodule *is a pair* $\left(M, \rho^M\right)$ *where*

- M *is a* k*-vector space*
- $\rho^M : M \to M \otimes C$ *is a* k*-linear map such that*

$$(M \otimes \Delta) \circ \rho^M = \left(\rho^M \otimes C\right) \circ \rho^M \quad \text{and} \quad r_M \circ (M \otimes \varepsilon) \circ \rho^M = M. \tag{2.1}$$

A left C-comodule *is a pair* $\left(N, {}^N\rho\right)$ *where*

- N *is a* k*-vector space*
- ${}^N\rho : N \to C \otimes N$ *is a* k*-linear map such that*

$$(\Delta \otimes N) \circ {}^N\rho = \left(C \otimes {}^N\rho\right) \circ {}^N\rho \quad \text{and} \quad l_N \circ (\varepsilon \otimes N) \circ {}^N\rho = N. \tag{2.2}$$

Definition 2.2. *Let* (C, Δ, ε) *be a coalgebra and let* $\left(M, \rho^M\right)$ *be a right* C*-comodule. We define, by recursion, a sequence* $\left(\rho_n^M\right)_{n \geq 1}$ *by setting*

$$\rho_1^M = \rho^M \quad \text{and} \quad \rho_n^M = \left(\rho^M \otimes C^{n-1}\right) \circ \rho_{n-1}^M \quad \text{for every } n \in \mathbb{N}, n \geq 2.$$

Proposition 2.3. *Let* (C, Δ, ε) *be a coalgebra and let* $\left(M, \rho^M\right)$ *be a right* C*-comodule. Then*

$$\rho_n^M = \left(M \otimes C^{t-1} \otimes \Delta \otimes C^{n-1-t}\right) \circ \rho_{n-1}^M \quad \text{for every } n, t \in \mathbb{N}, n \geq 2 \quad \text{and } 1 \leq t \leq n-1. \tag{2.3}$$

Proof. It is similar to that of Lemma 1.15. □

Notation 2.4. *Let* $\left(M, \rho^M\right)$ *be a right* C*-comodule. For every* $x \in M$ *we will write*

$$\rho^M(x) = \sum x_{(0)} \otimes x_{(1)}$$

or even

$$\rho^M(x) = \sum x_0 \otimes x_1.$$

Note that, using this notation, equalities in (2.1) *can be rewritten as*

$$\sum x_{(0)} \otimes x_{(1)_1} \otimes x_{(1)_2} = \sum x_{(0)_{(0)}} \otimes x_{(0)_{(1)}} \otimes x_{(1)} \quad \text{and} \quad \sum x_{(0)} \varepsilon\left(x_{(1)}\right) = x$$

for every $x \in M$.

Notation 2.5. *More generally, for any* $n \in \mathbb{N}$, $n \geq 1$ *we write*

$$\rho_n^M(x) = \sum x_{(0)} \otimes \ldots \otimes x_{(n)}$$

Using this notation, equality (2.3) *gives rise to*

$$\sum x_{(0)} \otimes \ldots \otimes x_{(n)} = \sum x_{(0)} \otimes \ldots \otimes x_{(t-1)} \otimes x_{(t)_1} \otimes x_{(t)_2} \otimes x_{(t+1)} \ldots \otimes x_{(n-1)}.$$

Definition 2.6. *Let* (C, Δ, ε) *be a coalgebra and let* $\left(N, {}^N\rho\right)$ *be a left* C*-comodule. We define, by recursion, a sequence* $\left({}^N\rho_n\right)_{n\geq 1}$ *by setting*

$${}^N\rho_1 = {}^N\rho \quad \text{and} \quad {}^N\rho_n = \left(C^{n-1} \otimes {}^N\rho\right) \circ {}^N\rho_{n-1} \quad \text{for every } n \in \mathbb{N}, n \geq 2.$$

Notation 2.7. *Let* $\left(N, {}^N\rho\right)$ *be a left* C*-comodule. For every* $x \in N$ *we will write*

$${}^N\rho(x) = \sum x_{(-1)} \otimes x_{(0)}$$

or even

$${}^N\rho(x) = \sum x_{-1} \otimes x_0.$$

Note that, using this notation, equalities in (2.2) *can be rewritten as*

$$\sum x_{(-1)_1} \otimes x_{(-1)_2} \otimes x_{(0)} = \sum x_{(-1)} \otimes x_{(0)_{(-1)}} \otimes x_{(0)_{(0)}} \quad \text{and} \quad \sum \varepsilon\left(x_{(-1)}\right) x_{(0)} = x$$

for every $x \in N$.

Notation 2.8. *More generally, for any* $n \in \mathbb{N}$, $n \geq 1$ *we write*

$${}^N\rho_n(x) = \sum x_{(-n)} \otimes \ldots \otimes x_{(0)}$$

Using this notation, an equality analogous to (2.3) *gives rise to*

$$\sum x_{(-n)} \otimes \ldots \otimes x_{(0)} = \sum x_{(-n+1)} \otimes \ldots \otimes x_{(-t-1)} \otimes x_{(-t)_1} \otimes x_{(-t)_2} \otimes x_{(-t+1)} \ldots \otimes x_{(0)}.$$

Remarks 2.9. *1) Both for right and for left comodules, using the same criteria involved in the case of coalgebras, others formulas can be deduced.*

2) Both for right and for left comodules, sometimes we will need to use as brackets the symbols [] *or even* $\langle\rangle$.

Definitions 2.10. *Let (C,Δ,ε) be a k-coalgebra and let $\left(M_1,\rho^{M_1}\right)$ and $\left(M_2,\rho^{M_2}\right)$ be right C-comodules. A k-linear map $f : M_1 \to M_2$ is called a* morphism of (right) comodules *(or* right colinear map*) if*

$$(f\otimes C)\circ\rho^{M_1}=\rho^{M_2}\circ f$$

i.e. if

$$\sum f(x_0)\otimes x_1=\sum f(x)_0\otimes f(x)_1 \quad \text{for every } x\in M_1.$$

We will denote by $\mathcal{M}^C$ the category of right C-comodules.

Let (C,Δ,ε) be a k-coalgebra and let $\left(N_1,{}^{N_1}\rho\right)$ and $\left(N_2,{}^{N_2}\rho\right)$ be left C-comodules. A k-linear map $f : N_1 \to N_2$ is called a morphism of (left) comodules *(or* left colinear map*) if*

$$(C\otimes f)\circ{}^{N_1}\rho={}^{N_2}\rho\circ f$$

i.e. if

$$\sum x_{-1}\otimes f(x_0)=\sum f(x)_{-1}\otimes f(x)_0 \quad \text{for every } x\in N_1.$$

We will denote by ${}^C\mathcal{M}$ the category of left C-comodules.

Exercise 2.11. *Let (C,Δ,ε) be a coalgebra and let $\left(M,\rho^M\right)$ be a right C-comodule. Prove that ρ^M is injective.*

Exercise 2.12. *Let $f : \left(M_1,\rho^{M_1}\right)\to\left(M_2,\rho^{M_2}\right)$ be a comodule morphism and assume that f is bijective. Sow that f^{-1} is a comodule morphism.*

Definition 2.13. *A subspace L of a right C-comodule $\left(M,\rho^M\right)$ is called a C-subcomodule if*

$$\rho^M(L)\subseteq L\otimes C.$$

In this case L itself becomes in a natural way a right C-comodule by setting

$$\rho^L=\left(\left(\rho^M\right)_{|L}\right)^{L\otimes C}.$$

In this way the natural inclusion $i_L : L\to M$ becomes automatically a morphism of comodules.

Remark 2.14. *An analogous definition hold for left C-comodules.*

Example 2.15. *Any coalgebra C can be regarded as a right C-comodule by setting $\rho^C=\Delta$. The subcomodules of this particular comodule are just the right coideals of C.*

Exercise 2.16. *Let $f : M_1\to M_2$ be a morphism of right C-comodules. Prove that $Ker(f)$ is a subcomodule of M_1 and $\mathrm{Im}(f)$ is a subcomodule of M_2.*

Theorem 2.17. *(**The Fundamental Theorem of the Quotient Comodule**) Let $\left(M,\rho^M\right)$ be a right C-comodule, let L be a subcomodule of M and let $p=p_L : M\to M/L$ be the canonical projection. Then M/L can be endowed by a unique comodule structure (called* quotient comodule*) such that p becomes a comodule morphism. Moreover given any morphism $f : M\to M'$ of right C-comodules such that $L\subseteq \operatorname{Ker}(f)$, there exists a unique comodule morphism $\overline{f} : M/L\to M'$ such that $f=\overline{f}\circ p$.*

Proof. Since $\rho^M(L)\subseteq L\otimes C$, we get that $\left[(p\otimes C)\circ\rho^M\right](L)=\{0\}$. Hence there exists a unique k-linear map $\rho^{M/L} : M/L\to M/L\otimes C$ such that $\rho^{M/L}\circ p=(p\otimes C)\circ\rho^M$ and we have

$$\begin{aligned}(M/L\otimes\Delta)\circ\rho^{M/L}\circ p &= (M/L\otimes\Delta)\circ(p\otimes C)\circ\rho^M\\ &= (p\otimes\Delta)\circ\rho^M=(p\otimes C\otimes C)\circ(M\otimes\Delta)\circ\rho^M\\ &= (p\otimes C\otimes C)\circ\left(\rho^M\otimes C\right)\circ\rho^M\\ &= \left((p\otimes C)\circ\rho^M\otimes C\right)\circ\rho^M\\ &= \left(\rho^{M/L}\circ p\otimes C\right)\circ\rho^M=\left(\rho^{M/L}\otimes C\right)\circ(p\otimes C)\circ\rho^M\\ &= \left(\rho^{M/L}\otimes C\right)\circ\rho^{M/L}\circ p.\end{aligned}$$

Since p is surjective, we get that $(M/L\otimes\Delta)\circ\rho^{M/L}=\left(\rho^{M/L}\otimes C\right)\circ\rho^{M/L}$. Let us compute

$$\begin{aligned}r_{M/L}\circ(M/L\otimes\varepsilon)\circ\rho^{M/L}\circ p &= r_{M/L}\circ(M/L\otimes\varepsilon)\circ(p\otimes C)\circ\rho^M\\ &= r_{M/L}\circ(p\otimes k)\circ(M\otimes\varepsilon)\circ\rho^M\\ \overset{(1.1)}{=} p\circ r_M\circ(M\otimes\varepsilon)\circ\rho^M &= p\end{aligned}$$

Since p is surjective, we get that $r_{M/L}\circ(M/L\otimes\varepsilon)\circ\rho^{M/L}=\mathrm{Id}_{M/L}$ and hence $\left(M/L,\rho^{M/L}\right)$ is a right C-comodule. Note that p becomes automatically a comodule morphism.

Let now $f : M\to M'$ be a comodule morphism and assume that L is contained in $Ker(f)$. Then there exists a unique k-linear map $\overline{f} : M/L\to M'$ such that $\overline{f}\circ p=f$. Let us check that $\overline{f}$ is a comodule morphism. Indeed we have

$$\begin{aligned}\left(\overline{f}\otimes C\right)\circ\rho^{M/L}\circ p &= \left(\overline{f}\otimes C\right)\circ(p\otimes C)\circ\rho^M=(f\otimes C)\circ\rho^M=\rho^{M'}\circ f\\ &= \rho^{M'}\circ\overline{f}\circ p.\end{aligned}$$

Since p is surjective we deduce that $\left(\overline{f}\otimes C\right)\circ\rho^{M/L}=\rho^{M'}\circ\overline{f}$. □

Exercise 2.18. *Let $(L_i)_{i\in I}$ be a family of subcomodules of a right comodule $\left(M,\rho^M\right)$. Show that both $\sum_{i\in I}L_i$ and $\bigcap_{i\in I}L_i$ are subcomodules of M.*

2.19. *Let C be a coalgebra and let M be a k-vector space. Let $W\subseteq M^*$ and let $ev_{M,W} : M\otimes W\to k$ be the evaluation map. For every k-linear map $\rho : M\to M\otimes C$ set*

$$\mu_\rho : C^*\otimes M\overset{\tau_{C^*,M}}{\longrightarrow} M\otimes C^*\overset{\rho\otimes C^*}{\longrightarrow} M\otimes C\otimes C^*\overset{M\otimes ev_{C,C^*}}{\longrightarrow} M\otimes k\overset{r_M}{\longrightarrow} M.$$

Lemma 2.20. *Using the notation of 2.19, let*

$$\theta := m_k \circ (ev_{C,C^*} \otimes ev_{C,C^*}) \circ (C \otimes \tau_{C,C^*} \otimes C^*) : C \otimes C \otimes C^* \otimes C^* \to k.$$

Then the map

$$\Theta : \mathrm{Hom}\,(M, M \otimes C \otimes C) \to \mathrm{Hom}\,(M \otimes C^* \otimes C^*, M \otimes k) :$$

defined by setting

$$\Theta\,(\gamma) = (M \otimes \theta) \circ (\gamma \otimes C^* \otimes C^*) \quad \textit{for every } \gamma \in \mathrm{Hom}\,(M, M \otimes C \otimes C)$$

is injective.

Proof. Note that, for every $x \in M, c, d \in C$ and $f, g \in C^*$, we have

$$(M \otimes \theta)\,(x \otimes c \otimes d \otimes f \otimes g) = x \otimes f\,(c)\,g\,(d) = [M \otimes (m_k \circ (f \otimes g))]\,(x \otimes c \otimes d)\,. \tag{2.4}$$

Let $\gamma \in \mathrm{Hom}\,(M, M \otimes C \otimes C)$ and let $x \in M, f, g \in C^*$. Let us compute

$$\begin{aligned} \Theta\,(\gamma)\,(x \otimes f \otimes g) &= [(M \otimes \theta) \circ (\gamma \otimes C^* \otimes C^*)]\,(x \otimes f \otimes g) \\ = (M \otimes \theta)\,(\gamma\,(x) \otimes f \otimes g) &\overset{(2.4)}{=} [M \otimes (m_k \circ (f \otimes g))]\,(\gamma\,(x))\,. \end{aligned}$$

Let $\gamma, \xi \in Hom\,(M, M \otimes C \otimes C)$ and assume that $\Theta\,(\gamma) = \Theta\,(\xi)$. From the foregoing, we deduce that, for every $x \in M, f, g \in C^*$, we have

$$[M \otimes (m_k \circ (f \otimes g))]\,(\gamma\,(x)) = [M \otimes (m_k \circ (f \otimes g))]\,(\xi\,(x))\,. \tag{2.5}$$

Now assume that there exists an $x \in M$ such that

$$y = \gamma\,(x) - \xi\,(x) \neq 0.$$

Let $(e_i)_{i \in I}$ be a basis of C. Then there exist $x_{i,j} \in M, i, j \in F$ where F is a finite subset of I such that

$$y = \sum_{i,j \in F} x_{i,j} \otimes e_i \otimes e_{j.}$$

Let $(e_i^*)_{i \in I}$ be the dual system of $(e_i)_{i \in I}$. Then for any $s, t \in F$ we get

$$(M \otimes (m_k \circ (e_s^* \otimes e_t^*))) \left(\sum_{i,j \in F} x_{i,j} \otimes e_i \otimes e_{j.} \right) = x_{s,t}.$$

Since $y \neq 0$, there exist s_0, t_0 such that $x_{s_0,t_0} \neq 0$. This contradicts (2.5). □

The proof of the following theorem is mostly due to Alessandro Ardizzoni. We thank him for this great help.

Theorem 2.21. *Using the notation of 2.19, we have that*

$$(M,\rho)\ \text{ is a right } C\text{-comodule } \Longleftrightarrow (M,\mu_\rho)\ \text{ is a left } C^*\text{-module.}$$

Proof. Set $ev = ev_{C,C^*}$. Let us prove that

$$ev \circ (C \otimes m_{C^*}) = m_k \circ (ev \otimes ev) \circ (C \otimes \tau_{C,C^*} \otimes C^*) \circ (\Delta_C \otimes C^* \otimes C^*) \tag{2.6}$$

Let $c \in C, f, g \in C^*$. We compute

$$[ev \circ (C \otimes m_{C^*})]\,(c \otimes f \otimes g) = (f * g)\,(c)$$

and

$$\begin{aligned}&[m_k \circ (ev \otimes ev) \circ (C \otimes \tau_{C,C^*} \otimes C^*) \circ (\Delta_C \otimes C^* \otimes C^*)]\,(c \otimes f \otimes g)\\ &= m_k\,(ev \otimes ev)\,(C \otimes \tau_{C,C^*} \otimes C^*)\left(\sum c_1 \otimes c_2 \otimes f \otimes g\right) =\\ &= m_k\,(ev \otimes ev)\left(\sum c_1 \otimes f \otimes c_2 \otimes g\right) = \sum f\,(c_1)\,g\,(c_2)\,.\end{aligned}$$

By definition of $f * g$ we deduce (2.6). From this we get that

$$\begin{aligned}\mu_\rho \circ (m_{C^*} \otimes M) &= r_M \circ (M \otimes ev) \circ (\rho \otimes C^*) \circ \tau_{C^*,M} \circ (m_{C^*} \otimes M)\\ &\overset{(1.3)}{=} r_M \circ (M \otimes ev) \circ (\rho \otimes C^*) \circ (M \otimes m_{C^*}) \circ \tau_{C^* \otimes C^*,M}\\ &= r_M \circ (M \otimes ev \circ (C \otimes m_{C^*})) \circ (\rho \otimes C^* \otimes C^*) \circ \tau_{C^* \otimes C^*,M}\\ &\overset{(2.6)}{=} r_M \circ (M \otimes [m_k \circ (ev \otimes ev) \circ (C \otimes \tau_{C,C^*} \otimes C^*) \circ (\Delta_C \otimes C^* \otimes C^*)])\\ &\qquad \circ (\rho \otimes C^* \otimes C^*) \circ \tau_{C^* \otimes C^*,M}\\ &= r_M \circ (M \otimes [m_k \circ (ev \otimes ev) \circ (C \otimes \tau_{C,C^*} \otimes C^*)])\\ &\qquad \circ (M \otimes \Delta_C \otimes C^* \otimes C^*) \circ (\rho \otimes C^* \otimes C^*) \circ \tau_{C^* \otimes C^*,M}\\ &= r_M \circ (M \otimes [m_k \circ (ev \otimes ev) \circ (C \otimes \tau_{C,C^*} \otimes C^*)])\\ &\qquad \circ [(M \otimes \Delta_C) \circ \rho \otimes C^* \otimes C^*] \circ \tau_{C^* \otimes C^*,M}\end{aligned}$$

and hence we have

$$\begin{aligned}\mu_\rho \circ (m_{C^*} \otimes M) \;=\;& r_M \circ (M \otimes [m_k \circ (ev \otimes ev) \circ (C \otimes \tau_{C,C^*} \otimes C^*)])\\ & \circ [(M \otimes \Delta_C) \circ \rho \otimes C^* \otimes C^*] \circ \tau_{C^* \otimes C^*,M}\end{aligned} \tag{2.7}$$

Now it is easy to check that

$$(\tau_{C^*,M} \otimes C^*) \circ \tau_{C^*,C^* \otimes M} = (M \otimes \tau_{C^*,C^*}) \circ \tau_{C^* \otimes C^*,M}. \tag{2.8}$$

We compute

$$\mu_\rho \circ (C^* \otimes \mu_\rho) = r_M \circ (M \otimes ev) \circ (\rho \otimes C^*) \circ \tau_{C^*,M} \circ (C^* \otimes \mu_\rho)$$

$$\overset{(1.3)}{=} r_M \circ (M \otimes ev) \circ (\rho \otimes C^*) \circ (\mu_\rho \otimes C^*) \circ \tau_{C^*,C^*\otimes M} = r_M \circ (M \otimes ev) \circ (\rho \otimes C^*) \circ$$
$$(r_M \otimes C^*) \circ (M \otimes ev \otimes C^*) \circ (\rho \otimes C^* \otimes C^*) \circ (\tau_{C^*,M} \otimes C^*) \circ \tau_{C^*,C^*\otimes M}$$
$$\overset{(2.8)}{=} r_M \circ (M \otimes ev) \circ (\rho \otimes C^*) \circ$$
$$(r_M \otimes C^*) \circ (M \otimes ev \otimes C^*) \circ (\rho \otimes C^* \otimes C^*) \circ (M \otimes \tau_{C^*,C^*}) \circ \tau_{C^*\otimes C^*,M}$$
$$= r_M \circ (M \otimes ev) \circ (\rho \circ r_M \otimes C^*) \circ$$
$$(M \otimes ev \otimes C^*) \circ (M \otimes C \otimes \tau_{C^*,C^*}) \circ (\rho \otimes C^* \otimes C^*) \circ \tau_{C^*\otimes C^*,M}$$
$$\overset{(1.1)}{=} r_M \circ (M \otimes ev) \circ$$
$$(r_{M\otimes C} \circ (\rho \otimes k) \otimes C^*) \circ (M \otimes ev \otimes C^*) \circ (M \otimes C \otimes \tau_{C^*,C^*}) \circ (\rho \otimes C^* \otimes C^*) \circ \tau_{C^*\otimes C^*,M}$$
$$\overset{1.2}{=} r_M \circ (M \otimes ev) \circ (M \otimes r_C \otimes C^*)$$
$$\circ (\rho \otimes k \otimes C^*) \circ (M \otimes ev \otimes C^*) \circ (M \otimes C \otimes \tau_{C^*,C^*}) \circ (\rho \otimes C^* \otimes C^*) \circ \tau_{C^*\otimes C^*,M}$$
$$= r_M \circ (M \otimes ev) \circ (M \otimes r_C \otimes C^*) \circ (M \otimes C \otimes ev \otimes C^*) \circ$$
$$\circ (\rho \otimes C \otimes C^* \otimes C^*) \circ (M \otimes C \otimes \tau_{C^*,C^*}) \circ (\rho \otimes C^* \otimes C^*) \circ \tau_{C^*\otimes C^*,M}$$
$$= r_M \circ (M \otimes ev) \circ (M \otimes r_C \otimes C^*) \circ (M \otimes C \otimes ev \otimes C^*) \circ$$
$$\circ (M \otimes C \otimes C \otimes \tau_{C^*,C^*}) \circ (\rho \otimes C \otimes C^* \otimes C^*) \circ (\rho \otimes C^* \otimes C^*) \circ \tau_{C^*\otimes C^*,M}$$
$$= r_M \circ (M \otimes [ev \circ (r_C \otimes C^*) \circ (C \otimes ev \otimes C^*) \circ (C \otimes C \otimes \tau_{C^*,C^*})])$$
$$\circ ((\rho \otimes C) \circ \rho \otimes C^* \otimes C^*) \circ \tau_{C^*\otimes C^*,M}$$

Now it is easy to prove that

(2.9)
$$[ev \circ (r_C \otimes C^*) \circ (C \otimes ev \otimes C^*) \circ (C \otimes C \otimes \tau_{C^*,C^*})] = [m_k (ev \otimes ev) (C \otimes \tau_{C,C^*} \otimes C^*)] .$$

In fact, for every $c, d \in C$, $f, g \in C^*$ we have

$$\begin{aligned} & [ev \circ (r_C \otimes C^*) \circ (C \otimes ev \otimes C^*) \circ (C \otimes C \otimes \tau_{C^*,C^*})] (c \otimes d \otimes f \otimes g) \\ = \; & [ev \circ (r_C \otimes C^*) \circ (C \otimes ev \otimes C^*)] (c \otimes d \otimes g \otimes f) = g (d) f (c) \end{aligned}$$

and

$$[m_k \circ (ev \otimes ev) \circ (C \otimes \tau_{C,C^*} \otimes C^*)] (c \otimes d \otimes f \otimes g) = m_k (ev \otimes ev) (c \otimes d \otimes g \otimes f) = f (c) g (d) .$$

Thus we obtain

$$\text{(2.10)} \quad \mu_\rho \circ (C^* \otimes \mu_\rho) = r_M \circ (M \otimes [m_k \circ (ev \otimes ev) \circ (C \otimes \tau_{C,C^*} \otimes C^*)]) \circ ((\rho \otimes C) \circ \rho \otimes C^* \otimes C^*) \circ \tau_{C^*\otimes C^*,M}.$$

Now, for every $m \in M, c \in C$, we have

$$\text{(2.11)} \qquad [r_M \circ (M \otimes ev)] (m \otimes c \otimes \varepsilon) = m \cdot \varepsilon (c) = [r_M \circ (M \otimes \varepsilon)] (m \otimes c)$$

so that

$$\left(\mu_\rho \circ (u_{C^*} \otimes M) \circ l_M^{-1}\right)(x) = \mu_\rho(\varepsilon \otimes x) = \left[r_M \circ (M \otimes ev) \circ (\rho \otimes C^*) \circ \tau_{C^*,M}\right](\varepsilon \otimes x) =$$
$$= \left[r_M \circ (M \otimes ev) \circ (\rho \otimes C^*)\right](x \otimes \varepsilon) = \left[r_M \circ (M \otimes ev)\right](\rho(x) \otimes \varepsilon)$$
$$\overset{(2.11)}{=} \left[r_M \circ (M \otimes \varepsilon)\right](\rho(x)) = \left[r_M \circ (M \otimes \varepsilon) \circ \rho\right](x)$$

and hence we get

$$\mu_\rho \circ (u_{C^*} \otimes M) \circ l_M^{-1} = r_M \circ (M \otimes \varepsilon) \circ \rho \tag{2.12}$$

$\Rightarrow$) Since $(\rho \otimes C) \circ \rho = (M \otimes \Delta) \circ \rho$, from (2.7) and from (2.10) we get $\mu_\rho \circ (m_{C^*} \otimes M) = \mu_\rho \circ (C^* \otimes \mu_\rho)$. On the other hand, since $r_M \circ (M \otimes \varepsilon) \circ \rho = \mathrm{Id}_M$, from (2.12) we get $\mu_\rho \circ (u_{C^*} \otimes M) \circ l_M^{-1} = \mathrm{Id}_M$.

$\Leftarrow$) Conversely, since $\mu_\rho \circ (m_{C^*} \otimes M) = \mu_\rho \circ (C^* \otimes \mu_\rho)$, from (2.7) and (2.10), we get

$$\begin{aligned} (2.13) \quad & (M \otimes [m_k \circ (ev \otimes ev) \circ (C \otimes \tau_{C,C^*} \otimes C^*)]) \circ [(M \otimes \Delta_C) \circ \rho \otimes C^* \otimes C^*] \\ = \quad & (M \otimes [m_k \circ (ev \otimes ev) \circ (C \otimes \tau_{C,C^*} \otimes C^*)]) \circ ((\rho \otimes C) \circ \rho \otimes C^* \otimes C^*). \end{aligned}$$

Set $\gamma = (M \otimes \Delta_C) \circ \rho$ and $\xi = (\rho \otimes C) \circ \rho$. Using the notations of Lemma 2.20 this means that

$$\Theta(\gamma) = \Theta(\xi).$$

Since Θ is injective, we deduce that $\gamma = \xi$.

Since $\mu_\rho \circ (u_{C^*} \otimes M) \circ l_M^{-1} = \mathrm{Id}_M$, from (2.12) we get $r_M \circ (M \otimes \varepsilon) \circ \rho = \mathrm{Id}_M$. □

Proposition 2.22. *The assignment* $\left(M, \rho^M\right) \mapsto \left(M, \mu_{\rho^M}\right)$ *gives rise to a functor* $H : \mathcal{M}^C \to {}_{C^*}\mathcal{M}$

Proof. Let $\gamma : M \to M'$ be a comodule morphism. Given $f \in C^*$ and $x \in M$ let us compute

$$\gamma(f \cdot x) = \gamma\left(\sum x_0 f(x_1)\right) = \sum \gamma(x_0) f(x_1) = \left(\sum (\gamma(x))_0 f((\gamma(x))_1)\right) = f \cdot (\gamma(x)).$$

From this we deduce that γ is a morphism of left C^*-modules. □

2.23. *Let* M *be a vector space. The map* $\zeta : M \times C \to \mathrm{Hom}(C^*, M)$ *defined by setting*

$$[\zeta((x, c))](f) = xf(c) \quad \textit{for every } x \in M, c \in C, f \in C^*$$

is bilinear so that it gives rise to a k*-linear map* $\alpha_M : M \otimes C \to \mathrm{Hom}(C^*, M)$ *such that*

$$(\alpha_M(x \otimes c))(f) = xf(c) \quad \textit{for every } x \in M, c \in C, f \in C^*.$$

Proposition 2.24. *Within the assumptions and notations of 2.23, the map* $\alpha_M : M \otimes C \to \mathrm{Hom}(C^*, M)$ *is injective.*

Proof. Let $z = \sum_{i=1,\dots,n} x_i \otimes c_i \in M \otimes C$ and suppose that $z \neq 0$ and $\alpha_M(z) = 0$. We can assume, w.l.o.g. that $c_1, \dots, c_n$ are linearly independent and that $x_1 \neq 0$. Let $c_i^* \in C^*$ such that $c_i^*(c_j) = \delta_{i,j}$. Then

$$0 = \alpha_M(z)(c_1^*) = \alpha_M\left(\sum_{i=1}^{n} x_i \otimes c_i\right)(c_1^*) = \sum_{i=1}^{n} x_i c_1^*(c_i) = x_1 \neq 0,$$

contradiction. □

2.25. *Let $(M, {}^M\mu)$ be a left C^*-module. Then we can consider the k-linear map $\beta_M : M \to \mathrm{Hom}(C^*, M)$ defined by setting*

$$\beta_M(x) = r_x : C^* \to M \quad \textit{where} \quad r_x(f) = f \cdot x.$$

Definition 2.26. *A left C^*-module $(M, {}^M\mu)$ is called* rational *when there exists a k-linear map $\delta^M : M \to M \otimes C$ such that*

$$\alpha_M \circ \delta^M = \beta_M.$$

We will denote by $Rat({}_{C^*}\mathcal{M})$ the full subcategory of ${}_{C^*}\mathcal{M}$ whose objects are exactly the rational modules.

Remark 2.27. *Note that if $\delta, \delta' : M \to M \otimes C$ satisfy $\alpha_M \circ \delta = \beta_M = \alpha_M \circ \delta'$, then, since α_M is injective, we get $\delta = \delta'$. Thus we will write $(M, {}^M\mu, \delta^M) \in Rat({}_{C^*}\mathcal{M})$ to specify the unique map δ^M such that $\alpha_M \circ \delta^M = \beta_M$.*

Proposition 2.28. *Let $(M, {}^M\mu)$ be a left C^*-module. M is rational if and only if for any $x \in M$ there exist $n \in \mathbb{N}$, $n \geq 1$, $y_1, \dots, y_n \in M$ and $c_1, \dots, c_n \in C$ such that*

$$f \cdot x = \sum_{i=1}^{n} y_i f(c_i) \quad \textit{for any } f \in C^*.$$

In this case

$$\delta^M(x) = \sum_{i=1}^{n} y_i \otimes c_i \quad \textit{for any } x \in M.$$

Proof. Assume that M is rational and let $\delta^M : M \to M \otimes C$ such that $\alpha_M \circ \delta^M = \beta_M$. For $x \in M$ let

$$\delta^M(x) = \sum_{i=1}^{n} y_i \otimes c_i \quad \text{where } n \in \mathbb{N}, n \geq 1, y_1, \dots, y_n \in M, c_1, \dots, c_n \in C.$$

Then, for any $f \in C^*$, we have

$$f \cdot x = [\beta_M(x)](f) = \left[\alpha_M\left(\delta^M(x)\right)\right](f) = \alpha_M\left(\sum_{i=1}^{n} y_i \otimes c_i\right)(f) = \sum_{i=1}^{n} y_i f(c_i)$$

Conversely assume that for any $x \in M$ there exist $n \in \mathbb{N}$, $n \geq 1$, $y_1, \ldots, y_n \in M$ and $c_1, \ldots, c_n \in C$ such that $f \cdot x = \sum_{i=1}^{n} y_i f(c_i)$ for any $f \in C^*$. Then, given $x \in M$, for any $f \in C^*$, we have

$$[\beta_M(x)](f) = f \cdot x = \sum_{i=1}^{n} y_i f(c_i) = \alpha_M\left(\sum_{i=1}^{n} y_i \otimes c_i\right)(f)$$

i.e.

$$\beta_M(x) = \alpha_M\left(\sum_{i=1}^{n} y_i \otimes c_i\right).$$

Since α_M is injective, we define a map $\delta^M : M \to M \otimes C$ by setting

$$\delta^M(x) = \sum_{i=1}^{n} y_i \otimes c_i \quad \text{for any } x \in M.$$

Then

$$\alpha_M\left(\delta^M(x)\right) = \alpha_M\left(\sum_{i=1}^{n} y_i \otimes c_i\right) = \beta_M(x)$$

so that $\alpha_M \circ \delta^M = \beta_M$. Since α_M is injective and both α_M and β_M are k-linear, it follows that δ^M is k-linear too. □

Lemma 2.29. *Using the assumptions and notations of Proposition 2.22, for every* $\left(M, \rho^M\right) \in \mathcal{M}^C$ *we have that* $\left(M, \mu_{\rho^M}, \rho^M\right)$ *is a rational module. Therefore* $\operatorname{Im}(H) \subseteq$ *Rat* $({}_{C^*}\mathcal{M})$.

Proof. Let $\left(M, \rho^M\right)$ be a right C-comodule and let us consider the associated left C^*-module $\left(\left(M, \mu_{\rho^M}\right)\right)$. Then, for every $x \in M$ and $f \in C^*$, we compute

$$\left[\left(\alpha_M \circ \rho^M\right)(x)\right](f) = \left[\alpha_M\left(\sum x_0 \otimes x_1\right)\right](f) = \sum x_0 f(x_1) = f \cdot x = [\beta_M(x)](f).$$

Therefore we deduce that

$$\alpha_M \circ \rho^M = \beta_M.$$

□

Theorem 2.30. *The assignment* $\left(M, \rho^M\right) \mapsto \left(M, \mu_{\rho^M}, \rho^M\right)$ *gives rise to a category isomorphism* $\Gamma : \mathcal{M}^C \to$ *Rat* $({}_{C^*}\mathcal{M})$.

Proof. In view of Lemma 2.29, the image of the functor $H : \mathcal{M}^C \to {}_{C^*}\mathcal{M}$ in Proposition 2.22 is contained in *Rat* $({}_{C^*}\mathcal{M})$ and hence we can consider the functor $\Gamma = H^{|Rat({}_{C^*}\mathcal{M})}$.

Now assume that $\left(M, {}^M\mu, \delta^M\right)$ is rational. For $x \in M$, let

$$\delta^M(x) = \sum_{i=1}^{n} y_i \otimes c_i \quad \text{where } n \in \mathbb{N}, n \geq 1, y_1, \ldots, y_n \in M, c_1, \ldots, c_n \in C.$$

Then, for any $f \in C^*$ we have

$$\mu_{\delta^M}(f \otimes x) = \left[r_M \circ (M \otimes ev) \circ \left(\delta^M \otimes C^*\right) \circ \tau_{C^*,M}\right](f \otimes x) =$$
$$[r_M \circ (M \otimes ev)]\left(\left(\sum_{i=1}^n y_i \otimes c_i\right) \otimes f\right) = \sum_{i=1}^n y_i f(c_i) = \alpha_M\left(\sum_{i=1}^n y_i \otimes c_i\right)(f) =$$
$$= \left[\alpha_M\left(\delta^M(x)\right)\right](f) = [\beta_M(x)](f) = f \cdot x = {}^M\mu(f \otimes x)$$

Thus

$$\mu_{\delta^M} = {}^M\mu \tag{2.14}$$

and hence, by Theorem 2.21, we deduce that $\left(M, \delta^M\right)$ is a right C-comodule.

Now we want to prove that the assignment $\left(M, {}^M\mu, \delta^M\right) \mapsto \left(M, \delta^M\right)$ gives rise to a functor $\Lambda : Rat\left({}_{C^*}\mathcal{M}\right) \to \mathcal{M}^C$. Thus let $\left(M, {}^M\mu, \delta^M\right)$ and $\left(M', {}^{M'}\mu, \delta^{M'}\right)$ be rational modules and let $\gamma : M \to M'$ be a morphism of left C^*-modules. We will prove that $\gamma : \left(M, \delta^M\right) \to \left(M', \delta^{M'}\right)$ is a morphism of comodules. For any $t \in M, c \in C, f \in C^*$ we have

$$[\alpha_M((t \otimes c))](f) = tf(c)$$

so that

$$\begin{aligned} \{[\alpha_{M'} \circ (\gamma \otimes C)](t \otimes c)\}(f) &= [\alpha_{M'}(\gamma(t) \otimes c)](f) = \gamma(t) f(c) \\ &= \gamma(tf(c)) = \gamma[\alpha_M(t \otimes c)(f)] \end{aligned}$$

and hence

$$\{[\alpha_{M'} \circ (\gamma \otimes C)](t \otimes c)\}(f) = \gamma[\alpha_M(t \otimes c)(f)]. \tag{2.15}$$

Now, for every $x \in M$ and $f \in C^*$, we have

$$\begin{aligned} \left\{\left[\alpha_{M'} \circ (\gamma \otimes C) \circ \delta^M\right](x)\right\}(f) &= \left\{[\alpha_{M'} \circ (\gamma \otimes C)]\left(\delta^M(x)\right)\right\}(f) \\ \overset{(2.15)}{=} \gamma\left[\alpha_M\left(\delta^M(x)\right)(f)\right] &= \gamma[\beta_M(x)(f)] = \gamma(f \cdot x) \end{aligned}$$

and

$$\left\{\left[\alpha_{M'} \circ \delta^{M'} \circ \gamma\right](x)\right\}(f) = (\beta_{M'}(\gamma(x)))(f) = f \cdot \gamma(x).$$

Since γ is a morphism of left C^*-modules, for every $x \in M$ and $f \in C^*$, we obtain that

$$\left\{\left[\alpha_{M'} \circ (\gamma \otimes C) \circ \delta^M\right](x)\right\}(f) = \left\{\left[\alpha_{M'} \circ \delta^{M'} \circ \gamma\right](x)\right\}(f)$$

and hence

$$\alpha_{M'} \circ (\gamma \otimes C) \circ \delta^M = \alpha_{M'} \circ \delta^{M'} \circ \gamma.$$

Since $\alpha_{M'}$ is injective we get

$$(\gamma \otimes C) \circ \delta^M = \delta^{M'} \circ \gamma.$$

Hence we obtain a functor $\Lambda : Rat\left({}_{C^*}\mathcal{M}\right) \to \mathcal{M}^C$ such that

$$\Lambda\left(M, {}^M\mu, \delta^M\right) = \left(M, \delta^M\right) \quad \text{and } \Lambda\left(f\right) = f \text{ for any morphism } f \text{ in } {}_{C^*}\mathcal{M}.$$

Let us prove that the functors Γ and Λ give rise to an isomorphism of categories between $\mathcal{M}^C$ and $Rat\left({}_{C^*}\mathcal{M}\right)$.

Let $\left(M, \rho^M\right) \in \mathcal{M}^C$. Then $\Gamma\left(M, \rho^M\right) = \left(M, \mu_{\rho^M}, \rho^M\right)$ and hence $\Lambda\left(\Gamma\left(M, \rho^M\right)\right) = \left(M, \rho^M\right)$. Conversely, let $\left(M, {}^M\mu, \delta^M\right) \in Rat\left({}_{C^*}\mathcal{M}\right)$. Then $\Lambda\left(M, {}^M\mu, \delta^M\right) = \left(M, \delta^M\right)$ and hence $\Gamma\left(\Lambda\left(M, {}^M\mu, \delta^M\right)\right) = \Gamma\left(M, \delta^M\right) = \left(M, \mu_{\delta^M}, \delta^M\right) \overset{2.14}{=} \left(M, {}^M\mu, \delta^M\right)$. □

Exercise 2.31. *Let C be a coalgebra and let $f : M \to N$ be an isomorphism in ${}_{C^*}\mathcal{M}$. Show that, if M is rational, also N is rational.*

Theorem 2.32. *Let C be a coalgebra. The full subcategory $Rat\left({}_{C^*}\mathcal{M}\right)$ of ${}_{C^*}\mathcal{M}$ is closed under submodules, quotients and direct sums.*

Proof. Let $\left(M, {}^M\mu, \delta^M\right) \in Rat\left({}_{C^*}\mathcal{M}\right)$ and let L be a C^*-submodule of M.

Since M is rational, by Proposition 2.28, for every $l \in L$, there exist $n \in \mathbb{N}$, $n \geq 1$, $y_1, \ldots, y_n \in M$ and $c_1, \ldots, c_n \in C$ such that

$$f \cdot l = \sum_{i=1}^{n} y_i f\left(c_i\right) \quad \text{for any } f \in C^*.$$

We can assume $c_1, \ldots, c_n$ linearly independent and denote by c_j^* the elements of C^* defined by $c_j^*\left(c_i\right) = \delta_{i,j}$. Then we obtain

$$L \ni c_j^* \cdot l = y_j \quad \text{for every } j = 1_1, \ldots, n.$$

Hence, by Proposition 2.28, we conclude that L is rational with $\delta^L = \left(\left(\delta^M\right)_{|L}\right)^{|L\otimes C}$.

Now we apply again Proposition 2.28 to get that, for every $x \in M$, there exist $n \in \mathbb{N}$, $n \geq 1$, $y_1, \ldots, y_n \in M$ and $c_1, \ldots, c_n \in C$ such that

$$f \cdot x = \sum_{i=1}^{n} y_i f\left(c_i\right) \quad \text{for any } f \in C^*.$$

Then

$$f \cdot (x + L) = (f \cdot x) + L = \left(\sum_{i=1}^{n} y_i f\left(c_i\right)\right) + L = \sum_{i=1}^{n} \left(y_i + L\right) f\left(c_i\right)$$

and hence, using one more time Proposition 2.28, we conclude that M/L is rational.

Let now $\left(M_i, {}^{M_i}\mu, \delta^{M_i}\right)_{i\in I}$ be a family in $Rat\left({}_{C^*}\mathcal{M}\right)$. Let

$$\psi : \bigoplus_{i\in I} \left(M_i \otimes C\right) \to \left(\bigoplus_{i\in I} M_i\right) \otimes C$$

be the natural isomorphism, i.e. for every $t_i \in M_i$ and $c_i \in C$ we have

$$\psi\left((t_i \otimes c_i)_{i\in I}\right) = \sum_{i\in I} \varepsilon_i(t_i) \otimes c_i = \sum_{i\in I} (\varepsilon_i \otimes C)(t_i \otimes c_i).$$

Set

$$\delta^{\oplus_{i\in I} M_i} = \psi \circ \left(\oplus_{i\in I} \delta^{M_i}\right).$$

Then, for every $(x_i)_{i\in I} \in \bigoplus_{i\in I} M_i$, we get

$$\left[\psi \circ \left(\oplus_{i\in I}\delta^{M_i}\right)\right](x_i)_{i\in I} = \psi\left((\delta_{M_i}(x_i))_{i\in I}\right) = \sum_{i\in I} (\varepsilon_i \otimes C)(\delta_{M_i}(x_i)).$$

Let $(x_i)_{i\in I} \in \bigoplus_{i\in I} M_i, c \in C, f \in C^*$ and let us compute

$$\begin{aligned}\left[\left(\alpha_{\oplus_{i\in I}M_i}\right) \circ (\varepsilon_i \otimes C)(x_i \otimes c)\right](f) &= \left[\left(\alpha_{\oplus_{i\in I}M_i}\right)(\varepsilon_i(x_i) \otimes c)\right](f) = \varepsilon_i(x_i) f(c) \\ &= \varepsilon_i(x_i f(c)) = \varepsilon_i\left[\alpha_{M_i}(x_i \otimes c)(f)\right].\end{aligned}$$

Hence we deduce that

$$\begin{aligned}&\left[\left(\alpha_{\oplus_{i\in I}M_i} \circ \delta^{\oplus_{i\in I}M_i}\right)(x_i)_{i\in I}\right](f) = \left[\left(\alpha_{\oplus_{i\in I}M_i} \circ \psi \circ \left(\oplus_{i\in I}\delta^{M_i}\right)\right)(x_i)_{i\in I}\right](f) \\ &= \left[\alpha_{\oplus_{i\in I}M_i}\left(\sum_{i\in I}(\varepsilon_i \otimes C)(\delta_{M_i}(x_i))\right)\right](f) = \sum_{i\in I} \varepsilon_i\left\{\left[\alpha_{M_i}(\delta_{M_i}(x_i))\right](f)\right\} \\ &= \sum_{i\in I} \varepsilon_i\left[(\beta_{M_i}(x_i))(f)\right] = \sum_{i\in I} \varepsilon_i(f \cdot x_i) = f \cdot (x_i)_{i\in I} = \left[\beta_{\oplus_{i\in I}M_i}\left((x_i)_{i\in I}\right)\right](f)\end{aligned}$$

i.e.

$$\alpha_{\oplus_{i\in I}M_i} \circ \delta^{\oplus_{i\in I}M_i} = \beta_{\oplus_{i\in I}M_i}.$$

□

Theorem 2.33. *Let $\left(M, \rho^M\right)$ be a right C-comodule and let $x \in M$. Then C^*x is the minimal subcomodule of M containing x. Moreover $\dim_k(C^*x) < \infty$.*

Proof. The first assertion follows from Theorem 2.32 and Theorem 2.30. Let $x \in M$ and write $\rho(x) = \sum_{i=1,\ldots,n} y_i \otimes c_i$. Then

$$f \cdot x = \sum_{i=1}^{n} y_i f(c_i) \in \sum_{i=1}^{n} k y_i \quad \text{for any } f \in C^*$$

so that

$$C^*x \leq \sum_{i=1}^{n} k y_i.$$

□

Theorem 2.34. *Let* $\left(M, {}^{M}\mu\right) \in {}_{C^*}\mathcal{M}$ *and let* $Rat\,(M) = \{L \leq_{C^*} M \mid L \in Rat\,({}_{C^*}\mathcal{M})\}$. *Set*

$$rat\,(M) = \sum_{L \in Rat(M)} L. \tag{2.16}$$

Then $rat\,(M) \in {}_{C^*}\mathcal{M}$ *and it is the maximal submodule of* M *which is a rational module. Moreover if* $f : M \to M'$ *is a morphism in* ${}_{C^*}\mathcal{M}$, *then*

1) $f\,(rat\,(M)) \subseteq rat\,(M')$,

2) $Ker\left(f_{|rat(M)}\right) = rat\,(\mathrm{Ker}\,(f))$.

Proof. For every $L \in Rat\,(M)$, let $i_L : L \to M$ be the canonical inclusion and let Φ be the codiagonal morphism of the family $(i_L)_{L \in Rat(M)}$:

$$\Phi : \bigoplus_{L \in Rat(M)} L \to M.$$

Then $\mathrm{Im}\,(\Phi) = \sum_{L \in Rat(M)} L$ and, in view of Theorem 2.32, we obtain that $\mathrm{Im}\,(\Phi) \in Rat\,({}_{C^*}\mathcal{M})$.

Let now $f : M \to M'$ be a morphism in ${}_{C^*}\mathcal{M}$. Then $f\,(rat\,(M))$ is a quotient of $rat\,(M)$ and hence, by Theorem 2.32, $f\,(rat\,(M)) \in Rat\,(M')$. Moreover, by the same Theorem we have that any C^*-submodule of $rat\,(M)$ is rational so that $Ker(f) \cap rat\,(M) \subseteq rat\,(\mathrm{Ker}\,(f))$ and we get

$$\mathrm{Ker}\left(f_{|rat(M)}\right) = \mathrm{Ker}\,(f) \cap rat\,(M) \subseteq rat\,(\mathrm{Ker}\,(f)) \subseteq \mathrm{Ker}\,(f) \cap rat\,(M)\,.$$

□

Proposition 2.35. *Let* $\left(M, {}^{M}\mu\right) \in {}_{C^*}\mathcal{M}$. *Then*

$$rat\,(M) = \beta_M^{\leftarrow}\left(\alpha_M\,(M \otimes C)\right)$$

Proof. Let L be a C^*-submodule of M and assume that $\left(L, \mu^L, \delta^L\right) \in Rat\,({}_{C^*}\mathcal{M})$. Then $\beta_L = \alpha_L \circ \delta_L$. Let $i_L : L \to M$ be the canonical inclusion. Then

$$\mathrm{Hom}\,(C^*, i_L) \circ \alpha_L = \alpha_M \circ (i_L \otimes C)\,. \tag{2.17}$$

and

$$\beta_M \circ i_L = \mathrm{Hom}\,(C^*, i_L) \circ \beta_L \tag{2.18}$$

Hence

$$\beta_M \circ i_L \overset{(2.18)}{=} \mathrm{Hom}\,(C^*, i_L) \circ \beta_L = \mathrm{Hom}\,(C^*, i_L) \circ \alpha_L \circ \delta_L \overset{(2.17)}{=} \alpha_M \circ (i_L \otimes C) \circ \delta_L$$

so that

$$\beta_M \circ i_L = \alpha_M \circ (i_L \otimes C) \circ \delta_L \tag{2.19}$$

$$\beta_M (L) = \beta_M \circ i_L (L) = [\alpha_M \circ (i_L \otimes C) \circ \delta_L] (L) \subseteq \alpha_M (M \otimes C)$$

and hence

$$L \subseteq \beta_M^{\leftarrow} \beta_M (L) \subseteq \beta_M^{\leftarrow} (\alpha_M (M \otimes C)).$$

Conversely, let us prove that $X = \beta_M^{\leftarrow} (\alpha_M (M \otimes C))$ is a rational H^*-submodule of M. Let $g \in C^*$ and let $x \in X$. Then there exist $n \in \mathbb{N}, n \geq 1$, $x_1, \ldots, x_n \in M$ and $c_1, \ldots, c_n \in C$ such that

$$\beta_M (x) = \alpha_M \left(\sum_{i=1}^{n} x_i \otimes c_i \right).$$

i.e.

$$[\beta_M (x)] (f) = \left[\alpha_M \left(\sum_{i=1}^{n} x_i \otimes c_i \right) \right] (f) \quad \text{for every } f \in C^*$$

so that

$$f \cdot x = [\beta_M (x)] (f) = \left[\alpha_M \left(\sum_{i=1}^{n} x_i \otimes c_i \right) \right] (f) = \sum_{i=1}^{n} x_i f (c_i) \quad \text{for every } f \in C^*$$

and hence we get

$$f \cdot x = \sum_{i=1}^{n} x_i f (c_i) \quad \text{for every } f \in C^*. \tag{2.20}$$

Thus we have

$$[\beta_M (gx)] (f) = f (gx) = (f * g) x \overset{(2.20)}{=} \sum_{i=1}^{n} x_i [(f * g) (c_i)]$$

$$= \sum_{i=1}^{n} \sum x_i f [(c_i)_1] g [(c_i)_2] = \left[\alpha_M \left(\sum_{i=1}^{n} x_i \otimes c_{i_1} g [(c_i)_2] \right) \right] (f)$$

i.e.

$$[\beta_M (gx)] (f) = \left[\alpha_M \left(\sum_{i=1}^{n} x_i \otimes c_{i_1} g [(c_i)_2] \right) \right] (f) \quad \text{for every } f \in C^*$$

which means that

$$\beta_M (gx) = \alpha_M \left(\sum_{i=1}^{n} x_i \otimes c_{i_1} g [(c_i)_2] \right) \in \alpha_M (M \otimes C)$$

and hence we get that $gx \in X$. Therefore X is a left C^*-submodule of M.

Thus we can apply to the left C^*-module X Proposition 2.28. Since, for any $x \in X$ there exist $n \in \mathbb{N}, n \geq 1$, $x_1, \ldots, x_n \in M$ and $c_1, \ldots, c_n \in C$ such that (2.20) holds, we conclude, in view of Proposition 2.28, that X is rational. □

Theorem 2.36. *Let (C,Δ,ε) be a finite dimensional coalgebra. Then $Rat\left({}_{C^*}\mathcal{M}\right)={}_{C^*}\mathcal{M}$*

Proof. In view of Theorem 2.34, we have only to prove that $C^*\in Rat\left({}_{C^*}\mathcal{M}\right)$.

Let $n\in\mathbb{N},n\geq 1$ and let $e_1,\ldots e_n$ be a basis of C. Let $e_1^*,\ldots e_n^*$ the corresponding dual basis. Then, for every $f\in C^*$

$$f=\sum_{i=1}^{n}e_i^*f\left(e_i\right)$$

and hence, given $\gamma\in C^*$, for every $f\in C^*$ we have

$$f\cdot\gamma=\left(\sum_{i=1}^{n}e_i^*f\left(e_i\right)\right)\cdot\gamma=\sum_{i=1}^{n}\left(e_i^*\cdot\gamma\right)f\left(e_i\right).$$

Since $e_i^*\cdot\gamma\in C^*$ for every $i=1\ldots n$, in view of 2.28, we conclude. □

Definition 2.37. *Let R be a ring and let $M\in{}_R\mathcal{M}$. The* Wisbauer category *$\sigma\left[M\right]$ is the smallest full subcategory of ${}_R\mathcal{M}$ which contains M and is closed under submodules, quotients and direct sums.*

PROPOSAL FOR A DEEPER UNDERSTANDING: Introduce the concept of Grothendieck category. Prove that $Rat\left({}_{C^*}\mathcal{M}\right)$ is a Grothendieck category and that

$$Rat\left({}_{C^*}\mathcal{M}\right)=\sigma\left({}_{C^*}C\right).$$

Notation 2.38. *We will denote by Vec_k the category of k-vector spaces i.e. of symmetric k-bimodules.*

Proposition 2.39. *Let (C,Δ,ε) be a coalgebra, let $V\in Vec_k$ and $\left(M,\rho^M\right)\in\mathcal{M}^C$. Then the assignments $V\mapsto\left(V\otimes M,V\otimes\rho^M\right)$ and $f\mapsto f\otimes M$ define a functor $F_M:Vec_k\to\mathcal{M}^C$.*

Proof. We compute

$$\begin{aligned}\left(V\otimes M\otimes\Delta\right)\circ\left(V\otimes\rho^M\right) &= V\otimes\left[\left(M\otimes\Delta\right)\circ\rho^M\right]=V\otimes\left[\left(\rho^M\otimes C\right)\circ\rho^M\right]\\ &= \left(\left(V\otimes\rho^M\right)\otimes C\right)\circ\left(V\otimes\rho^M\right)\end{aligned}$$

and

$$\begin{aligned}&r_{V\otimes M}\circ\left(V\otimes M\otimes\varepsilon\right)\circ\left(V\otimes\rho^M\right)\overset{(1.2)}{=}\left(V\otimes r_M\right)\circ\left(V\otimes M\otimes\varepsilon\right)\circ\left(V\otimes\rho^M\right)\\ =\;&\left(V\otimes\left[r_M\circ\left(M\otimes\varepsilon\right)\circ\left(\rho^M\right)\right]\right)=V\otimes M.\end{aligned}$$

Moreover, for any k-linear map $f:V\to V'$ we have

$$\left(f\otimes M\otimes C\right)\circ\left(V\otimes\rho^M\right)=\left(f\otimes\rho^M\right)=\left(V'\otimes\rho^M\right)\circ\left(f\otimes M\right).$$

□

Theorem 2.40. *Let (C, Δ, ε) be a coalgebra. The functor $F = F_C : Vec_k \to \mathcal{M}^C$ is a right adjoint of the forgetful functor $U : \mathcal{M}^C \to Vec_k$.*

Proof. Let $\left(M, \rho^M\right) \in \mathcal{M}^C$. Then $\rho^M : \left(M, \rho^M\right) \to (M \otimes C, M \otimes \Delta)$ is a comodule morphism. Indeed we have

$$\left(\rho^M \otimes C\right) = (M \otimes \Delta) \circ \rho^M.$$

Let us check that the family $\left(\rho^M\right)_{M \in \mathcal{M}^C}$ gives rise to a functorial morphism

$$\rho : \mathrm{Id}_{\mathcal{M}^C} \to FU.$$

Let $f : M \to M'$ be a right comodule morphism. This means that

$$(f \otimes C) \circ \rho^M = \rho^{M'} \circ f$$

and this is what is needed for ρ to be a functorial morphsim.

Let now V be a vector space and set

$$\epsilon_V = r_V \circ (V \otimes \varepsilon) : V \otimes C \to V.$$

Let us check that the family $(\epsilon_V)_{V \in Vec_k}$ gives rise to a functorial morphism

$$\epsilon : UF \to \mathrm{Id}_{Vec_k}.$$

In fact, given a k-linear map $h : V \to V'$, we have

$$\begin{aligned} h \circ \epsilon_V &= h \circ r_V \circ (V \otimes \varepsilon) \overset{(1.1)}{=} r_{V'} \circ (h \otimes k) \circ (V \otimes \varepsilon) = r_{V'} \circ (h \otimes \varepsilon) \\ &= r_{V'} \circ (V' \otimes \varepsilon) \circ (h \otimes C) = \epsilon_{V'} \circ (h \otimes C). \end{aligned}$$

Let us prove that ρ and ϵ fulfill the requirements for being the unit, resp. the counit, for an adjunction (U, F). Thus let $\left(M, \rho^M\right) \in \mathcal{M}^C$, let $V \in Vec_k$ and let us compute

$$\epsilon_{U(M,\rho^M)} \circ U\left(\rho^M\right) = r_M \circ (M \otimes \varepsilon) \circ \rho^M = \mathrm{Id}_M = \mathrm{Id}_{U(M,\rho^M)}$$

and

$$\begin{aligned} F(\epsilon_V) \circ \rho_{F(V)} &= F(r_V \circ (V \otimes \varepsilon)) \circ (V \otimes \Delta) = (r_V \otimes C) \circ (V \otimes \varepsilon \otimes C) \circ (V \otimes \Delta) \\ &\overset{(1.2)}{=} (V \otimes l_C) \circ (V \otimes \varepsilon \otimes C) \circ (V \otimes \Delta) = \mathrm{Id}_{V \otimes C}. \end{aligned}$$

□

Corollary 2.41. *For any k-vector space V, $F(V)$ is an injective object in $\mathcal{M}^C$.*

Proof. In view of Theorem 2.40, the functor $\mathrm{Hom}_{\mathcal{M}^C}(-, F(V))$ is isomorphic to the functor $\mathrm{Hom}(U(-), V)$. Since U and $\mathrm{Hom}(-, V)$ are exact functors, we conclude. □

Proposition 2.42. *Let (C, Δ, ε) be a coalgebra. Then (C, Δ) is an injective cogenerator of $\mathcal{M}^C$.*

Proof. By Corollary 2.41, we have that $F(k)$ is an injective object in $\mathcal{M}^C$. Now $l_C : F(k) = (k \otimes C, k \otimes \Delta) \to (C, \Delta)$ is colinear. In fact, by (1.1), we have

$$(l_C \otimes C) \circ (k \otimes \Delta) = l_{C \otimes C} \circ (k \otimes \Delta) = \Delta \circ l_C.$$

Thus l_C is an isomorphism in $\mathcal{M}^C$ and hence C is an injective object in $\mathcal{M}^C$. Let now $\left(M, \rho^M\right) \in \mathcal{M}^C$ and let $\lambda : M \to k^{(X)}$ be an isomorphism of vector spaces. It is easy to check that the usual isomorphism $\psi : k^{(X)} \otimes C \to C^{(X)}$ is a colinear map from $F\left(k^{(X)}\right)$ into $(C, \Delta)^{(X)}$. Since $\rho_M : \left(M, \rho^M\right) \to FU(M) \simeq F\left(k^{(X)}\right)$ is an injective colinear map, we conclude. □

Definitions 2.43. *Let $(C, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$ be coalgebras. A C-D-bicomodule is a triple $\left(M, {}^M\rho, \rho^M\right)$ such that $\left(M, {}^M\rho\right) \in {}^C\mathcal{M}$, $\left(M, \rho^M\right) \in \mathcal{M}^D$ and*

$$\left({}^M\rho \otimes D\right) \circ \rho^M = \left(C \otimes \rho^M\right) \circ {}^M\rho. \tag{2.21}$$

A k-linear map $f : M \to M'$ between two C-D-bicomodules is called a morphism of C-D-bicomodules if it is both left C-colinear and right D-colinear. The category of C-D-bicomodules we will denoted by ${}^C\mathcal{M}^D$

Proposition 2.44. *Let $(C, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$ be coalgebras and let $\left(M, {}^M\rho\right) \in {}^C\mathcal{M}$ and $\left(N, \rho^N\right) \in \mathcal{M}^D$. Then $\left(M \otimes N, {}^M\rho \otimes N, M \otimes \rho^N\right) \in {}^C\mathcal{M}^D$.*

Proof. By Proposition 2.39, $\left(M \otimes N, {}^M\rho \otimes N\right) \in {}^C\mathcal{M}$ and $\left(M \otimes N, M \otimes \rho^N\right) \in \mathcal{M}^D$. Since we also have

$$\left({}^M\rho \otimes N \otimes D\right) \circ \left(M \otimes \rho^N\right) = \left({}^M\rho \otimes \rho^N\right) = \left(C \otimes M \otimes \rho^N\right) \circ \left({}^M\rho \otimes N\right),$$

we conclude. □

Remark 2.45. *From the foregoing, we deduce that* (2.21) *can be read both as*

- $\rho^M : \left(M, {}^M\rho\right) \to \left(M \otimes D, {}^M\rho \otimes D\right)$ is a morphism in ${}^C\mathcal{M}$ (and hence in ${}^C\mathcal{M}^D$) or
- ${}^M\rho : \left(M, \rho^M\right) \to \left(C \otimes M, C \otimes \rho^M\right)$ is a morphism in $\mathcal{M}^D$ (and hence in ${}^C\mathcal{M}^D$).

Remark 2.46. *Let $D = \left(k, \Delta_k = r_k^{-1}, \varepsilon_k = \mathrm{Id}_k\right)$. Then ${}^C\mathcal{M}^D = {}^C\mathcal{M}$.*

Definition 2.47. *Let $(C, \Delta_C, \varepsilon_C)$, $(D, \Delta_D, \varepsilon_D)$, $(E, \Delta_E, \varepsilon_E)$ be coalgebras and let $\left(M, {}^M\rho, \rho^M\right) \in {}^D\mathcal{M}^C$ and $\left(N, {}^N\rho, \rho^N\right) \in {}^C\mathcal{M}^E$. The* cotensor product *of the comodules M and N is the k-subspace $M \square_C N$ of $M \otimes N$ defined by setting*

$$M \square_C N = \mathrm{Ker}\left(\rho^M \otimes N - M \otimes {}^N\rho\right).$$

Lemma 2.48. *Let $L, M, N \in Vec_k$ and assume that $L \leq M$. Let $i_L : L \to M$ and $i_{L\otimes N} : L \otimes N \to M \otimes N$ be the canonical inclusions. Then*

$$i_{L\otimes N} = i_L \otimes N.$$

Proof. Since the functor $\otimes N$ is left exact we get that

$$i_L \otimes N : L \otimes N \to M \otimes N$$

is injective and hence it coincides with the canonical inclusion $i_{L\otimes N}$. $\square$

Proposition 2.49. *Let $(C, \Delta_C, \varepsilon_C)$, $(D, \Delta_D, \varepsilon_D)$ and $(E, \Delta_E, \varepsilon_E)$ be coalgebras. The assignment $(M, N) \mapsto M\square_C N$ defines a left exact functor*

$$\square_C : {}^D\mathcal{M}^C \times {}^C\mathcal{M}^E \to {}^D\mathcal{M}^E.$$

Proof. Since $\otimes E$ is an exact functor, we have that $(M\square_C N)\otimes E = Ker\big(\rho^M \otimes N \otimes E - M \otimes {}^N\rho \otimes E\big)$. Since $N \in {}^C\mathcal{M}^E$, we have $\big({}^N\rho \otimes E\big) \circ \rho^N = \big(C \otimes \rho^N\big) \circ {}^N\rho$. From this, it follows that

$$\begin{aligned}&\big(\rho^M \otimes N \otimes E - M \otimes {}^N\rho \otimes E\big) \circ \big(M \otimes \rho^N\big) = \big(\rho^M \otimes \rho^N - M \otimes \big[\big({}^N\rho \otimes E\big) \circ \rho^N\big]\big)\\ &= \big(\rho^M \otimes \rho^N - M \otimes \big[\big(C\otimes\rho^N\big) \circ {}^N\rho\big]\big) = \big(M \otimes C \otimes \rho^N\big) \circ \big(\rho^M \otimes N - M \otimes {}^N\rho\big).\end{aligned}$$

Therefore

$$\begin{aligned}&\big(\rho^M \otimes N \otimes E - M \otimes {}^N\rho \otimes E\big) \circ \big(M \otimes \rho^N\big) \circ i_{M\square_C N} =\\ &= \big(M \otimes C \otimes \rho^N\big) \circ \big(\rho^M \otimes N - M \otimes {}^N\rho\big) \circ i_{M\square_C N} = 0\end{aligned}$$

and hence

$$\big(\rho^M \otimes N \otimes E - M \otimes {}^N\rho \otimes E\big) \circ \big(M \otimes \rho^N\big) \circ i_{M\square_C N} = 0.$$

Hence there exists a unique map $\rho^{M\square_C N} : M\square_C N \to (M\square_C N) \otimes E$ such that

$$(i_{M\square_C N} \otimes E) \circ \rho^{M\square_C N} = \big(i_{(M\square_C N)\otimes E}\big) \circ \rho^{M\square_C N} = \big(M \otimes \rho^N\big) \circ i_{M\square_C N} \tag{2.22}$$

where $i_{M\square_C N}$ and $i_{(M\square_C N)\otimes E}$ denote the obvious canonical inclusions. Then we compute

$$\begin{aligned}(i_{M\square_C N} \otimes E \otimes E) \circ \big(\rho^{M\square_C N} \otimes E\big) \circ \rho^{M\square_C N} &= \big(\big([i_{M\square_C N} \otimes E] \circ \rho^{M\square_C N}\big) \otimes E\big) \circ \rho^{M\square_C N}\\ &= \big(\big[\big(M \otimes \rho^N\big) \circ i_{M\square_C N}\big] \otimes E\big) \circ \rho^{M\square_C N} =\\ &= \big(\big(M \otimes \rho^N\big) \otimes E\big) \circ (i_{M\square_C N} \otimes E) \circ \rho^{M\square_C N} =\\ &= \big(\big(M \otimes \rho^N\big) \otimes E\big) \circ \big(M \otimes \rho^N\big) \circ i_{M\square_C N}\\ &= \big(M \otimes \big(\rho^N \otimes E\big) \circ \rho^N\big) \circ i_{M\square_C N}\\ &= \big(M \otimes (N \otimes \Delta_E) \circ \rho^N\big) \circ i_{M\square_C N}\\ &= (M \otimes N \otimes \Delta_E) \circ \big(M \otimes \rho^N\big) \circ i_{M\square_C N}\\ &= (M \otimes N \otimes \Delta_E) \circ (i_{M\square_C N} \otimes E) \circ \rho^{M\square_C N}\\ &= (i_{M\square_C N} \otimes \Delta_E) \circ \rho^{M\square_C N}\\ &= (i_{M\square_C N} \otimes E \otimes E) \circ (M\square_C N \otimes \Delta_E) \circ \rho^{M\square_C N}.\end{aligned}$$

Since $i_{M\square_C N} \otimes E \otimes E$ is injective, we conclude that $\rho^{M\square_C N}$ is coassociative.

Let us compute

$$\begin{aligned}
i_{(M\square_C N)} \circ r_{(M\square_C N)} \circ (M\square_C N \otimes \varepsilon_E) \circ \rho^{M\square_C N} &\overset{(1.1)}{=} \\
= r_{M\otimes N} \circ (i_{M\square_C N} \otimes k) \circ (M\square_C N \otimes \varepsilon_E) \circ \rho^{M\square_C N} \\
= r_{M\otimes N} \circ (i_{M\square_C N} \otimes \varepsilon_E) \circ \rho^{M\square_C N} \\
= r_{M\otimes N} \circ (M \otimes N \otimes \varepsilon_E) \circ (i_{M\square_C N} \otimes E) \circ \rho^{M\square_C N} \\
= r_{M\otimes N} \circ (M \otimes N \otimes \varepsilon_E) \circ \left(M \otimes {}^N\rho\right) \circ i_{M\square_C N} \\
= r_{M\otimes N} \circ \left(M \otimes \left[(N \otimes \varepsilon_E) \circ {}^N\rho\right]\right) \circ i_{M\square_C N} \\
\overset{(1.2)}{=} (M \otimes r_N) \left(M \otimes \left[(N \otimes \varepsilon_E) \circ {}^N\rho\right]\right) \circ i_{M\square_C N} \\
= \left(M \otimes \left[r_N \circ (N \otimes \varepsilon_E) \circ {}^N\rho\right]\right) \circ i_{M\square_C N} \\
= (M \otimes N) \circ i_{M\square_C N} = i_{M\square_C N}.
\end{aligned}$$

Since $i_{M\square_C N}$ is injective, we conclude that $\left(M\square_C N, \rho^{M\square_C N}\right) \in \mathcal{M}^E$. An analogous procedure endows $M\square_C N$ with a left D-comodule structure uniquely defined by

$$(D \otimes i_{M\square_C N}) \circ {}^{M\square_C N}\rho = \left(i_{D\otimes(M\square_C N)}\right) \circ {}^{M\square_C N}\rho = \left({}^M\rho \otimes N\right) \circ i_{M\square_C N}.$$

Let us prove that $\left(M\square_C N, {}^{M\square_C N}\rho, \rho^{M\square_C N}\right) \in {}^D\mathcal{M}^E$ i.e. that

$$\left({}^{M\square_C N}\rho \otimes E\right) \circ \rho^{M\square_C N} = \left(D \otimes \rho^{M\square_C N}\right) \circ {}^{M\square_C N}\rho.$$

Let us compute

$$\begin{aligned}
(D \otimes i_{M\square_C N} \otimes E) \circ \left({}^{M\square_C N}\rho \otimes E\right) \circ \rho^{M\square_C N} &= \left({}^M\rho \otimes N \otimes E\right) \circ (i_{M\square_C N} \otimes E) \circ \rho^{M\square_C N} \\
&= \left({}^M\rho \otimes N \otimes E\right) \circ \left(M \otimes \rho^N\right) \circ i_{M\square_C N} \\
&= \left({}^M\rho \otimes \rho^N\right) \circ i_{M\square_C N} \\
&= \left(D \otimes M \otimes \rho^N\right) \circ \left({}^M\rho \otimes N\right) \circ i_{M\square_C N} \\
&= \left(D \otimes M \otimes \rho^N\right) \circ (D \otimes i_{M\square_C N}) \circ {}^{M\square_C N}\rho \\
&= \left(D \otimes \left[\left(M \otimes \rho^N\right) \circ i_{M\square_C N}\right]\right) \circ {}^{M\square_C N}\rho \\
&= \left(D \otimes \left[(i_{M\square_C N} \otimes E) \circ \rho^{M\square_C N}\right]\right) \circ {}^{M\square_C N}\rho \\
&= (D \otimes i_{M\square_C N} \otimes E) \circ \left(D \otimes \rho^{M\square_C N}\right) \circ {}^{M\square_C N}\rho.
\end{aligned}$$

Since $D \otimes i_{M\square_C N} \otimes E$ is injective, we get that $M\square_C N \in {}^D\mathcal{M}^E$.

Let now $f : M \to M'$ and $g : N \to N'$ be morphism in ${}^D\mathcal{M}^C$ and in ${}^C\mathcal{M}^E$ respectively. Let us prove that $\left(\rho^{M'} \otimes N' - M' \otimes {}^{N'}\rho\right) \circ (f \otimes g) \circ i_{M\square_C N} = 0$.

We compute

$$
\begin{aligned}
\left(\rho^{M'} \otimes N'\right) \circ (f \otimes g) \circ i_{M\Box_C N} &= \left(\rho^{M'} \otimes N'\right) \circ (f \otimes N') \circ (M \otimes g) \circ i_{M\Box_C N} \\
&= (f \otimes C \otimes N') \circ \left(\rho^{M} \otimes N'\right) \circ (M \otimes g) \circ i_{M\Box_C N} \\
&= (f \otimes C \otimes N') \circ (M \otimes C \otimes g) \circ \left(\rho^{M} \otimes N\right) \circ i_{M\Box_C N} \\
&= (f \otimes C \otimes N') \circ (M \otimes C \otimes g) \circ \left(M \otimes {}^{N}\rho\right) \circ i_{M\Box_C N} \\
&= (f \otimes C \otimes N') \circ \left(M \otimes \left[(C \otimes g) \circ {}^{N}\rho\right]\right) \circ i_{M\Box_C N} \\
&= (f \otimes C \otimes N') \circ \left(M \otimes \left[{}^{N'}\rho \circ g\right]\right) \circ i_{M\Box_C N} \\
&= \left(M' \otimes {}^{N'}\rho\right) \circ (f \otimes g) \circ i_{M\Box_C N}.
\end{aligned}
$$

Hence $\left(\rho^{M'} \otimes N' - M' \otimes {}^{N'}\rho\right) \circ (f \otimes g) \circ i_{M\Box_C N} = 0$. Therefore there exists a unique map $(f\Box_C g) : M\Box_C N \to M'\Box_C N'$ such that

$$i_{M'\Box_C N'} \circ (f\Box_C g) = (f \otimes g) \circ i_{M\Box_C N}. \tag{2.23}$$

It is now easy to check that, in this way we get a functor $\Box_C : {}^{D}\mathcal{M}^{C} \times {}^{C}\mathcal{M}^{E} \to {}^{D}\mathcal{M}^{E}$. Let us check it is left exact. Let

$$0 \to N' \xrightarrow{f} N \xrightarrow{g} N'' \to 0$$

be an exact sequence in ${}^{C}\mathcal{M}^{E}$ and let $M \in {}^{D}\mathcal{M}^{C}$. Then we can consider the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M\Box_C N' & \xrightarrow{M\Box_C f} & M\Box_C N & \xrightarrow{M\Box_C g} & M\Box_C N'' & & \\
 & & \downarrow{\scriptstyle i_{M\Box_C N'}} & & \downarrow{\scriptstyle i_{M\Box_C N}} & & \downarrow{\scriptstyle i_{M\Box_C N''}} & & \\
0 & \longrightarrow & M \otimes N' & \xrightarrow{M\otimes f} & M \otimes N & \xrightarrow{M\otimes g} & M \otimes N'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \gamma_{M,N'}} & & \downarrow{\scriptstyle \gamma_{M,N}} & & \downarrow{\scriptstyle \gamma_{M,N'}} & & \\
0 & \longrightarrow & M \otimes C \otimes N' & \xrightarrow{M\otimes C\otimes f} & M \otimes C \otimes N & \xrightarrow{M\otimes C\otimes g} & M \otimes C \otimes N'' & \longrightarrow & 0
\end{array}
$$

where, for each M, N we have

$$\gamma_{M,N} = \rho^{M} \otimes N - M \otimes {}^{N}\rho.$$

Note that the first row of the diagram is exact in view of the Snake's Lemma. □

Lemma 2.50. *Let C be a coalgebra and let M be a right C-comodule. Then*

$$M \simeq M\Box_C C \tag{2.24}$$

Proof. Since $M\Box_C C = Ker\left(\rho^{M} \otimes C - M \otimes \Delta\right)$ and since $\left(\rho^{M} \otimes C - M \otimes \Delta\right) \circ \rho^{M} = 0$ and ρ^{M} is injective, there exists a unique isomorphism $\phi_M : M \to M\Box_C C$ such that

$$i_{M\Box_C C} \circ \phi_M = \rho^{M}.$$

□

Proposition 2.51. *Let* $\varphi : C \to D$ *be a coalgbra morphism and let* $\left(M, \rho^M\right) \in \mathcal{M}^C$*. Set*

$$\rho_D^M = (M \otimes \varphi) \circ \rho^M \tag{2.25}$$

Then $\left(M, \rho_D^M\right) \in \mathcal{M}^D$ *and the assignment* $\left(M, \rho^M\right) \mapsto \left(M, \rho_D^M\right)$ *yields an exact functor*

$$(-)_\varphi : \mathcal{M}^C \to \mathcal{M}^D$$

Proof. Let us compute

$$\begin{aligned}
\left(\rho_D^M \otimes D\right) \circ \rho_D^M &\overset{(2.25)}{=} (M \otimes \varphi \otimes D) \circ \left(\rho^M \otimes D\right) \circ (M \otimes \varphi) \circ \rho^M \\
= (M \otimes \varphi \otimes D) \circ \left(\rho^M \otimes \varphi\right) \circ \rho^M &= (M \otimes \varphi \otimes D) \circ (M \otimes C \otimes \varphi) \circ \left(\rho^M \otimes C\right) \circ \rho^M \\
= (M \otimes \varphi \otimes \varphi) \circ (M \otimes \Delta_C) \circ \rho^M &= (M \otimes [(\varphi \otimes \varphi) \circ \Delta_C]) \circ \rho^M \overset{\varphi \mathrm{iscoalgmorphism}}{=} \\
= (M \otimes [\Delta_D \circ \varphi]) \circ \rho^M &= (M \otimes \Delta_D) \circ (M \otimes \varphi) \circ \rho^M \overset{(2.25)}{=} (M \otimes \Delta_D) \circ \rho_D^M
\end{aligned}$$

and

$$\begin{aligned}
r^M \circ (M \otimes \varepsilon_D) \circ \rho_D^M &= r^M \circ (M \otimes \varepsilon_D) \circ (M \otimes \varphi) \circ \rho^M \\
= r^M \circ (M \otimes (\varepsilon_D \circ \varphi)) \circ \rho^M &\overset{\varphi \mathrm{iscoalgmorphism}}{=} r^M \circ (M \otimes \varepsilon_C) \circ \rho^M = M.
\end{aligned}$$

Thus we get that $\left(M, \rho_D^M\right) \in \mathcal{M}^D$. Let now $f : M \to M'$ be a morphism in $\mathcal{M}^C$ and let us check that $f : \left(M, \rho_D^M\right) \to \left(M', \rho_D^{M'}\right)$ is a morphism in $\mathcal{M}^D$. In fact we have

$$\begin{aligned}
(f \otimes D) \circ \rho_D^M &\overset{(2.25)}{=} (f \otimes D) \circ (M \otimes \varphi) \circ \rho^M = (f \otimes \varphi) \circ \rho^M = \\
= (M' \otimes \varphi) \circ (f \otimes C) \circ \rho^M &\overset{f \mathrm{iscolin}}{=} (M' \otimes \varphi) \circ \rho^{M'} \circ f = \rho_D^{M'} \circ f.
\end{aligned}$$

□

Lemma 2.52. *Let* C, D *and* E *be coalgebras and let* $\varphi : C \to D$ *be a coalgebra morphism. Let* $\left(M, {}^M\rho, \rho^M\right) \in {}^E\mathcal{M}^C$*. Then* $\left(M, {}^M\rho, \rho_D^M\right) \in {}^E\mathcal{M}^D$*.*

Proof. Since $\left(M, {}^M\rho, \rho^M\right) \in {}^E\mathcal{M}^C$ we have that $\left({}^M\rho \otimes C\right) \circ \rho^M = \left(E \otimes \rho^M\right) \circ {}^M\rho$. Let us compute

$$\begin{aligned}
\left({}^M\rho \otimes D\right) \circ \rho_D^M &\overset{(2.25)}{=} \left({}^M\rho \otimes D\right) \circ (M \otimes \varphi) \circ \rho^M = \left({}^M\rho \otimes \varphi\right) \circ \rho^M \\
= (E \otimes M \otimes \varphi) \circ \left({}^M\rho \otimes C\right) \circ \rho^M &\overset{(2.21)}{=} (E \otimes M \otimes \varphi) \circ \left(E \otimes \rho^M\right) \circ {}^M\rho \\
&\overset{(2.25)}{=} \left(E \otimes \rho_D^M\right) \circ {}^M\rho
\end{aligned}$$

□

Theorem 2.53. *Let $\varphi : C \to D$ be a coalgebra morphism and let us consider C endowed with its D-C-bicomodule structure:*

$$\left(C, {}_D^C\rho = (\varphi \otimes C) \circ \Delta_C, \Delta_C\right) \in {}^D\mathcal{M}^C.$$

For any $\left(N, \rho^N\right) \in \mathcal{M}^D$ we set

$$N^\varphi = \left(N \square_D C, \rho^{N \square_D C} = N \square_D \Delta_C\right) \in \mathcal{M}^C \tag{2.26}$$

Then the assignment $\left(N, \rho^N\right) \mapsto N^\varphi$ yields a functor

$$(-)^\varphi : \mathcal{M}^D \to \mathcal{M}^C$$

which is a right adjoint of $(-)_\varphi$.

Proof. By Proposition 2.49, we have only to prove the adjunction statement. Let $\left(M, \rho^M\right) \in \mathcal{M}^C$ and let us compute

$$\begin{aligned}\left(\rho_D^M \otimes C\right) \circ \rho^M &\overset{(2.25)}{=} (M \otimes \varphi \otimes C) \circ \left(\rho^M \otimes C\right) \circ \rho^M = \\ &\overset{M\text{iscomod}}{=} (M \otimes \varphi \otimes C) \circ (M \otimes \Delta_C) \circ \rho^M = \\ &= (M \otimes (\varphi \otimes C) \circ \Delta_C) \circ \rho^M \overset{(2.25)}{=} \left(M \otimes {}_D^C\rho\right) \circ \rho^M.\end{aligned}$$

Therefore there exists a linear map $\gamma_M : M \to (M_\varphi)^\varphi = M_\varphi \square_D C$ such that

$$\rho^M = i_{M_\varphi \square_D C} \circ \gamma_M. \tag{2.27}$$

Let us prove that γ_M is a morphism in $\mathcal{M}^C$. We compute

$$\begin{aligned}\left(i_{M_\varphi \square_D C} \otimes C\right) \circ \rho^{(M_\varphi)^\varphi} \circ \gamma_M &= \left(i_{M_\varphi \square_D C} \otimes C\right) \circ \rho^{M_\varphi \square_D C} \circ \gamma_M \\ &\overset{(2.22)}{=} [M_\varphi \otimes \Delta_C] \circ i_{M_\varphi \square_D C} \circ \gamma_M \\ &\overset{(2.27)}{=} (M \otimes \Delta_C) \circ \rho^M = \\ &= \left(\rho^M \otimes C\right) \circ \rho^M \overset{(2.27)}{=} \left[\left(i_{M_\varphi \square_D C} \circ \gamma_M\right) \otimes C\right] \\ &= \left(i_{M_\varphi \square_D C} \otimes C\right) \circ (\gamma_M \otimes C) \circ \rho^M.\end{aligned}$$

Now we prove that $(\gamma_M)_{M \in \mathcal{M}^C}$ yields a functorial morphism $\gamma : \mathrm{Id}_{\mathcal{M}^C} \to \left((-)_\varphi\right)^\varphi$. Thus let $f : \left(M, \rho^M\right) \to \left(M', \rho^{M'}\right)$ be a morphism in $\mathcal{M}^C$ and let us compute

$$\begin{aligned}i_{M'_\varphi \square_D C} \circ \left((f)_\varphi\right)^\varphi \circ \gamma_M &= i_{M'_\varphi \square_D C} \circ (f \square_D C) \circ \gamma_M \overset{(2.23)}{=} (f \otimes C) \circ i_{M_\varphi \square_D C} \circ \gamma_M = \\ &\overset{(2.27)}{=} (f \otimes C) \circ \rho^M \overset{f\text{iscol}}{=} \rho^{M'} \circ f \overset{(2.27)}{=} i_{M'_\varphi \square_D C} \circ \gamma_{M'} \circ f.\end{aligned}$$

Since $i_{M'_\varphi \square_D C}$ is injective, we conclude.

Now, for every $\left(N,\rho^{N}\right)\in\mathcal{M}^{D}$, let us consider the map

$$\lambda_{N}=r_{N}\circ(N\otimes\varepsilon_{C})\circ i_{N\square_{D}C}:(N^{\varphi})_{\varphi}=(N\square_{D}C)_{\varphi}\to N \tag{2.28}$$

and let us prove it is a morphism in $\mathcal{M}^{D}$. Thus, let us compute

$$\begin{aligned}
(\lambda_{N}\otimes D)\circ\rho^{(N^{\varphi})_{\varphi}} &\overset{(2.28)}{=} (r_{N}\otimes D)\circ(N\otimes\varepsilon_{C}\otimes D)\circ(i_{N\square_{D}C}\otimes D)\circ\rho^{(N^{\varphi})_{\varphi}}\\
&\overset{(2.25)}{=} (r_{N}\otimes D)\circ(N\otimes\varepsilon_{C}\otimes D)\circ(i_{N\square_{D}C}\otimes D)\circ(N\square_{D}C\otimes\varphi)\circ\rho^{N\square_{D}C}\\
&= (r_{N}\otimes D)\circ(N\otimes\varepsilon_{C}\otimes D)\circ(N\otimes C\otimes\varphi)\circ(i_{N\square_{D}C}\otimes C)\circ\rho^{N\square_{D}C}\\
&\overset{(2.22)}{=} (r_{N}\otimes D)\circ(N\otimes\varepsilon_{C}\otimes D)\circ(N\otimes C\otimes\varphi)\circ(N\otimes\Delta_{C})\circ i_{N\square_{D}C}\\
&\overset{\varphi\text{iscoalgmrph}}{=} (r_{N}\otimes D)\circ(N\otimes(\varepsilon_{D}\circ\varphi)\otimes D)\circ(N\otimes C\otimes\varphi)\circ(N\otimes\Delta_{C})\circ i_{N\square_{D}C}\\
&= (r_{N}\otimes D)\circ(N\otimes\varepsilon_{D}\otimes D)\circ(N\otimes\varphi\otimes\varphi)\circ(N\otimes\Delta_{C})\circ i_{N\square_{D}C}\\
&= (r_{N}\otimes D)\circ(N\otimes\varepsilon_{D}\otimes D)\circ(N\otimes[(\varphi\otimes\varphi)\circ\Delta_{C}])\circ i_{N\square_{D}C}\\
&\overset{\varphi\text{iscoalgmrph}}{=} (r_{N}\otimes D)\circ(N\otimes\varepsilon_{D}\otimes D)\circ(N\otimes[\Delta_{D}\circ\varphi])\circ i_{N\square_{D}C}\\
&= (r_{N}\otimes D)\circ(N\otimes\varepsilon_{D}\otimes D)\circ(N\otimes\Delta_{D})\circ(N\otimes\varphi)\circ i_{N\square_{D}C}\\
&\overset{(1.2)}{=} (N\otimes l_{D})\circ(N\otimes\varepsilon_{D}\otimes D)\circ(N\otimes\Delta_{D})\circ(N\otimes\varphi)\circ i_{N\square_{D}C}\\
&= (N\otimes[l_{D}\circ(\varepsilon_{D}\otimes D)\circ\Delta_{D}])\circ(N\otimes\varphi)\circ i_{N\square_{D}C}\\
&\overset{D\text{iscoalg}}{=} (N\otimes\varphi)\circ i_{N\square_{D}C}\\
&\overset{D\text{iscoalg}}{=} (N\otimes[r_{D}\circ(D\otimes\varepsilon_{D})\circ\Delta_{D}])\circ(N\otimes\varphi)\circ i_{N\square_{D}C}\\
&= (N\otimes r_{D})\circ(N\otimes(D\otimes\varepsilon_{D})\circ\Delta_{D})\circ(N\otimes\varphi)\circ i_{N\square_{D}C}\\
&\overset{(1.2)}{=} r_{N\otimes D}\circ(N\otimes D\otimes\varepsilon_{D})\circ(N\otimes[\Delta_{D}\circ\varphi])\circ i_{N\square_{D}C}\\
&\overset{\varphi\text{iscoalgmrph}}{=} r_{N\otimes D}\circ(N\otimes D\otimes\varepsilon_{D})\circ(N\otimes[(\varphi\otimes\varphi)\circ\Delta_{C}])\circ i_{N\square_{D}C}\\
&= r_{N\otimes D}\circ(N\otimes D\otimes\varepsilon_{D})\circ(N\otimes\varphi\otimes\varphi)\circ(N\otimes\Delta_{C})\circ i_{N\square_{D}C}\\
&= r_{N\otimes D}\circ(N\otimes D\otimes(\varepsilon_{D}\circ\varphi))\circ(N\otimes\varphi\otimes C)\circ(N\otimes\Delta_{C})\circ i_{N\square_{D}C}\\
&\overset{\varphi\text{iscoalgmrph}}{=} r_{N\otimes D}\circ(N\otimes D\otimes\varepsilon_{C})\circ(N\otimes(\varphi\otimes C)\circ\Delta_{C})\circ i_{N\square_{D}C}\\
&= r_{N\otimes D}\circ(N\otimes D\otimes\varepsilon_{C})\circ\left(N\otimes{}_{D}^{C}\rho\right)\circ i_{N\square_{D}C}\\
&\overset{(\text{defcot})}{=} r_{N\otimes D}\circ(N\otimes D\otimes\varepsilon_{C})\circ\left(\rho^{N}\otimes C\right)\circ i_{N\square_{D}C}=r_{N\otimes D}\circ\left(\rho^{N}\otimes\varepsilon_{C}\right)\circ i_{N\square_{D}C}\\
&= r_{N\otimes D}\circ\left(\rho^{N}\otimes k\right)\circ(N\otimes\varepsilon_{C})\circ i_{N\square_{D}C}\overset{(1.1)}{=}\rho^{N}\circ r_{N}\circ(N\otimes\varepsilon_{C})\circ i_{N\square_{D}C}\overset{(2.28)}{=}\rho^{N}\circ\lambda_{N}
\end{aligned}$$

Let us prove that $(\lambda_{N})_{N\in\mathcal{M}^{D}}$ yields a functorial morphism

$$\lambda:((-)^{\varphi})_{\varphi}=(-\square_{D}C)_{\varphi}\to\mathrm{Id}_{\mathcal{M}^{D}}.$$

Hence let $h : N \to N'$ be a morphism in $\mathcal{M}^D$ and let us compute

$$\begin{aligned}\lambda_{N'} \circ (h\Box_D C) &\overset{(2.28)}{=} r_{N'} \circ (N' \otimes \varepsilon_C) \circ i_{N'\Box_D C} \circ (h\Box_D C) = \\ &\overset{(2.23)}{=} r_{N'} \circ (N' \otimes \varepsilon_C) \circ (h \otimes C) \circ i_{N\Box_D C} \\ &= r_{N'} \circ (h \otimes k) \circ (N \otimes \varepsilon_C) \circ i_{N\Box_D C} \\ &\overset{(1.1)}{=} h \circ r_N \circ (N \otimes \varepsilon_C) \circ i_{N\Box_D C} \overset{(2.28)}{=} h \circ \lambda_N.\end{aligned}$$

Let us prove that γ and λ give rise to an adjunction. Given $\left(M, \rho^M\right) \in \mathcal{M}^C$, let us compute

$$\begin{aligned}&\lambda_{M_\varphi} \circ (\gamma_M)_\varphi \overset{(2.28)}{=} r_M \circ (M \otimes \varepsilon_C) \circ i_{M_\varphi\Box_D C} \circ \gamma_M \overset{(2.27)}{=} r_M \circ (M \otimes \varepsilon_C) \circ \rho^M \\ = \;& \mathrm{Id}_M = \mathrm{Id}_{M_\varphi}.\end{aligned}$$

Given $\left(N, \rho^N\right) \in \mathcal{M}^D$, let us compute

$$\begin{aligned}i_{N\Box_D C} \circ (\lambda_N)^\varphi \circ \gamma_{N^\varphi} &\overset{(2.28)}{=} i_{N\Box_D C} \circ (r_N \circ (N \otimes \varepsilon_C) \circ i_{N\Box_D C})^\varphi \circ \gamma_{N^\varphi} = \\ &\overset{(2.26)}{=} i_{N\Box_D C} \circ [(r_N \circ (N \otimes \varepsilon_C) \circ i_{N\Box_D C}) \Box_D C] \circ \gamma_{N\Box_D C} = \\ &\overset{(2.23)}{=} [(r_N \circ (N \otimes \varepsilon_C) \circ i_{N\Box_D C}) \otimes C] \circ i_{(N\Box_D C)_\varphi \Box_D C} \circ \gamma_{N\Box_D C} = \\ &\overset{(2.27)}{=} [(r_N \circ (N \otimes \varepsilon_C) \circ i_{N\Box_D C}) \otimes C] \circ \rho^{N\Box_D C} = \\ &= [(r_N \circ (N \otimes \varepsilon_C)) \otimes C] \circ (i_{N\Box_D C} \otimes C) \circ \rho^{N\Box_D C} = \\ &\overset{(2.22)}{=} [(r_N \circ (N \otimes \varepsilon_C)) \otimes C] \circ \left(N \otimes \rho^C\right) \circ i_{N\Box_D C} \\ &= [(r_N \circ (N \otimes \varepsilon_C)) \otimes C] \circ (N \otimes \Delta_C) \circ i_{N\Box_D C} \\ &= (r_N \otimes C) \circ (N \otimes (\varepsilon_C \otimes C) \circ \Delta_C) \circ i_{N\Box_D C} \\ &\overset{(1.2)}{=} (N \otimes l_C) \circ (N \otimes (\varepsilon_C \otimes C) \circ \Delta_C) \circ i_{N\Box_D C} \\ &\overset{C \text{iscoalg}}{=} i_{N\Box_D C}\end{aligned}$$

□

Exercise 2.54. *Apply Theorem 2.53 to the particular case when the coalgebra morphism is* $\varepsilon_C : (C, \Delta_C, \varepsilon_C) \to \left(k, \Delta_k = r_k^{-1} = l_k^{-1}, \varepsilon_k = \mathrm{Id}_k\right)$. *(See 1.26. Show that* $(-)_{\varepsilon_C} : \mathcal{M}^C \to \mathcal{M}^k = Vec_k$ *is just the forgetful functor* U *and* $(-)^{\varepsilon_C} : \mathcal{M}^k = Vec_k \to \mathcal{M}^C$ *is just the functor* F_C*. Therefore Theorem 2.40 can be obtained as a particular case of Theorem 2.53.*

Chapter 3

Bialgebras and Hopf Algebras

Theorem 3.1. *Let us consider a* 5*th-uple* $(B, m_B, u_B, \Delta_B, \varepsilon_B)$ *such that* (B, m_B, u_B) *is an algebra,* $(B, \Delta_B, \varepsilon_B)$ *is a coalgebra. The following assertions are equivalent:*

(a) *The maps* Δ_B *and* ε_B *are algebra morphisms.*

(b) *The maps* m_B *and* u_B *are coalgebra morphisms.*

Proof. Recall that, in view of (1.8), we have

$$\Delta_{B\otimes B} = (B \otimes \tau_{B,B} \otimes B) \circ (\Delta_B \otimes \Delta_B) \quad \text{and} \quad \varepsilon_{B\otimes B} = l_k \circ (\varepsilon_B \otimes \varepsilon_B) .$$

Analogously

$$m_{B\otimes B} = (m_B \otimes m_B) \circ (B \otimes \tau_{B,B} \otimes B) \quad \text{and} \quad u_{B\otimes B} = (u_B \otimes u_B) \circ (l_k)^{-1} .$$

Δ_B is an algebra morphism means

$$m_{B\otimes B} \circ (\Delta_B \otimes \Delta_B) = \Delta_B \circ m_B \quad \text{and} \quad \Delta_B \circ u_B = u_{B\otimes B}$$

i.e.

$$(m_B \otimes m_B) \circ (B \otimes \tau_{B,B} \otimes B) \circ (\Delta_B \otimes \Delta_B) = \Delta_B \circ m_B \tag{3.1}$$

and

$$\Delta_B \circ u_B \circ l_k = u_B \otimes u_B \tag{3.2}$$

ε_B is an algebra morphism means

$$\varepsilon_B \circ m_B = m_k \circ (\varepsilon_B \otimes \varepsilon_B) = l_k \circ (\varepsilon_B \otimes \varepsilon_B) \tag{3.3}$$

and

$$\varepsilon_B \circ u_B = u_k = \mathrm{Id}_k. \tag{3.4}$$

m_B is a coalgebra morphism means

$$\Delta_B \circ m_B = (m_B \otimes m_B) \circ \Delta_{B \otimes B} \quad \text{and} \quad \varepsilon_B \circ m_B = \varepsilon_{B \otimes B}$$

i.e.

$$\Delta_B \circ m_B = (m_B \otimes m_B) \circ (B \otimes \tau_{B,B} \otimes B) \circ (\Delta_B \otimes \Delta_B) \tag{3.5}$$

and

$$\varepsilon_B \circ m_B = l_k \circ (\varepsilon_B \otimes \varepsilon_B) \tag{3.6}$$

u_B is a coalgebra morphism means

$$\Delta_B \circ u_B = (u_B \otimes u_B) \circ \Delta_{k \otimes k} \quad \text{and} \quad \varepsilon_B \circ u_B = \varepsilon_k$$

i.e.

$$\Delta_B \circ u_B \circ l_k = u_B \otimes u_B \tag{3.7}$$

and

$$\varepsilon_B \circ u_B = \mathrm{Id}_k. \tag{3.8}$$

Since (3.1) = (3.5), (3.2) = (3.7), (3.3) = (3.6) and (3.4) = (3.8), we conclude. □

Definition 3.2. *A* bialgebra *over k is a 5th-uple $(B, m_B, u_B, \Delta_B, \varepsilon_B)$ such that (B, m_B, u_B) is an algebra, $(B, \Delta_B, \varepsilon_B)$ is a coalgebra and the equivalent conditions in Theorem 3.1 hold.*

Remark 3.3. *Using the sigma notation,* (3.1) *can be written as*

$$\sum (a \cdot b)_1 \otimes (a \cdot b)_1 = \Sigma a_1 b_1 \otimes a_2 b_2, \tag{3.9}$$

(3.2) *can be written as*

$$\sum (1_B)_1 \otimes (1_B)_2 = 1_B \otimes 1_B, \tag{3.10}$$

(3.3) *can be written as*

$$\varepsilon_B (ab) = \varepsilon_B (a) \cdot \varepsilon_B (b) \tag{3.11}$$

(3.4) *can be written as*

$$\varepsilon_B (1_B) = 1_k. \tag{3.12}$$

Definition 3.4. *Let $(H, m_H, u_H, \Delta_H, \varepsilon_H)$ be a bialgebra. Set $H^c = (H, \Delta_H, \varepsilon_H)$ and $H^a = (H, m_H, u_H)$. A linear map $S : H \to H$ is called an* antipode *for H if S is an inverse for Id_H in the convolution algebra $\mathrm{Hom}(H^c, H^a)$ i.e.*

$$S * \mathrm{Id}_H = u_H \circ \varepsilon_H = \mathrm{Id}_H * S.$$

This means that, for every $h \in H$

$$\sum S (h_1) \cdot h_2 = \varepsilon_H (h) 1_H = \sum h_1 \cdot S (h_2) . \tag{3.13}$$

Remark 3.5. *If a bialgebra has an antipode, then this antipode is unique. (Why?)*

Definition 3.6. *An Hopf algebra is a* $6th$*-uple* $(H, m_H, u_H, \Delta_H, \varepsilon_H, S)$ *where* $(H, m_H, u_H, \Delta_H, \varepsilon_H)$ *i.e. a bialgebra and* S *is an antipode for* H.

Theorem 3.7. *Let* $(H, m, u, \Delta, \varepsilon, S)$ *be a Hopf algebra. Then:*

1) $S(gh) = S(h) S(g)$ *for every* $g, h \in H$.

2) $S(1_H) = 1_H$.

3) $\Delta(S(h)) = \sum S(h_2) \otimes S(h_1)$ *for every* $h \in H$.

4) $\varepsilon(S(h)) = \varepsilon(h)$ *for every* $h \in H$.

Properties 1) and 2) mean that S *is an algebra antihomomorphism. Properties 3) and 4) mean tha* S *is a coalgebra antihomomorphism.*

Proof. 1) Let $g, h \in H$ and let us compute

$$\begin{aligned} S(gh) &= S\left(\sum g_1 \varepsilon(g_2) h\right) = S\left[\left(\sum g_1 h\right) \varepsilon(g_2)\right] = \sum S(g_1 h_1 \varepsilon(h_2)) \varepsilon(g_2) \\ &= \sum S(g_1 h_1 \varepsilon(h_2)) g_{2_1} S(g_{2_2}) = \sum S(g_{1_1} h_1 \varepsilon(h_2)) g_{1_2} S(g_2) \\ &= \sum S(g_{1_1} h_1) g_{1_2} \varepsilon(h_2) S(g_2) = \sum S(g_{1_1} h_1) g_{1_2} h_{2_1} S(h_{2_2}) S(g_2) = \\ &= \sum S(g_{1_1} h_{1_1}) g_{1_2} h_{1_2} S(h_2) S(g_2) \overset{(3.9)}{=} \sum S((g_1 h_1)_1)(g_1 h_1)_2 S(h_2) S(g_2) = \\ &\overset{(3.13)}{=} \sum \varepsilon(g_1 h_1) S(h_2) S(g_2) \overset{(3.11)}{=} \sum \varepsilon(g_1) \varepsilon(h_1) S(h_2) S(g_2) \\ &= \sum S(\varepsilon(h_1) h_2) S(\varepsilon(g_1) g_2) = S(h) S(g) \end{aligned}$$

2) We know that

$$(S * \mathrm{Id}_H)(1_H) = (u \circ \varepsilon)(1_H)$$

Since $\Delta(1_H) = 1_H \otimes 1_H$ and $\varepsilon(1_H) = 1_k$, this means that

$$S(1_H) \cdot 1_H = u(1_k)$$

and hence

$$S(1_H) = 1_H.$$

3) In this proof, for every $h \in H$ we will simply write $\Delta(h) = h_1 \otimes h_2$, summation understood.

Let $h \in H$. Since

$$\varepsilon(h) 1_H = S(h_1) h_2$$

we get

$$\varepsilon(h) 1_H \otimes 1_H = \varepsilon(h) \Delta(1_H) = \Delta(\varepsilon(h) 1_H) = \Delta(S(h_1) h_2) = S(h_1)_1 h_{2_1} \otimes S(h_1)_2 h_{2_2}$$

so that

$$\varepsilon(h)\,1_H \otimes 1_H = S(h_1)_1\, h_{2_1} \otimes S(h_1)_2\, h_{2_2} \tag{3.14}$$

and hence

$$\begin{aligned}
[S(h)]_1 \otimes [S(h)]_2 &= [S(h_1)]_1\, \varepsilon(h_2) \otimes [S(h_1)]_2 = [S(h_1)]_1\, h_{2_1} S(h_{2_2}) \otimes [S(h_1)]_2 \\
&= [S(h_1)]_1\, h_2 S(h_3) \otimes [S(h_1)]_2 \\
&= [S(h_1)]_1\, h_{2_1} \varepsilon(h_{2_2})\, S(h_3) \otimes [S(h_1)]_2 \\
&= [S(h_1)]_1\, h_2 S(h_4) \otimes [S(h_1)]_2\, \varepsilon(h_3) \\
&= [S(h_1)]_1\, h_2 S(h_5) \otimes [S(h_1)]_2\, h_3 S(h_4) \\
&= \left[S\left(h_{1_{1_1}}\right)\right]_1 h_{1_{1_{2_1}}} S(h_2) \otimes \left[S\left(h_{1_{1_1}}\right)\right]_2 \left(h_{1_{1_{2_2}}}\right) S(h_{1_2}) \\
&\overset{(3.14)}{=} \varepsilon(h_{1_1})\, S(h_2) \otimes S(h_{1_2}) = S(h_2) \otimes S(\varepsilon(h_{1_1})\, h_{1_2}) = S(h_2) \otimes S(h_1)
\end{aligned}$$

4) Let $h \in H$. We compute

$$\varepsilon(S(h)) = \varepsilon\left(S\left(\sum \varepsilon(h_1)\, h_2\right)\right) = \sum \varepsilon(h_1)\, \varepsilon(S(h_2)) \overset{(3.11)}{=}$$
$$= \varepsilon(h_1 S(h_2)) = \varepsilon\left(\sum h_1 S(h_2)\right) = \varepsilon(\varepsilon(h)) = \varepsilon(1_H) = \varepsilon(h)\, 1_k = \varepsilon(h)\,.$$

□

Proposition 3.8. *Let $(H, m, u, \Delta, \varepsilon, S)$ be a Hopf algebra. Then the following statements are equivalent:*

(a) $\sum S(h_2)\, h_1 = \varepsilon(h)\, 1_H$ *for every* $h \in H$.

(b) $\sum h_2 S(h_1) = \varepsilon(h)\, 1_H$ *for every* $h \in H$.

(c) $S \circ S = \mathrm{Id}_H$.

Proof. In this proof, for every $h \in H$ we will simply write $\Delta(h) = h_1 \otimes h_2$, summation understood.

$(a) \Rightarrow (c)$ Let $h \in H$. From (a) we deduce that

$$\varepsilon(h)\, 1_H = S(S(h_2)\, h_1) = S(h_1)\, [(S \circ S)(h_2)] \tag{3.15}$$

and hence we get

$$\begin{aligned}
h &= h_1 \varepsilon(h_2) \overset{(3.15)}{=} h_1 S(h_{2_1})\, [(S \circ S)(h_{2_2})] = h_{1_1} S(h_{1_2})\, [(S \circ S)(h_2)] \\
&= \varepsilon(h_1)\, [(S \circ S)(h_2)] = (S \circ S)(\varepsilon(h_1)\, h_2) = (S \circ S)(h)\,.
\end{aligned}$$

$(c) \Rightarrow (a)$ Let $h \in H$. Then

$$\varepsilon(h)\, 1_H = S(\varepsilon(h)\, 1_H) = S[S(h_1)\, h_2] = S(h_2)\, S(S(h_1)) = S(h_2)\, h_1$$

$(b) \Rightarrow (c)$ Let $h \in H$. From (b) we deduce that

$$\varepsilon(h)\, 1_H = S(h_2 S(h_1)) = [(S \circ S)(h_1)]\, S(h_2) \tag{3.16}$$

and hence we get

$$\begin{aligned} h &= \varepsilon(h_1)\, h_2 \overset{(3.16)}{=} [(S \circ S)(h_{1_1})]\, S(h_{1_2})\, h_2 = [(S \circ S)(h_1)]\, S(h_{2_1})\, h_{2_2} \\ &= [(S \circ S)(h_1)]\, \varepsilon(h_2) = [(S \circ S)(h_1 \varepsilon(h_2))] = (S \circ S)(h). \end{aligned}$$

$(c) \Rightarrow (b)$ Let $h \in H$. Then

$$\varepsilon(h)\, 1_H = S(\varepsilon(h)\, 1_H) = S[h_1 S(h_2)] = S(S(h_2))\, S(h_1) = h_2 S(h_1)$$

□

Corollary 3.9. *Let $(H, m, u, \Delta, \varepsilon, S)$ be a Hopf algebra. If H is either commutative or cocommutative, then $S^2 = \mathrm{Id}_H$.*

Proof. Assume that H is commutative. Then, for every $h \in H$, we have

$$\varepsilon(h)\, 1_H = \sum h_1 S(h_2) = \sum S(h_2)\, h_1.$$

Assume that H is cocommutative. Then, for every $h \in H$, we have

$$\varepsilon(h)\, 1_H = \sum h_1 S(h_2) = \sum h_2 S(h_1).$$

□

Proposition 3.10. *Let $(H, m_H, u_H, \Delta_H, \varepsilon_H, S_H)$ be a finite dimensional Hopf algebra. Then $(H^*, m_{H^*}, u_{H^*}, \Delta_{H^*}, \varepsilon_{H^*}, S_{H^*})$ is an Hopf algebra where $\Delta_{H^*} : H^* \overset{(m_H)^*}{\longrightarrow} (H \otimes H)^* \overset{\Lambda_{H,H}^{-1}}{\longrightarrow} H^* \otimes H^*$ and $\varepsilon_{H^*} : H^* \overset{(u_H)^*}{\longrightarrow} k^* \overset{ev_1}{\simeq} k$*
$m_{H^*} : H^* \otimes H^* \overset{\Lambda_{H,H}}{\longrightarrow} (H \otimes H)^* \overset{(\Delta_H)^*}{\longrightarrow} H^*$ and $u_{H^*} = k \overset{(ev_1)^{-1}}{\simeq} k^* \overset{(\varepsilon_H)^*}{\longrightarrow} H^*$
and $S_{H^*} : H^* \overset{(S_H)^*}{\longrightarrow} H^*$.

Proof. By Proposition 1.40 we know that $(H^*, \Delta_{H^*}, \varepsilon_{H^*})$ is a coalgebra and by Proposition 1.46 we know that (H^*, m_{H^*}, u_{H^*}) is an algebra. For every $f, g \in H^*$ and $x, y \in H$, we compute

$$\begin{aligned} (f * g)(xy) &= \sum f((xy)_1)\, g((xy)_2) = \sum f(x_1 y_1)\, g(x_2 y_2) \\ &= \sum f_1(x_1)\, f_2(y_1)\, g_1(x_2)\, g_2(y_2) = \sum f_1(x_1)\, g_1(x_2)\, f_2(y_1)\, g_2(y_2) \\ &= \sum (f_1 * g_1)(x)\, (f_2 * g_2)(y). \end{aligned}$$

Since $\Delta_{H^*}(f * g) = \sum (f * g)_1 \otimes (f * g)_2$ is uniquely determined by

$$(f * g)(xy) = \sum [(f * g)_1(x)]\, [(f * g)_2(y)] \quad \text{for every } x, y \in H$$

we deduce that

$$\sum (f * g)_1 \otimes (f * g)_2 = \sum (f_1 * g_1) \otimes (f_2 * g_2).$$

We also compute

$$\Delta_{H^*}(1_{H^*}) = \Delta_{H^*}(\varepsilon_H) = \sum (\varepsilon_H)_1 \otimes (\varepsilon_H)_2$$

which is uniquely determined by

$$\varepsilon_H(xy) = \sum (\varepsilon_H)_1(x)(\varepsilon_H)_2(y) \quad \text{for every } x, y \in H.$$

Since

$$\varepsilon_H(xy) = \varepsilon_H(x)\varepsilon_H(y) \quad \text{for every } x, y \in H$$

we deduce that

$$\Delta_{H^*}(\varepsilon_H) = \varepsilon_H \otimes \varepsilon_H.$$

Therefore Δ_{H^*} is an algebra morphism. For every $f, g \in H^*$, let us compute

$$\varepsilon_{H^*}(f * g) = (f * g)(1_H) = f(1_H)g(1_H) = \varepsilon_{H^*}(f)\varepsilon_{H^*}(g).$$

We have also

$$\varepsilon_{H^*}(1_{H^*}) = \varepsilon_{H^*}(\varepsilon_H) = \varepsilon_H(1_H) = 1_k.$$

Therefore also ε_{H^*} is an algebra morphism.

Let now $f \in H^*$ and let us compute

$$(S_{H^*} * \mathrm{Id}_{H^*})(f) = \sum S_{H^*}(f_1) * f_2 = \sum (f_1 \circ S_H) * f_2.$$

For every $x \in H$ we compute

$$\left[\sum (f_1 \circ S_H) * f_2\right](x) = \sum (f_1 \circ S_H)(x_1) f_2(x_2) =$$

$$= \sum f_1(S_H(x_1)) f_2(x_2) = \sum f[S_H(x_1)x_2] = f(\varepsilon_H(x)) = f(1_H)\varepsilon_H(x) = \varepsilon_{H^*}(f)\varepsilon_H(x).$$

We deduce that

$$(S_{H^*} * \mathrm{Id}_{H^*})(f) = \sum (f_1 \circ S_H) * f_2 = \varepsilon_{H^*}(f)\varepsilon_H = (u_{H^*} \circ \varepsilon_{H^*})(f) \quad \text{for every } f \in H^*$$

i.e. that

$$S_{H^*} * \mathrm{Id}_{H^*} = u_{H^*} \circ \varepsilon_{H^*}.$$

The proof that $\mathrm{Id}_{H^*} * S_{H^*} = u_{H^*} \circ \varepsilon_{H^*}$ is similar. □

Proposition 3.11. *Let $(H, m_H, u_H, \Delta_H, \varepsilon_H, S_H)$ be a finite dimensional Hopf algebra and let $\omega : H \to H^{**}$ the natural isomorphism:*

$$\omega(x)(f) = f(x) \quad \textit{for any } f \in H^* \textit{ and } x \in H.$$

Then

$$\omega : (H, m_H, u_H, \Delta_H, \varepsilon_H, S_H) \to (H^*, m_{H^*}, u_{H^*}, \Delta_{H^*}, \varepsilon_{H^*}, S_{H^*})$$

is an isomorphism of Hopf algebras.

Proof. Let α and $\beta \in H^{**}$. Then

$$(\alpha * \beta)(f) = \sum \alpha(f_1)\beta(f_2) \quad \text{for any } f \in H^*$$

where $\Delta(f) = \sum f_1 \otimes f_2$ is uniquely determined by

$$f(ab) = \sum f_1(a) f_2(b) \quad \text{for any } a, b \in H.$$

We have

$$[\omega(x) * \omega(y)](f) = \sum f_1(x) f_2(y) = f(xy) = \omega(xy)(f) \quad \text{for any } f \in H^* \text{and } x, y \in H.$$

and hence $\omega(x) * \omega(y) = \omega(xy)$. Now we know that

$$\Delta_{H^{**}}(\omega(x)) = \sum [\omega(x)]_1 \otimes [\omega(x)]_2$$

uniquely determined by

$$\omega(x)(f * g) = \sum [\omega(x)]_1(f)[\omega(x)]_2(g) \quad \text{for any } f, g \in H^* \text{and } x \in H.$$

We compute

$$\sum \omega(x_1)(f)\omega(x_2)(g) = \sum f(x_1) g(x_2) = (f * g)(x) \quad \text{for any } f, g \in H^* \text{and } x \in H.$$

Since

$$\omega(x)(f * g) = (f * g)(x) \quad \text{for any } f, g \in H^* \text{and } x \in H$$

we conclude that

$$\sum [\omega(x)]_1(f)[\omega(x)]_2(g) = \sum [\omega(x_1)(f)][\omega(x_2)(g)] \quad \text{for any } f, g \in H^* \text{and } x \in H$$

i.e. that

$$\sum [\omega(x)]_1 \otimes [\omega(x)]_2 = \sum \omega(x_1) \otimes \omega(x_2) \quad \text{for any } x \in H.$$

Moreover we have

$$[\omega(1_H)](f) = f(1_H) = 1_{H^{**}}(f) \quad \text{for any } f \in H^*$$

and

$$[\varepsilon_{H^{**}} \circ \omega](x) = \varepsilon_{H^{**}}(\omega(x)) = \omega(x)(\varepsilon_H) = \varepsilon_H(x) \quad \text{for any } x \in H.$$

Hence $\omega(1_H) = 1_{H^{**}}$ and $\varepsilon_{H^*} \circ \omega = \varepsilon_H$. □

Definition 3.12. *Let* $(A, m_A, u_A, \Delta_A, \varepsilon_A)$ *and* $(B, m_B, u_B, \Delta_B, \varepsilon_B, S_B)$ *be bialgebras. A* k*-linear map* $f : A \to B$ *is called a* bialgebra morphism *if* $f : (A, m_A, u_A) \to (B, m_B, u_B)$ *is an algebra homomorphism and* $f : (A, \Delta_A, \varepsilon_A) \to (B, \Delta_B, \varepsilon_B)$ *is a coalgebra homomorphism.*

Definition 3.13. *Let* $(A, m, u, \Delta, \varepsilon)$ *be a bialgebra. A vector subspace* I *of* A *is called a* bi-ideal *of* A *if*

- I *is an ideal of the algebra* (A, m, u),
- I *is a coideal of the coalgebra* (A, Δ, ε).

Theorem 3.14. ***(The Fundamental Theorem of the Quotient Bialgebra)*** *Let* $(A, m, u, \Delta, \varepsilon, S)$ *be a Hopf algebra, let* I *be a bi-ideal of* A*.and let* $p = p_I : A \to A/I$ *be the canonical projection. Then* A/I *can be endowed by a unique bialgebra structure (*called quotient bialgebra*) such that* p *becomes a morphism of bialgebras. Moreover given any bialgebra morphism* $f : A \to L$ *such that* $I \subseteq \mathrm{Ker}(f)$*, there exists a unique bialgebra morphism* $\overline{f} : A/I \to L$ *such that* $f = \overline{f} \circ p$.

Proof. Exercise. □

Definition 3.15. *Let* $(H, m_H, u_H, \Delta_H, \varepsilon_H, S_H)$ *and* $(B, m_B, u_B, \Delta_B, \varepsilon_B, S_B)$ *be Hopf algebras. A* k*-linear map* $f : A \to B$ *is called a* Hopf algebra morphism *if it is a bialgebra morphism.*

Proposition 3.16. *Let* $(H, m_H, u_H, \Delta_H, \varepsilon_H, S_H)$ *and* $(B, m_B, u_B, \Delta_B, \varepsilon_B, S_B)$ *be Hopf algebras and let* $f : H \to B$ *be a Hopf algebra morphism. Then* $S_B \circ f = f \circ S_H$.

Proof. For every $x \in H$, let us compute

$$[(S_B \circ f) * f](x) = \sum S_B(f(x_1)) \cdot_B f(x_2) = \sum S_B([f(x)]_1) \cdot_B [f(x)]_2 = \varepsilon_B(f(x)) 1_B = \varepsilon_H(x) 1_B.$$

Thus we get that

$$(S_B \circ f) * f = 1_{\mathrm{Hom}(H,B)}. \tag{3.17}$$

For every $x \in H$, let us also compute

$$\begin{aligned}[f * (f \circ S_H)](x) = \sum f(x_1) \cdot_B f(S_H(x_2)) = f\left[\sum x_1 \cdot_H S_H(x_2)\right] = f(\varepsilon_H(x)) = \varepsilon_H(x) f(1_H) \\ = \varepsilon_H(x) 1_B.\end{aligned}$$

Thus we get that

$$f * (f \circ S_H) = 1_{\mathrm{Hom}(H,B)}. \tag{3.18}$$

From (3.17) and (3.18) we deduce that f is invertible in $\mathrm{Hom}(H, B)$ and that its two-sided inverse is

$$S_B \circ f = f \circ S_H.$$

□

Definition 3.17. *Let* $(H, m, u, \Delta, \varepsilon, S)$ *be a Hopf algebra. A vector subspace* I *of* H *is called a* Hopf ideal *of* H *if*

- *I is an ideal of the algebra (H, m, u),*
- *I is a coideal of the coalgebra (H, Δ, ε) and*
- *$S(I) \subseteq I$.*

Theorem 3.18. ***The Fundamental Theorem of the Quotient Hopf Algebra)*** *Let $(H, m, u, \Delta, \varepsilon, S)$ be a Hopf algebra, let I be a Hopf ideal of H.and let $p = p_I : H \to H/I$ be the canonical projection. Then H/I can be endowed by a unique Hopf algebra structure (*called quotient Hopf algebra*) such that p becomes a morphism of Hopf algebras. Moreover given any Hopf algebra morphism $f : H \to L$ such that $I \subseteq \mathrm{Ker}(f)$, there exists a unique Hopf algebra morphism $\overline{f} : H/I \to L$ such that $f = \overline{f} \circ p$.*

Proof. Exercise. □

Exercise 3.19. *From Sweedler's book we quote these exercise for practicing sigma notation. Let $(H, m, u, \Delta, \varepsilon, S)$ be an Hopf algebra and let $h, f, g \in H$. Show that*

$$h_1 S(h_2) \otimes h_3 = 1_H \otimes h$$

$$S(h_1) h_2 \otimes h_3 = 1_H \otimes h$$
$$h_1 \otimes S(h_2) h_3 = h \otimes 1_H$$
$$h_1 \otimes h_2 S(h_3) = h \otimes 1_H$$
$$h_1 \otimes \ldots \otimes h_{i-1} \otimes h_i S(h_{i+1}) \otimes h_{i+2} \otimes \ldots \otimes h_n = h_1 \otimes \ldots \otimes h_{n-2}$$
$$h_1 \otimes \ldots \otimes h_{i-1} \otimes S(h_i) h_{i+1} \otimes h_{i+2} \otimes \ldots \otimes h_n = h_1 \otimes \ldots \otimes h_{n-2}$$
$$h_1 S(g_1 f h_2) g_2 = \varepsilon(gh) S(f)$$

$$\begin{aligned} & h_1 \otimes \ldots \otimes h_{i-1} \otimes \Delta S(h_i) \otimes h_{i+1} \ldots \otimes h_{n-1} \\ = \; & h_1 \otimes \ldots \otimes h_{i-1} \otimes S(h_{i+1}) \otimes S(h_i) \otimes h_{i+2} \ldots \otimes h_n \end{aligned}$$

$$(1_H \otimes S(h_1) h_2) [\Delta S(h_3)] = \Delta S(h)$$
$$(1_H \otimes S(h_3) h_1) [\Delta S(h_2)] = (S \otimes S) \Delta(h)$$

Chapter 4

Hopf Modules

Throughout this section $H = (H, m, u, \Delta, \varepsilon, S)$ will be a Hopf algebra.

Proposition 4.1. *Let* $\left(M, \mu^M\right), \left(N, \mu^N\right) \in \mathcal{M}_H$. *Set*

$$\mu^{M\otimes N} : M\otimes N\otimes H \overset{M\otimes N\otimes\Delta}{\longrightarrow} M\otimes N\otimes H\otimes H \overset{M\otimes\tau_{N,H}\otimes H}{\longrightarrow} M\otimes H\otimes N\otimes H \overset{\mu^M\otimes\mu^N}{\longrightarrow} M\otimes N.$$

Then $\left(M \otimes N, \mu^{M\otimes N}\right) \in \mathcal{M}_H$.

Proof. It is easy to check that

$$[\tau_{N,H\otimes H} \otimes H] \circ [N \otimes H \otimes \tau_{H,H}] = [H \otimes \tau_{N\otimes H,H}] \circ [\tau_{N,H} \otimes H \otimes H] \tag{4.1}$$

$$
\begin{aligned}
\mu^{M\otimes N}\circ(M\otimes N\otimes m) &= \left(\mu^M\otimes\mu^N\right)\circ(M\otimes\tau_{N,H}\otimes H)\circ(M\otimes N\otimes\Delta)\circ(M\otimes N\otimes m)\\
&= \left(\mu^M\otimes\mu^N\right)\circ(M\otimes\tau_{N,H}\otimes H)\circ(M\otimes N\otimes[\Delta\circ m]) =\\
&\overset{(3.1)}{=} \left(\mu^M\otimes\mu^N\right)\circ(M\otimes\tau_{N,H}\otimes H)\circ(M\otimes N\otimes m\otimes m)\\
&\quad\circ(M\otimes N\otimes H\otimes\tau_{H,H}\otimes H)\circ(M\otimes N\otimes\Delta\otimes\Delta)\\
&= \left(\mu^M\otimes\mu^N\right)\circ(M\otimes\tau_{N,H}\circ(N\otimes m)\otimes m)\\
&\quad\circ(M\otimes N\otimes H\otimes\tau_{H,H}\otimes H)\circ(M\otimes N\otimes\Delta\otimes\Delta)\\
&\overset{(1.3)}{=} \left(\mu^M\otimes\mu^N\right)\circ(M\otimes(m\otimes N)\circ\tau_{N,H\otimes H}\otimes m)\\
&\quad\circ(M\otimes N\otimes H\otimes\tau_{H,H}\otimes H)\circ(M\otimes N\otimes\Delta\otimes\Delta)\\
&= \left(\mu^M\otimes\mu^N\right)\circ(M\otimes m\otimes N\otimes m)\circ(M\otimes\tau_{N,H\otimes H}\otimes H\otimes H)\\
&\quad\circ(M\otimes N\otimes H\otimes\tau_{H,H}\otimes H)\circ(M\otimes N\otimes\Delta\otimes\Delta)\\
&= \left(\mu^M\circ(M\otimes m)\otimes\mu^N\circ(N\otimes m)\right)\circ(M\otimes\tau_{N,H\otimes H}\otimes H\otimes H)\\
&\quad\circ(M\otimes N\otimes H\otimes\tau_{H,H}\otimes H)\circ(M\otimes N\otimes\Delta\otimes\Delta)\\
&= \left(\mu^M\circ\left(\mu^M\otimes H\right)\otimes\mu^N\circ\left(\mu^N\otimes H\right)\right)\circ(M\otimes\tau_{N,H\otimes H}\otimes H\otimes H)\\
&\quad\circ(M\otimes N\otimes H\otimes\tau_{H,H}\otimes H)\circ(M\otimes N\otimes\Delta\otimes\Delta)\\
&= \left(\mu^M\otimes\mu^N\right)\circ\left(\mu^M\otimes H\otimes\mu^N\otimes H\right)\\
&\quad\circ(M\otimes\tau_{N,H\otimes H}\otimes H\otimes H)\circ(M\otimes N\otimes H\otimes\tau_{H,H}\otimes H)\circ(M\otimes N\otimes\Delta\otimes\Delta)\\
&\overset{(4.1)}{=} \left(\mu^M\otimes\mu^N\right)\circ\left(\mu^M\otimes H\otimes\mu^N\otimes H\right)\circ(M\otimes H\otimes\tau_{N\otimes H,H}\otimes H)\circ(M\otimes\tau_{N,H}\otimes H\otimes H\otimes H)\\
&\quad\circ(M\otimes N\otimes\Delta\otimes\Delta)\\
&= \left(\mu^M\otimes\mu^N\right)\circ\left(\mu^M\otimes\left(H\otimes\mu^N\right)\circ\tau_{N\otimes H,H}\otimes H\right)\circ(M\otimes\tau_{N,H}\otimes H\otimes H\otimes H)\\
&\quad\circ(M\otimes N\otimes\Delta\otimes\Delta)\\
&\overset{(1.3)}{=} \left(\mu^M\otimes\mu^N\right)\circ\left(\mu^M\otimes\tau_{N,H}\circ\left(\mu^N\otimes H\right)\otimes H\right)\circ(M\otimes\tau_{N,H}\otimes H\otimes H\otimes H)\\
&\quad\circ(M\otimes N\otimes\Delta\otimes\Delta)\\
&= \left(\mu^M\otimes\mu^N\right)\circ(M\otimes\tau_{N,H}\otimes H)\circ\left(\mu^M\otimes\mu^N\otimes H\otimes H\right)\circ(M\otimes\tau_{N,H}\otimes H\otimes H\otimes H)\\
&\quad\circ(M\otimes N\otimes H\otimes H\otimes\Delta)\circ(M\otimes N\otimes\Delta\otimes H)\\
&= \left(\mu^M\otimes\mu^N\right)\circ(M\otimes\tau_{N,H}\otimes H)\circ(M\otimes N\otimes\Delta)\circ\\
&\quad\circ\left(\mu^M\otimes\mu^N\otimes H\right)\circ(M\otimes\tau_{N,H}\otimes H\otimes H)\circ(M\otimes N\otimes\Delta\otimes H)\\
&= \mu^{M\otimes N}\circ\left(\mu^{M\otimes N}\otimes H\right)
\end{aligned}
$$

It is easy to prove that

$$
(r_M\otimes N\otimes k)\circ(M\otimes\tau_{N,k}\otimes k)\circ\left(M\otimes N\otimes l_K^{-1}\right)=\mathrm{Id}_{M\otimes N\otimes k} \tag{4.2}
$$

We have

$$\mu^{M\otimes N}\circ(M\otimes N\otimes u)=\left(\mu^M\otimes\mu^N\right)\circ(M\otimes\tau_{N,H}\otimes H)\circ(M\otimes N\otimes\Delta)\circ(M\otimes N\otimes u)$$
$$\overset{(3.7)}{=}\left(\mu^M\otimes\mu^N\right)\circ(M\otimes\tau_{N,H}\otimes H)\circ(M\otimes N\otimes u\otimes u)\circ\left(M\otimes N\otimes l_K^{-1}\right)=$$
$$=\left(\mu^M\otimes\mu^N\right)\circ(M\otimes H\otimes N\otimes u)\circ(M\otimes\tau_{N,H}\otimes k)\circ(M\otimes N\otimes u\otimes k)\circ\left(M\otimes N\otimes l_K^{-1}\right)=$$
$$=\left(\mu^M\otimes\mu^N\right)\circ(M\otimes H\otimes N\otimes u)\circ(M\otimes\tau_{N,H}\circ(N\otimes u)\otimes k)\left(M\otimes N\otimes l_K^{-1}\right)=$$
$$\overset{(1.3)}{=}\left(\mu^M\otimes\mu^N\right)\circ(M\otimes H\otimes N\otimes u)\circ(M\otimes(u\otimes N)\circ\tau_{N,k}\otimes k)\circ\left(M\otimes N\otimes l_K^{-1}\right)$$
$$=\left(\mu^M\otimes\mu^N\right)\circ(M\otimes H\otimes N\otimes u)\circ(M\otimes u\otimes N\otimes k)$$
$$\circ(M\otimes\tau_{N,k}\otimes k)\circ\left(M\otimes N\otimes l_K^{-1}\right)$$
$$=\left(\mu^M(M\otimes u)\otimes N\right)\circ\left(M\otimes k\otimes\mu^N\circ(N\otimes u)\right)\circ(M\otimes\tau_{N,k}\otimes k)\circ\left(M\otimes N\otimes l_K^{-1}\right)$$
$$=(r_M\otimes N)\circ(M\otimes k\otimes r_N)\circ(M\otimes\tau_{N,k}\otimes k)\circ\left(M\otimes N\otimes l_K^{-1}\right)$$
$$=(M\otimes r_N)\circ(r_M\otimes N\otimes k)\circ(M\otimes\tau_{N,k}\otimes k)\circ\left(M\otimes N\otimes l_K^{-1}\right)$$
$$\overset{(4.2)}{=}(M\otimes r_N)=r_{M\otimes N}.$$

Let us prove the same statement directly. For all $x\in M, y\in N$ and $a\in H$, we have

$$\begin{aligned}\mu^{M\otimes N}(x\otimes y\otimes a)&=\left[\left(\mu^M\otimes\mu^N\right)\circ(M\otimes\tau_{N,H}\otimes H)\circ(M\otimes N\otimes\Delta)\right](x\otimes y\otimes a)\\&=\sum xa_1\otimes ya_2\end{aligned}$$

so that, for all $x\in M, y\in N$ and $a,b\in H$ we deduce that

$$\left[\mu^{M\otimes N}\circ(M\otimes N\otimes m)\right](x\otimes y\otimes a\otimes b)=(x\otimes y)(ab)$$
$$=\sum x(ab)_1\otimes y(ab)_2=\sum x(a_1b_1)\otimes y(a_2b_2)=\sum(xa_1)b_1\otimes(ya_2)b_2$$
$$=\left[\sum(xa_1)\otimes(ya_2)\right]b=[(x\otimes y)a]b=\left[\mu^{M\otimes N}\circ\left(\mu^{M\otimes N}\otimes H\right)\right](x\otimes y\otimes a\otimes b)$$

and

$$\left[\mu^{M\otimes N}\circ(M\otimes N\otimes u)\right](x\otimes y\otimes 1_k)=(x\otimes y)1_H=x1_H\otimes y1_H=x\otimes y=r_{M\otimes N}(x\otimes y\otimes 1_k).$$

□

Proposition 4.2. *Let* $\left(M,\rho^M\right),\left(N,\rho^N\right)\in\mathcal{M}^H$. *Set*

$$\rho^{M\otimes N}:M\otimes N\overset{\rho^M\otimes\rho^N}{\longrightarrow}M\otimes H\otimes N\otimes H\overset{M\otimes\tau_{H,N}\otimes H}{\longrightarrow}M\otimes N\otimes H\otimes H\overset{M\otimes N\otimes m}{\longrightarrow}M\otimes N\otimes H.$$

Then $\left(M\otimes N,\rho^{M\otimes N}\right)\in\mathcal{M}^H$.

Proof. The Proof is dual to that of Proposition 4.1 and is left to the reader. □

Definition 4.3. *A* right H-Hopf module *is a triple* $\left(M,\mu^M,\rho^M\right)$ *where*

- $\left(M,\mu^M\right)\in\mathcal{M}_H$,

- $(M, \rho^M) \in \mathcal{M}^H$ *and*
- $\rho^M : M \to M \otimes H$ *satisfies*

$$\rho^M \circ \mu^M = \left(\mu^M \otimes m\right) \circ \left(M \otimes \tau_{H,H} \otimes H\right) \circ \left(\rho^M \otimes \Delta\right) \tag{4.3}$$

which means that

$$\sum (x \cdot h)_{(0)} \otimes (x \cdot h)_{(1)} = \sum x_{(0)} h_1 \otimes x_{(1)} h_2 \quad \text{for every } x \in M \text{ and } h \in H.$$

Proposition 4.4. *Given a triple (M, μ^M, ρ^M), where $(M, \mu^M) \in \mathcal{M}_H$ and $(M, \rho^M) \in \mathcal{M}^H$, the following assertions are equivalent*

(a) (M, μ^M, ρ^M) *is a right H-Hopf module.*

(b) $\rho^M : (M, \mu^M) \to (M \otimes H, \mu^{M \otimes H})$ *is a morphism in $\mathcal{M}_H$.*

(c) $\mu^M : (M \otimes H, \rho^{M \otimes H}) \to (M, \rho^M)$ *is a morphism in $\mathcal{M}^H$.*

Proof. $\rho^M : (M, \mu^M) \to (M \otimes H, \mu^{M \otimes H})$ is a morphism in $\mathcal{M}_H$ means that

$$\begin{aligned} \rho^M \circ \mu^M &= \mu^{M \otimes H} \circ \left(\rho^M \otimes H\right) \\ &= \left(\mu^M \otimes m\right) \circ \left(M \otimes \tau_{H,H} \otimes H\right) \circ \left(M \otimes H \otimes \Delta\right) \circ \left(\rho^M \otimes H\right) \\ &= \left(\mu^M \otimes m\right) \circ \left(M \otimes \tau_{H,H} \otimes H\right) \circ \left(\rho^M \otimes \Delta\right). \end{aligned}$$

$\mu^M : (M \otimes H, \rho^{M \otimes H}) \to (M, \rho^M)$ is a morphism in $\mathcal{M}^H$ means that

$$\begin{aligned} \rho^M \circ \mu^M &= \left(\mu^M \otimes H\right) \circ \rho^{M \otimes H} = \left(\mu^M \otimes H\right) \circ \left(M \otimes H \otimes m\right) \circ \left(M \otimes \tau_{H,H} \otimes H\right) \circ \left(\rho^M \otimes \Delta\right) \\ &= \left(\mu^M \otimes m\right) \circ \left(M \otimes \tau_{H,H} \otimes H\right) \circ \left(\rho^M \otimes \Delta\right). \end{aligned}$$

□

Definition 4.5. *Let (M, μ^M, ρ^M) and $(M', \mu^{M'}, \rho^{M'})$ be right H-Hopf modules. A linear map $f : M \to M'$ is called a* morphism of right H-Hopf modules *if it is both a module and a comodule morphism. We will denote by $\mathcal{M}_H^H$ the category of right H-Hopf modules.*

Proposition 4.6. *Let $V \in Vec_k$ and let $M \in \mathcal{M}_H^H$. Then $(V \otimes M, V \otimes \mu^M, V \otimes \rho^M) \in \mathcal{M}_H^H$. Moreover the assignment $V \mapsto (V \otimes M, V \otimes \mu^M, V \otimes \rho^M)$ and $f \mapsto f \otimes M$ yield a functor*

$$F_M : Vec_k \to \mathcal{M}_H^H$$

Proof. By Proposition 2.39, we know that $(V \otimes M, V \otimes \rho^M) \in \mathcal{M}^H$. On the other hand it is easy to show that $(V \otimes M, V \otimes \mu^M) \in \mathcal{M}_H$. Let us check the compatibility relation:

$$\rho^{V \otimes M} \circ \mu^{V \otimes M} = \left(\mu^{V \otimes M} \otimes m\right) \circ \left(V \otimes M \otimes \tau_{H,H} \otimes H\right) \circ \left(\rho^{V \otimes M} \otimes \Delta\right).$$

We compute

$$\begin{gathered}\left(\mu^{V\otimes M}\otimes m\right)\circ(V\otimes M\otimes\tau_{H,H}\otimes H)\circ\left(\rho^{V\otimes M}\otimes\Delta\right)=\\ \left(V\otimes\mu^{M}\otimes m\right)\circ(V\otimes M\otimes\tau_{H,H}\otimes H)\circ\left(V\otimes\rho^{M}\otimes\Delta\right)=\\ =\left(V\otimes\left[\left(\mu^{M}\otimes m\right)\circ(M\otimes\tau_{H,H}\otimes H)\circ\left(\rho^{M}\otimes\Delta\right)\right]\right)=\\ \overset{M\in\mathcal{M}_H^H}{=}V\otimes\left(\rho^{M}\circ\mu^{M}\right)=\left(V\otimes\rho^{M}\right)\circ\left(V\otimes\mu^{M}\right)=\rho^{V\otimes M}\circ\mu^{V\otimes M}.\end{gathered}$$

Let now $f:V\to V'$ be a k-linear map. We compute

$$\begin{aligned}((f\otimes M)\otimes H)\circ\rho^{V\otimes M}&=((f\otimes M)\otimes H)\circ\left(V\otimes\rho^{M}\right)=\\ &=\left(f\otimes\rho^{M}\right)=\\ &=\left(V'\otimes\rho^{M}\right)\circ(f\otimes M)=\rho^{V'\otimes M}\circ(f\otimes M)\end{aligned}$$

and

$$\begin{aligned}(f\otimes M)\circ\mu^{V\otimes M}&=(f\otimes M)\circ\left(V\otimes\mu^{M}\right)\\ &=\left(f\otimes\mu^{M}\right)\\ &=\left(V'\otimes\mu^{M}\right)\circ(f\otimes M\otimes H).\end{aligned}$$

Thus $F_M(f)$ is a morphism in $\mathcal{M}_H^H$. □

Lemma 4.7. $(H,m,\Delta)\in\mathcal{M}_H^H$.

Proof. We know that $(H,m)\in\mathcal{M}^H$ and $(H,\Delta)\in\mathcal{M}_H$. Moreover

$$\Delta\circ m\overset{3.9}{=}(m\otimes m)\circ(H\otimes\tau_{H,H}\otimes H)\circ(\Delta\otimes\Delta).$$

□

Definition 4.8. *Let* $\left(M,\rho^M\right)\in\mathcal{M}^H$. *Set*

$$M^{coH}=\left\{x\in M\mid\rho^M(x)=x\otimes 1_H.\right\}$$

M^{coH} *is called the* subspace of coinvariants *in* M.

Remark 4.9. *Let* $\left(M,\rho^M\right)\in\mathcal{M}^H$ *and let* $\lambda_M:M\to M\otimes H$ *be the linear map defined by setting*

$$\lambda_M(x)=x\otimes 1_H\quad\text{for every } x\in M.$$

Then

$$M^{coH}=\operatorname{Ker}\left(\rho^M-\lambda_M\right)$$

Note that, if $f:M\to M'$ *is a* k*-linear map, then*

$$(f\otimes H)\circ\lambda^M=\lambda_{M'}\circ f\tag{4.4}$$

Proposition 4.10. *The assignment $M \mapsto M^{coH}$ yields a functor*

$$(-)^{coH} : \mathcal{M}^H \to Vec_k$$

Proof. Let $i_{M^{coH}} : M^{coH} \to M$ be the canonical inclusion. Then we compute

$$\left(\rho^{M'} - \lambda_{M'}\right) \circ f \circ i_{M^{coH}} = \rho^{M'} = \left[(f \otimes H) \circ \rho^M\right] \circ i_{M^{coH}} - \lambda_{M'} \circ f \circ i_{M^{coH}}$$
$$= \left[(f \otimes H) \circ \lambda^M\right] \circ i_{M^{coH}} - \lambda_{M'} \circ f \circ i_{M^{coH}} \overset{(4.4)}{=} 0$$

It follows that there is a unique linear map $f^{coH} : M^{coH} \to M'^{coH}$ such that

$$i_{M'^{coH}} \circ f^{coH} = f \circ i_{M^{coH}}.$$

It is now easy to check that this gives rise to a functor. □

Theorem 4.11. *(The Fundamental Theorem of Hopf Modules) Let H be a Hopf algebra and let*

$$G : \mathcal{M}_H^H \to Vec_k$$

be the restriction to $\mathcal{M}_H^H$ of the functor $(-)^{coH}$ introduced in Proposition 4.10. Let

$$F = F_H : Vec_k \to \mathcal{M}_H^H$$

be the functor defined in Proposition 4.6. Then (G, F) is an equivalence of categories.

Proof. Let $\left(M, \mu^M, \rho^M\right) \in \mathcal{M}_H^H$. We compute, for $x \in M$

$$\rho^M \left(\sum x_{(0)} S\left(x_{(1)}\right)\right) \overset{(4.3)}{=} \sum \left(x_{(0)}\right)_{(0)} \left(S\left(x_{(1)}\right)\right)_1 \otimes \left(x_{(0)}\right)_{(1)} \left(S\left(x_{(1)}\right)\right)_2$$
$$= \sum \left(x_{(0)}\right)_{(0)} S\left(x_{(1)_2}\right) \otimes \left(x_{(0)}\right)_{(1)} S\left(x_{(1)_1}\right) = \sum x_{(0)} S\left(x_{(3)}\right) \otimes x_{(1)} S\left(x_{(2)}\right)$$
$$= \sum x_{(0)} S\left(x_{(2)}\right) \otimes x_{(1)_1} S\left(x_{(1)_2}\right) = \sum x_{(0)} S\left(x_{(2)}\right) \otimes \varepsilon\left(x_{(1)}\right) 1_H$$
$$= \sum x_{(0)} S\left(\varepsilon\left(x_{(1)}\right) x_{(2)}\right) \otimes 1_H = \sum x_{(0)} S\left(x_{(1)}\right) \otimes 1_H.$$

This means that $\sum x_{(0)} S\left(x_{(1)}\right) \in M^{coH}$ for every $x \in M$. Thus we may define a map

$$P : M \to M^{coH} \quad \text{by setting } P(x) = \sum x_{(0)} S\left(x_{(1)}\right) \quad \text{for every } x \in M.$$

Let us define a map $\alpha_M : M^{coH} \otimes H \to M$ by setting

$$\alpha_M (x \otimes h) = x \cdot h \quad \text{for every } x \in M^{coH} \text{ and } h \in H.$$

Let us define a map $\beta_M : M \to M^{coH} \otimes H$ by setting

$$\beta_M = (P \otimes \mathrm{Id}_H) \circ \rho^M \quad \text{i.e}$$

$$\beta_M (x) = \sum P\left(x_{(0)}\right) \otimes x_{(1)} = \sum x_{(0)_{(0)}} S\left(x_{(0)_{(1)}}\right) \otimes x_{(1)} = \sum x_{(0)} S\left(x_{(1)}\right) \otimes x_{(2)}$$

for every $x \in M$. Given $x \in M^{coH}$ and $h \in H$, we compute

$$\beta_M(\alpha_M(x \otimes h)) = \left[(P \otimes \mathrm{Id}_H) \circ \rho^M\right](x \cdot h) =$$
$$= \sum (xh)_{(0)} S\left((xh)_{(1)}\right) \otimes (xh)_{(2)} \overset{(4.3)}{=} \sum x_{(0)} h_1 S\left(x_{(1)} h_2\right) \otimes \left(x_{(2)} h_3\right)$$

Since $x \in M^{coH}$ we have that $\sum x_{(0)} \otimes x_{(1)} = x \otimes 1_H$ from which we deduce that $\sum x_{(0)} \otimes x_{(1)} \otimes x_{(2)} = x \otimes 1_H \otimes 1_H$. Therefore we get

$$\beta_M(\alpha_M(x \otimes h)) = \sum x h_1 S(h_2) \otimes h_3 = x \sum \varepsilon(h_1) \otimes h_2 = x \otimes h.$$

We deduce that $\beta_M \circ \alpha_M = \mathrm{Id}_{M^{coH} \otimes H}$. Given $x \in M$, we also compute

$$(\alpha_M \circ \beta_M)(x) = \sum x_{(0)} S\left(x_{(1)}\right) x_{(2)} = \sum x_{(0)} S\left(x_{(1)_1}\right) x_{(1)_2} = \sum x_{(0)} \varepsilon\left(x_{(1)}\right) = x.$$

Let us prove that α_M is a morphism in $\mathcal{M}_H^H$. Let $x \in M^{coH}$ and $h \in H$. We compute

$$\alpha_M((x \otimes h) \cdot t) = \alpha_M(x \otimes ht) = x \cdot (ht) = (x \cdot h) \cdot t = \alpha_M(x \otimes h) \cdot t$$

and

$$\rho^M(\alpha_M(x \otimes h)) = \rho^M(x \cdot h) = \sum x_{(0)} h_1 \otimes x_{(1)} h_2 = \sum x h_1 \otimes h_2$$
$$= (\alpha_M \otimes H) \sum (x \otimes h_1 \otimes h_2) = (\alpha_M \otimes H)\left(\rho^{M \otimes H}(x \otimes h)\right).$$

For any k-vector space V, let us define

$$\gamma_V : (V \otimes H)^{coH} \to V \quad \text{by setting } \gamma_V\left(\sum_{i=1}^n v_i \otimes h_i\right) = \sum_{i=1}^n v_i \varepsilon(h_i)$$

for every $\sum_{i=1}^n v_i \otimes h_i \in (V \otimes H)^{coH}$. Let us also define a map

$$\delta_V : V \to (V \otimes H)^{coH} \quad \text{by setting } \delta_V(v) = v \otimes 1_H \in (V \otimes H)^{coH} \text{ for every } v \in V.$$

Then, for every $v \in V$ we have that

$$\gamma_V(\delta_V(v)) = v\varepsilon(1_H) = v$$

Let now $\sum_{i=1}^n v_i \otimes h_i \in (V \otimes H)^{coH}$. Then we get that

$$\sum_{i=1}^n v_i \otimes (h_i)_1 \otimes (h_i)_2 = \sum_{i=1}^n v_i \otimes h_i \otimes 1_H.$$

and hence we obtain

$$\begin{aligned} \delta_V\left(\gamma_V\left(\sum_{i=1}^n v_i \otimes h_i\right)\right) &= \sum_{i=1}^n v_i \varepsilon(h_i) \otimes 1_H = \sum_{i=1}^n v_i \otimes \varepsilon(h_i) 1_H = \sum_{i=1}^n v_i \otimes [\varepsilon((h_i)_1)](h_i)_2 \\ &= \sum_{i=1}^n v_i \otimes h_i. \end{aligned}$$

We give as an exercise to the reader to check that both the family $(\alpha_M)_{M \in \mathcal{M}_H^H}$ and $(\gamma_V)_{V \in Vec_k}$ yield functorial morphisms between the appropriate functors. □

Exercise 4.12. *Let* $\left(M, \rho^M\right) \in \mathcal{M}^H$ *and consider* $\left(k, (u \otimes k)\, l_k^{-1}\right) \in {}^H\mathcal{M}$*. Prove that*

$$M \square_H k \simeq M^{coH}.$$

Hint: use the isomorphism in (2.24).

Chapter 5

Integrals for bialgebras

Definition 5.1. *An* augmented k-algebra *is a* $4th$*-uple* (A, m_A, u_A, π) *where:*

- (A, m_A, u_A) *is a* k*-algebra.*
- $\pi : A \to k$ *is a* k*-algebra morphism*

π *is called the* augmentation *of* A*.*

Definitions 5.2. *Let* $A = (A, m_A, u_A, \pi)$ *be an augmented algebra and let* $x \in A$*. We say that*

- x *is a* left integral in A *if*

$$a \cdot_A x = \pi(a)\, x, \text{ for every } a \in A.$$

  In this case x *is called a* total left integral *if* $\pi(x) = 1_k$*.*

- x *is a* right integral in A *if*

$$x \cdot_A a = x\pi(a), \text{ per ogni } a \in A.$$

  In this case x *is called a* total right integral *if* $\pi(x) = 1_k$*.*

The set of all left integrals in A *will be denote by* $\int_l = \int_l(A)$*.*
The set of all right integrals in A *will be denoted by* $\int_r = \int_r(A)$*.*
We will say that A *is* unimodular *whenever* $\int_l = \int_r$*. In this case an element of* $\int_l = \int_r$ *will be simply called an integral.*

Remark 5.3. $\int_l$ *and* $\int_r$ *are* k*-vector subspaces of* A*. Thus they are called* space of left, resp. right, integrals in A.

Definition 5.4. *Let* (A, m_A, u_A, π) *and* $(A', m_{A'}, u_{A'}, \pi')$ *be augmented algebras. A linear map* $f : A \to A'$ *is called a* morphism of augmented algebras *if* f *is a morphism of algebras and* $\pi' \circ f = \pi$*.*

Proposition 5.5. *Let $f : (A, m_A, u_A, \pi) \to (A', m_{A'}, u_{A'}, \pi')$ be a surjective morphism of augmented algebras. Then*

$$f\left(\int_l (A)\right) \subseteq \int_l (A').$$

Proof. Let $t \in \int_l (A)$ and let $a \in A$. We compute

$$f(a) \cdot f(t) = f(a \cdot t) = f(\pi(a) t) = \pi(a) f(t) = (\pi' \circ f)(a) f(t) = \pi'(f(a)) f(t).$$

□

Proposition 5.6. *Let (A, m_A, u_A, π) be an augmented algebra. Then $\int_l (A)$ and $\int_r (A)$ are two-sided ideals of A.*

Proof. Let $\alpha \in A$ and $x \in \int_l (A)$. We have to prove that

$$\alpha x \in \int_l (A) \qquad \text{and} \qquad x\alpha \in \int_l (A).$$

For every $a \in A$ we compute

$$a(\alpha x) = (a\alpha) x \overset{x \text{isleftint}}{=} \pi(a\alpha) x = \pi(a) \pi(\alpha) x = \pi(a)(\pi(\alpha) x) \overset{x \text{isleftint}}{=} \pi(a)(\alpha x).$$

This means that αx is a left integral in A. We also compute

$$a(x\alpha) = (ax)\alpha \overset{x \text{isleftint}}{=} (\pi(a) x)\alpha = \pi(a)(x\alpha).$$

which means that also $x\alpha$ is a left integral in A.

The proof for $\int_r (A)$ is analogous. □

5.7. *Let $(H, m_H, u_H, \Delta_H, \varepsilon_H)$ be a bialgebra. Then*

- *$(H, m_H, u_H, \varepsilon_H)$ is an augmented algebra. A left integral in H is an element $t \in H$ such that*

  $$h \cdot_H t = \varepsilon_H(h) t, \text{ for every } h \in H.$$

  It is also total if $\varepsilon_H(t) = 1_K$.

- *$(H^*, m_{H^*}, u_{H^*}, \pi_{H^*})$ is an augmented algebra where $\pi_{H^*} : H^* \to k$ is defined by setting*

  $$\pi_{H^*}(f) = f(1_H) \text{ for every } f \in H^*.$$

  A left integral in H^* is an element $\lambda \in H^*$ such that

  $$f * \lambda = \pi_{H^*}(f) \lambda, \text{ for every } f \in H^*$$

  i.e.

  $$f * \lambda = f(1_H) \lambda, \text{ for every } f \in H^*.$$

  In this case λ is a total integral if $\pi_{H^*}(\lambda) = 1_K$ i.e. $\lambda(1_H) = 1_k$.

Lemma 5.8. *(The well-known Lemma) Let V be a k-vector space and let $x, y \in V$. Then*

$$x = y \Leftrightarrow f(x) = f(y) \text{ for every } f \in V^*.$$

Proposition 5.9. *Let $(H, m, u, \Delta, \varepsilon)$ be a bialgebra and let $\lambda \in H^*$. Then we have that*

1) *λ is a left integral in H^* if and only if*

$$\sum h_1 \lambda(h_2) = 1_H \lambda(h) \quad \text{for every } h \in H. \tag{5.1}$$

2) *λ is a right integral in H^* if and only if*

$$\sum \lambda(h_1) h_2 = 1_H \lambda(h) \quad \text{for every } h \in H. \tag{5.2}$$

Proof. 1) Let $\lambda \in H^*$. Then λ is a left integral in H^* if and only if $f * \lambda = f(1_H)\lambda$, for every $f \in H^*$ which means that

$$(f * \lambda)(h) = f(1_H)\lambda(h) \quad \text{for every } f \in H^* \text{ and } h \in H.$$

We compute

$$(f * \lambda)(h) = \sum f(h_1)\lambda(h_2) = f\left(\sum h_1 \lambda(h_2)\right)$$

and

$$f(1_H)\lambda(h) = f(1_H \lambda(h)).$$

Thus λ is a left integral in H^* if and only if

$$f\left(\sum h_1 \lambda(h_2)\right) = f(1_H \lambda(h)) \quad \text{for every } h \in H \text{ and } f \in H^*.$$

In view of Lemma 5.8 this happens if and only if

$$\sum h_1 \lambda(h_2) = 1_H \lambda(h) \quad \text{for every } h \in H.$$

2) The proof is analogous. □

Proposition 5.10. *Let $(H, m, u, \Delta, \varepsilon, S)$ be a Hopf algebra and let $t \in H$. Then*

1) *If t is a left integral in H then $S(t)$ is a right integral in H.*

2) *If t is a total left integral in H, then $t = S(t)$.*

1′) *If t is a right integral in H then $S(t)$ is a left integral in H.*

2′) *If t is a total right integral in H, then $t = S(t)$.*

Proof. 1) We have to show that

$$S(t) \cdot h = \varepsilon(h) S(t) \text{ for every } h \in H.$$

Since t is a left integral in H we have

$$h \cdot t = \varepsilon(h) t \text{ for every } h \in H.$$

We compute

$$\begin{aligned} S(t)\cdot h &= S(t)\cdot\left(\sum\varepsilon(h_1)h_2\right)=\sum S[\varepsilon(h_1)t]h_2 \overset{\text{tisleftint}}{=} \sum S(h_1\cdot t)\cdot h_2 \\ &= \sum S(t)\cdot S(h_1)h_2 = S(t)\cdot\sum S(h_1)h_2 = S(t)\varepsilon(h)=\varepsilon(h)S(t). \end{aligned}$$

2) We compute

$$\begin{aligned} S(t) &= 1_k S(t) \overset{\text{tistotal}}{=} \varepsilon(t)S(t) \overset{S(t)\text{ is a right int}}{=} S(t)t \overset{t\text{ is a left int}}{=} \varepsilon[S(t)]t = \\ &= \varepsilon(t)t \overset{\text{tistotal}}{=} 1_k t = t. \end{aligned}$$

□

Corollary 5.11. *Let $(H,m,u,\Delta,\varepsilon,S)$ be a Hopf algebra and let $t\in H$. The following statements are equivalent:*

(a) t is a left total integral in H.

(b) t is a right total integral in H.

If there is a left total integral in H, then

$$\int_l(H)=\int_r(H)=kt.$$

In particular H is unimodular.

Proof. $(a)\Rightarrow(b)$ In view of Proposition 5.10, $t=S(t)$ is a right integral in H.

$(b)\Rightarrow(a)$ is analogous.

Assume now that t is a left (and hence right) total integral and let $x\in\int_l(H)$ be a left integral in H. Then

$$x=1_k x \overset{\text{tistotal}}{=} \varepsilon(t)x \overset{\text{xisleftint}}{=} tx \overset{\text{tisrightint}}{=} t\varepsilon(x)\in kt$$

so that

$$\int_l(H)\subseteq kt.$$

An analogous proof shows that $\int_r(H)=kt$. □

Proposition 5.12. *Let $(H,m,u,\Delta,\varepsilon,S)$ be a Hopf algebra and let $\lambda\in H^*$. Then*

1) If λ is a left integral in H^*, then $\lambda\circ S$ is a right integral in H^*.

2) If λ is a total left integral in H^*, then $\lambda=\lambda\circ S$.

1′) If λ is a right integral in H^*, then $\lambda\circ S$ is a left integral in H^*.

2′) If λ is a total right integral in H^*, then $\lambda=\lambda\circ S$.

Proof. 1) In view of Proposition 5.9, we have to show that

$$\sum[(\lambda\circ S)(h_1)]h_2 = 1_H[(\lambda\circ S)(h)] \quad \text{for every } h\in H$$

We compute

$$\begin{aligned}
1_H\left[(\lambda\circ S)(h)\right] &= 1_H\lambda\left[S\left(\sum h_1\varepsilon(h_2)\right)\right]\\
&= 1_H\sum\lambda\left[S\left(h_1\varepsilon(h_2)\right)\right] = 1_H\sum\lambda\left[S(h_1)\right]\varepsilon(h_2)\\
&= \sum\lambda\left[S(h_1)\right]\varepsilon(h_2)1_H = \sum\lambda\left[S(h_1)\right]\left[\sum S(h_2)h_3\right]\\
&= \sum\lambda\left[S(h_1)\right]S(h_2)h_3 = \sum\lambda\left[S(h_{1_1})\right]S(h_{1_2})h_2\\
&= \sum\lambda\left[\left[S(h_1)\right]_2\right]\left[S(h_1)\right]_1h_2 = \sum\left[S(h_1)\right]_1\lambda\left[\left[S(h_1)\right]_2\right]h_2\\
&\overset{(5.1)}{=} \sum 1_H\lambda\left[S(h_1)\right]h_2 = \sum\lambda\left[S(h_1)\right]h_2.
\end{aligned}$$

2) Since, in view of 1), $\lambda\circ S\in H^*$ is a right integral in H^*, we have that

$$(\lambda\circ S)*\lambda = (\lambda\circ S)\left[\lambda(1_H)\right] \overset{\lambda\text{is tot}}{=} \lambda\circ S$$

and since λ is a left integral we have that

$$(\lambda\circ S)*\lambda = \left[(\lambda\circ S)(1_H)\right]\lambda = \lambda\left(S(1_H)\right)\lambda = \lambda(1_H)\lambda \overset{\lambda\text{is tot}}{=} \lambda$$

so that we get

$$\lambda\circ S = \lambda.$$

□

Corollary 5.13. *Let $(H, m, u, \Delta, \varepsilon, S)$ be a Hopf algebra and let $\lambda\in H^*$. The following statements are equivalent:*

(a) λ is a total left integral in H^*.

(b) λ is a total right integral in H^*.

If there is a left total integral in H^*, then

$$\int_l(H^*) = \int_r(H^*) = k\lambda.$$

In particular H^* is unimodular.

Proof. $(a)\Rightarrow(b)$ In view of Proposition 5.12, $\lambda=\lambda\circ S$ is a right integral in H^*.

$(b)\Rightarrow(a)$ is analogous.

Assume now that λ is a left (and hence right) total integral and let $\chi\in\int_l(H^*)$ be a left integral in H^*. Then

$$\chi = 1_k\chi \overset{\lambda\text{istotal}}{=} \lambda(1_H)\chi \overset{\chi\text{isleftint}}{=} \lambda*\chi \overset{\lambda\text{isrightint}}{=} \lambda\chi(1_H)\in k\lambda$$

so that

$$\int_l(H^*)\subseteq k\lambda.$$

An analogous proof shows that $\int_r(H^*) = k\lambda$. □

5.1 H^{*rat}

5.14. *Let* $(H, m_H, u_H, \Delta_H, \varepsilon_H, S_H)$ *be a Hopf algebra. We know from 1.45 that* (H^*, m_{H^*}, u_{H^*}) *is an algebra. In particular* $(H^*, m_{H^*}) \in {}_{H^*}\mathcal{M}$ *and we can consider* $H^{*rat} = rat\left({}_{H^*}H^*\right)$*. In view of Theorem 2.30,* H^{*rat} *is a right* H*-comodule with respect to*

$$\rho = \delta_{H^{*rat}} : H^{*rat} \longrightarrow H^{*rat} \otimes H.$$

Then for every $\chi \in H^{*rat}$ *and* $f \in H^*$ *we have*

$$f * \chi = [\beta_{H^{*rat}}(\chi)](f) = [(\alpha_{H^{*rat}} \circ \rho)(\chi)](f) = \sum \chi_0 f(\chi_1)$$

so that

$$f * \chi = \sum \chi_0 f(\chi_1) \tag{5.3}$$

where

$$\rho(\chi) = \sum \chi_0 \otimes \chi_1 \quad \text{for every } \chi \in H^{*rat}.$$

Since H *is a right* H*-module via* m_H*, we have that* H^* *has a left* H*-module structure defined by*

$${}_H H^* = \mathrm{Hom}\left({}_k H_H, {}_k k\right).$$

For every $h \in H$ *and* $f \in H^*$ *we will write* $h \rightharpoonup f = h \cdot f$*. The we have*

$$(h \rightharpoonup f)(x) = f(xh) \quad \text{for all } h, x \in H \text{ and } f \in H^*.$$

Since $S = S_H : H \to H$ *is an algebra antihomomorphism, by setting*

$$f \leftharpoondown h = S(h) \rightharpoonup f$$

we obtain a right H*-module structure on* H^**. Explicitly we have*

$$(f \leftharpoondown h)(x) = (S(h) \rightharpoonup f)(x) = f(xS(h))$$

i.e.

$$(f \leftharpoondown h)(x) = f(xS(h)) \quad \text{for all } h, x \in H \text{ and } f \in H^*.$$

Theorem 5.15. H^{*rat} *is a right* H*-submodule of the right* H*-module* $(H^*, \leftharpoondown)$
Let $\mu : H^{*rat} \otimes H \to H^{*rat}$ *the induced right* H*-module structure on* H^{*rat}*.*
Then $(H^{*rat}, \mu, \rho) \in \mathcal{M}_H^H$ *is a right* H*-Hopf module.*

Proof. First of all let us recall that, in view of Proposition 2.35, we know that

$$H^{*rat} = \beta_{H^*}^{\leftarrow}\left(\alpha_{H^*}\left(H^* \otimes H\right)\right).$$

Thus to prove that H^{*rat} is a right H-submodule of the right H-module $(H^*, \leftharpoondown)$ we will prove that

$$\chi \leftharpoondown h \in \beta_{H^*}^{\leftarrow}\left(\alpha_{H^*}\left(H^* \otimes H\right)\right) = H^{*rat} \quad \text{for any } h \in H \text{ and } \chi \in H^{*rat}.$$

Actually we will prove that

$$\beta_{H^*}(\chi \leftharpoondown h) = \alpha_{H^*} \sum [(\chi_0 \leftharpoondown h_1) \otimes \chi_1 h_2] \tag{5.4}$$

which means that

$$[\beta_{H^*}(\chi \leftharpoondown h)](f) = \left\{\alpha_{H^*} \sum [(\chi_0 \leftharpoondown h_1) \otimes \chi_1 h_2]\right\}(f) \quad \text{for any } f \in H^*$$

i.e. that

$$f * (\chi \leftharpoondown h) = \sum (\chi_0 \leftharpoondown h_1) \cdot f(\chi_1 h_2) \quad \text{for any } f \in H^*.$$

This amounts to prove that

$$[f * (\chi \leftharpoondown h)](x) = \sum (\chi_0 \leftharpoondown h_1)(x) \cdot f(\chi_1 h_2) \quad \text{for any } f \in H^* \text{ and } x \in H.$$

Let us compute

$$\begin{aligned}
&\sum (\chi_0 \leftharpoondown h_1)(x) \cdot f(\chi_1 h_2) \overset{\text{def}\leftharpoondown}{=} \sum \chi_0 (xS(h_1)) \cdot f(\chi_1 h_2) \\
&\overset{\text{def}\rightharpoonup}{=} \sum \chi_0 (xS(h_1)) \cdot [(h_2 \rightharpoonup f)(\chi_1)] \overset{(5.3)}{=} \sum [(h_2 \rightharpoonup f) * \chi](xS(h_1)) \\
&= \sum [(h_2 \rightharpoonup f)(xS(h_1))_1] \cdot [\chi (xS(h_1))_2] = \sum [(h_2 \rightharpoonup f)(x_1 S(h_{1_2}))] \cdot [\chi(x_2 S(h_{1_1}))] \\
&= \sum [(h_3 \rightharpoonup f)(x_1 S(h_2))] \cdot [\chi(x_2 S(h_1))] = \sum [f(x_1 S(h_2) h_3)] \cdot [\chi(x_2 S(h_1))] \\
&= \sum [f(x_1 1_H \varepsilon(h_2))] \cdot [\chi(x_2 S(h_1))] = \sum f(x_1) \cdot [\chi(x_2 S(h_1 \varepsilon(h_2)))] \\
&= \sum f(x_1) \cdot [\chi(x_2 S(h))] \overset{\text{def}\leftharpoondown}{=} f(x_1)[(\chi \leftharpoondown h)(x_2)] = [f * (\chi \leftharpoondown h)](x).
\end{aligned}$$

Thus form (5.4) is proved.

Let $L = H^{*rat}$ and let $i_L : L \to H^*$ be the canonical inclusion. By (2.19) we have that

$$\beta_{H^*} \circ i_L = \alpha_{H^*} \circ (i_L \otimes H) \circ \rho.$$

Thus we obtain

$$\begin{aligned}
&(\alpha_{H^*} \circ (i_L \otimes H)) \left[\sum (\chi_0 \leftharpoondown h_1) \otimes \chi_1 h_2\right] = \alpha_{H^*} \left[\sum (\chi_0 \leftharpoondown h_1) \otimes \chi_1 h_2\right] = \\
&\overset{(5.4)}{=} (\beta_{H^*} \circ i_L)(\chi \leftharpoondown h) = \\
&\overset{(2.19)}{=} [\alpha_{H^*} \circ (i_L \otimes H) \circ \rho](\chi \leftharpoondown h) = (\alpha_{H^*} \circ (i_L \otimes H))[\rho(\chi \leftharpoondown h)]
\end{aligned}$$

and hence we get

$$\rho(\chi \leftharpoondown h) = \sum (\chi_0 \leftharpoondown h_1) \otimes \chi_1 h_2$$

which means that $(H^{*rat}, \mu, \rho) \in \mathcal{M}_H^H$ is a right H-Hopf module. □

Proposition 5.16. *Let $(H, m_H, u_H, \Delta_H, \varepsilon_H, S_H)$ be a Hopf algebra. Then*

1) $\int_l (H^*)$ *is a submodule of* ${}_{H^*}H^{*rat}$.

2) $(H^{*rat})^{coH} = \int_l (H^*)$.

3) *The map* $\alpha = \alpha_{H^{*rat}} : \int_l (H^*) \otimes H \longrightarrow H^{*rat}$ *defined by setting*

$$\alpha(\lambda \otimes h) = \lambda \leftharpoondown h \quad \textit{for every } \lambda \in \int_l (H^*) \ \textit{ and } h \in H$$

is an isomorphism in $\mathcal{M}_H^H$.

Proof. 1) and 2) By Proposition 5.6, $\int_l (H^*)$ is a two-sided ideal in H^*. In particular $\int_l (H^*)$ is a left H^*-submodule of H^*. Thus we may apply Proposition 2.28. Since, for any $\lambda \in \int_l (H^*)$ we have

$$f * \lambda = f(1_H)\lambda = \lambda f(1_H) \quad \text{for any } f \in H^*$$

we deduce that $X = \int_l (H^*)$ is a rational left H^*-module and that

$$\delta_X : X \longrightarrow X \otimes H \quad \text{is defined by setting } \delta_X(\lambda) = \lambda \otimes 1_H$$

so that $X \subseteq (H^{*rat})^{coH}$.
Conversely let $\chi \in (H^{*rat})^{coH}$. Then $\rho(\chi) = \chi \otimes 1_H$ and hence

$$f * \chi \overset{(5.3)}{=} \sum \chi_0 f(\chi_1) = \chi f(1_H) = f(1_H)\chi \quad \text{for every } f \in H^*$$

so that $\chi \in \int_l (H^*)$.
3) Apply now Theorem 4.11. □

Corollary 5.17. $\int_l (H^*) = \{0_{H^*}\}$ *if and only if* $H^{*rat} = \{0_{H^*}\}$.

Proof. By Proposition 5.16 we have that

$$\int_l (H^*) \otimes H \simeq H^{*rat}.$$

□

Proposition 5.18. *Let* $(H, m_H, u_H, \Delta_H, \varepsilon_H, S_H)$ *be a Hopf algebra and assume that*

$$\int_l (H^*) \neq \{0_{H^*}\} .$$

Then S_H *is injective.*

Proof. Let $\lambda \in \int_l (H^*)$, $\lambda \neq 0$ and let $h \in H$ such that $S_H(h) = 0$. By Proposition 5.16, the map α is an isomorphism. Since

$$\alpha(\lambda \otimes h) = \lambda \leftharpoondown h = S_H(h) \rightharpoonup \lambda = 0 \rightharpoonup \lambda = 0$$

and $\lambda \neq 0$ we conclude that $h = 0$. □

Proposition 5.19. *Let* $(H, m_H, u_H, \Delta_H, \varepsilon_H, S_H)$ *be a finite dimensional Hopf algebra. Then*

1) $\dim_k \int_l (H^*) = 1$

2) S_H *is bijective.*

Proof. 1) By Theorem 2.36 we have that

$$Rat\,({}_{H^*}\mathcal{M}) = {}_{H^*}\mathcal{M}.$$

and hence we get that $H^{*rat} = H^*$. Then, from Proposition 5.16 we deduce that

$$\int_l (H^*) \otimes H \simeq H^*$$

and hence

$$dim\,(H) = dim\,(H^*) = dim\left(\int_l (H^*) \otimes H\right) = dim\left(\int_l (H^*)\right) \cdot dim\,(H)$$

which implies that $dim\left(\int_l (H^*)\right) = 1$. Then, in view of Proposition 5.18, we obtain that S_H is injective and hence bijective as H has finite dimension. □

Lemma 5.20. *Let* H *be a finite dimensional Hopf algebra and consider the dual Hopf algebra* H^**. Then the space of left integrals in this Hopf algebra coincide with the space of left integrals in the augmented algebra* (H^*, π_{H^*}).

Proof. Since the algebra structure is the same, we have only to point out that $\varepsilon_{H^*} = \pi_{H^*}$. □

Lemma 5.21. *Let* H *be a finite dimensional Hopf algebra and let* $\omega : H \to H^{**}$ *the natural isomorphism. Then*

$$\omega\left(\int_l (H)\right) = \int_l (H^{**}).$$

Moreover

$$\int_l (H^{**}) = \{\alpha \in H^{**} \mid \alpha \textit{ is a left integral in the dual of the Hopf algebra } H^*\}$$

Proof. By Proposition 3.11, $\omega : H \to H^{**}$ is a Hopf algebra isomorphism. In particular $\omega : (H, \varepsilon_H) \to (H^{**}, \varepsilon_{H^{**}})$ is an isomorphism of augmented Hopf algebras. Apply now Proposition 5.5.

The last statement follows by Lemma 5.20. □

Proposition 5.22. *Let H be a finite dimensional Hopf algebra. Then*

$$\dim_k \int_l (H) = 1.$$

Moreover given a $t \in \int_l (H)$,$t \neq 0$, we have that

$$H = H^* t.$$

Proof. By Proposition 3.10, H^* is a finite dimensional Hopf algebra. Hence, by Proposition 5.19 and Lemma 5.21 we conclude.
Let $t \in \int_l (H), t \neq 0$. Then for every $x \in H$ there exists an $f \in H^*$ such that

$$\alpha_{H^{**}} (\omega(t) \otimes f) = \omega(t) \leftharpoondown f = \omega(x).$$

We compute

$$\begin{aligned}
[\omega(t) \leftharpoondown f](g) &= \omega(t)(g * S_{H^*} f) = \omega(t)(g * f \circ S_H) = (g * f \circ S_H)(t) \\
&= \sum g(t_1) f(S_H(t_2)) = g\left(\sum t_1 f(S_H(t_2))\right) = \omega\left(\sum t_1 f(S_H(t_2))\right)(g) \\
&= \omega(f \circ S_H \cdot t)(g)
\end{aligned}$$

so that $[\omega(t) \leftharpoondown f] = \omega(f \circ S_H \cdot t)$. Hence we deduce that $\omega(x) = \omega(f S_H \cdot t)$ which means that $x = f \circ S_H \cdot t \in H^* t$. □

5.2 Semisemplicity and Cosemisemplicity

Lemma 5.23. *Let H be a Hopf algebra Then we have*

1) $\sum \lambda(xS(y_1))y_2 = \sum x_1 \lambda(x_2 S(y))$ *for every* $\lambda \in \int_l (H^*)$, $x, y \in H$.

2) $\sum t_1 \otimes S(t_2)h = \sum ht_1 \otimes S(t_2)$ *for every* $t \in \int_l (H)$, $h \in H$.

Proof. 1) Let $\lambda \in \int_l (H^*)$ and $x, y \in H$. We compute

$$\begin{aligned}
\sum x_1 \lambda(x_2 S(y)) = \sum x_1 \lambda(x_2 S[\varepsilon(y_2) y_1]) &= \sum x_1 \varepsilon(y_2) \lambda(x_2 S(y_1)) \\
&= \sum x_1 [S(y_2) y_3] \lambda(x_2 S(y_1)) \\
&= \sum x_1 S(y_{1_2}) y_2 \lambda(x_2 S(y_{1_1})) \\
&= \sum x_1 [S(y_1)]_1 y_2 \lambda(x_2 [S(y_1)]_2) \\
&= \sum [(xS(y_1)]_1 y_2 \lambda([xS(y_1)]_2) \\
&= \sum [(xS(y_1)]_1 \lambda([xS(y_1)]_2) y_2 \\
&\overset{(5.1)}{=} \sum 1_H \lambda(xS(y_1)) y_2 \\
&= \sum \lambda(xS(y_1)) y_2.
\end{aligned}$$

2) Let $t \in \int_l (H)$ and $x \in H$. We compute

$$\sum \varepsilon(x)\, t_1 \otimes t_2 = \varepsilon(x)\, \Delta(t) = \Delta(\varepsilon(x)\, t) \overset{\text{tleftint}}{=} \Delta(xt) = \sum (xt)_1 \otimes (xt)_2$$

so that

$$\sum \varepsilon_H(x)\, t_1 \otimes t_2 = \sum (xt)_1 \otimes (xt)_2 . \tag{5.5}$$

We compute

$$\begin{aligned}\sum t_1 \otimes S(t_2)h &= \sum t_1 \otimes S(t_2)\varepsilon(h_1)\, h_2 = \sum \varepsilon(h_1)\, t_1 \otimes S(t_2)h_2 \\ &\overset{(5.5)}{=} \sum (h_1 t)_1 \otimes S((h_1 t)_2)h_2 = \sum h_{1_1} t_1 \otimes S(h_{1_2} t_2)h_2 \\ &= \sum h_1 t_1 \otimes S(h_2 t_2)h_3 = \sum h_1 t_1 \otimes S(t_2)S(h_2)h_3 \\ &= \sum h_1 t_1 \otimes S(t_2)\varepsilon(h_2) = \sum h t_1 \otimes S(t_2).\end{aligned}$$

□

Definition 5.24. *A k-algebra A is called* left (resp. right) semisimple *if it is left (resp. right) semisimple as a ring i.e. if every left (resp. right) A-module is projective. If A is both right and left semisimple, we will simlpy say that A* is semisimple.

Theorem 5.25 (Maschke's Theorem)**.** *Let H be a Hopf algebra*
The following statements are equivalent:

(a) *H is a left semisimple Hopf algebra.*

(a') *H is a right semisimple Hopf algebra.*

(b) *There exists a total left integral t in H.*

(c) *There exists a left integral t in H such that $\varepsilon_H(t) \neq 0$.*

Proof. $(b) \Rightarrow (c)$ It is trivial.
$(c) \Rightarrow (b)$. Let $t \in H$ be a left integral such that $\varepsilon_H(t) \neq 0$. Set

$$t' := \frac{1}{\varepsilon_H(t)} t.$$

Then t' è is a (left) total integral in H.
$(a) \Rightarrow (b)$ The map

$$\varepsilon_H : H \to k$$

is an algebra morphism. Hence k can be endowed with a left H-module structure defined by setting

$$h \cdot x = \varepsilon_H(h)\, x \quad \text{for every } h \in H \text{ and } x \in k.$$

Note that ε_H becomes automatically a left H-module morphism. Since H is a semisimple algebra, k is a projective left H-module so that, being ε_H surjective, there exists a left H-module morphism $\tau : k \to H$ such that the following diagram is commutative:

$$\begin{array}{ccc} & & k \\ & \tau \swarrow & \downarrow \mathrm{Id}_k \\ H & \xrightarrow{\varepsilon_H} & k \to 0 \end{array}$$

We set

$$t = \tau(1_k).$$

We have that

$$\varepsilon_H(t) = \varepsilon_H(\tau(1_k)) = \mathrm{Id}_k(1_k) = 1_k.$$

For any $h \in H$ let us compute

$$h \cdot t = h \cdot \tau(1_k) = \tau(h \cdot 1_k) = \tau(\varepsilon_H(h) \cdot 1_k) = \varepsilon_H(h) \cdot \tau(1_k) = \varepsilon_H(h) \cdot t.$$

We deduce that t is a total left integral in H.
$(b) \Rightarrow (a)$ Let $t \in H$ be a total left integral in H and let P be a left H-module. Let

$$\pi : M \longrightarrow N$$

be a surjective morphism of left H-modules and let $f : P \to N$ be a morphism of left H-modules.
We seek for a left H-module morphism $\overline{f}$ rendering the following diagram commutative.

$$\begin{array}{ccc} & & P \\ & \overline{f} \swarrow & \downarrow f \\ M & \xrightarrow{\pi} & N \end{array}$$

Since k is a field there exists a k-linear map $\gamma : N \longrightarrow M$ rendering the following diagram commutative

$$\begin{array}{ccc} & & N \\ & \gamma \swarrow & \downarrow \mathrm{Id}_N \\ M & \xrightarrow{\pi} & N \end{array}$$

i.e. such that $\pi \circ \gamma = \mathrm{Id}_N$. (Why?)
We define a map

$$\sigma : N \longrightarrow M \quad \text{by setting } \sigma(x) = \sum t_1 \gamma(S_H(t_2)x) \quad \text{for every } x \in N.$$

We have

$$\begin{aligned} \pi(\sigma(x)) &= \sum \pi[t_1 \gamma(S_H(t_2)x)] \overset{\pi \text{is} H\text{-lin}}{=} \sum t_1 \pi(\gamma(S_H(t_2)x)) = \sum t_1 S_H(t_2)x \\ &= \varepsilon_H(t)x = x. \end{aligned}$$

Thus we obtain that $\pi \circ \sigma = \mathrm{Id}_N$.
Now we will check that σ is a left H-module morphism. In view of Lemma 5.23, we have that

$$\sum t_1 \otimes S_H(t_2)h = \sum ht_1 \otimes S_H(t_2) \quad \text{for every } t \in \int_l (H) \text{ and } h \in H.$$

Thus we obtain

$$\sigma(hx) = \sum t_1 \gamma(S_H(t_2)hx) = \sum ht_1 \gamma(S_H(t_2)x) = h\sigma(x).$$

Now we set

$$\overline{f} = \sigma \circ f : P \to M.$$

Then $\overline{f}$ is a left H-module morphism and

$$\pi \circ \overline{f} = \pi \circ \sigma \circ f = f.$$

Since, by Corollary 5.11, any left total integral in H is a right total integral in H, the proof of $(a') \Leftrightarrow (b)$ is similar. □

Theorem 5.26. *Every semisimple Hopf algebra has finite dimension.*

Proof. In view of Theorem 5.25 there is a total left integral t in H. Now by Lemma 5.23, we have that

$$\sum t_1 \otimes S_H(t_2)h = \sum ht_1 \otimes S_H(t_2) \quad \text{for every } h \in H. \tag{5.6}$$

Let us write

$$\sum t_1 \otimes S_H(t_2) = \sum_{i=1}^{n} a_i \otimes b_i.$$

Then (5.6) rewrites as

$$\sum_{i=1}^{n} a_i \otimes b_i h = \sum_{i=1}^{n} ha_i \otimes b_i \quad \text{for every } h \in H. \tag{5.7}$$

Let $(e_i)_{i\in I}$ be a basis for H over k and let $(e_i^*)_{i\in I}$ be the dual basis. We have $e_j^*(e_i) = \delta_{ij}$ for every $i, j \in I$. Then for every $h \in H$ there is a finite subset $I(h)$ of I such that

$$x = \sum_{i\in I(h)} e_i^*(h)\, e_i.$$

We compute

$$\begin{aligned} h = h1_H = h\varepsilon_H(t)\,1_H = h\sum t_1 S_H(t_2) = h\sum_{i=1}^{n} a_i b_i \\ = \sum_{i=1}^{n} h a_i b_i = \sum_{i=1}^{n} h a_i \cdot \sum_{j\in I(b_i)} e_j^*(b_i)\, e_j \\ \overset{(5.7)}{=} \sum_{i=1}^{n} a_i \cdot \sum_{j\in I(b_i)} e_j^*(b_i h)\, e_j \\ = \sum_{i=1}^{n} \sum_{j\in I(b_i)} e_j^*(b_i h)\, a_i e_j. \end{aligned}$$

Hence

$$\{a_i e_j \mid i = 1, \ldots, n \text{ and } j \in I(b_i)\}$$

is a finite set of generators of H over k. □

Definition 5.27. *A coalgebra C is called* left (resp. right) cosemisimple *if every left (resp. right) C-comodule is injective.*

If C is both right and left cosemisimple, we will simply say that C is cosemisimple.

Theorem 5.28 (Dual Maschke's Theorem). *Let H be a Hopf algebra. The following statements are equivalent:*

(a) *H is a left cosemisimple Hopf algebra.*

(a') *H is a right cosemisimple Hopf algebra.*

(b) *There exists a left total integral λ in H^*.*

(c) *There exists a left integral λ in H^* such that $\lambda(1_H) \neq 0$.*

Proof. $(b) \Rightarrow (c)$ It is trivial.
$(c) \Rightarrow (b)$ Let $\lambda \in H^*$ be a left integral such that $\lambda(1_H) \neq 0$. Set

$$\lambda' := \frac{1}{\lambda(1_H)}\lambda.$$

Then λ' is a total left integral in H^*.
$(a) \Rightarrow (b)$ The map

$$u_H : k \to H : k \longmapsto k1_H$$

is a coalgebra morphism. Hence k can be endowed with a left H-comodule structure defined by setting

$${}^k\rho(x) = x1_H \otimes 1_k \quad \text{for every } x \in k.$$

Note that u_H becomes automatically a left H-comodule morphism. Since H is a left cosemisimple coalgebra, k is an injective left H-comodule so that, being u_H injective there exists a left H-comodule morphism $\lambda : k \to H$ such that the following diagram is commutative:

$$\begin{array}{ccc} k & \overset{u_H}{\longrightarrow} & H \\ \mathrm{Id}_k \downarrow & \swarrow \lambda & \\ k & & \end{array}$$

Then we have

$$\lambda(1_H) = \lambda(u_H(1_k)) = \mathrm{Id}_k(1_k) = 1_k.$$

Moreover, since λ is a left H-comodule morphism, we have that

$$(H \otimes \lambda) \circ \Delta_H = {}^k\rho \circ \lambda.$$

This means that

$$\sum h_1 \otimes \lambda(h_2) = \lambda(h) 1_H \otimes 1_k \quad \text{for every } h \in H$$

from which we deduce

$$\sum h_1 \lambda(h_2) = \lambda(h) 1_H \quad \text{for every } h \in H$$

Therefore λ is a total left integral in H^*.
$(b) \Rightarrow (a)$. Let $\lambda \in H^*$ be a total left integral in H^* and let E be a left H-comodule. Let

$$\sigma : M \longrightarrow N$$

be an injective morphism of left H-comodules and let $f : M \to E$ be a morphism of left H-comodules.
We seek for a left H-comodule morphism $\overline{f}$ rendering the following diagram commutative.

$$\begin{array}{ccc} M & \overset{\sigma}{\longrightarrow} & N \\ f \downarrow & \swarrow \overline{f} & \\ E & & \end{array}$$

Since k is a field, there exists a k-linear map $\gamma : N \longrightarrow M$ rendering the following diagram commutative

$$\begin{array}{ccc} M & \overset{\sigma}{\longrightarrow} & N \\ \mathrm{Id}_M \downarrow & \swarrow \gamma & \\ M & & \end{array}$$

i.e. such that $\gamma \circ \sigma = \mathrm{Id}_M$. (Why?)
We define a map

$$\pi : N \longrightarrow M \quad \text{by setting } \pi(y) = \sum \lambda\left[y_{-1} S_H((\gamma(y_0))_{-1})\right] (\gamma(y_0))_0 \quad \text{for every } y \in N.$$

Since σ is a morphism of left H-comodules, we have that

$$(H \otimes \sigma) \circ {}^M\rho = {}^N\rho \circ \sigma$$

which means that

$$\sum x_{-1}\otimes\sigma(x_0)=\sigma(x)_{-1}\otimes\sigma(x)_0 \quad \text{for every } x\in M.$$

We compute

$$(\pi\circ\sigma)(x)=\sum\lambda\left[\sigma(x)_{-1}S_H((\gamma(\sigma(x)_0))_{-1})\right](\gamma(\sigma(x)_0))_0$$
$$=\sum\lambda\left[x_{-1}S_H((\gamma(\sigma(x_0)))_{-1})\right](\gamma(\sigma(x_0)))_0=\sum\lambda\left[x_{-1}S_H((x_0)_{-1})\right](x_0)_0$$
$$=\sum\lambda\left[x_{-2}S_H(x_{-1})\right]x_0=\sum\lambda\left[x_{-1_1}S_H(x_{-1_2})\right]x_0=\sum\lambda\left[\varepsilon_H(x_{-1})1_H\right]x_0$$
$$=\sum\lambda(1_H)\,\varepsilon_H(x_{-1})x_0=\sum 1_k\varepsilon_H(x_{-1})x_0=x.$$

Thus we obtain that $\pi\circ\sigma=\mathrm{Id}_M$.
Let us prove that π is a morphism of left H-comodules, i.e. that

$$(H\otimes\pi)\circ{}^N\rho={}^M\rho\circ\pi$$

In view of Lemma 5.23, we have

$$\sum\lambda(xS_H(y_1))y_2=\sum x_1\lambda(x_2S_H(y)) \quad \text{for every } \lambda\in\int_l(H^*) \text{ and } x,y\in H.$$

Thus, for every $y\in N$, we obtain

$$\left[(H\otimes\pi)\circ{}^N\rho\right](y)=\sum y_{-1}\otimes\pi(y_0)$$
$$=\sum y_{-2}\otimes\lambda\left[y_{-1}S_H((\gamma(y_0))_{-1})\right](\gamma(y_0))_0$$
$$=\sum y_{-2}\lambda\left[y_{-1}S_H((\gamma(y_0))_{-1})\right]\otimes(\gamma(y_0))_0$$
$$=\sum y_{-1_1}\lambda\left[y_{-1_2}S_H((\gamma(y_0))_{-1})\right]\otimes(\gamma(y_0))_0$$
$$=\sum\lambda\left[y_{-1}S_H((\gamma(y_0))_{-1_1}\right](\gamma(y_0))_{-1_2}\otimes(\gamma(y_0))_0$$
$$=\sum\lambda\left[y_{-1}S_H((\gamma(y_0))_{-2}\right](\gamma(y_0))_{-1}\otimes(\gamma(y_0))_0$$
$$={}^M\rho\left\{\sum\lambda\left[y_{-1}S_H((\gamma(y_0))_{-1}\right](\gamma(y_0))_0\right\}$$
$$=\left({}^M\rho\circ\pi\right)(y)$$

Now we set

$$\overline{f}=f\circ\pi:N\to E.$$

Then $\overline{f}$ is a morphism of left H-comodules and

$$\overline{f}\circ\sigma=f\circ\pi\circ\sigma=f.$$

Since by Corollary 5.13, any left total integral in H^* is a right total integral in H^*, the proof of $(a')\Leftrightarrow(b)$ is similar. □

Corollary 5.29. *Let H be a finite dimensional Hopf algebra. Then*
H is semisimple $\Longleftrightarrow H^*$ is cosemisimple.
H is cosemisimple $\Longleftrightarrow H^*$ is semisimple.

Proof. Recall that, by Lemma 5.21

$$\omega\left(\int_l (H)\right) = \int_l (H^{**}) = \{\alpha \in H^{**} \mid \alpha \text{ is a left integral in the dual of the Hopf algebra } H^*\}$$

By Maschke Theorem 5.25, H is semisimple $\Longleftrightarrow$ there exists a left integral t in H such that $\varepsilon_H(t) \neq 0$.By Dual Maschke Theorem 5.28, H is cosemisimple $\Longleftrightarrow$ there exists a left integral λ in H^* such that $\lambda(1_H) \neq 0$.

Thus, by the foregoing we have that H^* is cosemisimple$\Longleftrightarrow$ there exists a left integral $t \in \int_l (H)$ such that $0 \neq \omega(t)(1_{H^*}) = \omega(t)(\varepsilon_H) = \varepsilon_H(t) \Longleftrightarrow H$ is semisimple.

Analogously H^* is semisimple $\Longleftrightarrow$ there exists a left integral λ in H^* such that $0 \neq \varepsilon_{H^*}(\lambda) = \lambda(1)$ i.e. H is cosemiusimple. $\square$

Chapter 6

Examples

6.1 kG

Let $(G, m_G, 1_G)$ be a multiplicative monoid. Then we can consider the monoid algebra kG (see Example 1.9). Recall that as a k-vector space it is just $k^{(G)}$ where the multiplication is defined by setting

$$(\alpha \cdot \beta)(x) = \sum_{\substack{z,w \in G \\ zw = x}} \alpha(z)\beta(w).$$

Then, for each $x \in G$, let e_x be the element of $k^{(G)}$ defined by

$$e_x(x) = 1_k \qquad \text{and} \qquad e_x(y) = 0_k \qquad \text{for every } y \in G, y \neq x.$$

Then, accordingly to 1.4, we write x **instead of** e_x for every $x \in G$ so that every element $\alpha \in k^{(G)}$ can be uniquely written, using the k-vector space structure of $k^{(G)}$, as

$$\alpha = \sum_{x \in Supp(\alpha)} \alpha(x)\, x.$$

Then the product in kG is uniquely defined by setting

$$x \cdot_{kG} y = x \cdot_G y$$

for every $x, y \in G$. In particular the identity 1_{kG} of kG is

$$1_{kG} = 1_G.$$

On the other hand, we can consider the grouplike coalgebra $(kG, \Delta_{kG}, \varepsilon_{kG})$ introduced in example 2a) 1.12. We have

$$\Delta_{kG}(x) = x \otimes x \quad \text{and} \quad \varepsilon_{kG}(x) = 1_k \quad \text{for every } x \in G.$$

Let us check that $(G, m_G, 1_G, \Delta_{kG}, \varepsilon_{kG})$ is a bialgebra. Indeed, we have:

$$\Delta_{kG}(xy) = xy \otimes xy = (x \otimes x)(y \otimes y) = \Delta_{kG}(x)\,\Delta_{kG}(y) \text{ for every } x, y \in G \quad \text{and } \Delta_{kG}(1_G) = 1_G \otimes$$

Moreover

$$\varepsilon_{kG}(xy) = 1_k = 1_k 1_k = \varepsilon_{kG}(x)\,\varepsilon_{kG}(y) \qquad \text{for every } x, y \in G \qquad \text{and } \varepsilon_{kG}(1_{kG}) = 1_k.$$

Assume now that G **is a group**. Then $(G, m_G, 1_G, \Delta_{kG}, \varepsilon_{kG}, S_{kG})$ is a Hopf algebra where

$$S_{kG}(g) = g^{-1} \text{ for every } g \in G.$$

In fact we have

$$(S_{kG} * \mathrm{Id}_{kG})(g) = g^{-1} \cdot g = 1_G = \varepsilon_{kG}(g) = g \cdot g^{-1} = (\mathrm{Id}_{kG} * S_{kG})(g) \text{ for every } g \in G.$$

Let $\lambda : kG \to k$ be the k-linear map defined by setting

$$\lambda(g) = \delta_{g,1_G} 1_k \quad \text{for every } g \in G.$$

Let us check that λ is a total left integral in $(kG)^*$. Let $f \in (kG)^*$ and, for every $x \in G$, let us compute

$$(f * \lambda)(x) = f(x)\,\lambda(x) = f(x)\,\delta_{x,1_G} = f(1_G)\,\delta_{x,1_G} = f(1_G)\,\lambda(x).$$

Thus we deduce that

$$f * \lambda = f(1_G)\,\lambda.$$

Moreover we have

$$\lambda(1_{kG}) = \lambda(1_G) = 1_k.$$

Thus, by The Dual Maschke's Theorem 5.28, kG is always a **cosemisimple** Hopf algebra.

Assume now that G is a **finite** group and let us set

$$t = \sum_{g \in G} g.$$

For every $x \in G$, we compute

$$x \cdot t = \sum_{g \in G} x \cdot g = \sum_{g \in G} g = t = 1_k t = \varepsilon_{kG}(x)\,t.$$

Therefore t is a left integral in kG. Since $t \neq 0_{kG}$, by Proposition 5.22, we know that $\int_l (H) = kt$. Thus we deduce, by Maschke's Theorem 5.25, that kG is also **semisimple** if and only if $\varepsilon_{kG}(t) \neq 0_k$. Therefore we compute

$$\varepsilon_{kG}(t) = \varepsilon_{kG}\left(\sum_{g \in G} g\right) = \sum_{g \in G} \varepsilon_{kG}(g) = |G|\,1_k.$$

Hence we conclude that, for a finite group G, kG is semisimple if and only if $char(k) \nmid |G|$.

When G is a **finite** group, by Proposition 3.10, $(kG)^*$ is also a Hopf algebra. Note that, since kG is a cocommutative Hopf algebra, $(kG)^*$ is a commutative Hopf algebra. Denote by $p_g : kG \to k$ the dual of the element $g \in G$, i.e. $p_g(h) = \delta_{g,h}$ for every $g, h \in G$. Then the p_g's, $g \in G$, are a basis of the k-vector space $(kG)^*$ and we have

$$(p_g * p_h)(x) = \delta_{g,x}\delta_{h,x}$$

so that

$$(p_g * p_h)(x) = 1_k \quad \text{if } g = x = h \quad \text{and } (p_g * p_h)(x) = 0_k \text{ otherwise, i.e.}$$

$$p_g * p_h = \delta_{g,h}p_g \quad \text{and} \quad \sum_{g\in G} p_g = \varepsilon_{kG} = 1_{(kG)^*}$$

which means that $(p_g)_{g\in G}$ is a complete system of orthogonal idempotents of the k-algebra $(kG)^*$. Moreover, for every $f \in (kG)^*$, we have

$$\Delta_{(kG)^*}(f) = \sum f_1 \otimes f_2$$

where $\sum f_1 \otimes f_2$ is uniquely defined by

$$f(gh) = \sum f_1(g) f_2(h) \text{ for every } g, h \in G.$$

Since the $p_g \otimes p_h, g, h \in G$ constitute a basis of $(kG)^* \otimes (kG)^*$, there exist elements $\alpha_{g,h} \in k$ such that

$$\Delta_{(kG)^*}(f) = \sum_{g,h\in G} \alpha_{g,h} p_g \otimes p_h$$

and hence

$$f(xy) = \sum_{g,h\in G} \alpha_{g,h} p_g(x) p_h(y) = \alpha_{x,y} \quad \text{for every } x, y \in G$$

so that

$$\Delta_{(kG)^*}(f) = \sum_{g,h\in G} f(gh) p_g \otimes p_h.$$

In particular, for $f = p_x$ we obtain

$$\Delta_{(kG)^*}(p_x) = \sum_{g,h\in G} p_x(gh) p_g \otimes p_h = \sum_{\substack{g,h\in G\\ gh=x}} p_g \otimes p_h = \sum_{g\in G} p_g \otimes p_{g^{-1}x}.$$

Moreover we have

$$\varepsilon_{(kG)^*}(p_x) = p_x(1_G) = \delta_{x,1_G} 1_k$$

and

$$\left[S_{(kG)^*}(f)\right](x) = [f \circ S_{kG}](x) = f\left(x^{-1}\right)$$

so that

$$\left[S_{(kG)^*}(p_g)\right](x) = p_g\left(x^{-1}\right) = \delta_{g,x^{-1}} 1_k = \delta_{g^{-1},x} 1_k = p_{g^{-1}}(x)$$

i.e.

$$S_{(kG)^*}(p_g) = p_{g^{-1}}.$$

Clearly, by the foregoing, $\lambda = p_{1_G}$ is a total integral in $(kG)^*$ so that, for a finite group G, $(kG)^*$ is always semisimple. Moreover, by means of Lemma 5.21, it is easy to prove that $(kG)^*$ is cosemisimple if and only if $char\,(k) \nmid |G|$.

We list all these result in the following theorem.

Theorem 6.1 (Classical Maschke's Theorem). *Let k be a field and let G be a group. Then*

- *the Hopf algebra kG is always cosemisimple.*
- *If G is a finite group, kG is semisimple if and only if $char\,(k) \nmid |G|$ if and only if $(kG)^*$ is cosemisimple.*
- *If G is a finite group, $(kG)^*$ is always semisimple.*

6.2 The Tensor Algebra

Let A be a ring and $M = {}_AM_A$ be a two-sided A-module.. Set

$$M^{\otimes_A^0} = A, \quad M^{\otimes_A^1} = M, \quad \text{and } M^{\otimes_A^n} = M^{\otimes_A^{n-1}} \otimes_A M \quad \text{for every } n \in \mathbb{N}, n \geq 2$$

and let

$$T_A(M) = \bigoplus_{n \in \mathbb{N}} M^{\otimes_A^n}.$$

For every $n \in \mathbb{N}$, let $i_n : M^{\otimes_A^n} \to T_A(M)$ be the obvious injective A-bimodule homomorphism. We define on $T = T_A(M)$ a multiplication by setting

$$\begin{aligned}
i_0(a) \cdot_T i_0(b) &= i_0(a \cdot_A b) \quad \text{for every } a, b \in A \\
i_0(a) \cdot_T i_n(x_1 \otimes_A \ldots \otimes_A x_n) &= i_n[(a \cdot_M x_1) \otimes_A \ldots \otimes_A x_n] \\
\text{for every } a &\in A, n \in \mathbb{N}, n \geq 1, x_1, \ldots, x_n \in M \\
i_n(x_1 \otimes_A \ldots \otimes_A x_n) \cdot_T i_0(a) &= i_n[x_1 \otimes_A \ldots \otimes_A (x_n.a)] \\
\text{for every } a &\in A, n \in \mathbb{N}, n \geq 1, x_1, \ldots, x_n \in M \\
i_m(x_1 \otimes_A \ldots \otimes_A x_m) \cdot_T i_n(y_1 \otimes_A \ldots \otimes_A y_n) &= i_{m+n}(x_1 \otimes_A \ldots \otimes_A x_m \otimes_A y_1 \otimes_A \ldots \otimes_A y_n) \\
\text{for every } m, n &\in \mathbb{N}, m, n \geq 1, x_1, \ldots, x_m, y_1, \ldots, y_n \in M
\end{aligned}$$

and extending it by linearity on T.

Lemma 6.2. *Let A be a bialgebra and let $h : A \to A^{op}$ be an algebra homomorphism. If, for $a, b \in A$, $(h * \mathrm{Id}_A)(a) = (u_A \circ \varepsilon_A)(a)$ and $(h * \mathrm{Id}_A)(b) = (u_A \circ \varepsilon_A)(b)$ then $(h * \mathrm{Id}_A)(ab) = (u_A \circ \varepsilon_A)(ab)$.*

Proof. Let us compute

$$\begin{aligned}(h * \mathrm{Id}_A)(ab) &= \sum h\left((ab)_1\right)(ab)_2 = \sum h(a_1 b_1)\, a_2 b_2 \overset{h\text{antialgmor}}{=} \sum h(b_1)\, h(a_1)\, a_2 b_2 \\ &= \sum h(b_1)\, \varepsilon_A(a)\, 1_A b_2 = \varepsilon_A(a) \sum h(b_1)\, b_2 = \varepsilon_A(a)\, \varepsilon_A(b)\, 1_A \\ &= \varepsilon_A(ab)\, 1_A = (u_A \circ \varepsilon_A)(ab).\end{aligned}$$

□

Theorem 6.3. *Let A be a ring and let $M = {}_A M_A$ be a two-sided A-module. Then, with respect to the structure defined above, $T_A(M)$ becomes a ring. Moreover $T_A(M)$ fulfills the following universal property. Let $f_0 : A \to B$ be a ring homomorphism and let $f_1 : M \to B$ be an A-bimodule homomorphism. Then there exists an algebra homomorphism $f : T_A(M) \to B$ such that*

$$f \circ i_0 = f_0 \quad \text{and} \quad f \circ i_1 = f_1.$$

Moreover f is unique with respect to this property.

Proof. For every $n \in \mathbb{N}, n \geq 2$, let us define

$$f_n : M^{\otimes_A^n} \to B$$

by setting

$$f_n(x_1 \otimes_A \ldots \otimes_A x_n) = f_1(x_1) \cdot_B \ldots \cdot_B f_1(x_n)$$
$$\text{for every } x_1 \otimes_A \ldots \otimes_A x_n \in M^{\otimes_A^n}.$$

Note that f_n is well defined since f_1 is a morphism of A-bimodules. Let $f : T = T_A(M) \to B$ be the codiagonal morphism of $(f_n)_{n \in \mathbb{N}}$. Then $f \circ i_j = f_j$ for every $j \in \mathbb{N}$. For every $a, b \in M^{\otimes_A^0} = A$, we compute

$$f(i_0(a) \cdot_T i_0(b)) = f(i_0(a \cdot_A b)) = f_0(a \cdot_A b) = f_0(a) \cdot_B f_0(b) = f(i_0(a)) \cdot_B f(i_0(b)).$$

For every $a \in M^{\otimes_A^0} = A$, for every $n \in \mathbb{N}, n \geq 1$ and for every $x_1 \otimes_A \ldots \otimes_A x_n \in M^{\otimes_A^n}$, we compute

$$\begin{aligned} f(i_0(a) \cdot_T i_n(x_1 \otimes_A \ldots \otimes_A x_n)) &= f(i_n[(a \cdot_M x_1) \otimes_A \ldots \otimes_A x_n]) = \\ = f_1(a \cdot_M x_1) \cdot_B \ldots \cdot_B f_1(x_n) &= [f_0(a) \cdot_B f_1(x_1)] \cdot_B \ldots \cdot_B f_1(x_n) = \\ = f_0(a) \cdot_B [f_1(x_1) \cdot_B \ldots \cdot_B f_1(x_n)] &= f[i_0(a)] \cdot_B f[i_n(x_1 \otimes_A \ldots \otimes_A x_n)]. \end{aligned}$$

Similarly, one gets

$$f(i_n(x_1 \otimes_A \ldots \otimes_A x_n) \cdot_T i_0(a)) = f[i_n(x_1 \otimes_A \ldots \otimes_A x_n)] \cdot_B f[i_0(a)].$$

For every $n, m \in \mathbb{N}, n, m \geq 1$ and for every $x_1 \otimes_A \ldots \otimes_A x_m \in M^{\otimes_A^m}$ and for every $y_1 \otimes_A \ldots \otimes_A y_n \in M^{\otimes_A^n}$, we compute

$$\begin{aligned} & f\left[i_m\left(x_1 \otimes_A \ldots \otimes_A x_m\right) \cdot_T i_n\left(y_1 \otimes_A \ldots \otimes_A y_n\right)\right]= \\ & =f\left[i_{m+n}\left(x_1 \otimes_A \ldots \otimes_A x_m \otimes_A y_1 \otimes_A \ldots \otimes_A y_n\right)\right] \\ & =f_1\left(x_1\right) \cdot_B \ldots \cdot_B f_1\left(x_m\right) \cdot_B f_1\left(y_1\right) \cdot_B \ldots \cdot_B f_1\left(y_n\right)= \\ & =f\left[i_m\left(x_1 \otimes_A \ldots \otimes_A x_m\right)\right] \cdot_B f\left[i_n\left(y_1 \otimes_A \ldots \otimes_A y_n\right)\right]. \end{aligned}$$

Let $g: T \rightarrow B$ be another algebra morphism such that $g \circ i_0=f_0$ and $g \circ i_1=f_1$. Then, for every $n \in \mathbb{N}, n \geq 2$, we compute

$$\begin{aligned} \left(g \circ i_n\right)\left(x_1 \otimes_A \ldots \otimes_A x_n\right) & =g\left(i_1\left(x_1\right) \cdot_T \ldots \cdot_T i_1\left(x_n\right)\right)=g\left(i_1\left(x_1\right)\right) \cdot_B \ldots \cdot_B g\left(i_1\left(x_n\right)\right) \\ & =f_1\left(x_1\right) \cdot_B \ldots \cdot_B f_1\left(x_n\right)=\left(f \circ i_n\right)\left(x_1 \otimes_A \ldots \otimes_A x_n\right). \end{aligned}$$

□

Assume now that $A=k$ is a field and that M is a k-vector space. In this case, we want to define a coalgebra structure on $T=T_k(M)$. To this aim, we will consider the algebra tensor product of T by itself. To avoid confusion, we will write this tensor product and his elements as

$$T \overline{\otimes} T, \quad x \overline{\otimes} y.$$

Set $f_0=\left(i_0 \overline{\otimes} i_0\right) \circ \Delta_k: k \rightarrow T \overline{\otimes} T$ where $\Delta_k=l_k^{-1}=r_k^{-1}$(see 1.26). Then f_0 is a bialgebra map.Let us consider the map $f_1: M \rightarrow T \overline{\otimes} T$ defined by setting

$$f_1(x)=i_1(x) \overline{\otimes} i_0\left(1_k\right)+i_0\left(1_k\right) \overline{\otimes} i_1(x), \quad \text{for every } x \in M.$$

Clearly f_1 is a k-linear map. Then, by the universal property of the tensor algebra, there exists a unique algebra map $\Delta_T: T \rightarrow T \overline{\otimes} T$ such that

$$\Delta_T \circ i_0=\left(i_0 \overline{\otimes} i_0\right) \circ \Delta_k \quad \text{and} \quad \Delta_T \circ i_1=f_1.$$

Always by the the universal property of the tensor algebra, there exists a unique algebra map $\varepsilon_T: T \rightarrow k$ such that

$$\varepsilon_T \circ i_0=\varepsilon_k=\mathrm{Id}_k \quad \text{and} \quad \varepsilon_T \circ i_1=0.$$

Let us check that $\left(T, \Delta_T, \varepsilon_T\right)$ is a bialgebra. We compute

$$\begin{gathered} {\left[\left(\mathrm{Id}_T \overline{\otimes} \Delta_T\right) \circ \Delta_T\right] \circ i_0=\left(\mathrm{Id}_T \overline{\otimes} \Delta_T\right) \circ\left(i_0 \overline{\otimes} i_0\right) \circ \Delta_k=\left[i_0 \overline{\otimes}\left(\Delta_T \circ i_0\right)\right] \circ \Delta_k} \\ =\left[i_0 \overline{\otimes}_H\left(\left(i_0 \overline{\otimes} i_0\right) \circ \Delta_k\right)\right] \circ \Delta_k=\left(i_0 \overline{\otimes} i_0 \overline{\otimes} i_0\right) \circ\left(k \otimes \Delta_k\right) \circ \Delta_k=\left(i_0 \overline{\otimes} i_0 \overline{\otimes} i_0\right) \circ\left(\Delta_k \otimes k\right) \circ \Delta_k= \\ {\left[\left(\left(i_0 \overline{\otimes} i_0\right) \circ \Delta_k\right) \overline{\otimes} i_0\right] \circ \Delta_k=\left[\left(\Delta_T \circ i_0\right) \overline{\otimes} i_0\right] \circ \Delta_k=} \\ =\left(\Delta_T \overline{\otimes} \mathrm{Id}_T\right) \circ\left(i_0 \overline{\otimes} i_0\right) \circ \Delta_k=\left[\left(\Delta_T \overline{\otimes} \mathrm{Id}_T\right) \circ \Delta_T\right] \circ i_0 \end{gathered}$$

so that, we obtain

$$\left[\left(\mathrm{Id}_T \overline{\otimes} \Delta_T\right) \circ \Delta_T\right] \circ i_0=\left[\left(\Delta_T \overline{\otimes} \mathrm{Id}_T\right) \circ \Delta_T\right] \circ i_0 \tag{6.1}$$

For every $x \in M$, we calculate

$$
\begin{gathered}
([(\mathrm{Id}_T \overline{\otimes} \Delta_T) \circ \Delta_T] \circ i_1)(x) = ((\mathrm{Id}_T \overline{\otimes} \Delta_T) \circ f_1)(x) \\
= (\mathrm{Id}_T \overline{\otimes} \Delta_T)[i_1(x) \overline{\otimes} i_0(1_k) + i_0(1_k) \overline{\otimes} i_1(x)] = \\
= i_1(x) \overline{\otimes} \Delta_T(i_0(1_k)) + i_0(1_k) \overline{\otimes} \Delta_T(i_1(x)) = \\
= i_1(x) \overline{\otimes} i_0(1_k) \overline{\otimes} i_0(1_k) + i_0(1_k) \overline{\otimes} (i_1(x) \overline{\otimes} i_0(1_k) + i_0(1_k) \overline{\otimes} i_0(1_k) \overline{\otimes} i_1(x)) = \\
= (i_1(x) \overline{\otimes} i_0(1_k) + i_0(1_k) \overline{\otimes} i_1(x)) \overline{\otimes} i_0(1_k) + i_0(1_k) \overline{\otimes} i_0(1_k) \overline{\otimes} i_1(x) = \\
= \Delta_T(i_1(x)) \overline{\otimes} i_0(1_k) + \Delta_T(i_0(1_k)) \overline{\otimes} i_1(x) = \\
= (\Delta_T \overline{\otimes} \mathrm{Id}_T)(i_1(x) \overline{\otimes} i_0(1_k) + i_0(1_k) \overline{\otimes} i_1(x)) = ((\Delta_T \overline{\otimes} \mathrm{Id}_T) \circ f_1)(x) = \\
= ([(\Delta_T \overline{\otimes} \mathrm{Id}_T) \circ \Delta_T] \circ i_1)(x)
\end{gathered}
$$

so that we obtain

$$
[(\mathrm{Id}_T \overline{\otimes} \Delta_T) \circ \Delta_T] \circ i_1 = [(\Delta_T \overline{\otimes} \mathrm{Id}_T) \circ \Delta_T] \circ i_1. \tag{6.2}
$$

By the uniqueness in the universal property of T, from (6.1) and (6.2) we deduce that

$$
(\mathrm{Id}_T \overline{\otimes} \Delta_T) \circ \Delta_T = (\Delta_T \overline{\otimes} \mathrm{Id}_T) \circ \Delta_T.
$$

Let us compute

$$
\begin{gathered}
(l_T \circ (\varepsilon_T \overline{\otimes} T) \circ \Delta_T) \circ i_0 = l_T \circ (\varepsilon_T \overline{\otimes} T) \circ (i_0 \overline{\otimes} i_0) \circ \Delta_k = l_T \circ ((\varepsilon_T \circ i_0) \overline{\otimes} i_0) \circ \Delta_k = \\
= l_T \circ (k \overline{\otimes} i_0) \circ (\varepsilon_k \overline{\otimes} k) \circ \Delta_k \overset{(1.1)}{=} i_0 \circ l_k \circ (\varepsilon_k \overline{\otimes} k) \circ \Delta_k = i_0
\end{gathered}
$$

so that we obtain

$$
(l_T \circ (\varepsilon_T \overline{\otimes} T) \circ \Delta_T) \circ i_0 = i_0. \tag{6.3}
$$

For every $x \in M$, we calculate

$$
\begin{gathered}
[(l_T \circ (\varepsilon_T \overline{\otimes} T) \circ \Delta_T) \circ i_1](x) = (l_T \circ (\varepsilon_T \overline{\otimes} T))(i_1(x) \overline{\otimes} i_0(1_k) + i_0(1_k) \overline{\otimes} i_1(x)) = l_T(\varepsilon_T(i_1(x)) \overline{\otimes} i_0(\\
= l_T(1_k \overline{\otimes} i_1(x)) = 1_k \cdot_T i_1(x) = i_1(x)
\end{gathered}
$$

so that we get

$$
(l_T \circ (\varepsilon_T \overline{\otimes} T) \circ \Delta_T) \circ i_1 = i_1. \tag{6.4}
$$

By the uniqueness in the universal property of T, from (6.3) and (6.4)we deduce that

$$
l_T \circ (\varepsilon_T \overline{\otimes} T) \circ \Delta_T = \mathrm{Id}_T.
$$

In a similar way one can prove that

$$
r_T \circ (T \overline{\otimes} \varepsilon_T) \circ \Delta_T = \mathrm{Id}_T.
$$

Thus $(T, \Delta_T, \varepsilon_T)$ is a coalgebra. By construction, both Δ_T and ε_T are algebra maps and hence we obtain that $(T, m_T, u_T, \Delta_T, \varepsilon_T)$ is a bialgebra.

Let us consider the linear map $h_1 : M \to T^{op}$ defined by setting

$$h_1(x) = i_1(-x), \quad \text{for every } x \in M$$

and consider

$$h_0 = i^{op} : k^{op} = k \to T^{op}.$$

Then, by the universal property of T, there exists a unique algebra morphism

$$S_T : T \to T^{op}$$

such that

$$S_T \circ i_0 = h_0 \quad \text{and} \quad S_T \circ i_1 = h_1.$$

Let us prove that $(T, m_T, u_T, \Delta_T, \varepsilon_T, S_T)$ is a Hopf algebra, that is

$$S_T * \mathrm{Id}_T = u_T \circ \varepsilon_T \quad \text{and} \quad \mathrm{Id}_T * S_T = u_T \circ \varepsilon_T.$$

By Lemma 6.2 it is sufficient to prove it for the elements $i_1(x)$, for every $x \in M$, that generate T

$$\begin{aligned} [(S_T * \mathrm{Id}_T) \circ i_1](x) &= S_T(i_1(x)) \cdot_T \mathrm{Id}_T(i_0(1_k)) + S_T(i_0(1_k)) \cdot_T \mathrm{Id}_T(i_1(x)) \\ &= h_1(x) \cdot_T i_0(1_k) + i_0(1_k) \cdot_T i_1(x) = i_1(-x) + i_1(x) = 0_T = u_T \circ \varepsilon_T \circ i_1(x) \end{aligned}$$

so that

$$(S_T * \mathrm{Id}_T) \circ i_1 = u_T \circ \varepsilon_T \circ i_1$$

and hence we deduce that

$$S_T * \mathrm{Id}_T = u_T \circ \varepsilon_T.$$

In a similar way one proves also that $\mathrm{Id}_T * S_T = u_T \circ \varepsilon_T$.

Remark 6.4. *Assume that M is a k-vector space of dimension n and let $x_1, \ldots, x_n$ be a basis of M. Set*

$$X_j = i_1(x_j) \quad \textit{for every } j = 1, \ldots, n.$$

Then $(x_{j_1} \otimes \ldots \otimes x_{j_t})_{j_s \in \{1,\ldots,n\}}$ is a basis of $M^{\otimes t}$ and hence

$$(X_{j_1} \cdot_T \cdots \cdot_T X_{j_t})_{j_s \in \{1,\ldots,n\}}$$

i.e. the "words" in $X_1, \ldots, X_n$ of length t, is a basis for $i_t(M^{\otimes t})$. Thus any element of $T = T_k(M)$ is a linear combination, with coefficients in k of the elements $(X_{j_1} \cdot_T \cdots \cdot_T X_{j_t})_{j_s \in \{1,\ldots,t\}}$ where t ranges in $\mathbb{N}$ i.e. is a linear combination of words in $X_1, \ldots, X_n$ of arbitrary length t.

When $n = 1$ we get that $T_k(M)$ can be identified with the polynomial ring $k[X]$.

When $n = 2$, writing $X = X_1$ and $Y = X_2$, we get that any element of $T_k(M)$ is a linear combination of elements of the form

$$X^{a_0} \cdot_T Y^{b_0} \cdot_T \cdots \cdot_T X^{a_s} \cdot_T Y^{b_s} \quad \textit{where } s \in \mathbb{N} \textit{ and } a_i, b_i \in \mathbb{N} \textit{ for every } i = 1, \ldots, s.$$

In general $T_k(M)$ can be thought as a polynomial ring in the noncommutative variables $X_1, \ldots, X_n$. For this reason it is also denoted by $k\{X_1, \ldots, X_n\}$.

6.3 The Symmetric Algebra

Let M be a vector space over the field k. For any $x, y \in M$ let us consider the element

$$s_{x,y} = i_2\left(x \otimes y - y \otimes x\right) = i_1\left(x\right) \cdot_T i_1\left(y\right) - i_1\left(y\right) \cdot_T i_1\left(x\right) \in T_k\left(M\right)$$

and let I be the two-sided ideal of $T_k\left(M\right)$ generated by all $s_{x,y}$ where x and y range in M. Let us check that I is a Hopf ideal of $T = T_k\left(M\right)$. Let $x, y \in M$ and let us compute

$$\begin{aligned}\Delta_T\left(s_{x,y}\right) &= \Delta_T\left(i_1\left(x\right)\right) \cdot \Delta_T\left(i_1\left(y\right)\right) - \Delta_T\left(i_1\left(y\right)\right) \cdot \Delta_T\left(i_1\left(x\right)\right) \\ &= \left[i_1\left(x\right)\overline{\otimes} i_0\left(1_k\right) + i_0\left(1_k\right)\overline{\otimes} i_1\left(x\right)\right]\left[i_1\left(y\right)\overline{\otimes} i_0\left(1_k\right) + i_0\left(1_k\right)\overline{\otimes} i_1\left(y\right)\right] + \\ &\quad - \left[i_1\left(y\right)\overline{\otimes} i_0\left(1_k\right) + i_0\left(1_k\right)\overline{\otimes} i_1\left(y\right)\right]\left[i_1\left(x\right)\overline{\otimes} i_0\left(1_k\right) + i_0\left(1_k\right)\overline{\otimes} i_1\left(x\right)\right]\end{aligned}$$
$$= \left[i_1\left(x\right) \cdot_T i_1\left(y\right) - i_1\left(y\right) \cdot_T i_1\left(x\right)\right]\overline{\otimes} i_0\left(1_k\right) + i_0\left(1_k\right)\overline{\otimes}\left[i_1\left(x\right) \cdot_T i_1\left(y\right) - i_1\left(y\right) \cdot_T i_1\left(x\right)\right] \in I\overline{\otimes}T + T\overline{\otimes}$$

and

$$\begin{aligned}\varepsilon_T\left(s_{x,y}\right) &= \varepsilon_T\left(i_1\left(x\right) \cdot_T i_1\left(y\right) - i_1\left(y\right) \cdot_T i_1\left(x\right)\right) \\ &= \left[\varepsilon_T \circ i_1\left(x\right)\right]\left[\varepsilon_T \circ i_1\left(y\right)\right] - \left[\varepsilon_T \circ i_1\left(y\right)\right]\left[\varepsilon_T \circ i_1\left(x\right)\right] = 0\end{aligned}$$

and also

$$\begin{aligned}S_T\left(s_{x,y}\right) &= S_T\left(i_1\left(x\right) \cdot_T i_1\left(y\right) - i_1\left(y\right) \cdot_T i_1\left(x\right)\right) \\ &= \left[S_T \circ i_1\left(y\right)\right] \cdot_T \left[S_T \circ i_1\left(x\right)\right] - \left[S_T \circ i_1\left(x\right)\right] \cdot_T \left[S_T \circ i_1\left(y\right)\right] =\end{aligned}$$
$$= \left[-i_1\left(y\right)\right] \cdot_T \left[-i_1\left(x\right)\right] - \left[-i_1\left(x\right)\right] \cdot_T \left[-i_1\left(y\right)\right] = i_1\left(y\right) \cdot_T i_1\left(x\right) - i_1\left(x\right) \cdot_T i_1\left(y\right) = -s_{x,y} \in I.$$

Thus, by Theorem 3.18, $T_k\left(M\right)/I$ is a Hopf algebra that will be denoted by $S_k\left(M\right)$ and called the *symmetric algebra* of M. Let $p : T_k\left(M\right) \to T_k\left(M\right)/I = S_k\left(M\right)$ be the canonical projection and let $j_n = p \circ i_n : M^{\otimes^n} \to S_k\left(M\right)$ for every $n \in \mathbb{N}$. We leave to the reader the proof of the following Theorem.

Theorem 6.5. *Let M be a vector space over the field k, let $\left(A, m_A, u_A\right)$ be a commutative k-algebra and let $f_1 : M \to A$ be a k-linear map. Then there exists a unique algebra map $f : S_k\left(M\right) \to A$ such that $f \circ j_0 = u_A$ and $f \circ j_1 = f_1$.*

Exercise 6.6. *Assume that M is a k-vector space of dimension n. Show that, in this case*

$$S_k\left(M\right) \simeq k\left[X_1, \ldots, X_n\right].$$

Proposition 6.7. *Let $\left(H, m, u, \Delta, \varepsilon\right)$ be a bialgebra. Assume that there exists a λ a left integral in H^* such that $\lambda\left(1_H\right) \neq 0$. Then*

$$P\left(H\right) = \left\{x \in H \mid \Delta\left(x\right) = x \otimes 1_H + 1_H \otimes x\right\} = \left\{0\right\}.$$

Proof. Let $x \in P(H)$. We compute

$$\begin{aligned}\sum x_1 \lambda(x_2) &= r_H (H \otimes \lambda) \left(\sum x_1 \otimes x_2\right) \\ &= r_H (H \otimes \lambda)(x \otimes 1_H + 1_H \otimes x) \\ &= r_H (H \otimes \lambda)(x \otimes 1_H) + r_H (H \otimes \lambda)(1_H \otimes x) \\ &= x\lambda(1_H) + 1_H \lambda(x).\end{aligned}$$

Then

$$x\lambda(1_H) + 1_H \lambda(x) \overset{(5.1)}{=} 1_H \lambda(x)$$

and hence

$$x\lambda(1_H) = 0$$

which implies, since $\lambda(1_H) \neq 0$, that $x = 0$. □

Remark 6.8. *Let $M \neq \{0\}$ be a k-vector space. Then, in view of Proposition 6.7, there exist no (left) total integrals both in $T_k(M)^*$ and in $S_k(M)^*$. In fact, we have that*

$$\{0\} \neq i_1(M) \subseteq P(T_k(M)) \quad \text{and} \quad \{0\} \neq j_1(M) \subseteq P(S_k(M)).$$

Thus, in view of Theorem 5.28, both $T_k(M)$ and $S_k(M)$ can never be cosemisimple.

6.4 Enveloping Algebra of a Lie Algebra.

Let us recall the following definition.

Definition 6.9. *A* Lie algebra *over a field k is a couple $(L, [\ ,\ ])$ where*

- *L is a k-vector space*
- *$[\ ,\ ] : L \times L \to L$ is a map such that*

  1) *$[\ ,\ ]$ is k-bilinear.*

  2) *$[x, x] = 0$ for every $x \in L$.*

  3) *$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for every $x, y, z \in L$. (Jacobi's Identity)*

Remark 6.10. *$[\ ,\ ]$ is , in general, non associative.*

Lemma 6.11. *Let $[\ ,\ ] : L \times L \to L$ be a k-bilinear map. Then, if $[\ ,\ ]$ fulfills 2) then it also fulfills*

2') *$[x, y] = -[y, x]$ for every $x, y \in L$.*

If char $(k) \neq 2$, *then 2) and 2′) are equivalent.*

Proof. Let $x, y \in L$. Then, by 2) and in view of the bilinearity of $[\ ,]$, we have

$$0 = [x+y, x+y] = [x,x] + [x,y] + [y,x] + [y,y] = [x,y] + [y,x]$$

from which we deduce $2'$). Conversely, assume that $2'$) holds and $\mathrm{char}(k) \neq 2$. Then from

$$[x,x] = -[x,x]$$

we deduce that

$$2[x,x] = 0 \quad \text{and hence, since } \mathrm{char}(k) \neq 2, \text{ that } [x,x] = 0 \text{ for every } x \in L.$$

□

Example 6.12. 1) *Let A be any k-algebra and let us consider the Lie algebra $A^- = (A, [\ ,])$ where $[\ ,]$ is defined by setting*

$$[x,y] = x \cdot_A y - y \cdot_A x \quad \textit{for every } x, y \in A.$$

In fact we have

$$\begin{aligned}[x,[y,z]] + [y,[z,x]] + [z,[x,y]] &= x \cdot_A [y \cdot_A z - z \cdot_A y] - [y \cdot_A z - z \cdot_A y] \cdot_A x \\ &\quad + y \cdot_A [z \cdot_A x - x \cdot_A z] - [z \cdot_A x - x \cdot_A z] \cdot_A y \\ &\quad + z \cdot_A [x \cdot_A y - y \cdot_A x] - [x \cdot_A y - y \cdot_A x] \cdot_A z \\ &= 0\end{aligned}$$

In particular, for $A = End_k(V)$, where V is a k-vector space, we have that A^- is denoted by $\mathfrak{gl}(V)$ and is called general linear algebra. *If $n \in \mathbb{N}, n \geq 1$, for $A = M_n(k)$, A^- is denoted by $\mathfrak{gl}_n(k)$. Let $e_{i,j}$ be the $n \times n$ matrix having 1_k in the (i,j) entry and 0_k elsewhere. Then $e_{i,j} \cdot e_{s,t} = \delta_{j,s} e_{i,t}$ and hence*

$$[e_{i,j}, e_{s,t}] = \delta_{j,s} e_{i,t} - \delta_{t,i} e_{s,j}.$$

2) *Let $\mathfrak{sl}_n(k)$ be the set of $n \times n$ matrices having trace 0_k. Given two $n \times n$ matrices a, b, we know that $Tr(ab) = Tr(ba)$ and $Tr(a+b) = Tr(a) + Tr(b)$. Hence $\mathfrak{gl}_n(k)$ induces a Lie algebra structure on $\mathfrak{sl}_n(k)$. This Lie algebra is called the* special linear algebra.

3) *Let $n = 2m$ and let*

$$s = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$$

where I_m is the identity matrix in $M_m(k)$. Let

$$\mathfrak{sp}_n(k) = \left\{ x \in M_n(k) \mid sx = -x^t s \right\}$$

where x^t denotes the transpose of the matrix x. It is easy to show that $\mathfrak{sp}_n(k) \subseteq \mathfrak{sl}_n(k)$ and that $\mathfrak{sl}_n(k)$ induces a Lie algebra structure on $\mathfrak{sp}_n(k)$. This Lie algebra is called the symplectic algebra.

Proposition 6.13. *Let $(L, [\ ,])$ be a Lie algebra over k and let I be the ideal of the tensor algebra $T = T_k(L)$ generated by all the elements of the form*

$$i_1([x,y]) - i_2(x \otimes y - y \otimes x) \quad \text{where } x, y \in L.$$

Then I is a Hopf ideal of T.

Proof. Set $l_{x,y} = i_1([x,y]) - i_2(x \otimes y - y \otimes x)$. For every $x, y \in L$, we compute

$$\begin{gathered}
\Delta_T(l_{x,y}) = \Delta_T(i_1([x,y])) - \Delta_T(i_1(x)) \cdot \Delta_T(i_1(y)) + \Delta_T(i_1(y)) \cdot \Delta_T(i_1(x)) = \\
= [i_1([x,y]) \overline{\otimes} i_0(1_k) + i_0(1_k) \overline{\otimes} i_1([x,y])] + \\
- [i_1(x) \overline{\otimes} i_0(1_k) + i_0(1_k) \overline{\otimes} i_1(x)] [i_1(y) \overline{\otimes} i_0(1_k) + i_0(1_k) \overline{\otimes} i_1(y)] + \\
+ [i_1(y) \overline{\otimes} i_0(1_k) + i_0(1_k) \overline{\otimes} i_1(y)] [i_1(x) \overline{\otimes} i_0(1_k) + i_0(1_k) \overline{\otimes} i_1(x)] = \\
= [i_1([x,y]) \overline{\otimes} i_0(1_k) + i_0(1_k) \overline{\otimes} i_1([x,y])] + [-i_1(x) \cdot_T i_1(y) + i_1(y) \cdot_T i_1(x)] \overline{\otimes} i_0(1_k) + \\
+ i_0(1_k) \overline{\otimes} [-i_1(x) \cdot_T i_1(y) + i_1(y) \cdot_T i_1(x)] = \\
= [i_1([x,y]) - i_1(x) \cdot_T i_1(y) + i_1(y) \cdot_T i_1(x)] \overline{\otimes} i_0(1_k) + \\
+ i_0(1_k) \overline{\otimes} [i_1([x,y]) - i_1(x) \cdot_T i_1(y) + i_1(y) \cdot_T i_1(x)] \in I \overline{\otimes} T + T \overline{\otimes} I.
\end{gathered}$$

We calculate also

$$\begin{aligned}
\varepsilon_T(l_{x,y}) &= \varepsilon_T(i_1([x,y]) - i_2(x \otimes y - y \otimes x)) \\
&= \varepsilon_T[i_1([x,y]) - i_1(x) \cdot_T i_1(y) + i_1(y) \cdot_T i_1(x)] \\
&= \varepsilon_T[i_1([x,y])] - [\varepsilon_T \circ i_1(x)][\varepsilon_T \circ i_1(y)] + [\varepsilon_T \circ i_1(y)][\varepsilon_T \circ i_1(x)] = 0
\end{aligned}$$

and

$$\begin{aligned}
S_T(l_{x,y}) &= S_T(i_1([x,y]) - i_1(x) \cdot_T i_1(y) + i_1(y) \cdot_T i_1(x)) \\
&= [S_T \circ i_1([x,y])] - [S_T \circ i_1(y)] \cdot_T [S_T \circ i_1(x)] + [S_T \circ i_1(x)] \cdot_T [S_T \circ i_1(y)] \\
&= -i_1([x,y]) + [i_1(y)] \cdot_T [-i_1(x)] + [-i_1(x)] \cdot_T [-i_1(y)] \\
&= -i_1([x,y]) - i_1(y) \cdot_T i_1(x) + i_1(x) \cdot_T i_1(y) = -l_{x,y} \in I.
\end{aligned}$$

□

Definition 6.14. *Let $(L, [\ ,])$ be a Lie algebra over k. The* enveloping algebra *of L is the quotient algebra $U(L)$ of the tensor algebra $T = T_k(L)$ modulo the ideal I generated by all the elements of the form*

$$i_1([x,y]) - i_2(x \otimes y - y \otimes x) \quad \text{where } x, y \in L.$$

Definition 6.15. *Let $(L, [\ ,])$ and $(L', [\ ,]')$ be Lie algebras over k. A k-linear map $f : L \to L'$ is called a morphism of Lie algebras if*

$$f([x,y]) = [f(x), f(y)]' \quad \text{for every } x, y \in L.$$

Theorem 6.16. *Let $(L, [\ ,])$ be a Lie algebra over k. Then the tensor algebra $T_k(L)$ induces a Hopf algebra structure on $U(L)$.*

Theorem 6.17. *(Universal Property of $U(L)$) Let $(L, [\ ,])$ be a Lie algebra over k and let A be a k-algebra. Given a morphism of Lie algebras $f : L \to A^-$ then there exists a unique morphism of algebras $\widehat{f} : U(L) \to A$ such that $\widehat{f} \circ j_L = f$. Here $j_L : L \to U(L)$ denotes the canonical map.*

Proof. By the universale property of the tensor algebra there exists a unique homomorphism of k-algebras $\widetilde{f} : T_k(L) \to A$ such that $\widetilde{f} \circ i_0 = \mathrm{Id}_k$ and $\widetilde{f} \circ i_1 = f$. Now, $U(L) = \frac{T_k(L)}{I}$ where I is the two-sided ideal of $T_k(L)$ generated by $i_1([x,y]) - i_2(x \otimes y - y \otimes x)$. We have to prove that $\widetilde{f}(I) = \{0\}$. Let us compute

$$\begin{aligned}
\widetilde{f}(i_1([x,y]) - i_2(x \otimes y - y \otimes x)) &= \widetilde{f}(i_1([x,y])) - \left(\widetilde{f}(i_1(x)\, i_1(y) - i_1(y)\, i_1(x))\right) \\
&= f([x,y]) - \left(\widetilde{f}(i_1(x))\, \widetilde{f}(i_1(y)) - \widetilde{f}(i_1(y))\, \widetilde{f}(i_1(x))\right) \\
&= f([x,y]) - (f(x) f(y) - f(y) f(x)) \\
&= f([x,y]) - [f(x), f(y)]^{A^-} = 0
\end{aligned}$$

so that there exists $\widehat{f} : \frac{T_k(L)}{I} = U(L) \to A$ such that $\widehat{f} \circ \pi = \widetilde{f}$ where $\pi : T_k(L) \to \frac{T_k(L)}{I} = U(L)$. Then $\widehat{f} \circ j_L = \widehat{f} \circ \pi \circ i_1 = \widetilde{f} \circ i_1 = f$. Assume that there exists another homomorphism of k-algebras $g : U(L) \to A$ such that $g \circ j_L = f$. Then $g \circ \pi \circ i_1 = g \circ j_L = f$ and $g \circ \pi \circ i_0 = \mathrm{Id}_k$ so that, by uniqueness of $\widetilde{f}$, $g \circ \pi = \widetilde{f} = \widehat{f} \circ \pi$. Since π is surjective we deduce that $g = \widehat{f}$. □

6.5 The Taft Algebra

Lemma 6.18. *Let $q \in k$. Let A be a k-algebra and $a, b \in A$ such that $ba = qab$. Then*

$$b^j a^i = q^{ij} a^i b^j \text{ for every } i, j \in \mathbb{N}. \tag{6.5}$$

Proof. First of all, let us prove that, for every $i \in \mathbb{N}$,

$$ba^i = q^i a^i b. \tag{6.6}$$

We proceed by induction on i. For $i = 0$ there is nothing to prove. Let us assume that the statement holds for some $i \in \mathbb{N}$ and let us prove it for $i+1$. Let us compute

$$ba^{i+1} = \left(ba^i\right) a \overset{\text{indhyp}}{=} \left(q^i a^i b\right) a = \left(q^i a^i\right) ba = \left(q^i a^i\right) qab = q^{i+1} a^{i+1} b.$$

Let us fix $i \in \mathbb{N}$ and let us prove the statement by induction on j. For $j = 0$ there is nothing to prove. Let us assume that the statement holds for some $j \in \mathbb{N}$ and let us prove it for $j+1$. Let us compute

$$b^{j+1} a^i = b\left(b^j a^i\right) \overset{\text{indhyp}}{=} b\left(q^{ij} a^i b^j\right) = q^{ij}\left(ba^i b^j\right) \overset{(6.6)}{=} q^{ij}\left(q^i a^i b\right) b^j = q^{ij+i} a^i b^{j+1} = q^{i(j+1)} a^i b^{j+1}.$$

□

Lemma 6.19. *Let $q \in k$. For every $n \in \mathbb{N}, n \geq 2$, let*
$c_{n,r} = \sum_{0 \leq m_1 \leq m_2 \leq \ldots \leq m_r \leq n-r} q^{m_1+m_2+\ldots+m_r}$ *for every $r \in \mathbb{N}, 1 \leq r \leq n-1$ and let $c_{n,n} = 1$.*
Then

$$c_{n+1,1} = 1 + q + \ldots + q^n = (c_{n,1} + q^n) \tag{6.7}$$

$$c_{n+1,r} = c_{n,r} + c_{n,r-1} q^{n+1-r} \text{ for } r = 2, \ldots, n-1 \tag{6.8}$$

$$c_{n+1,n} = \sum_{0 \leq m_1 \leq m_2 \leq \ldots \leq m_n \leq 1} q^{m_1+m_2+\ldots+m_n} = 1 + q + \ldots + q^n = 1 + q\,(c_{n,n-1})\,. \tag{6.9}$$

Proof. We have

$$c_{2,1} = \sum_{0 \leq m_1 \leq 1} q^{m_1} = 1 + q.$$

$$c_{3,1} = \sum_{0 \leq m_1 \leq 2} q^{m_1} = 1 + q + q^2 = \left(c_{2,1} + q^2\right).$$

Let us assume that, for some $n \geq 3$, (6.7) holds and let us prove it for $n+1$. We compute

$$c_{n+2,1} = \sum_{0 \leq m_1 \leq n+1} q^{m_1} = \sum_{0 \leq m_1 \leq n} q^{m_1} + q^{n+1} = c_{n+1,1} + q^n = 1 + q + \ldots + q^n + q^{n+1}.$$

Let us compute, for $r = 2, \ldots, n-1$,

$$\begin{aligned} c_{n+1,r} &= \sum_{0 \leq m_1 \leq m_2 \leq \ldots \leq m_r \leq n+1-r} q^{m_1+m_2+\ldots+m_r} \\ &= \sum_{0 \leq m_1 \leq m_2 \leq \ldots \leq m_r \leq n-r} q^{m_1+m_2+\ldots+m_r} + \sum_{0 \leq m_1 \leq m_2 \leq \ldots \leq m_r = n+1-r} q^{m_1+m_2+\ldots+m_r} \\ &= c_{n,r} + q^{n+1-r} \cdot \left(\sum_{0 \leq m_1 \leq m_2 \leq \ldots \leq m_{r-1} \leq n+1-r} q^{m_1+m_2+\ldots+m_{r-1}} \right) = c_{n,r} + c_{n,r-1} q^{n+1-r} \end{aligned}$$

and hence (6.8) is proved. Let us compute

$$\begin{aligned} c_{3,2} &= \sum_{0 \leq m_1 \leq m_2 \leq 1} q^{m_1+m_2} = \sum_{0 \leq m_1 \leq m_2 \leq 0} q^{m_1+m_2} + \sum_{0 \leq m_1 \leq m_2 = 1} q^{m_1+m_2} = 1 + q \cdot \left(\sum_{0 \leq m_1 \leq 1} q^{m_1} \right) \\ &= 1 + q\,(1+q) = 1 + q + q^2. \end{aligned}$$

Assume now that (6.9) holds for some $n \geq 2$ and let us prove it for $n+1$. Then

$$\begin{aligned} c_{n+2,n+1} &= \sum_{0 \leq m_1 \leq m_2 \leq \ldots \leq m_{n+1} \leq 1} q^{m_1+m_2+\ldots+m_{n+1}} \\ &= \sum_{0 \leq m_1 \leq m_2 \leq \ldots \leq m_{n+1} \leq 0} q^{m_1+m_2+\ldots+m_{n+1}} + \sum_{0 \leq m_1 \leq m_2 \leq \ldots \leq m_{n+1} = 1} q^{m_1+m_2+\ldots+m_{n+1}} \\ &= q^0 + q \cdot \left(\sum_{0 \leq m_1 \leq m_2 \leq \ldots \leq m_n \leq 1} q^{m_1+m_2+\ldots+m_n} \right) \\ &= 1 + q\,(1 + q + \ldots + q^n) = 1 + q + \ldots + q^{n+1}. \end{aligned}$$

□

Proposition 6.20. *Let $q \in k$, let A be a k-algebra and $a, b \in A$ such that $ba = qab$. Then*

$$(a+b)^n = a^n + \sum_{r=1}^{n-1} c_{n,r} a^{n-r} b^r + b^n$$

where

$$c_{n,r} = \sum_{0 \leq m_1 \leq m_2 \leq \ldots \leq m_r \leq n-r} q^{m_1+m_2+\ldots+m_r} \quad \textit{for every } n, r \in \mathbb{N}, n \geq 2, 1 \leq r \leq n-1.$$

Proof. For $n = 2$ we have

$$\begin{array}{rcl} (a+b)^2 & = & a^2 + ab + ba + b^2 = a^2 + (1+q)\,ab + b^2 \\ \text{Since } c_{2,1} & = & 1+q, \text{ we obtain } (a+b)^2 = a^2 + c_{2,1}ab + b^2. \end{array}$$

Let us assume that the statement holds for some $n \in \mathbb{N}, n \geq 2$ and let us prove it for $n+1$. We have

$$\begin{array}{rcl} (a+b)^{n+1} & = & (a+b)\left[a^n + \sum_{r=1}^{n-1} c_{n,r} a^{n-r} b^r + b^n\right] \\ & = & a^{n+1} + \sum_{r=1}^{n-1} c_{n,r} a^{(n+1)-r} b^r + ab^n + ba^n + \sum_{r=1}^{n-1} c_{n,r} ba^{n-r} b^r + b^{n+1}. \end{array}$$

Now we compute

$$\sum_{r=1}^{n-1} c_{n,r} ba^{n-r} b^r \overset{(6.5)}{=} \sum_{r=1}^{n-1} c_{n,r} q^{n-r} a^{n-r} b^{r+1} = \sum_{s=2}^{n} c_{n,s-1} q^{n+1-s} a^{n+1-s} b^s$$

so that we get

$$(a+b)^{n+1} = a^{n+1} + \sum_{r=1}^{n-1} c_{n,r} a^{(n+1)-r} b^r + ab^n + q^n a^n b + \sum_{s=2}^{n} c_{n,s-1} q^{n+1-s} a^{n+1-s} b^s + b^{n+1}$$

Now we calculate

$$\sum_{r=1}^{n-1} c_{n,r} a^{(n+1)-r} b^r + ab^n + q^n a^n b + \sum_{s=2}^{n} c_{n,s-1} q^{n+1-s} a^{n+1-s} b^s =$$

$$= (c_{n,1} + q^n)\, a^n b + \sum_{r=2}^{n-1} c_{n,r} a^{(n+1)-r} b^r + ab^n + \sum_{s=2}^{n-1} c_{n,s-1} q^{n+1-s} a^{n+1-s} b^s + c_{n,n-1} qab^n$$

$$= (c_{n,1} + q^n)\, a^n b + \sum_{r=2}^{n-1} \left(c_{n,r} + c_{n,r-1} q^{n+1-r}\right) a^{(n+1)-r} b^r + (1 + c_{n,n-1} q)\, ab^n$$

$$\overset{(6.7),(6.8),(6.9)}{=} c_{n+1,1} a^n b + \sum_{r=2}^{n-1} c_{n+1,r} a^{(n+1)-r} b^r + c_{n+1,n} ab^n$$

$$= \sum_{t=1}^{n} c_{n+1,t} a^{(n+1)-t} b^t$$

so that we get

$$(a+b)^{n+1} = a^{n+1} + \sum_{t=1}^{n} c_{n+1,t} a^{(n+1)-t} b^t + b^{n+1}.$$

□

Proposition 6.21. *Let $n \in \mathbb{N}, n \geq 2$ and let $q \in k$ be a primitive n-th root of unity. Then $c_{n,r} = 0$ for every $n, r \in \mathbb{N}, n \geq 2, 1 \leq r \leq n-1$.*

Proof. We have

$$c_{n,1} = 1 + q + ... + q^{n-1} = 0.$$

Assume that the statement holds for some $n \in \mathbb{N}, n \geq 2$, and every $r \in \mathbb{N}, 1 \leq r \leq n-1$ and let us prove it for $n+1$. In view of formula (6.8), we have

$$c_{n+1,r} = c_{n,r} + c_{n,r-1} q^{n+1-r} \text{ for } r = 2, ..., n-1$$

so that, in view of the induction assumption we obtain $c_{n+1,r} = 0$ for every $r = 2, ..., n-1$. Now we calculate

$$c_{n+1,n} \overset{6.9}{=} 1 + q + ... + q^n = 0 \text{ since } q \text{ is a primitive } n+1\text{-th root of unity.}$$

□

Corollary 6.22. *Let $n \in \mathbb{N}, n \geq 2$ and let $q \in k$ be a primitive n-th root of unity. Let A be a k-algebra and $a, b \in A$ such that $ba = qab$. Then*

$$(a+b)^n = a^n + b^n \text{ for every } n \in \mathbb{N}.$$

Proposition 6.23. *Let $q \in k$, let A be a k-algebra and $a, b \in A$ such that $ba = qab$. Then*

$$(ba)^n = q^{t_n} a^n b^n \text{ for every } n \in \mathbb{N}, n \geq 1. \tag{6.10}$$

where

$$t_n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Proof. Let us proceed by induction on $n \in \mathbb{N}, n \geq 1$. For $n = 1$ there is nothing to prove. Let us assume that the statement holds for some $n \in \mathbb{N}$ and let us prove it for $n + 1$.

$$(ba)^{n+1} = ba\,(ba)^n \overset{\text{indhyp}}{=} q^{t_n} (ba)(a^n b^n) = q^t \left(ba^{n+1}\right) b^n \overset{(6.5)}{=} q^{t_n} q^{n+1} a^{n+1} b^{n+1} = q^{t_{n+1}} a^{n+1} b^{n+1}.$$

□

Lemma 6.24. *Let A be a k-algebra. Assume that $a, x, y \in A$ and that $\Delta : A \to A \otimes A$ is a linear map such that*

$$\Delta(x) = x \otimes x, \quad \Delta(y) = y \otimes y \quad \text{and} \quad \Delta(a) = a \otimes x + y \otimes a$$

Then

$$[(\Delta \otimes A) \circ \Delta](x) = [(A \otimes \Delta) \circ \Delta](x) \quad [(\Delta \otimes A) \circ \Delta](y) = [(A \otimes \Delta) \circ \Delta](y)$$
$$\text{and} \quad [(\Delta \otimes A) \circ \Delta](a) = [(A \otimes \Delta) \circ \Delta](a).$$

Moreover if $\varepsilon : A \to k$ is such that $\varepsilon(x) = \varepsilon(y) = 1$ and $\varepsilon(a) = 0$ then

$$(l \circ (\varepsilon \otimes T) \circ \Delta)(x) = x \quad \text{and} \quad (l \circ (\varepsilon \otimes A) \circ \Delta)(a) = a$$

A similar result holds on the other side.

Proof. Clearly $[(\Delta \otimes A) \circ \Delta](x) = x \otimes x \otimes x = [(A \otimes \Delta) \circ \Delta](x)$. The same holds for y. We compute

$$\begin{aligned}
[(\Delta \otimes A) \circ \Delta](a) &= (\Delta \otimes A)(a \otimes x + y \otimes a) = \Delta(a) \otimes x + \Delta(y) \otimes a \\
&= a \otimes x \otimes x + y \otimes a \otimes x + y \otimes y \otimes a \\
[(A \otimes \Delta) \circ \Delta](a) &= (A \otimes \Delta)(a \otimes x + y \otimes a) = a \otimes \Delta(x) + y \otimes \Delta(a) \\
&= a \otimes x \otimes x + y \otimes a \otimes x + y \otimes y \otimes a
\end{aligned}$$

We compute

$$(l_A \circ (\varepsilon \otimes A) \circ \Delta)(a) = (l_A \circ (\varepsilon \otimes A))(a \otimes x + y \otimes a) = l_A(\varepsilon(a) \otimes x + \varepsilon(y) \otimes a) = a.$$

□

Let $n \in \mathbb{N}, n \geq 2$ and let $q \in k$ be a primitive n-th root of unity. Using the universal property of the tensor algebra, we define on the algebra $R = k\{X, Y\}$ an algebra homomorphism

$$\Delta_R : R \to R \otimes R$$

by setting

$$\Delta_R(X) = X \otimes X \quad \text{and} \quad \Delta_R(Y) = Y \otimes X + 1 \otimes Y.$$

Then by Lemma 6.24 we have that

$$[(\Delta_R \otimes R) \circ \Delta_R](X) = X \otimes X \otimes X = [(R \otimes \Delta_R) \circ \Delta_R](X)$$

and

$$[(\Delta_R \otimes R) \circ \Delta_R](Y) = [(R \otimes \Delta_R) \circ \Delta_R](Y)$$

so that we get

$$(\Delta_R \otimes R) \circ \Delta_R = (R \otimes \Delta_R) \circ \Delta_R.$$

Using again the universal property of the tensor algebra we define an algebra homomorphism

$$\varepsilon_R : R \to k$$

by setting

$$\varepsilon_R(X) = 1 \quad \text{and} \quad \varepsilon_R(Y) = 0.$$

By Lemma 6.24, we get

$$l_R \circ (\varepsilon_R \otimes T) \circ \Delta_R = \mathrm{Id}_R \quad \text{and} \quad r_R \circ (T \otimes \varepsilon_R) \circ \Delta_R = \mathrm{Id}_R$$

Hence $(R, \Delta_R, \varepsilon_R)$ is a bialgebra. Let now I be the two-sided ideal I of R spanned by the elements $X^n - 1, Y^n, YX - qXY$. I is a bi-ideal of R i.e. $\Delta_R(I) \subseteq I \otimes R + R \otimes I$ and $\varepsilon_R(I) = \{0\}$. Let $p = p_I : R \to R/I$ be the canonical projection. To prove that $\Delta_R(I) \subseteq I \otimes R + R \otimes I$ we can equivalently prove that $(p \otimes p) \circ \Delta_R = 0$. Let $x = X + I$ and $y = Y + I$, they fulfill the relations

$$x^n = 1, y^n = 0, yx = qxy.$$

Let us compute

$$\begin{aligned}
[(p \otimes p) \circ \Delta_R](X^n - 1_R) &= (p \otimes p)[\Delta_R(X)^n - \Delta_R(1_R)] \\
&= (p \otimes p)(X^n \otimes X^n) - (p \otimes p)(1_R \otimes 1_R) \\
&= (1_R + I) \otimes (1_R + I) - (1_R + I) \otimes (1_R + I) = 0.
\end{aligned}$$

We have

$$(y \otimes x)(1 \otimes y) = (y \otimes xy) \text{ and } (1 \otimes y)(y \otimes x) = y \otimes yx = y \otimes qxy = q(y \otimes xy).$$

Set

$a = y \otimes x$ and $b = 1 \otimes y$. Then we obtained that $ba = qba$.

Hence, by Corollary 6.22 we have that

$$(a+b)^n = a^n + b^n$$

and hence we obtain

$$\begin{aligned}
\left[(p\otimes p)\circ\Delta_R\right](Y^n) &= \left[\left[(p\otimes p)\circ\Delta_R\right](Y)\right]^n = \left[p(Y)\otimes p(X) + p(1)\otimes p(Y)\right]^n \\
&= \left[p(Y)\otimes p(X)\right]^n + \left[p(1)\otimes p(Y)\right]^n = p(Y^n)\otimes p(X^n) + p(1)\otimes p(Y^n) \\
&= 0.
\end{aligned}$$

Now let us calculate

$$\begin{aligned}
\left[(p\otimes p)\circ\Delta_R\right](YX - qXY) &= (p\otimes p)\left(\Delta_R(Y)\Delta_R(X) - q\Delta_R(X)\Delta_R(Y)\right) \\
&= (p\otimes p)\left((Y\otimes X + 1\otimes Y)(X\otimes X) - q(X\otimes X)(Y\otimes X + 1\otimes Y \right. \\
&= (p\otimes p)\left(YX\otimes X^2 + X\otimes YX - q\left(XY\otimes X^2 + X\otimes XY\right)\right) \\
&= yx\otimes x^2 + x\otimes yx - q\left(xy\otimes x^2 + x\otimes xy\right) \\
&= qxy\otimes x^2 + qx\otimes xy - q\left(xy\otimes x^2 + x\otimes xy\right) = 0.
\end{aligned}$$

Let us compute

$$\begin{aligned}
\varepsilon_R(X^n - 1) &= \varepsilon_R(X)^n - 1 = 1^n - 1 = 0 \\
\varepsilon_R(Y^n) &= \varepsilon_R(Y)^n = 0 \\
\varepsilon_R(YX - qXY) &= \varepsilon_R(Y)\varepsilon_R(X) - q\varepsilon_R(X)\varepsilon_R(Y) = 0.
\end{aligned}$$

Thus I is a bi-ideal of R. Let us use the universal property of R to define an algebra homomorphism $S : R \to R^{op}$ such that

$$S(X) = X^{n-1} \text{ and } S(Y) = -q^{-1}X^{n-1}Y.$$

Let us prove that $S(I) \subseteq I$ or equivalently that $p\circ S(I) = 0$. We compute

$$(p\circ S)(X^n - 1_R) = p\left((X^n)^{n-1} - 1\right) = (x^n)^{n-1} - 1 = 1 - 1 = 0.$$

Note that, by (6.5), we have $(x^{n-1})\,y = q^{-n+1}yx^{n-1}$. Thus, by applying (6.10) where $b = x^{n-1}, a = y$ we obtain $(ba)^n = (q^{-n+1})^{t_n} a^n b^n$ for every $n \in \mathbb{N}, n \geq 1$ which means that

$$\left(x^{n-1}y\right)^n = \left(q^{-n+1}\right)^{t_n} y^n \left(x^{n-1}\right)^n = 0. \tag{6.11}$$

Now we compute

$$(p\circ S)(Y^n) = \left[(p\circ S)(Y)\right]^n = \left[-q^{-1}x^{n-1}y\right]^n = (-1)^n q^{-n}\left(x^{n-1}y\right)^n \overset{(6.11)}{=} 0.$$

Let us calculate

$$\begin{aligned}
(p\circ S)(YX - qXY) &= p\left(X^{n-1}\cdot\left[-q^{-1}X^{n-1}Y\right] - q\left(-q^{-1}X^{n-1}Y\right)\cdot X^{n-1}\right) \\
&\overset{(6.5)}{=} -q^{-1}x^{n-1}x^{n-1}y + q^{n-1}x^{n-1}x^{n-1}y \\
&= 0.
\end{aligned}$$

Now we have that $yx = qxy$ so that, by (6.5) we have $y^j x^i = q^{ij} x^i y^j$ for every $i, j \in \mathbb{N}$. In particular $yx^{n-1} = q^{n-1} x^{n-1} y$ so that

$$-q^{-1} x^{n-1} x^{n-1} y + x^{n-1} y x^{n-1} = \left(-q^{-1} + q^{n-1}\right) \left(x^{n-1} x^{n-1} y\right) = 0.$$

Hence S induces an algebra homomorphism $S_{R/I} : R/I \to (R/I)^{op}$ such that

$$S_{R/I}(x) = x^{n-1} \text{ and } S_{R/I}(y) = -q^{-1} x^{n-1} y.$$

Let us check that $S_{R/I}$ is an antipode for the bialgebra R/I. By Lemma 6.2 it is enough to check this on x and y. Thus we compute

$$\begin{aligned}
\left(S_{R/I} * \mathrm{Id}_{R/I}\right)(x) &= S_{R/I}(x) \cdot x = x^{n-1} \cdot x = x^n = 1 = \left(u_{R/I} \circ \varepsilon_{R/I}\right)(x) \text{ and} \\
\left(S_{R/I} * \mathrm{Id}_{R/I}\right)(y) &= S_{R/I}(y)\, x + S_{R/I}(1)\, y = -q^{-1} x^{n-1} y x + y = -q^{-1} x^{n-1} q x y + y = -x^n y + y \\
&= -y + y = 0 = \left(u_{R/I} \circ \varepsilon_{R/I}\right)(y).
\end{aligned}$$

A similar computation shows that $\mathrm{Id}_{R/I} * S_{R/I} = u_{R/I} \circ \varepsilon_{R/I}$.

The Hopf algebra R/I is called the Taft algebra and denoted by $H_{n^2}(q)$. We list here its main properties.

$H_{n^2}(q)$ is generated by the elements x and y which fulfill the relations:

$$x^n = 1, y^n = 0, xy = qyx.$$

We have

$$\begin{aligned}
\Delta(x) &= x \otimes x, \varepsilon(x) = 1 \\
\Delta(y) &= y \otimes x + 1 \otimes y, \varepsilon(y) = 0 \\
S(x) &= x^{n-1}, S(y) = -q^{-1} x^{n-1} y.
\end{aligned}$$

For $n = 2$ the Taft algebra is also called Sweedler's 4-dimensional Hopf algebra. It was the first example of a noncommutative noncocommutative Hopf Algebra.

6.6 The divided power Hopf algebra

In Example 1) of 1.12, we have seen that on a vector space L over k with a basis $e_i, i \in \mathbb{N}$, one can define the so called divided power coalgebra by setting

$$\Delta(e_i) = \sum_{i+j=n} e_i \otimes e_j \text{ and } \varepsilon(e_i) = \delta_{i,0}.$$

Assume that $char(k) = 0$ and let us define an algebra structure on L by setting

$$e_m \cdot e_n = \binom{m+n}{m} e_{m+n}.$$

We compute

$$\begin{aligned}
e_s \cdot (e_m \cdot e_n) &= \binom{m+n}{m} e_s \cdot e_{m+n} = \binom{m+n}{m}\binom{s+m+n}{s} e_{s+m+n} \\
(e_s \cdot e_m) \cdot e_n &= \binom{s+m}{s} e_{s+m} \cdot e_n = \binom{s+m}{s}\binom{s+m+n}{s+m} e_{s+m+n}.
\end{aligned}$$

Since

$$\begin{aligned}
\binom{m+n}{m}\binom{s+m+n}{m+n} &= \frac{(m+n)!}{m!n!}\frac{(s+m+n)!}{s!\,(m+n)!} = \frac{(s+m+n)!}{m!n!s!} \\
\binom{s+m}{s}\binom{s+m+n}{s+m} &= \frac{(s+m)!}{s!m!}\frac{(s+m+n)!}{(s+m)!n!} = \frac{(s+m+n)!}{m!n!s!}
\end{aligned}$$

we deduce that the product is associative and the unit of the ring is $1_L = e_0$. Let us prove that L is a bialgebra. Let us compute

$$\begin{aligned}
\Delta(e_m \cdot e_n) &= \Delta\left(\binom{m+n}{m} e_{m+n}\right) = \binom{m+n}{m} \sum_{t+s=m+n} e_t \otimes e_s \\
\Delta(e_m)\,\Delta(e_n) &= \left(\sum_{i+j=m} e_i \otimes e_j\right)\left(\sum_{a+b=n} e_a \otimes e_b\right) = \sum_{\substack{i+j=m\\ a+b=n}} (e_i \cdot e_a) \otimes (e_j \cdot e_b) \\
&= \sum_{\substack{i+j=m\\ a+b=n}} \binom{i+a}{i}\binom{j+b}{j} e_{i+a} \otimes e_{j+b} = \sum_{\substack{i+j=m\\ a+b=n}} \binom{i+a}{a}\binom{j+b}{b} e_t \otimes e_s \\
&= \sum_{\substack{t+s=m+n\\ i+j=m}} \binom{t}{t-i}\binom{s}{s-j} e_t \otimes e_s
\end{aligned}$$

Since we have

$$\begin{aligned}
\binom{t}{t-i}\binom{s}{s-j} &= \frac{t!}{i!\,(t-i)!}\frac{s!}{j!\,(s-j)!} = \frac{t!}{i!\,(t-i)!}\frac{(m+n-t)!}{j!\,(m+n-t-j)!} \\
&= \frac{(m+n)!}{i!\,(t-i)!\,(m-i)!\,(n-(t-i))!} = \frac{(m+n)!}{m!n!}
\end{aligned}$$

we deduce that

$$\Delta(e_m \cdot e_n) = \Delta(e_m)\,\Delta(e_n).$$

Moreover we have

$$\Delta(1_L) = \Delta(e_0) = e_0 \otimes e_0 = 1_L \otimes 1_L.$$

$$\begin{aligned}
\varepsilon(e_m \cdot e_n.) &= \binom{m+n}{m} \varepsilon(e_{m+n}) = \binom{m+n}{m} \delta_{m+n,0} = \binom{m+n}{m} \delta_{m,0}\delta_{n,0} = \varepsilon(e_m)\,\varepsilon(e_n\ ) \\
\varepsilon(1_L) &= \varepsilon(e_0) = 1_k.
\end{aligned}$$

Let us define $S : L \to L$ recursively by setting

$$S(e_0) = S(1_L) = 1_L$$

and

$$S(e_n) = -\sum_{0 \le a \le n-1} S(e_a) e_{n-a}.$$

Let us check that S is an antipode for the bialgebra L. By Lemma 6.2 it is enough to check this on each e_n. We proceed by induction on n. Let us compute

$$(S * \mathrm{Id}_L)(e_0) = S(e_0) e_0 = 1_L = u_L \varepsilon_L (1_L) = u_L \varepsilon_L (e_0).$$

Let us assume that the statement holds for some $n \in \mathbb{N}$ and let us prove it for $n+1$.

$$\begin{aligned} (S * \mathrm{Id}_L)(e_{n+1}) &= \sum_{0 \le a \le n+1} S(e_a) e_{n+1-a} = S(e_{n+1}) e_0 + \sum_{0 \le a \le n} S(e_a) e_{n+1-a} \\ &= \left(-\sum_{0 \le a \le n} S(e_a) e_{n+1-a} \right) 1_L + \sum_{0 \le a \le n} S(e_a) e_{n+1-a} = 0. \end{aligned}$$

6.7 More Examples

Using the universal property of the tensor algebra, we define on the algebra $R = k\{X, Y\}$ an algebra homomorphism

$$\Delta_R : R \to R \otimes R$$

by setting

$$\Delta_R(X) = X \otimes X \quad \text{and} \quad \Delta_R(Y) = Y \otimes 1 + X \otimes Y.$$

Using again the universal property of the tensor algebra we define an algebra homomorphism

$$\varepsilon_R : R \to k$$

by setting

$$\varepsilon_R(X) = 1 \quad \text{and} \quad \varepsilon_R(Y) = 0.$$

By Lemma 6.24, we get that $(R, \Delta_R, \varepsilon_R)$ is a bialgebra. Let $q \in k$, $q \neq 0$ and let I be the two-sided ideal ideal of R generated by $XY - qYX$. Let us prove that I is a bi-ideal of R. We compute

$$\begin{aligned} \Delta_R(XY - qYX) &= \Delta_R(X) \Delta_R(Y) - q\Delta_R(Y) \Delta_R(X) \\ &= (X \otimes X)(Y \otimes 1 + X \otimes Y) - q(Y \otimes 1 + X \otimes Y)(X \otimes X) \\ &= XY \otimes X1 + XX \otimes XY - qYX \otimes X - qXX \otimes YX \\ &= (XY - qYX) \otimes X + XX \otimes (XY - qYX) \end{aligned}$$

and

$$\varepsilon_R(XY - qYX) = \varepsilon_R(X)\varepsilon_R(Y) - q\varepsilon_R(Y)\varepsilon_R(X) = 0.$$

Therefore R/I is a bialgebra. This bialgebra is denoted by $\mathcal{O}_q(k^2)$ and is called *quantum plane*. Let $x = X + I$ and $y = Y + I$. Then $\mathcal{O}_q(k^2)$ is generated by x and y which satisfy $xy = qyx$. Let $\mathcal{O} = \mathcal{O}_q(k^2)$. Then

$$\begin{aligned} \Delta_{\mathcal{O}}(x) &= x \otimes x, \quad \Delta_{\mathcal{O}}(y) = y \otimes 1 + x \otimes y \\ \varepsilon_{\mathcal{O}}(x) &= 1, \quad \varepsilon_{\mathcal{O}}(y) = 0. \end{aligned}$$

Let us consider $\mathfrak{sl}_2(k)$ the set of 2×2 matrices having trace 0_k.

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We compute

$$[e,f] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = h$$

$$[h,e] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = 2e$$

$$[h,f] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = -2f.$$

Then the enveloping algebra $U(\mathfrak{sl}_2(k))$ is the quotient of the polynomial ring in noncommutative variables $k\{E,F,K\}$ modulo the two-sided ideal I generated by

$$\begin{gathered} EF - FE - K \\ KE - EK - 2E \\ KF - FK + 2F \end{gathered}$$

For every $x \in U(\mathfrak{sl}_2(k))$ we have that

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \varepsilon(x) = 0 \text{ and } \quad S(x) = -x.$$

Let us consider the polynomial ring in noncommutative variables $R = k\{X,Y,Z,T\}$ and define on R a comultiplication Δ and a counit ε by setting

$$\begin{aligned} \Delta(X) &= 1 \otimes X + X \otimes Z, \quad \varepsilon(X) = 0 \\ \Delta(Y) &= T \otimes Y + Y \otimes 1, \quad \varepsilon(Y) = 0 \\ \Delta(Z) &= Z \otimes Z, \quad \varepsilon(Z) = 1 \\ \Delta(T) &= T \otimes T, \quad \varepsilon(T) = 1. \end{aligned}$$

By Lemma 6.24, we get that R is a bialgebra. Let now $q \in k, q \neq 0, q^2 \neq 1$ and let I be the two-sided ideal of R generated by

$$\begin{gathered} ZT-1, TZ-1, \\ XY-YX-\frac{Z-T}{q-q^{-1}} \\ ZX-q^2XZ \\ ZY-q^{-2}YZ. \end{gathered}$$

Let us prove that I is a bi-ideal of R. Let $p : R \to R/I$ be the canonical projection. We set $E = p(X), F = p(Y), K = p(Z)$ and $K' = p(T)$. Then in R/I we have

$$\begin{gathered} KK' = 1 = K'K, \text{i.e. } K \text{ is invertible and } K' \text{ is its two-sided inverse} \\ EF-FE = \frac{K-K'}{q-q^{-1}} \\ KE = q^2 EK \\ KF = q^{-2}FK. \end{gathered}$$

We compute

$$(p \circ \Delta)(ZT-1) = p\left[(Z \otimes Z)(T \otimes T) - 1 \otimes 1\right] = KK' \otimes K'K - 1 \otimes 1 = 0.$$

The computation for $TZ-1$ is similar.

$$\begin{aligned} & (p \circ \Delta)\left(XY-YX-\frac{Z-T}{q-q^{-1}}\right) \\ = p & \left[\begin{array}{c} (1 \otimes X + X \otimes Z)(T \otimes Y + Y \otimes 1) - (T \otimes Y + Y \otimes 1)(1 \otimes X + X \otimes Z) + \\ -\frac{1}{q-q^{-1}}(Z \otimes Z) + \frac{1}{q-q^{-1}}(T \otimes T) \end{array}\right] \\ & = (1 \otimes E + E \otimes K)(K' \otimes F + F \otimes 1) - (K' \otimes F + F \otimes 1)(1 \otimes E + E \otimes K) \\ & \quad -\frac{1}{q-q^{-1}}(K \otimes K) + \frac{1}{q-q^{-1}}(K' \otimes K') = \\ & = K' \otimes EF + F \otimes E + EK' \otimes KF + EF \otimes K - K' \otimes FE - K'E \otimes FK - F \otimes E - FE \otimes K + \\ & \quad -\frac{1}{q-q^{-1}}(K \otimes K) + \frac{1}{q-q^{-1}}(K' \otimes K') \\ & = K' \otimes [EF-FE] + [EF-FE] \otimes K + q^2 K'E \otimes q^{-2}FK - K'E \otimes FK \\ & \quad -\frac{1}{q-q^{-1}}(K \otimes K) + \frac{1}{q-q^{-1}}(K' \otimes K') \\ & = K' \otimes [EF-FE] + [EF-FE] \otimes K - \frac{1}{q-q^{-1}}(K \otimes K) + \frac{1}{q-q^{-1}}(K' \otimes K') \\ & = \frac{1}{q-q^{-1}}\left[K' \otimes (K-K') + (K-K') \otimes K - K \otimes K + K' \otimes K'\right] = 0 \end{aligned}$$

$$\begin{aligned}(p\circ\Delta)\left(ZX-q^2XZ\right) &= p\left[(Z\otimes Z)(1\otimes X+X\otimes Z)\right]-p\left[q^2(1\otimes X+X\otimes Z)(Z\otimes Z)\right]\\ &= (K\otimes K)(1\otimes E+E\otimes K)-q^2(K\otimes EK)-q^2\left(EK\otimes K^2\right)\\ &= K\otimes KE+KE\otimes K^2-q^2(K\otimes EK)-q^2\left(EK\otimes K^2\right)\\ &= K\otimes\left[KE-q^2EK\right]+\left[KE-q^2EK\right]\otimes K^2=0\end{aligned}$$

The computation for $ZY-q^{-2}YZ$ is similar.

Now we go back to R and we define an algebra homomorphism $S:R\to R^{op}$ by setting

$$S(X)=-XT, S(Y)=-ZY, S(Z)=T, S(T)=Z.$$

Let us prove that $S(I)\subseteq I$. Note that $KFEK'=q^{-2}FKq^2K'E=FE$. We compute

$$(p\circ S)(ZT-1)=p(ZT-1)=0.$$

The computation for $TZ-1$ is similar. We compute

$$\begin{aligned}(p\circ S)\left(XY-YX-\frac{Z-T}{q-q^{-1}}\right) &= p\left(+ZYXT-XTZY-\frac{T-Z}{q-q^{-1}}\right)\\ &= KFEK'-EF-\frac{K'-K}{q-q^{-1}}=FE-EF-\frac{K'-K}{q-q^{-1}}=0.\end{aligned}$$

Since $KE=q^2EK$ we have that $EK'-q^2K'E=0$ and hence

$$(p\circ S)\left(ZX-q^2XZ\right)=p\left(-XTT+q^2TXT\right)=-EK'K'+q^2K'EK'=0.$$

The computation for $ZY-q^{-2}YZ$ is similar. Now we want to check that S is an antipode. We compute

$$\begin{aligned}(S*\mathrm{Id})(E) &= S(1)E+S(E)K=E-EK'K=0=\varepsilon(E)1\\ (S*\mathrm{Id})(F) &= S(K')F+S(F)1=KF-KF=0=\varepsilon(F)1\\ (S*\mathrm{Id})(K) &= 1=\varepsilon(K)1\\ (S*\mathrm{Id})(K') &= 1=\varepsilon(K')1.\end{aligned}$$

The Hopf algebra R/I is called the *Quantized Enveloping Algebra of* $\mathfrak{sl}_2(k)$ and is denoted by $U_q(\mathfrak{sl}_2(k))$.

6.8 Gauss binomial coefficients

In this section we work inside $\mathbb{Q}(X,Y)$, the field of quotients of the polynomial ring in two variables, $\mathbb{Q}[X,Y]$. For all $a\in\mathbb{Z}$ we set

$$[a]=\frac{X^a-Y^a}{X-Y}. \tag{6.12}$$

Clearly we have that

$$[0] = 0.$$

Moreover

$$[a] = X^{a-1} + X^{a-2}Y + \cdots + X^2Y^{a-2} + Y^{a-1} \quad \text{for all } a \geq 1.$$

Define the Gauss binomial coefficients by

$$\begin{aligned} \begin{bmatrix} a \\ n \end{bmatrix} &= \frac{[a]\,[a-1]\cdots[a-n+1]}{[1]\,[2]\cdots[n]} \quad \text{for all } a, n \in \mathbb{Z}, n \geq 1 \text{ and} \\ \begin{bmatrix} a \\ 0 \end{bmatrix} &= 1 \quad \text{for all } a \in \mathbb{Z}. \end{aligned}$$

We have the following equalities

$$\begin{aligned} \begin{bmatrix} a \\ 1 \end{bmatrix} &= [a]\,, \begin{bmatrix} n \\ n \end{bmatrix} = 1 \text{ and} \\ \begin{bmatrix} a \\ n \end{bmatrix} &= 0 \text{ if } 0 \leq a < n. \end{aligned}$$

We also set

$$[0]! = 1 \quad \text{and } \; [n]! = [1]\,[2]\cdots[n] \quad \text{for all } n \in \mathbb{Z}, n \geq 1.$$

Thus

$$\begin{bmatrix} a \\ n \end{bmatrix} = \frac{[a]!}{[n]!\,[a-n]!} \quad \text{for all } a, n \in \mathbb{Z}, 0 \leq n \leq a.$$

$$X^{a+1} - Y^{a+1} = X^n\left(X^{a+1-n} - Y^{a+1-n}\right) + X^nY^{a+1-n} - Y^{a+1}$$

$$\frac{X^{a+1} - Y^{a+1}}{X^{a+1-n} - Y^{a+1-n}} = X^n + \frac{X^nY^{a+1-n} - Y^{a+1}}{X^{a+1-n} - Y^{a+1-n}}$$

$$\begin{aligned} \begin{bmatrix} a+1 \\ n \end{bmatrix} &= \frac{[a+1]!}{[n]!\,[a+1-n]!} = \frac{[a]!}{[n]!\,[a-n]!}\frac{[a+1]}{[a+1-n]} = \begin{bmatrix} a \\ n \end{bmatrix}\frac{X^{a+1} - Y^{a+1}}{X^{a+1-n} - Y^{a+1-n}} \\ &= \begin{bmatrix} a \\ n \end{bmatrix}X^n + \begin{bmatrix} a \\ n \end{bmatrix}\frac{X^nY^{a+1-n} - Y^{a+1}}{X^{a+1-n} - Y^{a+1-n}} \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} a \\ n \end{bmatrix}\frac{X^nY^{a+1-n} - Y^{a+1}}{X^{a+1-n} - Y^{a+1-n}} &= \frac{[a]!}{[n]!\,[a-n]!}\frac{X^nY^{a+1-n} - Y^{a+1}}{X^{a+1-n} - Y^{a+1-n}} \\ &= \frac{[a]!}{[n-1]!\,[a-n+1]!}\frac{[a-n+1]}{[n]}\frac{X^nY^{a+1-n} - Y^{a+1}}{X^{a+1-n} - Y^{a+1-n}} \\ &= \begin{bmatrix} a \\ n-1 \end{bmatrix}\frac{X^{a-n+1} - Y^{a-n+1}}{X^n - Y^n}\frac{X^nY^{a+1-n} - Y^{a+1}}{X^{a+1-n} - Y^{a+1-n}} \\ &= \begin{bmatrix} a \\ n-1 \end{bmatrix}\frac{X^nY^{a+1-n} - Y^{a+1}}{X^n - Y^n} \\ &= \begin{bmatrix} a \\ n-1 \end{bmatrix}Y^{a+1-n}\frac{X^n - Y^n}{X^n - Y^n} = \begin{bmatrix} a \\ n-1 \end{bmatrix}Y^{a+1-n} \end{aligned}$$

so that we get

$$\begin{bmatrix} a+1 \\ n \end{bmatrix} = \begin{bmatrix} a \\ n \end{bmatrix} X^n + \begin{bmatrix} a \\ n-1 \end{bmatrix} Y^{a+1-n}. \tag{6.13}$$

Note that

$$\begin{aligned}
\begin{bmatrix} a+1 \\ n \end{bmatrix} &= \begin{bmatrix} a \\ n \end{bmatrix} X^n + \begin{bmatrix} a \\ n-1 \end{bmatrix} Y^{a+1-n} \\
&= \begin{bmatrix} a \\ n-1 \end{bmatrix} \frac{[a-n+1]\, X^n}{[n]} + \begin{bmatrix} a \\ n-1 \end{bmatrix} Y^{a+1-n} \\
&= \begin{bmatrix} a \\ n-1 \end{bmatrix} \left(\frac{[a-n+1]\, X^n}{[n]} + Y^{a+1-n} \right) \\
&= \begin{bmatrix} a \\ n-1 \end{bmatrix} \left(\frac{X^{a-n+1} - Y^{a-n+1}}{X^n - Y^n} X^n + Y^{a+1-n} \right) \\
&= \begin{bmatrix} a \\ n-1 \end{bmatrix} \left(\frac{X^{a+1} - X^n Y^{a-n+1} + X^n Y^{a+1-n} - Y^{a+1}}{X^n - Y^n} \right) \\
&= \begin{bmatrix} a \\ n-1 \end{bmatrix} \left(\frac{X^{a+1} - Y^{a+1}}{X^n - Y^n} \right)
\end{aligned}$$

$$\begin{aligned}
\begin{bmatrix} a \\ n \end{bmatrix} Y^n + \begin{bmatrix} a \\ n-1 \end{bmatrix} X^{a+1-n} &= \begin{bmatrix} a \\ n-1 \end{bmatrix} \frac{[a-n+1]}{[n]} Y^n + \begin{bmatrix} a \\ n-1 \end{bmatrix} X^{a+1-n} \\
&= \begin{bmatrix} a \\ n-1 \end{bmatrix} \left(\frac{X^{a-n+1} - Y^{a-n+1}}{X^n - Y^n} Y^n + X^{a+1-n} \right) = \\
&= \begin{bmatrix} a \\ n-1 \end{bmatrix} \left(\frac{X^{a-n+1} Y^n - Y^{a+1}}{X^n - Y^n} + X^{a+1-n} \right) \\
&= \begin{bmatrix} a \\ n-1 \end{bmatrix} \left(\frac{X^{a-n+1} Y^n - Y^{a+1} + X^{a+1} - X^{a+1-n} Y^n}{X^n - Y^n} \right) \\
&= \begin{bmatrix} a \\ n-1 \end{bmatrix} \left(\frac{-Y^{a+1} + X^{a+1}}{X^n - Y^n} \right).
\end{aligned}$$

Therefore we get

$$\begin{bmatrix} a+1 \\ n \end{bmatrix} = \begin{bmatrix} a \\ n \end{bmatrix} X^n + \begin{bmatrix} a \\ n-1 \end{bmatrix} Y^{a+1-n} = \begin{bmatrix} a \\ n \end{bmatrix} Y^n + \begin{bmatrix} a \\ n-1 \end{bmatrix} X^{a+1-n} \tag{6.14}$$

$$\begin{bmatrix} a \\ n \end{bmatrix} = \frac{[a]!}{[n]!\,[a-n]!} = \frac{[a]!}{[n-1]!\,[a-n+1]!} \frac{[a-n+1]}{[n]} = \begin{bmatrix} a \\ n-1 \end{bmatrix} \frac{X^{a-n+1} - Y^{a-n+1}}{X^n - Y^n}.$$

Assume that $a, n \in \mathbb{N}, 0 \leq n \leq a$ and let us prove that $\begin{bmatrix} a \\ n \end{bmatrix} \in \mathbb{Z}[X, Y]$. Let us proceed by induction on n. Since $\begin{bmatrix} a \\ 0 \end{bmatrix} = 1$ the case $n = 0$ is trivial. Let us assume that the statement holds for some $n-1 \in \mathbb{N}$ and let us prove it for n. Let us proceed by induction on $a-n$. If $a-n=0$ then we have $\left[\begin{array}{c} a \\ a \end{array}\right] = \frac{[a]!}{[a]![0]!} = 1$. Let us assume that the statement holds for all a with $0 \leq n \leq a$ and $a-n=h$ and let us prove it for all a with $a \in \mathbb{N}, 0 \leq n \leq a$ and $a-n=h+1$. From (6.13) we deduce that

$$\left[\begin{array}{c} a \\ n \end{array}\right] = \left[\begin{array}{c} a-1 \\ n \end{array}\right] X^n + \left[\begin{array}{c} a-1 \\ n-1 \end{array}\right] Y^{a-n}. \tag{6.15}$$

Since $(a-1)-n=h$ and since the statement holds for $n-1$ and every $b \in \mathbb{N}, 0 \leq n-1 \leq b$, the conclusion follows.

Let $q \in k$ and let $\varphi : \mathbb{Z}[X,Y] \to k$ be the unique ring homomorphism such that $\varphi(X) = q$ and $\varphi(Y) = 1$. Set

$$\begin{array}{rcl} (n)_q & = & \varphi([n]) \text{ for every } n \in \mathbb{N}, n \geq 1 \\ (n)_q & = & \dfrac{q^n - 1}{q-1} \text{ for every } n \in \mathbb{N}, n \geq 1, q \neq 1 \\ (0)_q & = & 1 \end{array}$$

$$(n)!_q = (1)_q (2)_q \cdots (n)_q$$

and

$$\left(\begin{array}{c} n \\ h \end{array}\right)_q = \frac{(n)!_q}{(n-h)!_q (h)!_q} \quad \text{for all } n, h \in \mathbb{N}, 0 \leq h \leq n$$

Since $n, h \in \mathbb{N}, 0 \leq h \leq n$, from the above we have that $\left[\begin{array}{c} n \\ h \end{array}\right] \in \mathbb{Z}[X,Y]$ so that $\left(\begin{array}{c} n \\ h \end{array}\right)_q = \varphi\left(\left[\begin{array}{c} n \\ h \end{array}\right]\right)$. Then, from (6.15) we get that

$$\left(\begin{array}{c} n \\ h \end{array}\right)_q = \left(\begin{array}{c} n-1 \\ h \end{array}\right)_q q^h + \left(\begin{array}{c} n-1 \\ h-1 \end{array}\right)_q.$$

Let us prove that for every $n, r \in \mathbb{N}, n \geq 2$ and $1 \leq r \leq n$ we have that

$$c_{n,r} = \left(\begin{array}{c} n \\ r \end{array}\right)_q$$

where for every $n \in \mathbb{N}, n \geq 2$, let

$$c_{n,r} = \sum_{0 \leq m_1 \leq m_2 \leq \ldots \leq m_r \leq n-r} q^{m_1 + m_2 + \ldots + m_r} \text{ for every } r \in \mathbb{N}, 1 \leq r \leq n-1 \text{ and let } c_{n,n} = 1.$$

Let us proceed by induction n. For $n=2$ we have

$$c_{2,2} = q^0 = 1 = \left(\begin{array}{c} 2 \\ 2 \end{array}\right)_q \text{ and } c_{2,1} = 1 + q = \left(\begin{array}{c} 2 \\ 1 \end{array}\right)_q.$$

Let us assume that the statement holds for some $n \in \mathbb{N}, n \geq 2$, and let us prove it for $n+1$. From $(6.7), (6.8)$ and (6.9) we deduce that

$$c_{n+1,1} = 1 + q + ... + q^n = (c_{n,1} + q^n) = \begin{pmatrix} n \\ 1 \end{pmatrix}_q + q^n \tag{6.16}$$

$$c_{n+1,r} = c_{n,r} + c_{n,r-1} q^{n+1-r} = \begin{pmatrix} n \\ r \end{pmatrix}_q + \begin{pmatrix} n \\ r-1 \end{pmatrix}_q q^{n+1-r} \text{ for } r = 2, ..., n-1 \tag{6.17}$$

$$c_{n+1,n} = \sum_{0 \leq m_1 \leq m_2 \leq ... \leq m_n \leq 1} q^{m_1+m_2+...+m_n} = 1 + q + ... + q^n = 1 + q\,(c_{n,n-1}) = 1 + q \begin{pmatrix} n \\ n-1 \end{pmatrix}_q \tag{6.18}$$

hence we have to prove that

$$\begin{aligned} \begin{pmatrix} n+1 \\ 1 \end{pmatrix}_q &= \begin{pmatrix} n \\ 1 \end{pmatrix} + q^n \\ \begin{pmatrix} n+1 \\ r \end{pmatrix}_q &= \begin{pmatrix} n \\ r \end{pmatrix}_q + \begin{pmatrix} n \\ r-1 \end{pmatrix}_q q^{n+1-r} \text{for } r = 2, ..., n-1 \\ \begin{pmatrix} n+1 \\ n \end{pmatrix}_q &= 1 + q \begin{pmatrix} n \\ n-1 \end{pmatrix}_q \end{aligned}$$

The first and last equality are easily checked, while the second equality follows from (6.14), in fact

$$\begin{aligned} \begin{pmatrix} n+1 \\ r \end{pmatrix}_q &= \varphi\left(\begin{bmatrix} n+1 \\ r \end{bmatrix} \right) \\ &= \varphi\left(\begin{bmatrix} n \\ r \end{bmatrix} Y^r + \begin{bmatrix} n \\ r-1 \end{bmatrix} X^{n+1-r} \right) \\ &= \begin{pmatrix} n \\ r \end{pmatrix}_q + \begin{pmatrix} n \\ r-1 \end{pmatrix}_q q^{n+1-r}. \end{aligned}$$

Chapter 7

Bosonization

Let $(A, m_A, u_A, \Delta_A, \varepsilon_A)$ be a bialgebra and let $(H, m_H, u_H, \Delta_H, \varepsilon_H, S_H)$ be a Hopf algebra and suppose that

- $\sigma : H \hookrightarrow A$ embeds H as a Hopf subalgebra of A
- $\pi : A \to H$ is a Hopf algebra projection such that
- $\pi \circ \sigma = \mathrm{Id}_H$.

In this case we say that (A, H, σ, π) *is a bialgebra with a projection.* Whenever A is a Hopf algebra, we say that (A, H, σ, π) *is a Hopf algebra with a projection.*

Then A can be endowed with a natural H-bimodule structure by setting

$$h \cdot a = \sigma(h) \cdot_A a \quad \text{and} \quad a \cdot h = a \cdot_A \sigma(h) \quad \text{for every } h \in H \text{ and } a \in A$$

and with an H-bicomodule structure by setting

$${}^H\rho_A(a) = \sum \pi(a_1) \otimes a_2 \quad \text{and} \quad \rho_A^H(a) = \sum a_1 \otimes \pi(a_2) \quad \text{for every } a \in A.$$

Theorem 7.1. *Let (A, H, σ, π) be a bialgebra with a projection. Let $R := A^{co(H)} = \{a \in A \mid a_1 \otimes \pi(a_2) = a \otimes 1_H\}$. Consider the map*

$$\tau : A \to R, \tau(a) := \sum a \cdot_A \sigma S_H \pi(a_2).$$

Then τ is a well defined map and fulfills the following equalities

$$\begin{array}{llcl}
(7.1) & \Delta_A \tau(a) & = & \sum a_1 \cdot_A \sigma S_H \pi(a_3) \otimes \tau(a_2), \text{ for all } a \in A, \\
(7.2) & \pi\tau(a) & = & \varepsilon_A(a) 1_H, \text{ for all } a \in A, \\
(7.3) & \tau(r) & = & r, \text{ for all } r \in R \text{ (this says that } \tau \text{ is surjective)}, \\
(7.4) & \tau(a \cdot_A \sigma(h)) & = & \tau(a) \varepsilon_H(h), \text{ for all } a \in A, h \in H, \\
(7.5) & \Delta_A(r) & \in & A \otimes R, \text{ for all } r \in R, \\
(7.6) & \tau[a\tau(b)] & = & \tau(ab), \text{ for all } a, b \in A. \\
(7.7) & \pi(r) & = & \varepsilon_A(r) 1_H, \text{ for all } r \in R. \\
(7.8) & \sum \tau(a_1) \sigma\pi(a_2) & = & a, \text{ for all } a \in A.
\end{array}$$

Consider the following structures

$$ {}^{H}\rho_{R}(r) := \sum \pi(r_1)\otimes r_2, \qquad \Delta_R(r) = \sum r^1 \otimes r^2 := \sum \tau(r_1)\otimes r_2, \qquad \varepsilon_R(r) := \varepsilon_A(r), $$

$$ h \rightharpoonup r := \tau(\sigma(h)\cdot_A r) = \sigma(h_1)\, r\sigma S_H[(h_2)]. $$

Then

- $(R, \Delta_R, \varepsilon_R)$ *is a coalgebra and* $\tau : A \to R$ *is a coalgebra homomorphism.*
- $\left(\left(R, {}^{H}\rho_R\right), \Delta_R, \varepsilon_R\right)$ *is a left* H*-comodule coalgebra i.e.* Δ_R *and* ε_R *are morphisms of left* H*-comodules.*
- $((R, \rightharpoonup), \Delta_R, \varepsilon_R)$ *is a left* H*-module coalgebra. i.e.* Δ_R *and* ε_R *are morphisms of left* H*-modules.*

Proof. Define $\tau' : A \to A$ by setting $\tau'(a) := \sum a_1 \cdot_A \sigma S_H \pi(a_2)$ for every $a \in A$. We have

$$ \begin{aligned} \Delta_A \tau'(a) &= \sum \tau'(a)_1 \otimes \tau'(a)_2 \\ &= \sum a_{1_1} \cdot_A \sigma S_H \pi(a_2)_1 \otimes a_{1_2} \cdot_A \sigma S_H \pi(a_2)_2 \\ &= \sum a_{1_1} \cdot_A \sigma S_H \pi(a_{2_2}) \otimes a_{1_2} \cdot_A \sigma S_H \pi(a_{2_1}) \\ &= \sum a_1 \cdot_A \sigma S_H \pi(a_4) \otimes a_2 \cdot_A \sigma S_H \pi(a_3) \\ &= \sum a_1 \cdot_A \sigma S_H \pi(a_3) \otimes \tau'(a_2) \end{aligned} $$

and

$$ \pi\tau'(a) = \sum \pi[a_1 \cdot_A \sigma S_H \pi(a_2)] = \sum \pi[a_1] \cdot_H S_H \pi(a_2) = \sum \varepsilon_H \pi(a) 1_H = \varepsilon_A(a) 1_H $$

so that

$$ \begin{aligned} \rho_A^H(\tau'(a)) &= \sum \tau'(a)_1 \otimes \pi(\tau'(a)_2) = \sum a_1 \cdot_A \sigma S_H \pi(a_3) \otimes \pi\tau'(a_2) \\ &= \sum a_1 \cdot_A \sigma S_H \pi(a_2) \otimes 1_H = \tau'(a) \otimes 1_H. \end{aligned} $$

Therefore $\tau'(a) \in R$ and hence τ is well defined and (7.1) and (7.2) are proved.

Let us prove (7.3):

$$ \tau(r) = \sum r_1 \cdot_A \sigma S_H \pi(r_2) \overset{r\in R}{=} r \cdot_A \sigma S_H \pi(1_H) = r. $$

Let us prove (7.4):

$$ \begin{aligned} \tau(a \cdot_A \sigma(h)) &= \sum (a \cdot_A \sigma(h))_1 \cdot_A \sigma S_H \pi[(a \cdot_A \sigma(h))_2] \\ &= \sum a_1 \cdot_A \sigma(h_1) \cdot_A \sigma S_H \pi[a_2 \cdot_A \sigma(h_2)] \\ &= \sum a_1 \cdot_A \sigma(h_1) \cdot_A \sigma S_H[\pi(a_2) h_2] \\ &= \sum a_1 \cdot_A \sigma(h_1) \cdot_A \sigma S_H(h_2) \cdot_A \sigma S_H \pi(a_2) = \tau(a)\varepsilon_H(h). \end{aligned} $$

Let us prove (7.5):

$$A\otimes R\ni\sum r_1\otimes\tau\left(r_2\right)=\sum r_1\otimes r_{2_1}\cdot_A\sigma S_H\pi\left(r_{2_2}\right)=\sum r_{1_1}\otimes r_{1_2}\cdot_A\sigma S_H\pi\left(r_2\right)\overset{r\in R}{=}\sum r_1\otimes r_2=\Delta_A\left(r\right).$$

Let us prove (7.6):

$$\tau\left[a\tau\left(b\right)\right]=\sum\tau\left[ab_1\cdot_A\sigma S_H\pi\left(b_2\right)\right]\overset{(7.4)}{=}\tau\left(ab\right).$$

Note that (7.7) follows directly from (7.2) and (7.3).
Finally, let us prove (7.8). For every $a\in A$ we have

$$\begin{aligned}\sum\tau\left(a_1\right)\sigma\pi\left(a_2\right) &= \sum a_1\cdot_A\sigma S_H\pi\left(a_2\right)\sigma\pi\left(a_3\right)\\ &= \sum a_1\cdot_A\sigma\left(S_H\pi\left(a_2\right)\pi\left(a_3\right)\right)\\ &= a_1\varepsilon_A\left(a_2\right)=a.\end{aligned}$$

Now, for $a\in A$, we have

$$\begin{aligned}\Delta_R\tau\left(a\right)=\sum\tau\left(\tau\left(a\right)_1\right)\otimes\tau\left(a\right)_2 &\overset{(7.1)}{=}\sum\tau\left[a_1\cdot_A\sigma S_H\pi\left(a_3\right)\right]\otimes\tau\left(a_2\right)\\ &\overset{(7.4)}{=}\sum\tau\left(a_1\right)\otimes\tau\left(a_2\right)=\left(\tau\otimes\tau\right)\Delta_A\left(a\right)\end{aligned}$$

so that

$$\Delta_R\circ\tau=\left(\tau\otimes\tau\right)\circ\Delta_A.$$

Let us prove that $(R,\Delta_R,\varepsilon_R)$ is a coalgebra. First of all, note that, in view of (7.5), Δ_R is well defined. We have

$$\begin{aligned}\left(\Delta_R\otimes R\right)\circ\Delta_R\circ\tau &= \left(\Delta_R\otimes R\right)\circ\left(\tau\otimes\tau\right)\circ\Delta_A=\left(\tau\otimes\tau\otimes\tau\right)\circ\left(\Delta_A\otimes R\right)\circ\Delta_A,\\ \left(R\otimes\Delta_R\right)\circ\Delta_R\circ\tau &= \left(R\otimes\Delta_R\right)\circ\left(\tau\otimes\tau\right)\circ\Delta_A=\left(\tau\otimes\tau\otimes\tau\right)\circ\left(R\otimes\Delta_A\right)\circ\Delta_A\end{aligned}$$

which entail that $(\Delta_R\otimes R)\circ\Delta_R\circ\tau=(R\otimes\Delta_R)\circ\Delta_R\circ\tau$ whence $(\Delta_R\otimes R)\circ\Delta_R=(R\otimes\Delta_R)\circ\Delta_R$ (τ is surjective). Moreover

$$\varepsilon_R\tau\left(a\right)=\sum\varepsilon_A\left[a_1\cdot_A\sigma S_H\pi\left(a_2\right)\right]=\varepsilon_A\left(a\right).$$

Then

$$\begin{aligned}l_R\circ\left(\varepsilon_R\otimes R\right)\circ\Delta_R\circ\tau &= l_R\circ\left(\varepsilon_R\otimes R\right)\circ\left(\tau\otimes\tau\right)\circ\Delta_A=l_R\circ\left(K\otimes\tau\right)\circ\left(\varepsilon_A\otimes A\right)\circ\Delta_A\\ &= \tau\circ l_A\circ\left(\varepsilon_A\otimes A\right)\circ\Delta_A=\tau\\ r_R\circ\left(R\otimes\varepsilon_R\right)\circ\Delta_R\circ\tau &= r_R\circ\left(R\otimes\varepsilon_R\right)\circ\left(\tau\otimes\tau\right)\circ\Delta_A=r_R\circ\left(\tau\otimes K\right)\circ\left(A\otimes\varepsilon_A\right)\circ\Delta_A\\ &= \tau\circ r_A\circ\left(A\otimes\varepsilon_A\right)\circ\Delta_A=\tau\end{aligned}$$

so that $l_R\circ(\varepsilon_R\otimes R)\circ\Delta_R\circ\tau=\tau=r_R\circ(R\otimes\varepsilon_R)\circ\Delta_R\circ\tau$ and then $l_R\circ(\varepsilon_R\otimes R)\circ\Delta_R=\mathrm{Id}_R=r_R\circ(R\otimes\varepsilon_R)\circ\Delta_R$. Hence $(R,\Delta_R,\varepsilon_R)$ is a coalgebra and τ is a coalgebra homomorphism.

Let us prove $\left(\left(R,^{H}\rho_R\right),\Delta_R,\varepsilon_R\right)$ is a left H-comodule coalgebra. First we have

$$^{H}\rho_A(r)=\sum\pi(r_1)\otimes r_2\overset{(7.5)}{\in}H\otimes R$$

so that $\left(R,{}^{H}\rho_R\right)$ is a subcomodule of $\left(A,{}^{H}\rho_A\right)$. Moreover

$$\begin{aligned}
{}^{H}\rho_{R\otimes R}\Delta_R(r) &= {}^{H}\rho_{R\otimes R}\left(\sum\tau(r_1)\otimes r_2\right)\\
&= \sum\pi\left[\tau(r_1)_1\right]\pi(r_{2_1})\otimes\tau(r_1)_2\otimes r_{2_2}\\
&= \sum\pi\left[\tau(r_1)_1\right]\pi(r_2)\otimes\tau(r_1)_2\otimes r_3\\
&\overset{(7.1)}{=} \sum\pi\left[r_{1_1}\cdot_A\sigma S_H\pi(r_{1_3})\right]\pi(r_2)\otimes\tau(r_{1_2})\otimes r_3\\
&= \sum\pi\left[r_{1_1}\right]S_H\pi(r_{1_3})\pi(r_2)\otimes\tau(r_{1_2})\otimes r_3\\
&= \sum\pi(r_1)\otimes\tau(r_2)\otimes r_3\\
&= \sum\pi(r_1)\otimes\Delta_R(r_2)=(H\otimes\Delta_R)^{H}\rho_R(r).
\end{aligned}$$

Recall that k has a natural structure of left H-comodule defined by setting ${}^{H}\rho_k=(u_H\otimes k)\circ r_k^{-1}=r_H^{-1}\circ u_H$.

$$\begin{aligned}
(H\otimes\varepsilon_R)^{H}\rho_R(r) &= \sum\pi(r_1)\otimes\varepsilon_R(r_2)=\sum\pi(r_1)\otimes\varepsilon_A(r_2)\\
&= \pi(r)\otimes 1_K\overset{(7.3),(7.2)}{=}\varepsilon_A(r)1_H\otimes 1_K=\varepsilon_R(r)1_H\otimes 1_K={}^{H}\rho_k\varepsilon_R(r).
\end{aligned}$$

so that $\left(\left(R,^{H}\rho_R\right),\Delta_R,\varepsilon_R\right)$ is a left H-comodule coalgebra.

Let us prove that $((R,\rightharpoonup),\Delta_R,\varepsilon_R)$ is a left H-module coalgebra. First let us check that $\rightharpoonup: H\otimes R\to R$ defines a left action of H on R. We have, for every $h,k\in H$ and for every $r\in R$,

$$\begin{aligned}
k &\rightharpoonup (h\rightharpoonup r)=k\rightharpoonup\tau(\sigma(h)\cdot_A r)=\tau\left[\sigma(k)\cdot_A\tau(\sigma(h)\cdot_A r)\right]\overset{(7.6)}{=}\tau\left[\sigma(k)\cdot_A\sigma(h)\cdot_A r\right]=(kh) -\\
1_H &\rightharpoonup r=\tau(\sigma(1_H)\cdot_A r)=\tau(r)=r.
\end{aligned}$$

Let us prove that $\Delta_R: R\to R\otimes R$ is left H-linear where $R\otimes R$ is a left H-module via the diagonal action induced by $\rightharpoonup$. We have

$$\begin{aligned}
&\Delta_R(h\rightharpoonup r)=\Delta_R\tau\left[\sigma(h)\cdot_A r\right]\\
&= (\tau\otimes\tau)\Delta_A\left[\sigma(h)\cdot_A r\right]\\
&= \sum\tau(\sigma(h_1)\cdot_A r_1)\otimes\tau(\sigma(h_2)\cdot_A r_2)\\
&\overset{(7.6)}{=}\sum\tau(\sigma(h_1)\cdot_A\tau(r_1))\otimes\tau(\sigma(h_2)\cdot_A r_2)\\
&= \sum\left(h_1\rightharpoonup r^1\right)\otimes\left(h_2\rightharpoonup r^2\right)
\end{aligned}$$

and

$$\varepsilon_R(h\rightharpoonup r)=\varepsilon_R\tau(\sigma(h)\cdot_A r)=\varepsilon_A(\sigma(h)\cdot_A r)=\varepsilon_H(h)\varepsilon_R(r).$$

Thus $((R,\rightharpoonup),\Delta_R,\varepsilon_R)$ is a left H-module coalgebra. □

Proposition 7.2. *Using the assumptions and notations of Theorem 7.1 we have that*

- *R is a subalgebra of A.*
- $\Delta_R(1_R) = 1_R \otimes 1_R$.
- *$\varepsilon_R : R \to k$ is an algebra morphism.*
- *$\tau : A \to R$ is a morphism of left H-modules.*
- *For every $r, s \in R$ the following equality holds*

$$\Delta_R(r \cdot s) = \sum r^1 \left(r^2_{(-1)} \rightharpoonup s^1 \right) \otimes r^2_{(0)} s^2. \tag{7.9}$$

- *For every $h \in H$ and $r \in R$ the following equality holds*

$$^H\rho_R(h \rightharpoonup r) = \sum h_1 r_{(-1)} S_H(h_3) \otimes \left(h_2 \rightharpoonup r_{(0)} \right). \tag{7.10}$$

- Id_R *has an inverse in the convolution algebra* $\mathrm{Hom}(R^c, R^a)$ *whenever A is a Hopf algebra.*

Proof. Let $r, s \in R$. We compute

$$\begin{aligned} \rho_A^H(r \cdot_A s) &= \sum (r \cdot_A s)_1 \otimes \pi((r \cdot_A s)_2) = \sum r_1 \cdot_A s_1 \otimes \pi(r_2 \cdot_A s_2) \\ &= \sum (r_1 \cdot_A s_1) \otimes (\pi(r_2) \cdot_H \pi(s_2)) = r \cdot_A s \otimes 1_H. \end{aligned}$$

Hence we obtain that $r \cdot_A s \in R$. Moreover $1_A \in R$ and hence R is a subalgebra of A. Since $\varepsilon_R = \varepsilon_{A|R}$we deduce that ε_R is an algebra morphism. Moreover we have

$$\Delta_R(1_R) = \sum \tau((1_A)_1) \otimes (1_A)_2 = \sum (1_A)_1 \cdot_A \sigma S_H \pi((1_A)_2) \otimes (1_A)_3 = 1_R \otimes 1_R.$$

Let $h \in H$ and $r \in R$ and let us compute

$$\begin{aligned} \tau(\sigma(h) \cdot_A r) &= \sum (\sigma(h) \cdot_A r)_1 \cdot_A \sigma S_H \pi(\sigma(h) \cdot_A r)_2 \\ &= \sum (\sigma(h_1) \cdot_A r_1) \cdot_A \sigma S_H \pi(\sigma(h_2) \cdot_A r_2) \\ &= \sum \sigma(h_1) \cdot_A r_1 \cdot_A \sigma S_H \pi(r_2) \cdot_A \sigma S_H \pi(\sigma(h_2)) \\ &\overset{r \in R}{=} \sum \sigma(h_1) \cdot_A r \cdot_A \sigma S_H \pi(\sigma(h_2)) \\ &= h \rightharpoonup r \overset{(7.3)}{=} h \rightharpoonup \tau(r). \end{aligned}$$

Let us calculate

$$\sum r^1\left(r^2_{(-1)} \rightharpoonup s^1\right)\otimes r^2_{(0)}s^2 = \sum \tau\left(r_1\right)\left(\left(r_2\right)_{(-1)} \rightharpoonup s^1\right)\otimes \left(r_2\right)_{(0)}s^2 =$$
$$= \sum \tau\left(r_1\right)\left(\pi\left(r_{2_1}\right) \rightharpoonup s^1\right)\otimes r_{2_2}s^2 = \sum \tau\left(r_1\right)\left(\pi\left(r_2\right) \rightharpoonup s^1\right)\otimes r_3 s^2$$
$$= \sum \tau\left(r_1\right)\left(\pi\left(r_2\right) \rightharpoonup s^1\right)\otimes r_3 s^2 = \sum \tau\left(r_1\right)\left(\pi\left(r_2\right) \rightharpoonup \tau\left(s_1\right)\right)\otimes r_3 s_2 \overset{\tau \text{isleft} H\text{-lin}}{=}$$
$$= \sum \tau\left(r_1\right)\left(\tau\left(\sigma\pi\left(r_2\right)\cdot_A s_1\right)\right)\otimes r_3 s_2 = \sum r_1\cdot_A \sigma S_H\pi\left(r_2\right)\sigma\pi\left(r_3\right)s_1\cdot_A \sigma S_H\pi\left(\sigma\pi\left(r_4\right)s_2\right)\otimes r_5 s_3$$
$$= \sum r_1\cdot_A \sigma\left[S_H\pi\left(r_2\right)\pi\left(r_3\right)\right]s_1\cdot_A \sigma S_H\left[\pi\left(r_4\right)\pi\left(s_2\right)\right]\otimes r_5 s_3$$
$$= \sum r_1 s_1\cdot_A \sigma S_H\left[\pi\left(r_2\right)\pi\left(s_2\right)\right]\otimes r_3 s_3$$
$$= \sum r_1 s_1\cdot_A \sigma S_H\pi\left(r_2 s_2\right)\otimes r_3 s_3 = \sum \tau\left(r_1 s_1\right)\otimes r_2 s_2 = \Delta_R\left(rs\right).$$

Let us prove (7.10).

$$\begin{aligned}
{}^H\rho_R\left(h \rightharpoonup r\right) &= {}^H\rho_R\left(\sum \sigma\left(h_1\right)r\sigma\left(S_H\left(h_2\right)\right)\right)\\
&= \sum \pi\left[\left(\sum \sigma\left(h_1\right)r\sigma\left(S_H\left(h_2\right)\right)\right)_1\right]\otimes\left(\sum \sigma\left(h_1\right)r\sigma\left(S_H\left(h_2\right)\right)\right)_2\\
&= \sum \pi\sigma\left(h_{1_1}\right)\pi\left(r_1\right)\pi\sigma\left(\left(S_H\left(h_2\right)\right)_1\right)\otimes\sigma\left(h_{1_2}\right)r_2\sigma S_H\left(h_2\right)_2\\
&= \sum h_{1_1}\pi\left(r_1\right)S_H\left(h_{2_2}\right)\otimes\sigma\left(h_{1_2}\right)r_2\sigma S_H\left(h_{2_1}\right)\\
&= \sum h_1\pi\left(r_1\right)S_H\left(h_4\right)\otimes\sigma\left(h_2\right)r_2\sigma S_H\left(h_3\right)\\
&= \sum h_1 r_{(-1)}S_H\left(h_3\right)\otimes\left(h_2 \rightharpoonup r_0\right).
\end{aligned}$$

Assume now that A is a Hopf algebra with antipode S_A and consider the map $S : R \to R$ defined by setting

$$S\left(r\right) = \sum \tau\left(\sigma\pi\left(r_1\right)\cdot_A\left(\left[S_A\left(r_2\right)\right]\right)\right).$$

We compute

$$\begin{aligned}
\sum \tau\left(\sigma\pi\left(r_1\right)\cdot_A\left(\left[S_A\left(r_2\right)\right]\right)\right) &= \sum \sigma\pi\left(r_1\right)_1 S_A\left(r_2\right)_1\cdot_A \sigma S_H\pi\left[\sigma\pi\left(r_1\right)_2 S_A\left(r_2\right)_2\right]\\
&= \sum \sigma\pi\left(r_1\right)S_A\left(r_4\right)\cdot_A \sigma S_H\pi\left[\sigma\pi\left(r_2\right)S_A\left(r_3\right)\right]\\
&= \sum \sigma\pi\left(r_1\right)S_A\left(r_4\right)\cdot_A \sigma S_H\left[\pi\sigma\pi\left(r_2\right)\pi S_A\left(r_3\right)\right]\\
&= \sum \sigma\pi\left(r_1\right)S_A\left(r_4\right)\cdot_A \sigma S_H\left[\pi\left(r_2\right)\pi S_A\left(r_3\right)\right]\\
&= \sum \sigma\pi\left(r_1\right)S_A\left(r_4\right)\cdot_A \sigma S_H\pi\left[r_2 S_A\left(r_3\right)\right] = \sum \sigma\pi\left(r_1\right)S_A\left(r_2\right).
\end{aligned}$$

Therefore we get

$$S\left(r\right) = \sum \sigma\pi\left(r_1\right)S_A\left(r_2\right). \tag{7.11}$$

We compute

$$\begin{aligned}\sum r^1 S\left(r^2\right) &= \sum \tau\left(r_1\right) S\left(r_2\right) = \sum \tau\left(r_1\right) \sigma\pi\left(r_2\right) S_A\left(r_3\right) \overset{(7.8)}{=} \sum r_1 S_A\left(r_2\right) \\ &= \varepsilon_A\left(r\right) 1_A = \varepsilon_R\left(r\right)\end{aligned}$$

and

$$\begin{aligned}\sum S\left(r^1\right) r^2 &= \sum S\left(\tau\left(r_1\right)\right) r_2 \\ &= \sum \sigma\pi\left(\left(\tau\left(r_1\right)\right)_1\right) S_A\left(\left(\tau\left(r_1\right)\right)_2\right) r_2 \\ &\overset{(7.1)}{=} \sum \sigma\pi\left[\left(r_{1_1}\right) \cdot_A \sigma S_H \pi\left(r_{1_3}\right)\right] S_A\left(\tau\left(r_{1_2}\right)\right) r_2 \\ &= \sum \sigma\pi\left(r_{1_1}\right) \sigma\pi\sigma S_H \pi\left(r_{1_3}\right) S_A\left(\tau\left(r_{1_2}\right)\right) r_2 \\ &= \sum \sigma\pi\left(r_1\right) \sigma S_H \pi\left(r_3\right) S_A\left(\tau\left(r_2\right)\right) r_4 \\ &= \sum \sigma\pi\left(r_1\right) \sigma S_H \pi\left(r_3\right) S_A\left(r_{2_1} \cdot_A \sigma S_H \pi\left(r_{2_2}\right)\right) r_4 \\ &= \sum \sigma\pi\left(r_1\right) \sigma S_H \pi\left(r_4\right) S_A \sigma S_H \pi\left(r_3\right) S_A\left(r_2\right) r_5 \\ &= \sum \sigma\pi\left(r_1\right) \left[\sigma S_H \pi\left(r_3\right)\right]_1 S_A \left[\sigma S_H \pi\left(r_3\right)\right]_2 S_A\left(r_2\right) r_5 \\ &= \sum \sigma\pi\left(r_1\right) S_A\left(r_2\right) r_3 \\ &= \sigma\pi\left(r\right) = \sigma\left(\varepsilon_A\left(r\right) 1_H\right) = \varepsilon_R\left(r\right).\end{aligned}$$

□

7.3. *Let us consider the map* $\omega : R \otimes H \to A$ *defined by setting* $\omega\left(r \otimes h\right) = r \cdot_A \sigma\left(h\right)$ *and the map* $\omega^{'} : A \to R \otimes H$ *defined by setting* $\omega'\left(a\right) = \tau\left(a_1\right) \otimes \pi\left(a_2\right)$

Theorem 7.4. *Using the assumptions and notations above, we have that* $\omega : R \otimes H \to A$ *is bijective with inverse* ω'*.*

Proof. Let us compute, for every $r \in R$ and $h \in H$

$$\begin{aligned}\omega'\left[\omega\left(r \otimes h\right)\right] &= \omega'\left(r \cdot_A \sigma\left(h\right)\right) = \sum \tau\left(\left[r \cdot_A \sigma\left(h\right)\right]_1\right) \otimes \pi\left(\left[r \cdot_A \sigma\left(h\right)\right]_2\right) \\ &= \sum \tau\left(r_1 \cdot_A \sigma\left(h_1\right)\right) \otimes \pi\left(r_2 \cdot_A \sigma\left(h_2\right)\right) \overset{(7.4)}{=} \sum \tau\left(r_1\right) \otimes \pi\left(r_2\right) h \overset{r \in R}{=} \tau\left(r\right) \otimes h\end{aligned}$$

and, for every $a \in A$

$$\omega\left(\omega'\left(a\right)\right) = \sum \tau\left(a_1\right) \cdot_A \sigma\pi\left(a_2\right) \overset{(7.8)}{=} a.$$

□

7.5. *By using ω we can transfer the bialgebra structure of A to $R\otimes H$. Let us compute it. For every $r\in R$ and $h\in H$ we compute*

$$(\omega'\otimes\omega')\left(\Delta_A\left(\omega\left(r\otimes h\right)\right)\right)=(\omega'\otimes\omega')\,\Delta_A\left(r\cdot_A\sigma\left(h\right)\right)=(\omega'\otimes\omega')\sum\left(r_1\cdot_A\sigma\left(h_1\right)\right)\otimes\left(r_2\cdot_A\sigma\left(h_2\right)\right)$$
$$=\sum\omega'\left(r_1\cdot_A\sigma\left(h_1\right)\right)\otimes\omega'\left(r_2\cdot_A\sigma\left(h_2\right)\right)=\sum\omega'\left(r_1\cdot_A\sigma\left(h_1\right)\right)\otimes\omega'\left(\omega\left(r_2\otimes h_2\right)\right)$$
$$=\sum\omega'\left(r_1\cdot_A\sigma\left(h_1\right)\right)\otimes r_2\otimes h_2$$
$$=\sum\tau\left(r_{1_1}\cdot_A\sigma\left(h_{1_1}\right)\right)\otimes\pi\left(r_{1_2}\cdot_A\sigma\left(h_{1_2}\right)\right)\otimes r_2\otimes h_2$$
$$\overset{(7.4)}{=}\sum\tau\left(r_{1_1}\right)\otimes\pi\left(r_{1_2}\right)h_1\otimes r_2\otimes h_2=\sum\tau\left(r_1\right)\otimes\pi\left(r_2\right)h_1\otimes r_3\otimes h_2$$
$$=\sum\tau\left(r_1\right)\otimes\pi\left(r_{2_1}\right)h_1\otimes r_{2_2}\otimes h_2\overset{(7.5)(7.3)}{=}\sum\tau\left(r_1\right)\otimes\pi\left(\tau\left(r_2\right)_1\right)h_1\otimes\tau\left(r_2\right)_2\otimes h_2$$
$$=\sum\tau\left(r_1\right)\otimes\tau\left(r_2\right)_{(-1)}h_1\otimes\tau\left(r_2\right)_{(0)}\otimes h_2=\sum r^1\otimes\left(r^2\right)_{(-1)}h_1\otimes\left(r^2\right)_{(0)}\otimes h_2$$

and for every $r,s\in R$ and $h,t\in H$ we calculate

$$\omega'\left(m_A\left(\omega\left(r\otimes h\right)\omega\left(s\otimes t\right)\right)\right)=\omega'\left(r\cdot_A\sigma\left(h\right)\cdot_A s\cdot_A\sigma\left(t\right)\right)=$$
$$=\omega'\left(r\cdot_A\sigma\left(h_1\right)\cdot_A s\cdot_A\sigma S_H\left(h_2\right)\cdot_A\sigma\left(h_3\right)\cdot_A\sigma\left(t\right)\right)$$
$$=\omega'\left(r\cdot_A\left(h_1\rightharpoonup s\right)\cdot_A\sigma\left(h_2\right)\cdot_A\sigma\left(t\right)\right)$$
$$\overset{Rissubal+(7.3)}{=}\omega'\left(\tau\left(r\cdot_A\left(h_1\rightharpoonup s\right)\right)\cdot_A\sigma\left(h_2t\right)\right)$$
$$=\omega'\omega\left(r\cdot_A\left(h_1\rightharpoonup s\right)\otimes h_2t\right)$$
$$=r\cdot_A\left(h_1\rightharpoonup s\right)\otimes h_2t=r\cdot_R\left(h_1\rightharpoonup s\right)\otimes h_2t.$$

Lemma 7.6. *Assume that $(H,m_H,u_H,\Delta_H,\varepsilon_H,S_H)$ is a Hopf algebra, (R,m_R,u_R) is a k-algebra, $(R,\Delta_R,\varepsilon_R)$ is a k-coalgebra, $(R,\rightharpoonup)$ is a left H-module, $\left(R,{}^H\rho_R\right)$ is a left H-comodule such that*

- *$m_R,u_R,\Delta_R,\varepsilon_R$ are left H-linear,*
- *$m_R,u_R,\Delta_R,\varepsilon_R$ are left H-colinear.*

Then the following statements are equivalent

(a) ${}^H\rho_R\left(h\rightharpoonup r\right)=\sum h_1r_{(-1)}S_H\left(h_3\right)\otimes\left(h_2\rightharpoonup r_{(0)}\right)$ *for every $h\in H$ and $r\in R$.*

(b) $\sum\left(h_1\rightharpoonup r\right)_{(-1)}h_2\otimes\left(h_1\rightharpoonup r\right)_{(0)}=\sum h_1r_{(-1)}\otimes h_2\rightharpoonup r_{(0)}$ *for every $h\in H$ and $r\in R$.*

Proof. $(a)\Rightarrow(b)$ For every $h\in H$ and $r\in R$, we compute

$$\sum\left(h_1\rightharpoonup r\right)_{(-1)}h_2\otimes\left(h_1\rightharpoonup r\right)_{(0)}\overset{(a)}{=}\sum h_1r_{(-1)}S_H\left(h_3\right)h_2\otimes\left(h_2\rightharpoonup r_{(0)}\right)=\sum h_1r_{(-1)}\otimes h_2\rightharpoonup r_{(0)}.$$

$(b)\Rightarrow(a)$ For every $h\in H$ and $r\in R$, we compute

$$\begin{aligned}{}^H\rho_R\left(h\rightharpoonup r\right)&=\sum\left(h\rightharpoonup r\right)_{(-1)}\otimes\left(h\rightharpoonup r\right)_{(-2)}=\sum\left(h\rightharpoonup r\right)_{(-1)}h_2S\left(h_3\right)\otimes\left(h\rightharpoonup r\right)_{(-2)}\overset{(b)}{=}\\&=\sum h_1r_{(-1)}S\left(h_3\right)\otimes\left(h_2\rightharpoonup r_{(0)}\right).\end{aligned}$$

□

7.7. *Assume now that* $(H, m_H, u_H, \Delta_H, \varepsilon_H, S_H)$ *is a Hopf algebra,* (R, m_R, u_R) *is a* k*-algebra,* $(R, \Delta_R, \varepsilon_R)$ *is a* k*-coalgebra,* $(R, \rightharpoonup)$ *is a left* H*-module,* $\left(R, {}^H\rho_R\right)$ *is a left* H*-comodule such that*

1. $m_R, u_R, \Delta_R, \varepsilon_R$ *are left* H*-linear*
2. $m_R, u_R, \Delta_R, \varepsilon_R$ *are left* H*-colinear*
3. ${}^H\rho_R(h \rightharpoonup r) = \sum h_1 r_{(-1)} S_H(h_3) \otimes \left(h_2 \rightharpoonup r_{(0)}\right)$ *or equivalently (see Lemma 7.6)* $\sum (h_1 \rightharpoonup r)_{(-1)} h_2 \otimes (h_1 \rightharpoonup r)_{(0)} = \sum h_1 r_{(-1)} \otimes h_2 \rightharpoonup r_{(0)}$ *for every* $h \in H$ *and* $r \in R$.
4. $\Delta_R(1_R) = 1_R \otimes 1_R$,
5. $\Delta_R(r \cdot s) = \sum r^1 \left(r^2_{(-1)} \rightharpoonup s^1\right) \otimes r^2_{(0)} s^2$ *for every* $r, s \in R$.
6. $\varepsilon_R : R \to k$ *is an algebra morphism.*

Define a multiplication on $R \otimes H$ *by setting*

$$(r \otimes h) \cdot (s \otimes t) = \sum r \cdot_R (h_1 \rightharpoonup s) \otimes h_2 t$$

with unit $1_R \otimes 1_H$, *a comultiplication by setting*

$$\Delta(r \otimes h) = \sum r^1 \otimes \left(r^2\right)_{(-1)} h_1 \otimes \left(r^2\right)_{(0)} \otimes h_2$$

and a counit

$$\varepsilon(r \otimes h) = \varepsilon_R(r)\, \varepsilon_H(h).$$

Theorem 7.8. *Within the assumptions and definitions above* $R \otimes H$ *is a bialgebra.*

Proof. First of all, let us prove that $R \otimes H$ is an algebra. For every $r, s, w \in R$ and for every $h, t, l \in H$ we have

$$(r \otimes h) \cdot [(s \otimes t) \cdot (w \otimes l)] = (r \otimes h) \cdot \left(\sum s \cdot_R (t_1 \rightharpoonup w) \otimes t_2 l\right)$$

$$= \sum r \cdot_R (h_1 \rightharpoonup [s \cdot_R (t_1 \rightharpoonup w)]) \otimes h_2 t_2 l \overset{\text{mult}H\text{lin}}{=} \sum r \cdot_R [(h_1 \rightharpoonup s) \cdot_R (h_2 \rightharpoonup (t_1 \rightharpoonup w))] \otimes h_3 t_2 l$$

$$= \sum r \cdot_R [(h_1 \rightharpoonup s) \cdot_R (h_2 t_1 \rightharpoonup w)] \otimes h_3 t_2 l = \sum [r \cdot_R (h_1 \rightharpoonup s)] \cdot_R (h_2 t_1 \rightharpoonup w) \otimes h_3 t_2 l$$

$$= \left(\sum r \cdot_R (h_1 \rightharpoonup s) \otimes h_2 t\right) \cdot (w \otimes l) = [(r \otimes h) \cdot (s \otimes t)] \cdot (w \otimes l)$$

so that the multiplication is associative. Moreover

$$(r \otimes h) \cdot (1_R \otimes 1_H) = \sum r \cdot_R (h_1 \rightharpoonup 1_R) \otimes h_2 1_H \overset{u\,\text{is left}\,H\text{-lin}}{=} \sum r \cdot_R \varepsilon_H(h_1) \otimes h_2 1_H = r \otimes h$$

and

$$(1_R \otimes 1_H) \cdot (r \otimes h) = \sum 1_R \cdot_R (1_H \rightharpoonup r) \otimes 1_H h = r \otimes h.$$

Let us prove that $R \otimes H$ is a coalgebra. For every $r \in R$ and $h \in H$, we have

$$
\begin{aligned}
(\Delta \otimes R \otimes H)\, \Delta (r \otimes h) &= \sum \Delta \left(r^1 \otimes \left(r^2\right)_{(-1)} h_1 \right) \otimes \left(r^2\right)_{(0)} \otimes h_2 \\
&= \sum \left(r^1\right)^1 \otimes \left(\left(r^1\right)^2\right)_{(-1)} \left[\left(r^2\right)_{(-1)} h_1\right]_1 \otimes \left(\left(r^1\right)^2\right)_{(0)} \otimes \left[\left(r^2\right)_{(-1)} h_1\right]_2 \otimes \left(r^2\right)_{(0)} \otimes h_2 \\
&= \sum r^1 \otimes \left(r^2\right)_{(-1)} \left[\left(r^3\right)_{(-1)} h_1\right]_1 \otimes \left(r^2\right)_{(0)} \otimes \left[\left(r^3\right)_{(-1)} h_1\right]_2 \otimes \left(r^3\right)_{(0)} \otimes h_2 \\
&= \sum r^1 \otimes \left(r^2\right)_{(-1)} \left[\left(r^3\right)_{(-1)}\right]_1 h_1 \otimes \left(r^2\right)_{(0)} \otimes \left[\left(r^3\right)_{(-1)}\right]_2 h_2 \otimes \left(r^3\right)_{(0)} \otimes h_3 \\
&= \sum r^1 \otimes \left(r^2\right)_{(-1)} \left(r^3\right)_{(-2)} h_1 \otimes \left(r^2\right)_{(0)} \otimes \left(r^3\right)_{(-1)} h_2 \otimes \left(r^3\right)_{(0)} \otimes h_3
\end{aligned}
$$

$$
\begin{aligned}
(R \otimes H \otimes \Delta)\, \Delta (r \otimes h) &= \sum r^1 \otimes \left(r^2\right)_{(-1)} h_1 \otimes \Delta \left(\left(r^2\right)_{(0)} \otimes h_2 \right) \\
&= \sum r^1 \otimes \left(r^2\right)_{(-1)} h_1 \otimes \left(\left(r^2\right)_{(0)}\right)^1 \otimes \left(\left(\left(r^2\right)_{(0)}\right)^2 \right)_{(-1)} h_{2_1} \otimes \left(\left(\left(r^2\right)_{(0)} \right.\right. \\
&\overset{\Delta_R \text{left} H\text{-col}}{=} \sum r^1 \otimes \left(\left(r^2\right)^1\right)_{(-1)} \left(\left(r^2\right)^2\right)_{(-1)} h_1 \otimes \left(\left(r^2\right)^1\right)_{(0)} \otimes \left(\left(\left(r^2\right)^2 \right.\right. \\
&= \sum r^1 \otimes \left(r^2\right)_{(-1)} \left(r^3\right)_{(-1)} h_1 \otimes \left(r^2\right)_{(0)} \otimes \left(\left(r^3\right)_{(0)}\right)_{(-1)} h_2 \otimes \left(\left(r^3\right)_{(0)}\right)_{(0)} \\
&= \sum r^1 \otimes \left(r^2\right)_{(-1)} \left(r^3\right)_{(-2)} h_1 \otimes \left(r^2\right)_{(0)} \otimes \left(r^3\right)_{(-1)} h_2 \otimes \left(r^3\right)_{(0)} \otimes h_3
\end{aligned}
$$

so that $(\Delta \otimes R \otimes H) \circ \Delta = (R \otimes H \otimes \Delta) \circ \Delta$. Moreover

$$
\begin{aligned}
\left[r_{R \otimes H} \circ (R \otimes H \otimes \varepsilon) \circ \Delta \right] (r \otimes h) &= \sum r^1 \otimes \left(r^2\right)_{(-1)} h_1 \varepsilon_R \left(\left(r^2\right)_{(0)} \right) \varepsilon_H (h_2) \\
&= \sum r^1 \otimes \left(r^2\right)_{(-1)} \varepsilon_R \left(\left(r^2\right)_{(0)} \right) h_1 \varepsilon_H (h_2) \\
\overset{\varepsilon_R \text{isleft} H\text{-col}}{=} \sum r^1 \otimes \varepsilon_R \left(r^2\right) h &= r \otimes h
\end{aligned}
$$

and

$$
\begin{aligned}
\left[l_{R \otimes H} \circ (\varepsilon \otimes R \otimes H) \circ \Delta \right] (r \otimes h) &= \sum \varepsilon_R \left(r^1\right) \varepsilon_H \left(\left(r^2\right)_{(-1)} h_1 \right) \left(r^2\right)_{(0)} \otimes h_2 \\
&= \sum \varepsilon_R \left(r^1\right) \varepsilon_H \left(\left(r^2\right)_{(-1)} \right) \left(r^2\right)_{(0)} \otimes \varepsilon_H (h_1)\, h_2 \\
&= r \otimes h.
\end{aligned}
$$

so that $r_{R \otimes H} \circ (R \otimes H \otimes \varepsilon) \circ \Delta = R \otimes H = l_{R \otimes H} \circ (\varepsilon \otimes R \otimes H) \circ \Delta$. Let us check that the algebra structure and the coalgebra structure are compatible. In fact

$$\Delta\left[(r\otimes h)\cdot(s\otimes t)\right]=\Delta\left[\sum r\cdot_R(h_1\rightharpoonup s)\otimes h_2t\right]=$$

$$=\sum\left(r\cdot_R(h_1\rightharpoonup s)\right)^1\otimes\left(\left(r\cdot_R(h_1\rightharpoonup s)\right)^2\right)_{(-1)}h_2t_1\otimes\left(\left(r\cdot_R(h_1\rightharpoonup s)\right)^2\right)_{(0)}\otimes h_3t_2=$$

$$\overset{5)}{=}\sum r^1\cdot_R\left(r^2_{(-1)}\rightharpoonup(h_1\rightharpoonup s)^1\right)\otimes\left(r^2_{(0)}\cdot_R(h_1\rightharpoonup s)^2\right)_{(-1)}h_2t_1\otimes\left(r^2_{(0)}\cdot_R(h_1\rightharpoonup s)^2\right)_{(0)}\otimes h_3t_2$$

$$\overset{\Delta_R\text{isleft}H\text{-lin}}{=}\sum r^1\cdot_R\left(r^2_{(-1)}\rightharpoonup\left(h_1\rightharpoonup s^1\right)\right)\otimes\left(r^2_{(0)}\cdot_R\left(h_2\rightharpoonup s^2\right)\right)_{(-1)}h_3t_1\otimes\left(r^2_{(0)}\cdot_R\left(h_2\rightharpoonup s^2\right)\right)_{(0)}\otimes h_4t_2$$

$$=\sum r^1\cdot_R\left(\left[r^2_{(-1)}h_1\right]\rightharpoonup s^1\right)\otimes\left(r^2_{(0)}\cdot_R\left(h_2\rightharpoonup s^2\right)\right)_{(-1)}h_3t_1\otimes\left(r^2_{(0)}\cdot_R\left(h_2\rightharpoonup s^2\right)\right)_{(0)}\otimes h_4t_2$$

$$\overset{m_R\text{isleft}H\text{-col}}{=}\sum r^1\cdot_R\left(\left[r^2_{(-1)}h_1\right]\rightharpoonup s^1\right)\otimes r^2_{(0)_{(-1)}}\cdot_R\left(h_2\rightharpoonup s^2\right)_{(-1)}h_3t_1\otimes r^2_{(0)_{(0)}}\cdot_R\left(h_2\rightharpoonup s^2\right)_{(0)}\otimes h_4t_2$$

$$=\sum r^1\cdot_R\left(\left[r^2_{(-2)}h_1\right]\rightharpoonup s^1\right)\otimes r^2_{(-1)}\cdot_R\left(h_2\rightharpoonup s^2\right)_{(-1)}h_3t_1\otimes r^2_{(0)}\cdot_R\left(h_2\rightharpoonup s^2\right)_{(0)}\otimes h_4t_2$$

and

$$\begin{aligned}\Delta(r\otimes h)\cdot\Delta(s\otimes t) &= \left(\sum r^1\otimes\left(r^2\right)_{(-1)}h_1\otimes\left(r^2\right)_{(0)}\otimes h_2\right)\cdot\left(\sum s^1\otimes\left(s^2\right)_{(-1)}t_1\otimes\left(s^2\right)_{(0)}\otimes t_2\right)\\ &= \sum\left(r^1\otimes\left(r^2\right)_{(-1)}h_1\right)\cdot\left(s^1\otimes\left(s^2\right)_{(-1)}t_1\right)\otimes\left(\left(r^2\right)_{(0)}\otimes h_2\right)\cdot\left(\left(s^2\right)_{(0)}\otimes t_2\right)\\ &= \sum r^1\cdot_R\left(\left[\left(r^2\right)_{(-1)}h_1\right]_1\rightharpoonup s^1\right)\otimes\left[\left(r^2\right)_{(-1)}h_1\right]_2\left(s^2\right)_{(-1)}t_1\otimes\left(r^2\right)_{(0)}\cdot_R\left(\left(h_2\right)_1 \right.\\ &= \sum r^1\cdot_R\left(\left[\left(r^2\right)_{(-1)_1}h_1\right]\rightharpoonup s^1\right)\otimes\left(r^2\right)_{(-1)_2}h_2\left(s^2\right)_{(-1)}t_1\otimes\left(r^2\right)_{(0)}\cdot_R\left(h_3\rightharpoonup\left(s^2\right.\right.\\ &= \sum r^1\cdot_R\left(\left[\left(r^2\right)_{(-2)}h_1\right]\rightharpoonup s^1\right)\otimes\left(r^2\right)_{(-1)}h_2\left(s^2\right)_{(-1)}t_1\otimes\left(r^2\right)_{(0)}\cdot_R\left(h_3\rightharpoonup\left(s^2\right)\right.\\ &\overset{3bis}{=} \sum r^1\cdot_R\left(\left[\left(r^2\right)_{(-2)}h_1\right]\rightharpoonup s^1\right)\otimes\left(r^2\right)_{(-1)}\left(h_{2_1}\rightharpoonup\left(s^2\right)\right)_{(-1)}h_{2_2}t_1\otimes\left(r^2\right)_{(0)}\cdot_R\end{aligned}$$

Therefore $\Delta\left[(r\otimes h)\cdot(s\otimes t)\right]=\Delta(r\otimes h)\cdot\Delta(s\otimes t)$. Moreover

$\Delta_R(r\cdot s)=\sum r^1\cdot_R\left(r^2_{(-1)}\rightharpoonup s^1\right)\otimes r^2_{(0)}\cdot_R s^2$

${}^H\rho_R(h\rightharpoonup r)=\sum h_1r_{(-1)}S(h_3)\otimes\left(h_2\rightharpoonup r_{(0)}\right)$

$(h_1\rightharpoonup r)_{(-1)}h_2\otimes(h_1\rightharpoonup r)_{(0)}=h_1r_{(-1)}\otimes h_2\rightharpoonup r_{(0)}$

$$\begin{aligned}\Delta(1_R\otimes 1_H) &= \sum 1_R^1\otimes\left(1_R^2\right)_{(-1)}(1_H)_1\otimes\left(1_R^2\right)_{(0)}\otimes(1_H)_2\\ &\overset{4)}{=}.\sum 1_R\otimes(1_R)_{(-1)}1_H\otimes(1_R)_{(0)}\otimes 1_H\\ &\overset{u_R\text{isleft}H\text{-lin}}{=} 1_R\otimes 1_H\otimes 1_R\otimes 1_H\end{aligned}$$

and

$$\begin{aligned}\varepsilon\left[(r\otimes h)\cdot(s\otimes t)\right] &= \varepsilon\left[\sum r\cdot_R(h_1\rightharpoonup s)\otimes h_2t\right]=\sum\varepsilon_R\left(r\cdot_R(h_1\rightharpoonup s)\right)\varepsilon_H(h_2t)=\\ &= \varepsilon_R(r)\,\varepsilon_R(h\rightharpoonup s)\,\varepsilon_H(t)\overset{\varepsilon_R\text{isleft}H\text{-lin}}{=}\varepsilon_R(r)\,\varepsilon_H(h)\,\varepsilon_R(s)\,\varepsilon_H(t)=\varepsilon(r\otimes h)\,\varepsilon(s\otimes t)\end{aligned}$$

$$\varepsilon(1_R\otimes 1_H)=\varepsilon_R(1_R)\,\varepsilon_H(1_H).$$

$$\pi\left(\left[\left(r\otimes h\right)\cdot\left(s\otimes t\right)\right]\right) = \pi\left(\left[\sum r\cdot_{R}\left(h_{1}\rightharpoonup s\right)\otimes h_{2}t\right]\right)=\varepsilon_{R}\left(r\cdot_{R}\left(h_{1}\rightharpoonup s\right)\right)h_{2}t \overset{\varepsilon_{R}\text{isan}}{=}$$

$$\overset{\varepsilon_{R}\text{isleft}H\text{-lin}}{=} \varepsilon_{R}\left(r\right)\varepsilon_{H}\left(h_{1}\right)\varepsilon_{R}\left(s\right)h_{2}t = \varepsilon_{R}\left(r\right)h\varepsilon_{R}\left(s\right)t=\pi\left(r\otimes h\right)\pi\left(s\otimes t\right)$$

$$\pi\left(1_{R}\otimes1_{H}\right)=\varepsilon_{R}\left(1_{R}\right)1_{H}\overset{\varepsilon_{R}\text{isanalgmap}}{=}1_{k}1_{H}=1_{H}$$

Definire Π e poi la $S_{R\otimes H}$ Prendere dal file del 4.6

$$\left(\pi\otimes\pi\right)\Delta\left(r\otimes h\right) = \sum\pi\left[r^{1}\otimes\left(r^{2}\right)_{\left(-1\right)}h_{1}\right]\otimes\pi\left[\left(r^{2}\right)_{\left(0\right)}\otimes\right.$$

$$\overset{\varepsilon_{R}\text{isleft}H\text{-col}}{=}\sum\varepsilon_{R}\left(r^{1}\right)\varepsilon_{R}\left(r^{2}\right)h_{1}\otimes h_{2}\sum\varepsilon_{R}\left(r\right)h_{1}\otimes h_{2} = \varepsilon_{R}\left(r\right)\sum h_{1}\otimes h_{2}=\Delta_{H}\pi\left(r\otimes h\right)$$

$$\varepsilon_{H}\pi\left(r\otimes h\right)=\varepsilon_{H}\left(\varepsilon_{R}\left(r\right)h\right)=\varepsilon_{R}\left(r\right)\varepsilon_{H}\left(h\right)=\varepsilon\left(r\otimes h\right).$$

□

$${}^{H}\rho$$

Chapter 8

Some results on modules and rings

8.1. *We will use the following notations.*
Let V *be a vector space over a field* k *and let* $\{e_x\}_{x\in X}$ *be a basis of* V*.*
For every $x \in X$*, we will denote by* e_x^* *the element of* $V^* = \mathrm{Hom}\,(V,k)$ *defined by setting*

$$e_x^*(e_x) = 1 \quad \text{and} \quad e_x^*(e_y) = 0 \text{ for every } y \in X, y \neq x.$$

Let A *be a ring. We set:*
$\mathcal{L}(A) =$ *the lattice of subgroups of the abelian group* $(A, +, 0_A)$
$\mathcal{L}({}_A A) = \{I \in \mathcal{L}(A) \mid I \text{ is a left ideal of } A\}$
$\mathcal{L}(A_A) = \{I \in \mathcal{L}(A) \mid I \text{ is a right ideal of } A\}$
$\mathcal{L}({}_A A_A) = \{I \in \mathcal{L}(A) \mid I \text{ is a two-sided ideal } A\}$
$\Omega = \Omega(A) = \{\mathcal{M} \mid \mathcal{M} \text{ is a maximal two-sided ideal of } A\}$
$\Omega_l = \Omega_l(A) = \{L \mid L \text{ is a maximal left ideal of } A\}$
$\Omega_r = \Omega_r(A) = \{M \mid M \text{ is a maximal right ideal of } A\}$
${}_A\mathcal{S} = \{S \in {}_A\mathcal{M} \mid S \text{ is a simple left } A\text{-module}\}$
$\mathcal{S}_A = \{S \in \mathcal{M}_A \mid S \text{ is a simple right } A\text{-module}\}$
When A *is a* k*-algebra, we also set:*
$\Omega_f = \Omega_f(A) = \{m \in \Omega \mid dim_k(A/m) < \infty\}$
Let $M \in {}_A\mathcal{M}$*. We set* $\mathcal{L}({}_A M) = \{L \mid L \text{ is a submodule of } {}_A M\}$*. Let* $x \in M$*. Consider the right* A*-module morphism*

$$\begin{array}{cccc} \mu_x : & {}_A A & \longrightarrow & {}_A M \\ & a & \longmapsto & a\,x \end{array}.$$

We set $Ann_A(x) = \mathrm{Ker}(\mu_x)$*. Since* $\mathrm{Im}(\mu_x) = A\,x$*, in view of the First Isomorphism Theorem for Modules, we get that*

$$\begin{array}{cccc} \widehat{\mu_x} : & A/Ann_A(x) & \longrightarrow & Ax \\ & a + Ann_A\,(x) & \longmapsto & a\,x \end{array}.$$

is an isomorphism. Therefore we deduce that

$$L \in \Omega_l \Leftrightarrow A/L \in {}_A\mathcal{S}$$

and similarly

$$L \in \Omega_r \Leftrightarrow A/L \in \mathcal{S}_A$$

Recall that a ring A is called simple *whenever*

$$\mathcal{L}({}_A A_A) = \{\{0\}, A\} \ .$$

Therefore we have:

$$m \in \Omega \Leftrightarrow A/m \text{ is a simple ring.}$$

We also set

$$Ann_A(M) = \{a \in A \,|\, a\,M = 0\} = \bigcap_{x \in M} Ann_A(x) \ .$$

Note that $Ann_A(M) \in \mathcal{L}({}_A A_A)$.

$End(M_A)$ *will denote the ring of endomorphism of* M_A.

Module homomorphisms will be written to the side opposite to the one of scalars.

Lemma 8.2. ***(Schur's Lemma)*** *Let A be a ring and let S_A be a simple right A-module. Then $F = End(S_A)$ is a division ring.*

Proof. Let $f \in F$, $f \neq 0$. Then $\mathrm{Ker}(f) \subsetneqq S_A$ and hence $\mathrm{Ker}(f) = \{0\}$. Since $\{0\} \subsetneqq Im(f) \subseteq S_A$ we also get that $\mathrm{Im}(f) = S$. □

Lemma 8.3. *Let A be a ring and let ${}_A M$ be a left A-module. Set $B = End\,({}_A M)$. Then the map*

$$\begin{array}{rccl} \varphi_M : & A & \longrightarrow & End(M_B) \\ & a & \longmapsto & \begin{array}{ccc} M_B & \longrightarrow & M_B \\ x & \longmapsto & a\,x \end{array} \end{array}$$

is well defined and is a ring homomorphism.

Proof. Let $\varphi = \varphi_M$. Then, for every $a \in A$, for every $x \in M$ we have that

$$\varphi\,(a)\,(x\beta) = a\,(x\beta) \overset{\beta \in B = End({}_A M)}{=} (a\,x)\beta = [\varphi\,(a)\,(x)]\,\beta \quad \text{for every } \beta \in B$$

which means that $\varphi\,(a) \in End(M_B)$ and hence φ is well defined. Clearly φ is additive. Let us check it is multiplicative. Let $a, b \in A$, then we have

$$\varphi\,(ab)\,(x) = (a\,b)x = a(b\,x) = [\varphi\,(a) \cdot \varphi\,(b)]\,(x) \quad \text{for every } x \in M$$

which means that $\varphi(a\,b) = \varphi(a) \cdot \varphi(b)$. Clearly we also have $\varphi(1_A) = \mathrm{Id}_M$. Thus φ is a ring morphism. □

Lemma 8.4. *Let A be a ring, let $S \in {}_A\mathcal{S}$ be a simple left A-module, let $D = End({}_A S)$ and let $E = End\,(S_D)$. Let $n \in \mathbb{N}, n \geq 1$, let $x_1, ..., x_n \in S$ and let $\eta \in E$. Then there exists an $a \in A$ such that $\eta(x_i) = a \cdot x_i$ for every $i = 1, ..., n$.*

Proof. Let $x = (x_1, ..., x_n) \in S^n$ and assume that $z = (\eta(x_1), ..., \eta(x_n)) \in S^n \setminus Ax$. Since ${}_AS^n$ is a semisimple left A-module, there exists a submodule $H \leq {}_AS^n$ such that

$$S^n = H \oplus Ax.$$

Let

$$\Lambda : {}_AS^n = H \oplus A\,x \to_A S^n$$

such that $(y)\Lambda = 0$ for every $y \in A\,x$ and $(y)\Lambda = y$ for every $y \in H$.
Since $z \notin Ax$, we have that $z = h + \alpha x$ where $a \in A$, $h \in H$ and $h \neq 0$. We have $(z)\Lambda = (h)\Lambda + (\alpha x)\Lambda = h \neq 0$.
Let $\eta \in EndS_D$ and let us consider the map

$$\eta^n : S^n \to S^n.$$

Clearly $\eta^n \in End(S^n_D)$.
For every $i = 1, ..., n$ let

$$e_i : S \to S^n$$

denote the i-th embedding of S into S^n and let

$$p_i : S^n \to S$$

denote the i-th projection from S^n to S
Then, for every $x \in S^n$, we can write

$$x = \sum_{i=1}^{n} (x)p_i e_i$$

so that we get

$$0 \neq (z)\Lambda = \sum_{i=1}^{n} (z\Lambda)p_i e_i = \sum_{i=1}^{n} (\sum_{j=1}^{n} zp_j e_j)\Lambda p_i e_i \ .$$

For every $i, j = 1, ..., n$ set $e_j \Lambda p_i = \Lambda_{ij}$. Note that $\Lambda_{ij} \in End({}_AS) = D$ and hence we have

$$(z)\Lambda = \sum_{i=1}^{n} \sum_{j=1}^{n} zp_j \Lambda_{ij} e_i = \sum_{i=1}^{n} \sum_{j=1}^{n} \eta(x_j)\Lambda_{ij} e_i \ .$$

Since $\eta \in E = End(S_D)$ and $\Lambda_{ij} \in D$, for every $i, j = 1, ..., n$, we obtain that

$$\eta(x_j)\Lambda_{ij} = \eta(x_j \Lambda_{ij})$$

and hence

$$(z)\Lambda = \sum_{i=1}^{n} \sum_{j=1}^{n} \eta(x_j)\Lambda_{ij} e_i = \sum_{i=1}^{n} \sum_{j=1}^{n} \eta(x_j \Lambda_{ij}) e_i = \sum_{i=1}^{n} \eta(\sum_{j=1}^{n} x_j \Lambda_{ij}) e_i = \sum_{i=1}^{n} \eta(\sum_{j=1}^{n} x_j e_j \Lambda p_i) e_i \ .$$

Now for every $i = 1, ..., n$ we have

$$\sum_{j=1}^{n} x_j e_j \Lambda p_i = [(\sum_{j=1}^{n} x_j e_j)\Lambda] p_i = (x)\Lambda p_i = (0)\, p_i = 0$$

we get that

$$0 \neq z\Lambda = \sum_{i=1}^{n} \eta(\sum_{j=1}^{n} x_j e_j \Lambda p_i) e_i = 0\ .$$

Contradiction. □

Notations 8.5. *Let R be a ring and let $n \in \mathbb{N}$, $n > 0$. Given $t, s \in \mathbb{N}$ such that $1 \leq s, t \leq n$, we will denote by $e_{s,t}$ the element of $M_n(R)$ defined by setting*

$$(e_{s,t})_{u,v} = \delta_{s,u}\delta_{t,v} \text{ for every } s, v \in \mathbb{N}, 1 \leq t, s \leq n.$$

Clearly we have

$$e_{s,t} e_{u,v} = \delta_{t,u} e_{s,v} \text{ for every } s, t, u, v \in \mathbb{N}, 1 \leq s, t, u, v \leq n. \tag{8.1}$$

For every i, $1 \leq i \leq n$, we set

$$J_i = \sum_{\substack{s,t \\ t \neq i}} R e_{s,t}.$$

Lemma 8.6. *Let $A = M_n(R)$. For every i, $1 \leq i \leq n$ we have that*

$$J_i = Ann_A\,(e_{i,i})$$

and hence J_i is a left ideal of A. Moreover we have

$$\bigcap_{i=1}^{n} J_i = \{0\}\,.$$

Furthermore $J_i \in \Omega_s(A)$ whenever $R = D$ is a division ring.

Proof. From formula (8.1) we get that $J_i \subseteq Ann_A\,(e_{i,i})$. Conversely let $a = \sum_{s,t} r_{s,t} e_{s,t} \in Ann_A\,(e_{i,i})$. Since

$$0 = \sum_{s,t} r_{s,t} e_{s,t} e_{i,i} = \sum_{s,t} r_{s,t} \delta_{t,i} e_{s,i} = \sum_{s,t} r_{s,i} e_{s,i}$$

we deduce that $r_{s,i} = 0$ for every $s, 1 \leq s \leq n$. Therefore

$$a = \sum_{\substack{s,t \\ t \neq i}} r_{s,t} e_{s,t} \in J_i.$$

We have

$$\bigcap_{i=1}^{n} J_i = \bigcap_{i=1}^{n} Ann_A\left(e_{i,i}\right) \subseteq Ann_A\left(\sum_{i=1}^{n} e_{i,i}\right) = Ann_A\left(1_A\right) = \{0\}.$$

Assume now that $R = D$ is a division ring and let $a \in A \setminus J_i$. Then we have

$$\begin{aligned} a &= \sum_{s,t} r_{s,t}e_{s,t} = \sum_{\substack{s,t \\ t=i}} r_{s,t}e_{s,t} + \sum_{\substack{s,t \\ t\neq i}} r_{s,t}e_{s,t} \\ &= \sum_{s=1}^{n} \lambda_s e_{s,i} + b \text{ where } b = \sum_{\substack{s,t \\ t\neq i}} r_{s,t}e_{s,t} \in J_i \text{ and } \lambda_s = r_{s,i} \in D. \end{aligned}$$

Moreover, since $a \notin J_i$ there exists an s_0, $1 \leq s_0 \leq n$ such that $\lambda_{s_0} \neq 0$. This implies that

$$\begin{aligned} \left(\lambda_{s_0}\right)^{-1} e_{s_0,s_0} \cdot a &= \left(\lambda_{s_0}\right)^{-1} \sum_{s=1}^{n} \lambda_s e_{s_0,s_0} e_{s,i} + \left(\lambda_{s_0}\right)^{-1} e_{s_0,s_0} b = \left(\lambda_{s_0}\right)^{-1} \lambda_{s_0} e_{s_0,i} + \left(\lambda_{s_0}\right)^{-1} e_{s_0,s_0} b \\ &= e_{s_0,i} + \left(\lambda_{s_0}\right)^{-1} e_{s_0,s_0} b \end{aligned}$$

and hence

$$e_{s_0,i} = \left(\lambda_{s_0}\right)^{-1} e_{s_0,s_0} \cdot a - \left(\lambda_{s_0}\right)^{-1} e_{s_0,s_0} b \in Aa + J_i.$$

Since $Aa + J_i$ is a left ideal of A we get that

$$e_{t,i} = e_{t,s_0} e_{s_0,i} \in Aa + J_i \text{ for every } t = 1, ..., n.$$

On the other hand, if $t \neq i$, we know that $e_{s,t} \in J_i$ and hence we deduce that $e_{s,t} \in J_i$ for every $s, t = 1, ..., n$ so that

$$Aa + J_i = A.$$

This means that each J_i is a left maximal ideal of A. □

Lemma 8.7. *Let A be a ring and let M be a left A-module. Then the following conditions are equivalent:*

(a) *Every descending chain in $\mathcal{L}\left({}_A M\right)$ is stationary.*

(b) *Every non empty subset of $\mathcal{L}\left({}_A M\right)$ has a minimum.*

Proof. $(a) \Rightarrow (b)$. Let X be a non empty subset of $\mathcal{L}\left({}_A M\right)$. Since X is non-empty, there exists $L_0 \in X$. If X has no minimal element, then for each submodule L in X there is at least one submodule L' in X such that $L' \subsetneqq L$. By applying the Axiom of choice, for each $L \in X$ we can choose one such L'. Then, by recursion we construct a descending chain in X by setting: $L_1 = (L_0)'$ and $L_{n+1} = (L_n)'$. Contradiction.

$(b) \Rightarrow (a)$.Let $\left(L_n\right)_{n\in\mathbb{N}}$ be a descending chain of submodules of ${}_A M$. Then the set $\{L_n \mid n \in \mathbb{N}\}$ has a minimum element, say L_{n_0}. For every $n \geq n_0$ we have

$$L_{n_0} \subseteq L_n \subseteq L_{n_0}.$$

□

Definition 8.8. *Let A be a ring and let M be a left A-module. M is called* left artinian *if M satisfies one of the equivalent conditions of Lemma 8.7.*

Definition 8.9. *Let A be a ring. A is called* left artinian *if the left A-module ${}_AA$ is left artinian.*

Theorem 8.10. *Let A be a ring, let $S \in {}_A\mathcal{S}$ be a simple left A-module and let $D = End({}_AS)$. Let $P = Ann_A(S)$ and assume that A/P is left artinian. Then:*

1) *D is a division ring and $dim(S_D) < \infty$.*

2) *The canonical morphism $\varphi = \varphi_S : A \to End(S_D)$ is surjective.*

3) *$A/P \simeq End(S_D) \simeq M_n(D)$ where $n = dim_D(S_D)$.*

4) *$P = L_1 \cap ... \cap L_n$ where $n = dim_D(S_D)$ and $L_1, ..., L_n$ are left maximal ideals of A.*

Proof. 1) By Schur's Lemma 8.2, $D = End({}_AS)$ is a division ring.
Assume that $x_1, x_2, ..., x_n, ... \in S$ is a sequence of linearly independent elements of S_D. Let $E = End\,(S_D)$ and, for every $i \in \mathbb{N}, i \geq 1$, let $H_i = Ann_E(V_i), L_i = Ann_A\,(V_i)$ where $V_i = \{x_1, ..., x_i\}$. Then the H_i's form a strictly decreasing sequence of left ideals of E :

$$H_1 \supsetneqq H_2 \supsetneqq ... \supsetneqq H_n \supsetneqq ....$$

By Lemma 8.4, we have that also the L_i's form a strictly decreasing sequence of left ideals of A. Since $L_i \supseteq Ann_A(S) = P$ for every i, we can consider the left ideals L_i/P of A/P which form a strictly decreasing sequence of left ideals of A/P. Since A/P is left artinian, we get a contradiction. Hence $\dim_D S < \infty$.

2) Let $x_1, ..., x_n$ be a system of generators of S_D and $\eta \in E$. Then, by Lemma 8.4, there exists an $a \in A$ such that $\eta(x_i) = a\,x_i$ for every $i = 1, ..., n$. Let $x \in S$. Then there exists $\lambda_i \in D, i = 1, ..., n$, such that $x = \sum_{i=1}^n x_i\lambda_i$ so that

$$\eta(x) = \sum_{i=1}^{n} \eta(x_i\lambda_i) = \sum_{i=1}^{n} \eta(x_i)\lambda_i = \sum_{i=1}^{n} (a\,x_i)\lambda_i = a\,x\ .$$

We deduce that $\varphi(a) = \eta$ and hence that φ is surjective.

3) Since φ is surjective and $P = \mathrm{Ker}(\varphi)$ we get that $A/P \simeq E$.
Since $\dim_D\,(S_D) = n$ we get that $E \simeq M_n(D)$.

4) By Lemma 8.6 we know that in $M_n(D)$ we have that $\{0\} = J_1 \cap ... \cap J_n$ where $J_1, ..., J_n$ are left maximal ideals of $M_n(D)$. Since $A/P \simeq M_n(D)$ the ideals $J_1, ..., J_n$ lift to left maximal ideals $L_1, ..., L_n$ of A such that $L_1 \cap ... \cap L_n = P$. □

Corollary 8.11. *Let A be a simple left artinian ring. Then there exist an $n \in \mathbb{N}, n \geq 1$ and a division ring D such that $A \simeq M_n(D)$. Moreover there exist left maximal ideals $L_1, ..., L_n$ of A such that $\{0\} = L_1 \cap ... \cap L_n$.*

Proof. Since ${}_AA$ is left artinian, it contains a non zero left ideal ${}_AI$ such that

$$ {}_AI = \min\left\{L \mid L \leq {}_AA \text{ and } L \neq \{0\}\right\}. $$

Then ${}_AI$ is a simple left A-module and $Ann_A\left({}_AI\right) \subsetneqq A$. Since $Ann_A\left({}_AI\right)$ is a two-sided ideal of A we deduce that $Ann_A\left({}_AI\right) = \{0\}$. By Theorem 8.10, we get our conclusion. □

Corollary 8.12. *Let A be a k-algebra and let $m \in \Omega_f$. Then there exist an $n \in \mathbb{N}$ and a division ring D such that $A/m \simeq M_n(D)$. Moreover there exist $L_1, ..., L_n \in \Omega_s$ such that $m = L_1 \cap ... \cap L_n$.*

Proof. Since $m \in \Omega_f$, A/m is a simple ring and $\dim_k (A/m) < \infty$. Hence A/m is a simple left artinian ring. Apply now Corollary 8.11. □

Definition 8.13. *Let A be a ring. The* Jacobson radical *of A, which will be denoted by $J(A)$ or also by $Jac(A)$, is the intersection of all left maximal ideals of A, i.e.*

$$ Jac(A) = \bigcap_{L \in \Omega_s(A)} L \ . $$

Theorem 8.14. *Let A be a finite dimensional k-algebra. Then*

- *every maximal two-sided ideal of A is an intersection of a finite number of maximal left ideals of A.*
- *every maximal left ideal contains a maximal two-sided ideal of A.*

Therefore

$$ Jac(A) = \bigcap_{m \in \Omega(A)} m. $$

Proof. Let ${}_AS$ be a simple left A-module. Since A is a finitely dimensional k-algebra, we have that also $\dim_k \left(A/Ann_A\left(S\right)\right)$ is finite so that $A/Ann_A\left(S\right)$ is, in particular, a left artinian ring. Thus we can apply Theorem 8.10 to get that $D = End\left({}_AS\right)$ is a division ring, $n = \dim_D\left(S\right) < \infty$, $A/Ann_A\left(S\right) \simeq M_n(D)$ and $Ann_A\left(S\right) = L_1 \cap ... \cap L_n$ where $L_i \in \Omega_s(A)$ for every $i = 1, ..., n$.
Let now $m \in \Omega(A)$ and let T be a simple left A/m-module. Then T is a simple left A-module and $A \supsetneqq Ann_A\left(T\right) \supseteq m$ so that, since m is a maximal two-sided ideal, we have that $m = Ann_A(T)$. Then, by the foregoing, we deduce that there exists an $n \in \mathbb{N}, n \geq 1$ and $L_1, ..., L_n \in \Omega_s(A)$ such that $m = L_1 \cap ... \cap L_n$. Conversely, let $L \in \Omega_s(A)$. Then $S = A/L$ is a simple left A-module and we have that $L = Ann_A(x) \supseteq Ann_A(S)$ where $x = 1_A + L$. By the foregoing we know that $A/Ann_A(S) \simeq M_n(D)$ where D is a division ring. Thus $A/Ann_A(S)$ is a simple ring i.e. $Ann_A(S)$ is a maximal two-sided ideal of A. □

Theorem 8.15. *Let A be a simple ring and let I be a left ideal of A, $I \neq \{0\}$. Set $D = End({}_A I)$. Then the canonical morphism*

$$\begin{array}{rccc} \varphi = \varphi_I : & A & \longrightarrow & End(I_D) \\ & a & \longmapsto & \begin{array}{ccc} I_D & \longrightarrow & I_D \\ x & \longmapsto & a\,x \end{array} \end{array}$$

is an isomorphism.

Proof. Let us recall that, in view of Lemma 8.3, φ is a well defined ring homomorphism. Thus, since $\varphi(1_A) = 1_{End(I_D)}$, we get that $Ker(\varphi)$ is a proper two-sided ideal of A and hence, A being simple, we obtain that $Ker(\varphi) = \{0\}$.

Let $E = End(I_D)$. Let us show that

$$h \cdot \varphi(r) = h \circ \varphi(r) = \varphi(h(r)) \quad \text{for every } h \in E \text{ and } r \in I.$$

Let $x \in {}_A I$. Then the map

$$\begin{array}{rccc} \gamma_x : & I & \longrightarrow & I \\ & r & \longmapsto & r\,x\ . \end{array}$$

is well defined since I is a left ideal of A. Let $a \in A$ and $z \in I$. We compute

$$(a\,z)\gamma_x = (a\,z)x = a(z\,x) = a[(z)\gamma_x]$$

which means that $\gamma_x \in End({}_A I) = D$. Now let $h \in End(I_D)$ and $r \in I$. For every $x \in I$, we calculate

$$(h \cdot \varphi(r))(x) = (h \circ \varphi(r))(x) = h(\varphi(r)(x)) = h(rx) \overset{r \in I}{=} h((r)\gamma_x) \overset{r \in I \text{and} \gamma_x \in D \text{andstructureof} I_D}{=} h\,(r \cdot \gamma_x)$$
$$\overset{h \in E \text{and} \gamma_x \in D}{=} h\,(r) \cdot \gamma_x \overset{h(r) \in I \text{and} \gamma_x \in D \text{andstructureof} I_D}{=} (h(r))\gamma_x \overset{\text{def} \gamma_x}{=} h(r)x = \varphi(h(r))(x).$$

Therefore we get

$$(h \cdot \varphi(r))(x) = \varphi(h(r))(x)$$

for every $x \in I$, i.e.

$$h \cdot \varphi(r) = \varphi(h(r))$$

for every $h \in E$ and $r \in I$ which means that

$$E \cdot \varphi(I) \subseteq \varphi(I). \tag{8.2}$$

Since $I\,A \neq \{0\}$ and $I\,A$ is a two-sided ideal of A, which is a simple ring, we deduce that $I\,A = A$ and hence

$$\varphi(A) = \varphi(I\,A) = \varphi(I) \cdot \varphi(A). \tag{8.3}$$

Then we have

$$E \cdot \varphi(A) \overset{(8.3)}{=} E \cdot [\varphi(I) \cdot \varphi(A)] = [E \cdot \varphi(I)] \cdot \varphi(A) \overset{(8.2)}{\subseteq} \varphi(I) \cdot \varphi(A) \overset{(8.3)}{=} \varphi(A)\ .$$

Then $\varphi(A)$ is a left ideal of E. Since $1_E = \mathrm{Id}_I = \varphi(1_A) \in \varphi(A)$, we deduce that $\varphi(A) = E$ and thus φ is an isomorphism. $\square$

Chapter 9

The coradical

9.1. *Let C be a k-coalgebra and let $M \in \mathcal{M}^C$. Recall from Theorem 2.21 that, M has a natural structure of left C^*-module defined by setting*

$$f \cdot m = \sum m_0 f(m_1) \text{ for every } f \in C^* \text{and } m \in M.$$

Analogously every $M \in {}^C\mathcal{M}$ has a natural structure of right C^*-module defined by setting

$$m \cdot f = \sum f(m_{-1}) m_0 \text{ for every } f \in C^* \text{and } m \in M.$$

In particular C, being a right C-comodule, has a natural structure of left C^*-module which we will write as

$$f \rightharpoonup c = \sum c_1 f(c_2) \text{ for every } f \in C^* \text{and } c \in C.$$

Analogously C, being a left C-comodule, has a natural structure of right C^*-module which we will write as

$$c \leftharpoonup f = \sum f(c_1) c_2 \text{ for every } f \in C^* \text{and } c \in C.$$

It is easy to check that, with respect to this structures, C becomes a two-sided C^*-module.

Proposition 9.2. *Let M be a right C-comodule and let L be a subvector space of M. Then L is a right subcomodule of M if and only if L is a left C^*-submodule of M.*

Proof. Let $i_L : L \to M$ be the canonical inclusion. Assume that L is a right subcomodule of M. Then, by Proposition 2.22, $H(i_L) = i_L : L \to M$ is a morphism of left C^*-modules i.e. L is a left C^*-submodule of M. Conversely, assume that L is a left C^*-submodule of M. Then, by Theorem 2.32, $L \in Rat({}_{C^*}\mathcal{M})$ so that $i_L : L \to M$ is a morphism in $Rat({}_{C^*}\mathcal{M})$. By Theorem 2.30, $\Gamma^{-1}(i_L) = i_L : L \to M$ is a morphism in $\mathcal{M}^C$ i.e. L is a subcomodule of M □

Lemma 9.3. *Let C be a k-coalgebra and let D be a vector subspace of C. Then the following are equivalent*

(a) *D is a subcoalgebra of C.*

(b) *D is a right subcomodule (a right coideal) of C_C and a left subcomodule (left coideal) of ${}_CC$.*

(c) *D is a two-sided submodule of ${}_{C^*}C_{C^*}$.*

Proof. $(a) \Leftrightarrow (b)$. We have that

$$(D \otimes D) = (D \otimes C) \cap (C \otimes D).$$

$(b) \Leftrightarrow (c)$. It follows from 9.2. □

Corollary 9.4. *Let C be a k-coalgebra. Then C^*cC^* is a subcoalgebra of C, for every $c \in C$. C^*cC^* is the smallest subcoalgebra of C containing c. Moreover C^*cC^* is finitely dimensional*

Proof. Apply Proposition 9.2 and Lemma 9.3. By Theorem 2.33, C^*c is finitely dimensional. □

Definition 9.5. *Let C be a k-coalgebra and let $c \in C$. The subcoalgebra C^*cC^* is called* subcoalgebra of C generated by c.

Proposition 9.6. *Let C be a k-coalgebra. Then the set of subcoalgebras of C is closed under intersections and summations.*

Proof. Apply Lemma 9.3 and Theorem 2.32. □

Theorem 9.7. *Let C be a k-coalgebra.*

1) *For every right C-comodule M and every finite subset $\{m_1, ..., m_n\} \subset M$, there exists a finite dimensional right subcomodule N of M such that $\{m_1, ..., m_n\} \subseteq N$.*

2) *Let F be a subset of C, the subcoalgebra $\sum_{c \in F} C^*cC^*$ is the smallest subcoalgebra of C containing F. Clearly $\sum_{c \in F} C^*cC^*$ is finite dimensional whenever F is finite.*

Proof. The first assertion follows from Theorem 2.33.

Let now F be a subset of C. Then, by Corollary 9.4 and by Proposition 9.6, $\sum_{c \in F} C^*cC^*$ is the minimal subcoalgebra of C containing F. Since $\dim_k C^*cC^*$ is finite, the last assertion is trivial. □

Definition 9.8. *Let F be a subset of a k-coalgebra C. The subcoalgebra $\sum_{c \in F} C^*cC^*$ will be called* subcoalgebra of C generated by F.

Definition 9.9. *Let C be a k-coalgebra. We say that C is a* simple coalgebra *if $C \neq \{0\}$ and C does not contain any proper nonzero subcoalgebras.*

Definition 9.10. *Let C be a k-coalgebra and let M be a right C-comodule. We say that M is a* simple right C-comodule *if $M \neq \{0\}$ and M does not contain any nonzero proper subcomodule.*

Proposition 9.11. *1. Every simple coalgebra has finite dimension.*

2. Let C be a coalgebra. Every simple right C-comodule has finite dimension.

Proof. 1) Let D be a simple coalgebra and let $d \in D \backslash \{0\}$. By Thorem 9.7 there exists a finite dimensional subcoalgebra E of D which contains d. Since $\{0\} \neq E \subseteq D$ and D is a simple coalgebra we deduce that $E = D$.

2) Let M be a simple right C-comodule and let $m \in M$, $m \neq 0$. Then, by Theorem 9.7, there exists a finite subcomodule N of M which contains m. Since $\{0\} \neq N \subseteq M$ and M is a simple right C-comodule we deduce that $M = N$. □

Corollary 9.12. *Let C be a k-coalgebra. Then*

1) every simple subcoalgebra of C has finite dimension.

2) every simple right C-comodule has finite dimension.

Notations 9.13. *Let C be a k-coalgebra. For every subset X of C we set*

$$X^{\perp} = \{f \in C^* \mid f(x) = 0 \text{ for every } x \in X\}.$$

For every subset W of C^* we set

$$W^{\perp} = \{x \in C \mid f(x) = 0 \text{ for every } f \in W\}.$$

Lemma 9.14. *Then we have that*

1) $V^{\perp\perp} = V$ for every k-vector subspace V of C.

2) $Z^{\perp\perp} = Z$ for every subspace Z of C^* whenever $\dim_k C < \infty$.

Proof. 1) Let V be a k-vector subspace of C. It is clear that $V \subseteq V^{\perp\perp}$. Assume that $x \in V^{\perp\perp} \setminus V$. Then there exists a $c^* \in C^*$ such that $c^*(V) = 0$ and $c^*(x) \neq 0$. From $c^*(V) = 0$ we deduce that $c^* \in V^{\perp}$ and hence, since $x \in V^{\perp\perp}$ we get that $c^*(x) = 0$. Contradiction.

2) Assume now that $\dim_k C < \infty$ and let Z be a subspace of C^*. It is clear that $Z \subseteq Z^{\perp\perp}$. Assume that $h \in Z^{\perp\perp} \setminus Z$. Then there exists an $\alpha \in (C^*)^*$ such that $\alpha(Z) = 0$ and $\alpha(h) \neq 0$.Since C is finite dimensional there exists a $c \in C$ such that $\alpha(f) = f(c)$ for every $f \in C^*$. Therefore we get that $f(c) = 0$ for every $f \in Z$ and hence that $c \in Z^{\perp}$. This implies that $h(c) = 0$. On the other hand we have $0 \neq \alpha(h) = h(c)$. Contradiction. □

Proposition 9.15. *Let C be a k-coalgebra. Then*

1) L is a right (resp. left) coideal of $C \Leftrightarrow L^{\perp}$ is a right (resp. left) ideal of C^*.

2) If I is a right (resp. left) ideal of C^*, then $I^{\perp}$ is a right (resp. left) coideal of C. The converse is true whenever C has finite dimension.

3) D is a subcoalgebra of $C \Leftrightarrow D^{\perp}$ is a two-sided ideal of C^*.

Proof. 1)" $\Rightarrow$ " Let L be a right coideal of C and let $f \in L^{\perp}$ and $c^* \in C^*$. For any $x \in L$ we compute

$$(fc^*)(x) = (f * c^*)(x) = \sum f(x_1)c^*(x_2) = \sum f(x_1)\, c^*(x_2)\,.$$

Since L is a right coideal of C we have that

$$\Delta(x) = \sum x_1 \otimes x_2 \in L \otimes C$$

so that, since $f \in L^{\perp}$, we get that

$$(fc^*)(x) = \sum f(x_1)\, c^*(x_2) = 0$$

which means that $fc^* \in L^{\perp}$.

2) Let I be a right ideal of C^*. In view of Proposition 9.2 we have to prove that $I^{\perp}$ is a left C^*-submodule of C i.e. that

$$C^* \rightharpoonup I^{\perp} \subseteq I^{\perp}$$

Let $f \in C^*$, $c \in I^{\perp}$ and $g \in I$. Then $g * f \in I$ and hence

$$\begin{aligned} g(f \rightharpoonup c) &= g\left(\sum c_1 f(c_2)\right) = \sum g(c_1)\, f(c_2) \\ &= (g * f)(c) = 0. \end{aligned}$$

Therefore we deduce that $f \rightharpoonup c \in I^{\perp}$.

Assume now that $\dim_k C < \infty$ and let I be a subspace of C^* such that $I^{\perp}$ is a right coideal of C. Then, by 1)" $\Rightarrow$ " $I^{\perp\perp}$ is a right ideal of C^* and by Lemma 9.14 we have that $I = I^{\perp\perp}$.

1)" $\Leftarrow$ " Let L be a subspace of C such that $L^{\perp}$ is a right ideal of C^*. Then, by 2) $L^{\perp\perp}$ is a right coideal of C and by Lemma 9.14 we have that $L = L^{\perp\perp}$.

3) follows from 1) in view of Lemma 9.3. □

Corollary 9.16. *Let C be a finite dimensiona k-coalgebra. Then the assignment*

$$L \longmapsto L^{\perp}$$

defines a bijection between the right coideals of C and the right ideals of C^* which induces a bijection between the subcoalgebras of C and the two-sided ideals of C^*.

Proposition 9.17. *Let C be a k-coalgebra. Then C is a simple coalgebra if and only if C^* is a finite dimensional simple k-algebra.*

Proof. By Corollary 9.12 every simple subcoalgebra of C has finite dimension. On the other hand, $\dim_k C^* < \infty$ implies that $\dim_k C < \infty$. Apply then Corollary 9.16. □

Corollary 9.18. *Let D be a subcoalgebra of a k-coalgebra C. Then the following statements are equivalent.*

(a) D is a simple subcoalgebra of C.

(b) D^* is a finite dimensional simple algebra.

(c) $D^\perp$ is a two-sided maximal ideal of C^* of finite codimension.

Proof. $(a) \Leftrightarrow (b)$ follows from Proposition 9.17.
Let V be a vector subspace of C. From the exact sequence

$$0 \to V \longrightarrow C \longrightarrow C/V \to 0$$

we get the exact sequence

$$0 \to V^\perp \longrightarrow C^* \longrightarrow V^* \to 0. \tag{9.1}$$

$(a) = (b) \Rightarrow (c)$ Assume that D is a simple coalgebra. Then by Proposition 9.15 $D^\perp$ is a two-sided ideal of C^* and from (9.1) we deduce that $D^\perp$ is a maximal two-sided ideal of finite codimension.
$(c) \Rightarrow (b)$ Assume that $D^\perp$ is a two-sided maximal ideal C^* of finite codimension. From (9.1)

$$0 \to D^\perp \longrightarrow C^* \longrightarrow D^* \to 0$$

we deduce that D^* is a finite dimensional simple algebra. □

Definition 9.19. *Let C be a k-coalgebra. The* coradical C_0 *of C is the sum of all simple subcoalgebras of C.*

Definition 9.20. *Let C be a nonzero k-coalgebra. C is called* pointed *if all simple subcoalgebras of C are 1-dimensional.*

Definition 9.21. *Let C be a nonzero k-coalgebra. C is called* connected *if $\dim_k C_0 = 1$.*

Corollary 9.22. *Let C be a nonzero k-coalgebra. Then C contains a simple subcoalgebra and hence $C_0 \neq \{0\}$.*

Proof. Let $0 \neq c \in C$. Then by Corollary 9.4 $D = C^* c C^*$ is a finite dimensional subcoalgebra of C. Let I be a maximal two-sided ideal of D^*. Since D is finite dimensional, by 2) of Lemma 9.14 $I = \left(I^\perp\right)^\perp$. Then, by Corollary 9.18, we deduce that $I^\perp$ is a simple subcoalgebra of D and in particular of C. □

Proposition 9.23. *Let C be a k-coalgebra. Then the coradical C_0 of C is a subcoalgebra of C.*

Proof. By Proposition 9.6, the sum of subcoalgebras is a subcoalgebra. □

Proposition 9.24. *Let C be a k-coalgebra. The* 1*-dimensional subcoalgebras of C are exactly those of the form kg for $g \in G(C)$.*

Proof. Let D be a 1-dimensional subcoalgebra of C and let $e \in D, e \neq 0$. Then there exists $\lambda \in k$ such that $\Delta(e) = \lambda e \otimes e$. Hence we get that

$$e = \lambda \varepsilon(e) e$$

from which we deduce that $\lambda \varepsilon(e) = 1$.
Set $g = \lambda e$. Then we get that

$$\Delta(g) = \Delta(\lambda e) = \lambda \Delta(e) = \lambda(\lambda e \otimes e) = \lambda e \otimes \lambda e \quad \text{and} \quad \varepsilon(g) = \lambda \varepsilon(e) = 1.$$

Therefore $g \in G(C)$ and $kg = ke = D$.
The converse is trivial. □

Lemma 9.25. *Let D be a simple subcoalgebra of a k-coalgebra C and let C', C'' be nonzero subcoalgebras of C such that $D \subseteq C' + C''$. Then we have that either $D \subseteq C'$ or $D \subseteq C''$.*

Proof. Assume that $D \nsubseteq C'$. Then, since D is simple we get that $D \cap C' = \{0\}$ and hence that $D + C' = D \oplus C'$. Then there exists a $\gamma \in C^*$ such that

$$\gamma_{|D} = \varepsilon_D \quad \text{and} \quad \gamma_{|C'} = 0.$$

The, for every $d \in D$, we get that

$$\gamma \rightharpoonup d = \sum d_1 \gamma(d_2) \overset{\Delta(D) \subseteq D \otimes D}{=} \sum d_1 \varepsilon(d_2) = d.$$

On the other hand, from $D \subseteq C' + C''$ we deduce that

$$\Delta(D) \subseteq \Delta(C') + \Delta(C'') \subseteq C' \otimes C' + C'' \otimes C''$$

and since $\gamma_{|C'} = 0$ we obtain that $\gamma \rightharpoonup d \in C''$. □

Proposition 9.26. *Let $(C_i)_{i \in I}$ be a family of subcoalgebras of a k-coalgebra C and let D be a simple subcoalgebra of C. Then $D \subseteq \sum_{i \in I} C_i$ if and only if there exists an $i_0 \in I$ such $D \subseteq C_{i_0}$.*

Proof. Since D is simple, by 9.12 D has finite dimension so that if $D \subseteq \sum_{i \in I} C_i$ there exist $n \in \mathbf{N}, n \geq 1$ and $i_1, ..., i_n \in I$ such that $D \subseteq \sum_{j=1}^{n} C_{i_j}$. Since, by Proposition 9.6, the sum of subcoalgebras is a subcoalgebra, in view of Lemma 9.25, we conclude. □

Lemma 9.27. *Let $(D_i)_{i \in I}$ be a family of pairwise distinct simple subcoalgebras of a k-coalgebra C. Then.we have that*

$$\sum_{i \in I} D_i = \bigoplus_{i \in I} D_i.$$

Proof. Let us assume that there exists an $i \in I$ such that $D_i \cap \sum_{j \neq i} D_j \neq \{0\}$. Since, by Proposition 9.6, $D_i \cap \sum_{j \neq i} D_j$ is a subcoalgebra of the simple algebra D_i we get that

$$D_i \cap \sum_{j \neq i} D_j = D_i$$

so that $D_i \subseteq \sum_{j \neq i} D_j$. Then, by Lemma 9.26 there exists an $i_0 \in I \setminus \{i\}$ such tha $D_i \subseteq D_{i_0}$. Since D_{i_0} is a simple coalgebra we get $D_i = D_{i_0}$. Contradiction. □

Proposition 9.28. *Let $\mathcal{D}$ be the set of all simple subcoalgebras of a k-coalgebra C. Then*

$$C_0 = \bigoplus_{D \in \mathcal{D}} D$$

Proof. Apply Lemma 9.27. □

Proposition 9.29. *Let F and D be subcoalgebras of a k-coalgebra C. Then*

$$(F + D)_0 = F_0 + D_0.$$

Proof. Clearly we have that

$$F_0 + D_0 \subseteq (F + D)_0.$$

The converse inclusion follows by Proposition 9.26. □

Proposition 9.30. *Let C be a k-coalgebra. Then C is pointed $\Leftrightarrow C_0 = kG(C)$.*

Proof. Let $\mathcal{A}$ be the set of simple subcoalgebras of C.

" $\Rightarrow$ " Assume that C is pointed. Then, by Proposition 9.24 we get that $\mathcal{A} = \{kg \mid g \in G(C)\}$ and hence that

$$C_0 = \sum_{A \in \mathcal{A}} A = \sum_{g \in G(C)} kg = kG(C).$$

" $\Leftarrow$ " Conversely, assume that $C_0 = kG(C)$ and let D be a simple subcoalgebra of C. Then from $D \subseteq C_0 = kG(C)$, by Proposition 9.26 we deduce that there exists a $g \in G(C)$ such that $D \subseteq kg$ and hence $D = kg$. □

Definition 9.31. *Let C be a nonzero k-coalgebra. We say that C is an* irreducible coalgebra *if any two nonzero subcoalgebras of C have nonzero intersection.*

Lemma 9.32. *Let C be a k-coalgebra. Then C is irriducible $\Leftrightarrow$ C contains a unique simple subcoalgebra.*

Proof. " $\Rightarrow$ " By Corollary 9.22, C contains a simple subcoalgebra. Since the intersection of two distinct simple subcoalgebras is zero, C must contain an unique simple subcoalgebra.

" $\Leftarrow$ " Let D be the unique simple subcoalgebra of C. Then, by Corollary 9.4, D is contained in every nonzero subcoalgebra of C. □

Proposition 9.33. *Let C be a k-coalgebra. Then the following are equivalent*

(a) *C is pointed and irreducible*

(b) *C is pointed and $|G(C)| = 1$.*

(c) *C is connected.*

Proof. $(a) \Leftrightarrow (b)$ By Lemma 9.32 C is irreducible if and only if C contains an unique simple subcoalgebra. Since C is pointed, this means that C has a unique 1-dimensional subcoalgebra. By Proposition 9.24, this happens if and only if $|G(C)| = 1$.

$(c) \Leftrightarrow (b)$ Follows by 9.30. □

Definition 9.34. *Let R be a ring and let M be a left R-module. The socle $Soc({}_RM)$ of M is the sum of all simple left submodules of M.*

Proposition 9.35. *Let C be a simple k-coalgebra. Then $Soc(C_{C^*}) = C = Soc({}_{C^*}C)$.*

Proof. Since C is a simple coalgebra, by Corollary 9.12 C is finite dimensional and by Proposition 9.17 C^* is a finite dimensional simple k-algebra. Thus, by Corollary 8.12, there exists $n \in \mathbf{N}$ and $I_1, ..., I_n$ left maximal ideals of C^* such that

$$\{0\} = \bigcap_{j=1}^{n} I_j.$$

Since C is finite dimensional, we have that

$$C^\perp = \{0\} = \bigcap_{j=1}^{n} I_j = \bigcap_{j=1}^{n} I_j^{\perp\perp} = \left(\sum_{j=1}^{n} I_j^\perp\right)^\perp$$

and hence, by Lemma 9.14, we get that

$$C = C^{\perp\perp} = \left(\sum_{j=1}^{n} I_j^\perp\right)^{\perp\perp} = \sum_{j=1}^{n} I_j^\perp.$$

Since C is finite dimensional, by Proposition 9.15, every $I_j^\perp$ is a minimal left coideal of C and hence, by Proposition 9.2 it is a simple submodule of C_{C^*}. Therefore we get

$$C = \sum_{j=1}^{n} I_j^\perp \subseteq Soc(C_{C^*}) \subseteq C.$$

□

Lemma 9.36. *Let D be a subcoalgebra of a k-coalgebra C and let W be a vector subspace of D. Then W is a left D^*-submodule of D if and only if W is a left C^*-submodule of D.*

Proof. Let $i_D : D \to C$ be the canonical inclusion. Let $d \in D$ and let $g \in D^*$. Then there exists an element $f \in C^*$ such that $g = f \circ i_D$ so that we get

$$g \rightharpoonup d = \sum d_1 g(d_2) = \sum d_1 f(d_2) = f \rightharpoonup d \in C^* \rightharpoonup d.$$

Conversely, let $f \in C^*$. Then

$$f \rightharpoonup d = \sum d_1 f(d_2) = \sum d_1 (f \circ i_D)(d_2) \in D^* \rightharpoonup d.$$

□

Lemma 9.37. *Let C be a finite dimensional k-coalgebra. Then every simple left C^*-submodule of C is contained in a simple subcoalgebra of C.*

Proof. Let S be a simple left C^*-submodule of C. Then, by Proposition 9.2, S is a minimal left coideal of C.
By Proposition 9.15 $S^\perp$ is a left maximal ideal of C^*. Since C^* is finite dimensional, by Theorem 8.14 it contains a maximal two-sided ideal I of C^*. Then, by Lemma 9.14, $I = I^{\perp\perp}$ and hence, in view of Corollary 9.18 $I^\perp$ is a simple subcoalgebra of C. By Lemma 9.14 we have that

$$S = S^{\perp\perp} \subseteq I^\perp.$$

□

Proposition 9.38. *Let C be a k-coalgebra. Then*

$$C_0 = Soc({}_{C^*}C).$$

Proof. Let D be a simple subcoalgebra of C. Then, by Proposition 9.35 $D = Soc({}_{D^*}D)$. By Lemma 9.36, every simple left D^*-submodule of D is a simple C^*-submodule and hence $D \subseteq Soc({}_{C^*}C)$.
Conversely, let S be a simple left C^*-submodule of C. By Corollary 9.12, S has finite dimension. Let $x \in S$, $x \neq 0$. We have that

$$S = C^* x \subseteq C^* x\, C^*.$$

and $D = C^* x\, C^*$ has finite dimension. By Lemma 9.36 S is a simple left D^*-submodule of D. Since D is finite dimensional we can apply Lemma 9.37 and get that S is contained in a simple subcoalgebra E of D so that

$$S \subseteq E \subseteq C_0.$$

□

Lemma 9.39. *Let R be a ring and let L be a submodule of a left R-module M. Then*

$$Soc({}_R L) = Soc({}_R M) \cap L.$$

Proof. The simple submodules of ${}_RL$ are the simple submodules of ${}_RM$ which are contained in L. The inclusion $Soc\,({}_RL) \subseteq Soc\,({}_RM) \cap L$ is trivial. Conversely $Soc\,({}_RM) \cap L$ is a submodule of the semisimple left R-module $Soc\,({}_RM)$ and hence it is semisimple. Thus $Soc\,({}_RM) \cap L$ is a sum of simple modules which are contained in L so that $Soc\,({}_RM) \cap L \subseteq Soc\,({}_RL)$. □

Lemma 9.40. *Let D be a subcoalgebra of a k-coalgebra C. Then*

$$D_0 = C_0 \cap D.$$

Proof. By Proposition 9.38, we have that $D_0 = Soc\,({}_{D^*}D)$ and by Lemma 9.36 we have that $Soc\,({}_{D^*}D) = Soc\,({}_{C^*}D)$ so that, by Lemma 9.39 we obtain that

$$D_0 = Soc\,({}_{C^*}D) = Soc({}_{C^*}C) \cap D \overset{\text{Prop9.38}}{=} C_0 \cap D.$$

□

Proposition 9.41. *Let C be a finite dimensional k-coalgebra. Then*

$$C_0^{\perp} = Jac(C^*).$$

Proof. By Proposition 9.15 the maximal right ideals of C^* are exactly those of the form $L^{\perp}$ where L is a minimal right coideal of C i.e., by Proposition 9.2, a simple subcomodule of ${}_{C^*}C$. Let $\mathcal{S}$ denotes the set of simple submodules of ${}_{C^*}C$. Then we have

$$Jac(C^*) = \bigcap_{L \in \mathcal{S}} L^{\perp} = \left(\sum_{L \in \mathcal{S}} L \right)^{\perp} = C_0^{\perp}\ .$$

□

Lemma 9.42. *Let R be a ring and let $f \in R$. Then $f \in Jac(R) \Leftrightarrow$ for every $h \in R$, $1_R - hf$ has a left inverse in R.*

Proof. " $\Rightarrow$ " Since $1 = hf + (1 - hf)$, and $hf \in Jac(R)$ we have that $1 - hf$ is not contained in any left maximal ideal of R. By Krull's Lemma this means that

$$R(1 - hf) = R$$

i.e. $1 - hf$ has a left inverse.
" $\Leftarrow$ " Assume that $f \notin Jac(R)$. Then there exists a left maximal ideal L of R such that $f \notin L$ and hence

$$Rf + L = R.$$

Thus there exist an $h \in R$ and an $l \in L$ such that

$$hf + l = 1_R.$$

Then $1 - hf$ does not have any left inverse. □

Lemma 9.43. *Let R be a ring and let L be a left ideal of R such that every element of L is nilpotent R.*
Then $L \subseteq Jac(R)$.

Proof. Let $a \in L$ and let $x \in R$. Then $xa \in L$ and hence there exists an $n \in \mathbb{N}, n \geq 1$ such that $(xa)^n = 0$. Thus we obtain

$$(1 + xa + (xa)^2 + ... + (xa)^{n-1})(1 - xa) = 1 - (xa)^n = 1$$

and hence $1 - xa$ has a left inverse in R. Thus, by Lemma 9.42 we get that $a \in Jac(R)$. □

Lemma 9.44. *Let C and D be k-coalgebras. Then*

$$(C \otimes D)_0 \subseteq C_0 \otimes D_0.$$

Moreover if C and D are also pointed (resp. connected), then

$$(C \otimes D)_0 = C_0 \otimes D_0$$

and $C \otimes D$ is pointed (resp. connected).

Proof. Let $X \neq \{0\}$ be a simple subcoalgebra of $C \otimes D$. We have to prove that $X \subseteq C_0 \otimes D_0$. First of all let us show that we can assume that both C and D are finite dimensional. By Corollary 9.12 X is finitely dimensional. Let $\{v_1, ..., v_n\}$ be a basis of X. Since $X \leq C \otimes D$, for every $i = 1, ..., n$, there exists a finite subset F_i of C and a finite subset G_i of D such that

$$v_i = \sum_{c \in F_i, d \in G_i} c \otimes d$$

Let $C^{'}$ be the subcoalgebra of C generated by F_i and let $D^{'}$ be the subcoalgebra of D generated by G. Then both C' and D' are finite dimensional. We will show that $X \subseteq C'_0 \otimes D'_0$. Thus we may assume that both C and D are finite dimensional. Then we have the isomorphism

$$(C \otimes D)^* \cong C^* \otimes D^*$$

and by Proposition 9.41 we have that

$$C_0^{\perp} = Jac(C^*) \quad \text{and} \quad D_0^{\perp} = Jac(D^*).$$

Since C^* and D^* are finitely dimensional, by Nakayama's Lemma, there exist $m, n \in \mathbb{N}$, $m, n \geq 1$ such that

$$(C_0^{\perp})^n = \{0\} \text{ and } (D_0^{\perp})^m = \{0\}.$$

Clearly we may assume $n = m$.
Set

$$I = C_0^{\perp} \otimes D^* + C^* \otimes D_0^{\perp}$$

then I is a two-sided ideal of $C^* \otimes D^*$. Note that

$$\left(C_0^{\perp} \otimes D^*\right)\left(C^* \otimes D_0^{\perp}\right) = C_0^{\perp} \otimes D_0^{\perp} = \left(C^* \otimes D_0^{\perp}\right)\left(C_0^{\perp} \otimes D^*\right)$$

so that

$$I^{2n} = \sum_{i+j=2n} \binom{2n}{i} \left(C_0^{\perp} \otimes D^*\right)^i \left(C^* \otimes D_0^{\perp}\right)^j = \{0\}.$$

Therefore, by Lemma 9.43 and Theorem 8.14

$$I \subseteq Jac(C^* \otimes D^*) \subseteq P$$

for every two-sided maximal ideal P di $C^* \otimes D^*$. Therefore we deduce that

$$P^{\perp} \subseteq I^{\perp}$$

where $P^{\perp}$ is any simple subcoalgebra of $C \otimes D$. By Lemma 15.3, $I = (C_0 \otimes D_0)^{\perp}$ and hence, by Proposition 9.15 $I^{\perp} = C_0 \otimes D_0$ and it contains all simple subcoalgebras of $C \otimes D$. In particular we get that $X \subseteq I^{\perp} = (C_0 \otimes D_0)$.
Assume now that both C and D are pointed. Since

$$G(C) \otimes G(D) \subseteq G(C \otimes D)$$

we get that, in this case,

$$C_0 \otimes D_0 \subseteq (C \otimes D)_0$$

and hence

$$(C \otimes D)_0 \subseteq C_0 \otimes D_0 \subseteq (C \otimes D)_0.$$

Thus $(C \otimes D)_0 = C_0 \otimes D_0 = k\left(G\left(C\right)\right) \otimes kG\left(D\right) = kG(C \otimes D)$ so that $C \otimes D$ is pointed. $\square$

Chapter 10

The Coradical Filtration

Definition 10.1. *Let X and Y be subspaces of a k-coalgebra (C, Δ, ε). The* wedge product *of X and Y (in C)is defined by*

$$X \wedge_C Y = X \wedge Y = \operatorname{Ker}(C \xrightarrow{\Delta} C \otimes C \xrightarrow{\pi_X^C \otimes \pi_Y^C} C/X \otimes C/Y)$$

where π_X^C and π_Y^C are the canonical projections.

Lemma 10.2. *Let $f : C \to U$ and $g : C \to W$ be k-linear maps. Then*

$$\Delta^{\leftarrow}\left[C \otimes \operatorname{Ker}(g) + \operatorname{Ker}(f) \otimes C\right] = \operatorname{Ker}\left[(f \otimes g) \circ \Delta\right]. \tag{10.1}$$

Lemma 10.3. *Let X, Y, Z be subspaces of a k-coalgebra (C, Δ, ε).*

1)

$$X \wedge Y = \Delta^{\leftarrow}(C \otimes Y + X \otimes C). \tag{10.2}$$

2)

$$X \wedge Y = (X^{\perp} * Y^{\perp})^{\perp} \text{ where the product } X^{\perp} * Y^{\perp} \text{ is in } C^*. \tag{10.3}$$

3)

$$(X \wedge Y) \wedge Z = \operatorname{Ker}\left[(\pi_X \otimes \pi_Y \otimes \pi_Z) \circ \Delta_2\right] = X \wedge (Y \wedge Z). \tag{10.4}$$

4)
(10.5)
$D \wedge E$ is a subcoalgebra of C whenever D and E are subcoalgebras of C.

Proof. 1) We have

$$\begin{aligned} X \wedge Y &= \operatorname{Ker}((\pi_X \otimes \pi_Y) \circ \Delta) \overset{(10.1)}{=} \Delta^{\leftarrow}(C \otimes \operatorname{Ker}(\pi_Y) + \operatorname{Ker}(\pi_X) \otimes C) \\ &= \Delta^{\leftarrow}(C \otimes Y + X \otimes C). \end{aligned}$$

2) Let $z \in X \wedge Y = Ker((\pi_X \otimes \pi_Y) \circ \Delta)$. We compute

$$\begin{aligned}
(X^{\perp} * Y^{\perp})^{\perp} &= \{c \in C \mid (f * g)(c) = 0, \text{ for every } f \in X^{\perp}, g \in Y^{\perp}\} \\
&= \{c \in C \mid \sum f(c_1) g(c_2) = 0, \text{ for every } f \in X^{\perp}, g \in Y^{\perp}\} \\
&= \{c \in C \mid m_k(f \otimes g)\Delta(c) = 0, \text{ for every } f \in X^{\perp}, g \in Y^{\perp}\} \\
&\overset{m_k \text{isiso}}{=} \{c \in C \mid (f \otimes g)\Delta(c) = 0, \text{ for every } f \in X^{\perp}, g \in Y^{\perp}\} \\
&= \{c \in C \mid \Delta(c) \in \mathrm{Ker}(f \otimes g), \text{ for every } f \in X^{\perp}, g \in Y^{\perp}\} \\
&= \bigcap_{f \in X^{\perp}, g \in Y^{\perp}} \Delta^{\leftarrow} [\mathrm{Ker}(f \otimes g)]
\end{aligned}$$

so that

$$(X^{\perp} * Y^{\perp})^{\perp} = \bigcap_{f \in X^{\perp}, g \in Y^{\perp}} \Delta^{\leftarrow} [\mathrm{Ker}(f \otimes g)] .$$

Now $f \in X^{\perp}$ means that $f(X) = \{0\}$ i.e. $X \subseteq \mathrm{Ker}(f)$ and similarly $g \in Y^{\perp}$ means that $Y \subseteq \mathrm{Ker}(g)$. Thus

$$\begin{aligned}
X \wedge Y &= \Delta^{\leftarrow}(C \otimes Y + X \otimes C) \subseteq \bigcap_{f \in X^{\perp}, g \in Y^{\perp}} \Delta^{\leftarrow} [C \otimes \mathrm{Ker}(g) + \mathrm{Ker}(f) \otimes C] \\
&= \bigcap_{f \in X^{\perp}, g \in Y^{\perp}} \Delta^{\leftarrow} [\mathrm{Ker}(f \otimes g)] = (X^{\perp} * Y^{\perp})^{\perp}.
\end{aligned}$$

Let us prove the other inclusion. Let $(x_i)_{i \in I}$ be a basis of X and let $(x_j)_{j \in J}$, where $J \supseteq I$, be a basis of C. Analogously let $(y_l)_{l \in L}$ be a basis of Y and let $(y_t)_{t \in T}$, where $T \supseteq L$ a basis of C.

Let x_j^* and y_t^* the dual morphisms of x_j and y_t respectively. Then, for every $j \in J \backslash I$ we have that $x_j^* \in X^{\perp}$ and for every $t \in T \backslash L$ we have that $y_t^* \in Y^{\perp}$. Let $c \in (X^{\perp} * Y^{\perp})^{\perp}$. Then we can write

$$\Delta(c) = \sum_{j \in J, t \in T} \lambda_{jt} x_j \otimes y_t \text{ for some } \lambda_{j,t} \in k.$$

For every $(j_0, t_0) \in (J \backslash I) \times (T \backslash L)$ we have $x_{j_0}^* * y_{t_0}^* \in X^{\perp} * Y^{\perp}$ so that

$$0 = \left(x_{j_0}^* * y_{t_0}^*\right)(c) = \lambda_{j_0 t_0}$$

so that

$$\Delta(c) = \sum_{\substack{(j,t) \in J \times T \\ , j \in I \text{ or } t \in L}} \lambda_{jt} x_j \otimes y_t \in X \otimes C + C \otimes Y.$$

3) We compute

$$\begin{aligned}
(X \wedge Y) \wedge Z &= \Delta^{\leftarrow}\left[C \otimes Z + (X \wedge Y) \otimes C\right] \\
&= \Delta^{\leftarrow}\left[C \otimes \operatorname{Ker}(\pi_Z) + \operatorname{Ker}\left[(\pi_X \otimes \pi_Y) \circ \Delta\right] \otimes C\right] \\
&\overset{(10.1)}{=} \operatorname{Ker}\left[(\left[(\pi_X \otimes \pi_Y) \circ \Delta\right] \otimes \pi_Z) \circ \Delta\right] \\
&= \operatorname{Ker}\left[\left[(\pi_X \otimes \pi_Y \otimes \pi_Z)\right] \circ (\Delta \otimes C) \circ \Delta\right] \\
&= \operatorname{Ker}\left[\left[(\pi_X \otimes \pi_Y \otimes \pi_Z)\right] \circ (C \otimes \Delta) \circ \Delta\right] \\
&= \operatorname{Ker}\left[(\pi_X \otimes \left[(\pi_Y \otimes \pi_Z) \circ \Delta\right]) \circ \Delta\right] \\
&\overset{(10.1)}{=} \Delta^{\leftarrow}\left[C \otimes \operatorname{Ker}(\pi_X) + \operatorname{Ker}\left[(\pi_Y \otimes \pi_Z) \circ \Delta\right] \otimes C\right] \\
&= \Delta^{\leftarrow}\left[C \otimes X + (Y \wedge Z) \otimes C\right] \\
&= X \wedge (Y \wedge Z).
\end{aligned}$$

4) Let D and E be subcoalgebras of C. Then, by Proposition 9.15, $D^{\perp}$ and $E^{\perp}$ are two-sided ideals of C^* so that $D^{\perp} * E^{\perp}$ is a two-sided ideal of C^* and hence, by 2) and Proposition 9.15, $D \wedge E = (D^{\perp} * E^{\perp})^{\perp}$ is a subcoalgebra of C. □

Lemma 10.4. *Let D and E be subcoalgebras of a coalgebra C. Then*

$$D \subseteq D \wedge E \text{ and } E \subseteq D \wedge E.$$

Proof. Since D is a subcoalgebra of C we have

$$\Delta(D) \subseteq D \otimes D \subseteq D \otimes C \subseteq D \otimes C + C \otimes E$$

so that, by 1) of Lemma 10.3, we get

$$D \subseteq \Delta^{\leftarrow}(C \otimes E + D \otimes C) = D \wedge E$$

□

Lemma 10.5. *Let C be a k-coalgebra, D a subcoalgebra of C and E and F subcoalgebras of D. Then*

$$E \wedge_D F = (E \wedge_C F) \cap D.$$

Proof. We have that

$$E \wedge_D F = \operatorname{Ker}(D \xrightarrow{\Delta_D} D \otimes D \xrightarrow{\pi_E^D \otimes \pi_F^D} D/E \otimes D/F).$$

Let $i : D \to C$, $i_{D/E} : D/E \to C/E$ and $i_{D/F} : D/F \to C/F$ be the canonical inclusions. Then

$$\begin{aligned}
\left(i_{D/E} \otimes i_{D/F}\right) \circ \left(\pi_E^D \otimes \pi_F^D\right) \circ \Delta_D &= \left(\pi_E^C \circ i \otimes \pi_F^C \circ i\right) \\
&= \left(\pi_E^C \otimes \pi_F^C\right) \circ (i \otimes i) \circ \Delta_D \\
&= \left(\pi_E^C \otimes \pi_F^C\right) \circ \Delta_C \circ i
\end{aligned}$$

so that

$$\begin{aligned} E \wedge_D F &= \operatorname{Ker}\left[\left(\pi_E^D \otimes \pi_F^D\right) \circ \Delta_D\right] = \operatorname{Ker}\left[\left(i_{D/E} \otimes i_{D/F}\right) \circ \left(\pi_E^D \otimes \pi_F^D\right) \circ \Delta_D\right] \\ &= \operatorname{Ker}\left[\left(\pi_E^C \otimes \pi_F^C\right) \circ \Delta_C \circ i\right] = i^{\leftarrow}\left(E \wedge_C F\right) = \left(E \wedge_C F\right) \cap D. \end{aligned}$$

□

Lemma 10.6. *Let C be a k-coalgebra, D a subcoalgebra of C and E a subcoalgebra of D. Then*

$$E \wedge_C E \subseteq D \wedge_C D.$$

Proof. Let $\pi_D^E : C/E \to C/D$ be the canonical projection. Then

$$\pi_D^C = \pi_D^E \circ \pi_E^C$$

so that

$$\begin{aligned} D \wedge_C D &= \operatorname{Ker}\left[\left(\pi_D^C \otimes \pi_D^C\right) \circ \Delta_C\right] = \operatorname{Ker}\left\{\left[\left(\pi_D^E \circ \pi_E^C\right) \otimes \left(\pi_D^E \circ \pi_E^C\right)\right] \circ \Delta_C\right\} \\ &= \operatorname{Ker}\left\{\left[\left(\pi_D^E \otimes \pi_D^E\right) \circ \left(\pi_E^C \otimes \pi_E^C\right)\right] \circ \Delta_C\right\} \end{aligned}$$

$$\begin{aligned} E \wedge_C E &= \operatorname{Ker}\left[\left(\pi_E^C \otimes \pi_E^C\right) \circ \Delta_C\right] \subseteq \operatorname{Ker}\left\{\left[\left(\pi_D^E \otimes \pi_D^E\right) \circ \left(\pi_E^C \otimes \pi_E^C\right)\right] \circ \Delta_C\right\} \\ &= \operatorname{Ker}\left\{\left[\left(\pi_D^E \circ \pi_E^C\right) \otimes \left(\pi_D^E \circ \pi_E^C\right)\right] \circ \Delta_C\right\} \\ &= D \wedge_C D. \end{aligned}$$

□

We recall that the sequence $(\Delta_n)_{n \geq 1}$ was defined by recursion by setting

$$\Delta_1 = \Delta \quad \text{and} \quad \Delta_n = \left(\Delta \otimes I^{n-1}\right) \circ \Delta_{n-1} \quad \text{for every } n \in \mathbb{N}, n \geq 2$$

and that, by Theorem 1.17, for every $n, i, m \in \mathbb{N}, n \geq 2, 1 \leq i \leq n-1$ and $0 \leq m \leq n-i$,

$$\Delta_n = \left(I^m \otimes \Delta_i \otimes I^{n-i-m}\right) \circ \Delta_{n-i}.$$

Definition 10.7. *Let C be a k-coalgebra and let X be a vector subspace of C. We define $\bigwedge_C^n X = \bigwedge^n X$ as follows*

$$\bigwedge^n X = \operatorname{Ker}\left[\left(\pi_X^C\right)^{\otimes n} \circ \Delta_{n-1}\right] \text{ for every } n \in \mathbb{N} \text{ where } \Delta_{-1} = \Delta_0 = \left(\pi_X^C\right)^{\otimes 0} = \mathrm{Id}_C$$

so that $\bigwedge^0 X = \{0\}, \quad \bigwedge^1 X = X.$

Lemma 10.8. *Let C be a k-coalgebra and let X be a vector subspace of C. Then*

$$\bigwedge^a X \wedge \bigwedge^b X = \bigwedge^{a+b} X = \bigwedge^b X \wedge \bigwedge^a X \text{ for every } a, b \in \mathbb{N}, a, b \geq 1. \tag{10.6}$$

Proof. For every $a, b \in \mathbb{N}, a, b \geq 1$, we compute

$$\begin{aligned}
\bigwedge^{a} X \wedge \bigwedge^{b} X &= \Delta^{\leftarrow}\left[C \otimes \left(\bigwedge^{b} X\right) + \left(\bigwedge^{a} X\right) \otimes C\right] = \\
&= \Delta^{\leftarrow}\left[C \otimes \operatorname{Ker}\left[\left(\pi_X^C\right)^{\otimes b} \circ \Delta_{b-1}\right] + \operatorname{Ker}\left[\left(\pi_X^C\right)^{\otimes a} \circ \Delta_{a-1}\right] \otimes C\right] \\
&\overset{(10.1)}{=} \operatorname{Ker}\left(\left\{\left[\left(\pi_X^C\right)^{\otimes a} \circ \Delta_{a-1}\right] \otimes \left[\left(\pi_X^C\right)^{\otimes b} \circ \Delta_{b-1}\right]\right\} \circ \Delta\right) = \\
&= \operatorname{Ker}\left(\left\{\left[\left(\pi_X^C\right)^{\otimes a} \otimes \left(\pi_X^C\right)^{\otimes b}\right] \circ [\Delta_{a-1} \otimes \Delta_{b-1}]\right\} \circ \Delta\right) \\
&= \operatorname{Ker}\left[\left(\pi_X^C\right)^{\otimes a+b} \circ (\Delta_{a-1} \otimes \Delta_{b-1}) \circ \Delta\right] \\
&= \operatorname{Ker}\left[\left(\pi_X^C\right)^{\otimes a+b} \circ \left(C^{\otimes a} \otimes \Delta_{b-1}\right) \circ (\Delta_{a-1} \otimes C) \circ \Delta\right] \\
&\overset{Lemma (1.16)}{=} \operatorname{Ker}\left[\left(\pi_X^C\right)^{\otimes a+b} \circ \left(C^{\otimes a} \otimes \Delta_{b-1}\right) \circ \Delta_a\right] = \operatorname{Ker}\left[\left(\pi_X^C\right)^{\otimes a+b} \circ \Delta_{a+b}\right] \\
&= \bigwedge^{a+b} X.
\end{aligned}$$

□

Definition 10.9. *Let (C, Δ, ε) be a k-coalgebra. We define a sequence $(C_n)_{n \in \mathbb{N}}$ of subspaces of C as follows : for $n = -1$ we set $C_{-1} = \{0\}$, for $n = 0$ we let C_0 be the coradical of C and for each $n \in \mathbb{N}, n \geq 1$ we set*

$$C_n = \bigwedge^{n+1} C_0.$$

Theorem 10.10. *For every $n \in \mathbb{N}$, we have that*

1) $C_{a+b+1} = C_a \wedge C_b$ for every $a, b \in \mathbb{N}$.

2) C_n is a subcoalgebra of C, for every $n \in \mathbb{N}$.

3) $C_n \subseteq C_{n+1}$, for every $n \in \mathbb{N}$.

4) $\Delta(C_n) \subseteq \sum_{i=0}^{n} C_i \otimes C_{n-i}$, for every $n \in \mathbb{N}$.

5) $C = \bigcup_{n \geqslant 0} C_n$.

Proof. 1) We have

$$C_a \wedge C_b = \bigwedge^{a+1} C_0 \wedge \bigwedge^{b+1} C_0 \overset{(10.6)}{=} \bigwedge^{a+1+b+1} C_0 = C_{a+b+1}$$

2) We proceed by induction on $n \in \mathbb{N}$.For $n = 0$ we know that, by Proposition 9.6, C_0 is a subcoalgebra of C. Let us assume that there exists an $n \in \mathbf{N}, n \geq 1$ such

that C_{n-1} is a subcoalgebra of C. Then $C_n = C_0 \wedge C_{n-1}$, in view of 4) in Lemma 10.3, is a subcoalgebra of C.

3) By Lemma 10.4, for any subcoalgebra D and E we have

$$D \subseteq D \wedge E \text{ and } E \subseteq D \wedge E.$$

Then for every $n \in \mathbb{N}$

$$C_n \subseteq C_n \wedge C_0 \overset{1)}{=} C_{n+1}.$$

4) In view of 1) we get

$$C_n = (\bigwedge^{i} C_0) \wedge (\bigwedge^{n+1-i} C_0)$$

for every $1 \leq i \leq n$ so that, for every $1 \leq i \leq n$ we obtain

$$\begin{aligned}
\Delta(C_n) &= \Delta((\bigwedge^{i} C_0) \wedge (\bigwedge^{n+1-i} C_0)) \\
&= \Delta\left[\Delta^{\leftarrow}\left(C \otimes \bigwedge^{n+1-i} C_0 + \bigwedge^{i} C_0 \otimes C\right)\right] \\
&\subseteq C \otimes \bigwedge^{n+1-i} C_0 + \bigwedge^{i} C_0 \otimes C \\
&= C \otimes C_{n-i} + C_{i-1} \otimes C. \qquad (10.7)
\end{aligned}$$

Moreover, since C_n is a subcoalgebra of C, for $i = 0$ we have

$$\Delta(C_n) \subseteq C \otimes C_n + \{0\} \otimes C = C \otimes C_n$$

and for $i = n + 1$

$$\Delta(C_n) \subseteq C \otimes \{0\} + C_n \otimes C = C_n \otimes C.$$

Now, for every vector space V and for every ascending chain of subspaces

$$\{0\} = V_0 \subseteq V_1 \subseteq ... \subseteq V_n \subseteq ...$$

by Lemma 15.4 we have that

$$\bigcap_{i=0}^{n+1} (V \otimes V_{n+1-i} + V_i \otimes V) = \sum_{i=1}^{n+1} V_i \otimes V_{n+2-i}. \qquad (10.8)$$

Since we already know that, for every $0 \leq i \leq n + 1$

$$\Delta(C_n) \subseteq C \otimes C_{n-i} + C_{i-1} \otimes C$$

i.e.

$$\Delta(C_n) \subseteq \bigcap_{i=0}^{n+1} (C \otimes C_{n-i} + C_{i-1} \otimes C)$$

we can apply (10.8) for $V = C$ and $V_i = C_{i-1}$ and get that

$$\begin{aligned}\Delta(C_n) &\subseteq \bigcap_{i=0}^{n+1}(C \otimes C_{n-i} + C_{i-1} \otimes C) \\ &= \sum_{i=1}^{n+1} C_{i-1} \otimes C_{n+1-i} \\ &= \sum_{i=0}^{n} C_i \otimes C_{n-i}.\end{aligned}$$

5) In view of Theorem 9.7, C is the union of its finite dimensional subcoalgebras. Thus let D be a finite dimensional subcoalgebra of C and let us prove that there exists an $n \in \mathbb{N}$ such that $D \subseteq C_n$. Since D is finite dimensional we can apply Proposition 9.41 to get that $Jac(D^*) = D_0^{\perp}$ and that there exists an $n \in \mathbb{N}$ such that $(D_0^{\perp})^n = (D_0^{\perp})^{n+1}$ so that, by Nakayama's Lemma, we obtain that $(D_0^{\perp})^n = \{0\}$. Hence, by (10.3) we obtain that

$$D = \{0\}^{\perp} = ((D_0^{\perp})^n)^{\perp} = \bigwedge_D^n D_0.$$

Now, by Lemma 10.5, we get that

$$\bigwedge_D^n D_0 \subseteq \bigwedge_C^n D_0$$

and by Lemma 9.40, we have

$$D_0 = C_0 \cap D.$$

Hence, by Lemma 10.6, we deduce that

$$\bigwedge_C^n D_0 \subseteq \bigwedge_C^n C_0$$

so that we finally obtain that

$$D = \bigwedge_D^n D_0 \subseteq \bigwedge_C^n C_0 = C_{n-1}.$$

□

Lemma 10.11. *Let D be a subcoalgebra of a k-coalgebra C. Then*

$$D_n = C_n \cap D \textit{ for every } n \geq 0.$$

Proof. Let us proceed by induction on n. For $n=0$ the equality follows by Lemma 9.40. Assume now that the equality holds for some $n\in\mathbb{N}$ and let us prove it for $n+1$. We have

$$\begin{aligned} D\cap C_{n+1} &= D\cap\Delta_C^{\leftarrow}\left(C_n\otimes C+C\otimes C_0\right)=D\cap\Delta_C^{\leftarrow}\left[(D\otimes D)\cap\left(C_n\otimes C+C\otimes C_0\right)\right]\\ &\overset{\text{Lem15.5}}{=} D\cap\Delta_C^{\leftarrow}\left[(D\cap C_n)\otimes D+D\otimes(D\cap C_0)\right]\\ &\overset{\text{ind hyp}}{=} D\cap\Delta_C^{\leftarrow}\left(D_n\otimes D+D\otimes D_0\right)=\Delta_D^{\leftarrow}\left(D_n\otimes D+D\otimes D_0\right)=D_{n+1}.\end{aligned}$$

□

Lemma 10.12. *Let A a be k-algebra, let C be a k-coalgebra and let $f\in\mathrm{Hom}_k(C,A)$. If $f_{|C_0}=0$ then $f_{|C_n}^{n+1}=0$ for every $n\in\mathbb{N}$.*

Proof. Let us proceed by induction on $n\in\mathbb{N}$. For $n=0$ there is nothing to prove. Assume that $f_{|C_n}^{n+1}=0$ for some $n\in\mathbb{N}$ and let us prove that $f_{|C_{n+1}}^{n+2}=0$. We have that

$$C_{n+1}=C_0\wedge C_n==\Delta^{\leftarrow}(C\otimes C_n+C_0\otimes C).$$

Thus, for every $c\in C_{n+1}$ we can write

$$\begin{aligned}\Delta(c) &= \sum_{i=1}^{m}a_i\otimes b_i+\sum_{j=1}^{s}c_j\otimes d_j \text{ where } m,s\in\mathbb{N},a_i\in C,b_i\in C_n,c_j\in C_0,d_j\in C\\ \text{for every } i &= 1,\ldots,m \text{ and } j=1,\ldots,s\end{aligned}$$

so that

$$\begin{aligned} f^{n+2}(c) &= \left(f*f^{n+1}\right)(c)=\sum_{i=1}^{m}f(a_i)\cdot f^{n+1}(b_i)+\sum_{j=1}^{s}f(c_j)\cdot f^{n+1}(d_j)=\\ &= \sum_{i=1}^{m}f(a_i)\cdot 0+\sum_{j=1}^{s}0\cdot f^{n+1}(d_j)=0.\end{aligned}$$

□

Proposition 10.13. (*Takeuchi*) *Let A a be k-algebra and let C be a k-coalgebra. A map $f\in\mathrm{Hom}_k(C,A)$ is convolution invertible $\Leftrightarrow$ $f_{|C_0}$ is invertible in $\mathrm{Hom}_k(C_0,A)$.*

Proof. " $\Rightarrow$ " Let $g\in\mathrm{Hom}_k(C,A)$ be such that $f*g=u_A\circ\varepsilon_C=g*f$ i.e.

$$\sum f(c_1)\,g(c_2)=\varepsilon_C(c)\,1_A=\sum g(c_1)\,f(c_2)\ \text{ for every } c\in C.$$

Then we get

$$\sum f(c_1)\,g(c_2)=\varepsilon_C(c)\,1_A=\sum g(c_1)\,f(c_2)\ \text{ for every } c\in C_0$$

i.e. $f_{|C_0}*g_{|C_0}=u_A\circ\varepsilon_{C_0}=g_{|C_0}*f_{|C_0}$.

" $\Leftarrow$ " Let $h \in \mathrm{Hom}_k(C_0, A)$ be such that $f_{|C_0} * h = u_A \circ \varepsilon_{C_0} = h * f_{|C_0}$. Let W be a subvector space of C such that $C = C_0 \oplus W$ and extend h to a map $h' : C \to A$ by setting $h'(W) = 0$. Let $\chi = u_A \circ \varepsilon_C - f * h'$. Then $\chi_{|C_0} = 0$ so that, by Lemma 10.12, $\chi_{|C_n}^{n+1} = 0$ for every $n \in \mathbb{N}$ and hence $\sum_{n\in\mathbb{N}} \chi^n$ is by 5) in Theorem 10.10, well-defined on C and we have

$$(f * h') * \left(\sum_{n\in\mathbb{N}} \chi^n\right) = ((u_A \circ \varepsilon_C) - \chi) * \left(\sum_{n\in\mathbb{N}} \chi^n\right) = u_A \circ \varepsilon_C$$

so that $h' * \left(\sum_{n\in\mathbb{N}} \chi^n\right)$ is a right inverse for f. Similarly let $\gamma = u_A \circ \varepsilon_C - h' * f$. Then $\gamma_{|C_0} = 0$ so that, by Lemma 10.12, $\gamma_{|C_n}^{n+1} = 0$ for every $n \in \mathbb{N}$ and hence $\sum_{n\in\mathbb{N}} \gamma^n$ is by 5) in Theorem 10.10, well-defined on C and we have

$$\left(\sum_{n\in\mathbb{N}} \gamma^n\right) * (h' * f) = \left(\sum_{n\in\mathbb{N}} \gamma^n\right) * ((u_A \circ \varepsilon_C) - \gamma) = u_A \circ \varepsilon_C$$

so that $\left(\sum_{n\in\mathbb{N}} \gamma^n\right) * h'$ is a left inverse for f. □

Chapter 11

Algebra and Coalgebra Filtrations

Definition 11.1. *Let (C, Δ, ε) be a k-coalgebra. We say that a sequence $(V_n)_{n\in\mathbb{N}}$ of subspaces of C is a* coalgebra filtration of C *if*

1) $V_n \subseteq V_{n+1}$, *for every* $n \in \mathbb{N}$.

2) $\Delta V_n \subseteq \sum_{i=0}^{n} V_i \otimes V_{n-i}$, *for every* $n \in \mathbb{N}$.

3) $C = \cup_{n\geqslant 0} V_n$.

In this case we also say that the coalgebra C is filtered.

Definition 11.2. *Let (A, m, u) be a k-algebra. We say that a sequence $(V_n)_{n\in\mathbb{N}}$ of subspaces of A is an* algebra filtration of A *if*

1) $V_n \subseteq V_{n+1}$, *for every* $n \in \mathbb{N}$.

2) $1_A \in V_0$ *and* $V_i V_j \subseteq V_{i+j}$ *for every* $i, j \in \mathbb{N}$.

3) $A = \cup_{n\geqslant 0} V_n$.

In this case we also say that the algebra A is filtered.

Definition 11.3. *Let $(H, m, u, \Delta, \varepsilon, S)$ be a Hopf algebra over a field k. We say that a sequence $(V_n)_{n\in\mathbb{N}}$ of subspaces of H is a* Hopf algebra filtration of A *if*

1) $(V_n)_{n\in\mathbb{N}}$ *is a coalgebra filtration of* H;

2) $(V_n)_{n\in\mathbb{N}}$ *is an algebra filtration of* H;

3) $S(V_n) \subseteq V_n$ *for every* $n \in \mathbb{N}$.

Definition 11.4. *Let (C, Δ, ε) be a k-coalgebra and let $(C_n)_{n\in\mathbb{N}}$ be as in 10.9. Then, in view of Theorem 10.10, $(C_n)_{n\in\mathbb{N}}$ is a coalgebra filtration of C which is called* coradical filtration.

Example 11.5. *Let us provide an example of Hopf algebra filtration.*
Let us consider the usual polynomial ring $k[X]$ endowed with the usual Hopf algebra stucture

$$\Delta(X) = X \otimes 1 + 1 \otimes X \quad , \quad \varepsilon(X) = 0 \quad , \quad S(X) = -X.$$

Let us set, for every $n \in \mathbb{N}$,

$$A_n = k + kX + kX^2 + ... + kX^n$$

and let us show that (A_n) is a Hopf algebra filtration. Clearly we have

$$A_n \subseteq A_{n+1} \qquad , \bigcup_{n\geq 0} A_n = k[X] \qquad , S(A_n) \subseteq A_n \qquad \text{and } A_m A_n \subseteq A_{m+n} \text{ for all } m, n \in \mathbb{N}.$$

Let us show that

$$\Delta(A_n) \subseteq \sum_{i=0}^{n} A_i \otimes A_{n-i} \text{ for all } n \in \mathbb{N}.$$

We compute

$$\begin{aligned}
\Delta(A_n) &= \Delta(k + ... + kX^n) = k + k\Delta(X) + ... + k\Delta(X^n) \\
&= k + k\Delta(X) + ... + k\Delta(X)^n \\
&= k + k(X \otimes 1 + 1 \otimes X) + ... + k(X \otimes 1 + 1 \otimes X)^n \\
&= k + X \otimes k + k \otimes X + ... + k(\sum_{h=0}^{n} \binom{n}{h} (X \otimes 1)^h (1 \otimes X)^{n-h}) \\
&= k + X \otimes k + k \otimes X + ... + k(\sum_{h=0}^{n} \binom{n}{h} (X^h \otimes 1)(1 \otimes X^{n-h})) \\
&= k + X \otimes k + k \otimes X + ... + \sum_{h=0}^{n} \binom{n}{h} (kX^h \otimes X^{n-h}) \\
&\subseteq \sum_{h=0}^{n} (kX^h \otimes kX^{n-h}) \subseteq \sum_{h=0}^{n} A_h \otimes A_{n-h}.
\end{aligned}$$

Proposition 11.6. *Let $(V_n)_{n\in\mathbf{N}}$ be a coalgebra filtration of a k-coalgebra C. Then*

1) *each V_n is a subcoalgebra of C*

2)

$$\Delta V_n \subseteq V_0 \otimes V_n + V_n \otimes V_{n-1} \tag{11.1}$$

3) $C_0 \subseteq V_0$.

Proof. 1) and 2) From $\Delta V_n \subseteq \sum_{i=0}^{n} V_i \otimes V_{n-i}$ and $V_a \subseteq V_{a+1}$ for every $n, a \in \mathbb{N}$ we get that $\Delta V_n \subseteq V_n \otimes V_n$ and $\Delta V_n \subseteq V_0 \otimes V_n + \sum_{i=1}^{n} V_i \otimes V_{n-i} \subseteq V_0 \otimes V_n + V_n \otimes V_{n-1}$.

3) Let D be a simple subcoalgebra of C. In view of 1), it suffices to show that

$$D \cap V_0 \neq \{0\}.$$

Since $C = \bigcup_{k\in\mathbb{N}} V_k$ there exists a minimum n such that $D \cap V_n \neq \{0\}$. We will show that $n = 0$. Let $0 \neq d \in D \cap V_n$. Assume that $n > 0$. We have

$$\Delta(d) \in \Delta(V_n) \subseteq \sum_{i=0}^{n} V_i \otimes V_{n-i}$$

so that there exists $v_i \in V_i$ and $w_i \in V_{n-i}$, for every $i = 1, \ldots, n$ such that

$$\Delta(d) = \sum_{i=0}^{n} v_i \otimes w_i. \tag{11.2}$$

Let $(b_i)_{i\in I}$ be a basis of V_0 and let $(b_j)_{j\in J}$, where $J \supseteq I$, be a basis of C. Then we have

$$\Delta(d) = \sum_{j\in J} a_j \otimes b_j \text{ for some } a_j \in C, \text{ almost all } a_j = 0\ .$$

Then there exists a $j_0 \in J \setminus I$ such that $a_{j_0} \neq 0$. In fact, otherwise we would get $\Delta(d) \in C \otimes V_0$ and hence $d = l_C(\varepsilon \otimes C)\Delta(d) \in V_0$. Let $f = (b_{j_0})^* \in C^*$ i.e. $f(b_j) = \delta_{j_0 j}$ for every $j \in J$. Then

$$D \ni f \cdot d = \sum d_1 f(d_2) = \sum_{j\in J} a_j f(b_j) = a_{j_0} \neq 0.$$

Note that $f \in V_0^{\perp}$ and hence, in view of (11.2)

$$f \cdot d = \sum_{i=0}^{n} v_i f(w_i) = \sum_{i=0}^{n-1} v_i f(w_i) \in \sum_{i=0}^{n-1} V_i \subseteq V_{n-1}$$

so that

$$0 \neq f \cdot d \in D \cap V_{n-1}.$$

Contradiction. □

Corollary 11.7. *Let $f : C \longrightarrow D$ be a surjective morphism of k-coalgebras. Then $D_0 \subseteq f(C_0)$.*

Proof. Let $(C_n)_{n\in\mathbb{N}}$ be the coradical filtration of C and let us prove that $(V_n)_{n\in\mathbb{N}}$ with $V_n = f(C_n)$ is a coalgebra filtration of D. Clearly, since $C_n \subseteq C_{n+1}$ for every $n \in \mathbb{N}$,

$$V_n = f(C_n) \subseteq f(C_{n+1}) = V_{n+1} \text{ for every } n \in \mathbb{N}.$$

Since f is surjective

$$D = f(C) = f\left[\bigcup_{n\in\mathbb{N}} (C_n)\right] = \bigcup_{n\in\mathbb{N}} f(C_n) = \bigcup_{n\in\mathbb{N}} V_n$$

and since f is a coalgebra morphism, we have that

$$\Delta_D(V_n) = \Delta_D(f(C_n)) = (f\otimes f)(\Delta_C(C_n)) \subseteq (f\otimes f)\left(\sum_{i=0}^{n} C_i\otimes C_{n-i}\right) = \sum_{i=0}^{n} V_i\otimes V_{n-i}.$$

Then we can conclude by 3) of Proposition 11.6, that

$$D_0 \subseteq V_0 = f(C_0).$$

□

Corollary 11.8. *Let $f : C \longrightarrow D$ be a surjective morphism of k-coalgebras. Assume that $D \neq \{0\}$,*

1) *If C is pointed, also D is pointed.*

2) *If C is connected, also D is connected.*

Proof. 1) By Corollary 11.7, we have that $D_0 \subseteq f(C_0) = f(kG(C)) \subseteq kG(D) \subseteq D_0$.

2) By Corollary 11.7, we have that $\dim_k D_0 \leq \dim_k f(C_0) \leq 1$. By Corollary 9.22 we deduce that $\dim_k D_0 = 1$. □

Proposition 11.9. *Let C be a k-coalgebra, let $J = C_0^{\perp}$ in C^* and let $W = \Omega_f(C^*)$ the set of all two-sided ideals of C^* of finite codimension. Then*

1) $C_n = (J^{n+1})^{\perp}$, for every $n \in \mathbb{N}$

2) $J = Jac(C^*) = \bigcap_{M\in W} M$

3) $\bigcap_{n\geq 0} J^n = (0)$.By Proposition 11.9 we have $C_n = (J^{n+1})^{\perp}$

Proof. 1) By Lemma 9.14 we have $C_0 = C_0^{\perp\perp} = J^{\perp}$ so that 1) holds for $n = 0$. Assume now that 1) holds for some $n-1 \in \mathbb{N}, n \geq 1$ and let us prove it for n. We have

$$C_n \overset{(11.1)}{=} \Delta^{\leftarrow}(C\otimes C_{n-1} + C_0\otimes C) \overset{\text{indhyp}}{=} \Delta^{\leftarrow}\left(C\otimes (J^n)^{\perp} + J^{\perp}\otimes C\right) \overset{\text{Lemma15.3}}{=}$$
$$= \Delta^{\leftarrow}\left((J\otimes J^n)^{\perp}\right) = (J * J^n)^{\perp} = (J^{n+1})^{\perp}.$$

2) Let $f \in J$. Then, for every $n \in \mathbb{N}$, $f^{n+1} \in J^{n+1}$ and hence, by 1), $f^{n+1}(C_n) = 0$ so that it makes sense to consider the map g defined on C by setting

$$g = \sum_{n=0}^{\infty} f^n$$

where $f^0 = \varepsilon$. It is easy to show that $g = (\varepsilon - f)^{-1}$ in C^*. Let $f \in J$ and $h \in C^*$, then $hf \in J$ in fact $f(C_0) = 0$ and $(hf)(C_0) = (h * f)(C_0)$ so that $\varepsilon - hf$ has a left inverse. Hence, by Lemma 9.42, we get $f \in Jac(C^*)$. Therefore we obtain that $J \subseteq Jac(C^*)$. Now, by Corollary 8.12, every $M \in W$ is a finite intersection of $L \in \Omega_s$ so that we get

$$J \subseteq Jac(C^*) = \bigcap_{L \in \Omega_s} L \subseteq \bigcap_{M \in W} M.$$

Let $\{D_\alpha \mid \alpha \in A\}$ be the set of simple subcoalgebras of C. Then, by Corollary 9.18, every $D_a^\perp$ is a two-sided maximal ideal of C^* of finite codimension i.e. $D_a^\perp \in W$. Therefore we obtain

$$J \subseteq Jac(C^*) = \bigcap_{L \in \Omega_s} L \subseteq \bigcap_{M \in W} M \subseteq \bigcap_{\alpha \in A} D_\alpha^\perp = (\sum_{\alpha \in A} D_\alpha)^\perp = C_0^\perp = J$$

and hence

$$J = C_0^\perp = (\sum_{\alpha \in A} D_\alpha)^\perp = \bigcap_{\alpha \in A} D_\alpha^\perp = \bigcap_{M \in W} M.$$

3) Since, in view of 1), for every $n \in \mathbb{N}, n \geq 1$, we have $J^n \subseteq (J^n)^{\perp\perp} = (C_{n-1})^\perp$ we obtain

$$\bigcap_{n \geq 1} J^n \subseteq \bigcap_{n \geq 1} (J^n)^{\perp\perp} = \bigcap_{n \geq 1} (C_{n-1})^\perp = (\sum_{n \geq 1} C_{n-1})^\perp = C^\perp = \{0\}.$$

□

Lemma 11.10. *Let (H_n) be the coradical filtration of a Hopf algebra H. Then (H_n) is a Hopf algebra filtration of $H \Leftrightarrow H_0$ is a Hopf subalgebra of H.*

Proof. " $\Rightarrow$ " is trivial.

" $\Leftarrow$ " Let us show, by induction on $n \in \mathbb{N}$, that $S(H_n) \subseteq H_n$. For $n = 0$ this is trivial, since H_0 is a Hopf subalgebra of H. Assume that for some $n \in \mathbb{N}$, $n \geq 1$

$$S(H_i) \subseteq H_i \text{ for every } i < n.$$

By Theorem 3.7, we know that

$$\Delta S(H_n) = \tau(S \otimes S)\Delta(H_n).$$

where $\tau : H \otimes H \to H \otimes H$ denotes the usual flip. Then we get

$$\Delta S(H_n) = \tau(S \otimes S)\Delta(H_n) \overset{\text{by4)inTheo10.10}}{\subseteq} \tau\left(\sum_{i=0}^{n} S(H_i) \otimes S(H_{n-i})\right) =$$

$$= \sum_{i=0}^{n} S(H_{n-i}) \otimes S(H_i)$$

$$= \sum_{i=1}^{n-1} S(H_{n-i}) \otimes S(H_i) + S(H_0) \otimes S(H_n) + S(H_n) \otimes S(H_0)$$

$$\overset{\text{ind hyp}}{\subseteq} \sum_{i=1}^{n-1} H_{n-i} \otimes H_i + H_0 \otimes H + H \otimes H_0 \subseteq$$

$$\subseteq H \otimes H_{n-1} + H_0 \otimes H + H \otimes H_0 = H \otimes H_{n-1} + H_0 \otimes H$$

i.e.

$$\Delta S(H_n) \subseteq H \otimes H_{n-1} + H_0 \otimes H$$

so that

$$S(H_n) \subseteq \Delta^{\leftarrow}(H \otimes H_{n-1} + H_0 \otimes H) = H_n.$$

Let us show that

$$H_m H_n \subseteq H_{m+n} \text{ for every } m, n \in \mathbb{N}.$$

Assume $n = 0$ and let us prove this by induction on m. For $m = 0$ there is nothing to prove. Assume that, for some $m \geq 1$, we have $H_{m-1}H_0 \subseteq H_{m-1}$. Then we have

$$\begin{aligned}
\Delta(H_m H_0) &= \Delta(H_m)\Delta(H_0) \overset{11.1}{\subseteq} (H_0 \otimes H_m + H_m \otimes H_{m-1})(H_0 \otimes H_0) \subseteq \\
&\subseteq H_0^2 \otimes H + H \otimes H_{m-1}H_0 \\
&\subseteq H_0 \otimes H + H \otimes H_{m-1}
\end{aligned}$$

so that

$$H_m H_0 \subseteq \Delta^{\leftarrow}(H_0 \otimes H + H \otimes H_{m-1}) = H_m.$$

In a similar way we get that

$$H_0 H_n \subseteq H_n \text{ for every } n \geq 0.$$

Let us now sho that $H_m H_n \subseteq H_{m+n}$ by induction on $t = m + n$. If $t = 0$ then $m = 0 = n$ and there is nothing to prove. Assume now that the statement holds for some $t - 1 \geq 0$ and let us prove it for t. In view of the foregoing, we can assume that $m > 0$ and $n > 0$. We have

$$\begin{aligned}
&\Delta(H_m H_n) \overset{11.1}{\subseteq} (H_0 \otimes H_m + H_m \otimes H_{m-1})(H_0 \otimes H_n + H_n \otimes H_{n-1}) \\
\subseteq\ & H_0^2 \otimes H_m H_n + H_m H_0 \otimes H_{m-1}H_n + H_0 H_n \otimes H_m H_{n-1} + H_m H_n \otimes H_{m-1}H_{n-1} \\
\subseteq\ & H_0 \otimes H + H \otimes H_{m+n-1} + H \otimes H_{m+n-2} \\
\subseteq\ & H_0 \otimes H + H \otimes H_{m+n-1}
\end{aligned}$$

and hence

$$H_m H_n \subseteq \Delta^{\leftarrow}(H_0 \otimes H + H \otimes H_{m+n-1}) = H_{m+n}.$$

□

Chapter 12

Some Results on Connected Coalgebras

Definition 12.1. *Let C be a connected k-coalgebra with $G(C) = \{g\}$. We set*

$$P(C) = \{c \in C \mid \Delta(c) = c \otimes g + g \otimes c\}.$$

The elements of $P(C)$ will be called primitive elements of C.

Proposition 12.2. *Let C be a connected k-coalgebra with $G(C) = \{g\}$. Then*

$$P(C) \subseteq Ker(\varepsilon) \quad \text{and} \quad C_1 = kg \oplus P(C).$$

Proof. Let $x \in P(C)$. We compute

$$\begin{aligned} x = r_C(C \otimes \varepsilon)\Delta(x) = [r_C(C \otimes \varepsilon)](x \otimes g + g \otimes x) \\ = r_C(x \otimes \varepsilon(g) + g \otimes \varepsilon(x)) = x\varepsilon(g) + g\varepsilon(x) = x + g\varepsilon(x) \end{aligned}$$

so that we get $x = x + g\varepsilon(x)$ which implies that $\varepsilon(x) = 0$. Thus $P(C) \subseteq Ker(\varepsilon)$.

Note that

$$C_0 = kG(C) = kg$$

and denote by $\pi_{C_0} : C \to C/C_0$ the canonical projection. Then for every $x \in P(C)$ we have

$$\begin{aligned} (\pi_{C_0} \otimes \pi_{C_0})\Delta(x) &= (\pi_{C_0} \otimes \pi_{C_0})(x \otimes g + g \otimes x) \\ &= \pi_{C_0}(x) \otimes \pi_{C_0}(g) + \pi_{C_0}(g) \otimes \pi_{C_0}(x) \\ &= \pi_{C_0}(x) \otimes 0 + 0 \otimes \pi_{C_0}(x) = 0. \end{aligned}$$

Thus

$$P(C) \subseteq Ker((\pi_{C_0} \otimes \pi_{C_0})\Delta) = C_0 \wedge C_0 = C_1.$$

Now, by Theorem 10.10, we have that $kg = C_0 \subseteq C_1$ so that we get that $kg + P(C) \subseteq C_1$. Let $d = \lambda g \in kg \cap P(C), \lambda \in k$. Then we have

$$0 = \varepsilon(d) = \lambda\varepsilon(g) = \lambda$$

and hence the sum $kg + P(C)$ is direct. Let now $c \in C_1$ and set

$$d = c - \varepsilon(c)g.$$

Then $d \in C_1$. We compute

$$\varepsilon(d) = \varepsilon(c) - \varepsilon(c)\varepsilon(g) = \varepsilon(c) - \varepsilon(c) = 0.$$

Since $d \in C_1$ and by Theorem 10.10 $\Delta(C_1) \subseteq \sum_{i=0}^{1} C_i \otimes C_{1-i} = C_0 \otimes C_1 + C_1 \otimes C_0$ there exist $d_1, d_2 \in C_1$ such

$$\Delta(d) = d_1 \otimes g + g \otimes d_2$$

so that we get

$$\begin{aligned} 0 = \varepsilon(d) = m_k(\varepsilon \otimes \varepsilon)\Delta(d) &= \varepsilon(d_1)\varepsilon(g) + \varepsilon(g)\varepsilon(d_2) \\ &= \varepsilon(d_1) + \varepsilon(d_2) \end{aligned}$$

and also

$$d_1 + g\varepsilon(d_2) = d = \varepsilon(d_1)g + d_2.$$

Therefore we obtain

$$\begin{aligned} \Delta(d) &= d_1 \otimes g + g \otimes d_2 \\ &= (d - g\varepsilon(d_2)) \otimes g + g \otimes (d - \varepsilon(d_1)g) \\ &= d \otimes g - [g \otimes g(\varepsilon(d_2) + \varepsilon(d_1))] + g \otimes d \\ &= d \otimes g + g \otimes d \end{aligned}$$

i.e. $d = c - \varepsilon(c)g \in P(C)$ and hence $c = \varepsilon(c)g + d \in kg \oplus P(C)$. □

Definition 12.3. *Let C be a k-coalgebra. We set*

$$C_n^+ = C_n \cap Ker(\varepsilon).$$

Lemma 12.4. *Let C be a connected k-coalgebra with $G(C) = \{g\}$.*

1) *Then for every $n \in \mathbb{N}$, $n \geq 1$ and $c \in C_n$, we have that*

$$\Delta(c) = c \otimes g + g \otimes c + y \qquad \text{where } y \in C_{n-1} \otimes C_{n-1}.$$

2) *Then for every $n \in \mathbb{N}$, $n \geq 1$ and $c \in C_n^+$ we have that*

$$\Delta(c) = c \otimes g + g \otimes c + y \qquad \text{where } y \in C_{n-1}^+ \otimes C_{n-1}^+.$$

Proof. Let $c \in C_n$. By 4) of Theorem 10.10, we have that

$$\Delta(c) \in \sum_{i=0}^{n} C_i \otimes C_{n-i} = C_n \otimes C_0 + C_0 \otimes C_n + \sum_{i=1}^{n-1} C_i \otimes C_{n-i}.$$

Since $C_0 = kg$ we may write

$$\Delta(c) = a \otimes g + g \otimes b + w \text{ where } a, b \in C_n \text{ and } w \in C_{n-1} \otimes C_{n-1}.$$

We compute

$$\begin{aligned} c = r_C(C \otimes \varepsilon)\Delta(c) &= a\varepsilon(g) + g\varepsilon(b) + r_C(C \otimes \varepsilon)w \\ &= a + g\varepsilon(b) + r_C(C \otimes \varepsilon)w \\ &\in a + C_0 + C_{n-1} \subseteq a + C_{n-1}. \end{aligned}$$

Thus we deduce that $a - c = c^{'} \in C_{n-1}$. Analogously we have

$$\begin{aligned} c = l_C(\varepsilon \otimes C)\Delta(c) &= \varepsilon(a)g + \varepsilon(g)b + l_C(\varepsilon \otimes C)w \\ &= \varepsilon(a)g + b + l_C(\varepsilon \otimes C)w \\ &\in b + C_0 + C_{n-1} \subseteq b + C_{n-1} \end{aligned}$$

so that $b - c = c^{''} \in C_{n-1}$. Set

$$y = w + c^{'} \otimes g + g \otimes c^{''} \in C_{n-1} \otimes C_{n-1}.$$

Then we get

$$\begin{aligned} \Delta(c) &= a \otimes g + g \otimes b + w = a \otimes g + g \otimes b + y - c^{'} \otimes g - g \otimes c^{''} \\ &= \left(a - c^{'}\right) \otimes g + g \otimes \left(b - c^{''}\right) + y \\ &= c \otimes g + g \otimes c + y \quad \text{ where } y \in C_{n-1} \otimes C_{n-1}. \end{aligned}$$

Assume now that $c \in C_n^+$. We compute

$$\begin{aligned} r_C(C \otimes \varepsilon)(y) &= r_C(C \otimes \varepsilon)\Delta(c) - c\varepsilon(g) - g\varepsilon(c) \\ &= c - c - g\varepsilon(c) = 0 \end{aligned}$$

and also

$$\begin{aligned} l_C(\varepsilon \otimes C)y &= l_C(\varepsilon \otimes C)\Delta(c) - \varepsilon(c)g - \varepsilon(g)c \\ &= c - \varepsilon(c)g - c = 0. \end{aligned}$$

Thus we obtain that $y \in \mathrm{Ker}\,(C \otimes \varepsilon) = \mathrm{Ker}\,(\mathrm{Id}_C) \otimes C + C \otimes \mathrm{Ker}\,(\varepsilon) = C \otimes \mathrm{Ker}\,(\varepsilon)$ and also that $y \in \mathrm{Ker}\,(\varepsilon \otimes C) = \mathrm{Ker}\,(\varepsilon) \otimes C + C \otimes \mathrm{Ker}\,(\mathrm{Id}_C) = \mathrm{Ker}\,(\varepsilon) \otimes C$. We deduce that

$$y \in (Ker(\varepsilon) \otimes C) \cap (C \otimes Ker(\varepsilon)) = Ker(\varepsilon) \otimes Ker(\varepsilon)$$

and hence, by the foregoing, we obtain that

$$y \in (C_{n-1} \otimes C_{n-1}) \cap (Ker(\varepsilon) \otimes Ker(\varepsilon)) = C_{n-1}^{+} \otimes C_{n-1}^{+}.$$

□

Lemma 12.5. *Let C be a connected k-coalgebra with $G(C) = \{g\}$. Let $f : C \to D$ be a coalgebra morphism such that $f_{|P(C)}$ is injective. Then f is injective.*

Proof. We will show that $f_{|C_n}$ is injective for every $n \in \mathbb{N}$. We will proceed by induction on n. Since f is a coalgebra morphism we have $\varepsilon_D (f (g)) = (\varepsilon_D \circ f) (g) = \varepsilon_C (g) = 1$ and hence we deduce that $f(g) \neq 0$ and hence $f_{|C_0}$ is injective. Let us assume that $f_{|C_n}$ is injective for some $n \in \mathbb{N}$ and let $x \in C_{n+1} \cap \mathrm{Ker}(f)$. Now, by Lemma 12.4

$$\Delta(x) = x \otimes g + g \otimes x + y, \qquad \text{where } y \in C_n \otimes C_n$$

and hence

$$0 = \Delta(f(x)) = (f \otimes f)\Delta(x) = f(x) \otimes f(g) + f(g) \otimes f(x) + (f \otimes f)(y) = (f \otimes f)(y).$$

Since $f_{|C_n}$ is injective, also $f_{|C_n} \otimes f_{|C_n}$ is injective so that we deduce that $y = 0$. Thus $\Delta(x) = x \otimes g + g \otimes x$ so that $x \in P(C)$. Now, by hypothesis, $f_{|P(C)}$ is injective and hence we get that $x = 0$. □

Chapter 13

Separable algebras

We start by recalling the celebrated

Theorem 13.1. *(Wedderburn-Artin Theorem) Let R be a ring. ${}_RR$ is semisimple if and only if R is isomorphic to a direct product of rings, each isomorphic to a finite matrix ring $M_n(D)$ over a division ring D.*

By Wedderburn Artin Theorem it is clear that for a given ring R we have

$$_RR \text{ is semisimple} \Longleftrightarrow R_R \text{ is semisimple}$$

Definition 13.2. *Let R be a ring. R is called* semisimple *if ${}_RR$ is semisimple.*

Lemma 13.3. *Let R be a ring and assume that ${}_RR$ is artinian. Then there exists an $n \in \mathbb{N}$, $n \geq 1$ and maximal left ideals of R, $L_1, \dots, L_n$ such that*

$$L_1 \cap \cdots \cap L_n = \{0\}.$$

Proof. For every $F \in P_0(\Omega_l(R))$, let $J_F = \bigcap_{L \in F} L$ and let

$$X = \{J_F \mid F \in P_0(\Omega_l(R))\}.$$

Since ${}_RR$ is artinian, X has a minimal element. Let $F_0 \in P_0(\Omega_l(R))$ be such that J_{F_0} is a minimal element for X. Then, for every $L \in \Omega_l(R)$, we have that

$$J_{F_0} \cap L = J_{F_0 \cup \{X\}} \subseteq J_{F_0}$$

and hence, by the minimality of J_{F_0}, we obtain $J_{F_0} = J_{F_0} \cap L \subseteq L$. Thus we get that $J_{F_0} \subseteq Jac(R) \subseteq J_{F_0}$ and hence $J_{F_0} = Jac(R)$. □

Proposition 13.4. *Let R be a ring and assume that ${}_RR$ is artinian. Then the following statements are equivalent*

(*a*) *R is semisimple.*
(*b*) *$J(R) = \{0\}$*
(*c*) *R has no non-zero two-sided nilpotent ideal.*

Proof. $(a) \Rightarrow (b)$ is trivial in view of Wedderburn Artin Theorem.
$(b) \Rightarrow (c)$ is trivial since by Lemma 9.43 every nilpotent two-sided ideal is contained in $J(R) = \{0\}$

$(c) \Rightarrow (b)$ Since ${}_RR$ is artinian, there exists an $n \in \mathbb{N}$ such that

$$J(R)^n = J(R)^{n+1}$$

Since ${}_RR$ is noetherian (see [AF, Theorem 15.20]), ${}_RJ(R)^n$ is finitely generated and hence, by Nakayama's Lemma, we get that $J(R)^n = \{0\}$so that we get $J(R) = \{0\}$.

$(b) \Rightarrow (a)$ Since ${}_RR$ is artinian, by Lemma 13.3, there exists a finite number of maximal left ideals of R say $L_1, \ldots, L_n$ such that

$$L_1 \cap \cdots \cap L_n = \{0\}.$$

Thus ${}_RR$ embeds in the direct sum of a finite number of simple lef R-modules and hence (see [AF, Proposition 9.4]), it is semisimple. □

Corollary 13.5. *Let A be a finite dimensional algebra over a field k. Then*

A is semisimple $\Leftrightarrow J(A) = \{0\} \Leftrightarrow A$ contains no non-zero two-sided nilpotent ideal.

Proof. Since ${}_AA$ is artinian, just apply Proposition 13.4. □

Definition 13.6. *An algebra A over a field k is called* classically separable *if , for every field extension L of k, the Jacobson radical of the L-algebra $A_{(L)} = A \otimes_k L$ is zero.*

Proposition 13.7. *Let A be a finite dimensional algebra over a field k. Then the following are equivalent:*

(a) A is classically separable.

(b) For every field extension L of k, the L-algebra $A_{(L)}$ is semisimple.

(c) For every field extension L of k, the L-algebra $A_{(L)}$ contains no non-zero two-sided nilpotent ideal.

Proof. For every field extension L of k, we have that

$$\dim_L \left(A_{(L)}\right) = \dim_k (A) < \infty$$

Apply now Corollary 13.5. □

Proposition 13.8. *Let F be a finite field extension of a field k. Then*

F is a classically separable k-algebra $\Longleftrightarrow$ every $u \in F$ is separable over k.

Proof. $(\Rightarrow)$ Let $u \in F$, let f_u be the minimal polynomial of u over k and let L be a splitting field of f_u over k. Then

$$\frac{L[X]}{(f_u)} \cong k[u] \otimes_k L \subseteq F \otimes_k L.$$

Let

$$f_u = (X - \alpha_1)^{t_1} \cdots (X - \alpha_n)^{t_n}$$

where $\alpha_1, \ldots, \alpha_n$ are the distinct root of f_u in L. Then, by the Chinese Remainder's Theorem, we have a ring isomorphism

$$\frac{L[X]}{(f_u)} \cong \frac{L[X]}{\left((X - \alpha_1)^{t_1}\right)} \times \ldots \times \frac{L[X]}{\left((X - \alpha_n)^{t_n}\right)}$$

Thus, any $t_i > 1$ gives rise to a nilpotent ideal of $k[u] \otimes_k L$ and hence of $F \otimes_k L$. Since $\dim_L F \otimes_k L = \dim_k F < \infty$, the conclusion follows in view of Proposition 13.7.

($\Leftarrow$) Assume that every element $u \in F$ is separable over k. Then, by the Theorem of the Primitive Element, there exists an $u \in F$ such that $F = k(u)$ and the minimal polynomial f_u of u over k is separable over k. Let L be a field extension of k and let

$$f_u = h_1 \cdots h_t$$

be the factorization of f_u as a product of irreducible factors in $L[X]$. Let M be a spliting field of f_u over k. Then in $M[X]$ we can write

$$f_u = (X - \alpha_1) \cdots (X - \alpha_n)$$

where $\alpha_1, \ldots, \alpha_n$ are all distinct. Considering the field extension $L(\alpha_1, \ldots, \alpha_n)$, we deduce that $h_1, \ldots, h_t$ are two by two not associated. Then, by the Chinese Remainder's Theorem, we get

$$F \otimes_k L = k(u) \otimes_k L \cong \frac{k[X]}{(f_u)} \otimes_k L \cong \frac{L[X]}{(f_u)} \cong \frac{L[X]}{(h_1)} \times \ldots \times \frac{L[X]}{(h_t)}.$$

Since each $L[X] / (h_i)$ is a field, it follows that $F \otimes_k L$ contains no non-zero nilpotent ideal. $\square$

Definition 13.9. *Let R be a commutative ring. An R-algebra A is called separable if the multiplication map*

$$m_A : A \otimes_R A \to A$$

has a section σ (i.e. $m_A \sigma = \mathrm{Id}_A$) which is an A-bimodule homomorphism.

Proposition 13.10. *Let R be a commutative ring and let A be a separable R-algebra. Given a section σ of m_A which is an A-bimodule homomorphism, set*

$$e = \sigma(1_A) \quad \textit{and write} \quad e = \sum_{i=1}^{n} x_i \otimes_R y_i$$

for suitable $n \in \mathbb{N}$ and $x_i, y_i \in A$ for every $i = 1, \ldots, n$.

Then we have

$$m_A(e) = 1_A \quad i.e. \quad \sum_{i=1}^{n} x_i y_i = 1_A \tag{13.1}$$

and

$$ae = ea \quad i.e. \quad \sum_{i=1}^{n} a x_i \otimes_R y_i = \sum_{i=1}^{n} x_i \otimes_R y_i a \qquad \textit{for every } a \in A. \tag{13.2}$$

Proof. Equalities (13.1) and (13.2) follows directly from being σ an A-bimodule section of m_A.

□

Definition 13.11. *Let A be an algebra over a commutative ring R. An element $e \in A \otimes_R A$ is called a* separability element (*or also an* idempotent*)* for A *(over R) if e fulfills* (13.1) *and* (13.2)*.*

Proposition 13.12. *Let A be an algebra over a commutative ring R. Then*

A is a separable R-algebra $\Leftrightarrow$ $A \otimes_R A$ contains a separability element for A over R.

Moreover any separability element of A is an idempotent element of the ring $A \otimes_R A^{op}$.

Proof. Let e be a separability element for A and define a map

$$\sigma : A \to A \otimes_R A$$

by setting

$$\sigma(a) = ae.$$

Then σ is an A-bimodule homomorphism and a section of m_A. Write

$$e = \sum_{i=1}^{n} x_i \otimes_R y_i.$$

Then we have:

$$\begin{aligned} e &= \sigma(1_A) = \sigma\left(\sum_{i=1}^{n} x_i y_i\right) = \sigma\left(\sum_{i=1}^{n} x_i 1_A y_i\right) = \sum_{i=1}^{n} x_i \sigma(1_A) y_i = \\ &= \sigma(1_A) \cdot_{A \otimes_R A^{op}} \sigma(1_A) = e^2. \end{aligned}$$

The other implication is Proposition 13.10. □

Lemma 13.13. *Let A be a separable algebra over a commutative ring R. If L is a two-sided ideal of A then A/L is a separable R-algebra.*

Proof. Let $p : A \to A/L$ be the canonical projection. Let e be a separability element of A over R and let us prove that $\overline{e} = (p \otimes p)(e)$ is a separability element for A/L over R. We compute

$$m_{A/L}(\overline{e}) = \left[m_{A/L}(p \otimes p)\right](e) = [p \circ m_A](e) = p(1_A) = 1_{A/L}.$$

Write $e = \sum_{i=1}^{n} x_i \otimes_R y_i$ for suitable $n \in \mathbb{N}$ and $x_i, y_i \in A$ for every $i = 1, \ldots, n$. For every $a \in A$ we have

$$\begin{aligned}(a+L)\overline{e} = (a+L)[(p \otimes p)(e)] = (a+L)\left[\sum_{i=1}^{n}(x_i+L) \otimes_R (y_i+L)\right] = \\ = \sum_{i=1}^{n}(ax_i+L) \otimes_R (y_i+L) = (p \otimes p)(ae) \\ = (p \otimes p)(ea) = \sum_{i=1}^{n}(x_i+L) \otimes_R (y_ia+L) = \\ = \left[\sum_{i=1}^{n}(x_i+L) \otimes_R (y_i+L)\right](a+L) = [(p \otimes p)(e)](a+L) \\ = \overline{e}(a+L).\end{aligned}$$

□

Proposition 13.14. *Let R be a commutative ring and let $n \in \mathbb{N}$, $n \geq 1$. Then the matrix ring $M_n(R)$ is a separable R-algebra.*

Proof. Let $e_{i,j} \in M_n(R) = A$ be the matrix defined by

$$(e_{i,j})_{(i,j)} = 1_R \quad \text{and} \quad (e_{i,j})_{(h,k)} = 0 \text{ for every } (h,k) \neq (i,j)$$

and set

$$e = \sum_{i=1}^{n} e_{i,1} \otimes_R e_{1,i}$$

Then

$$m_R(e) = \sum_{i=1}^{n} e_{i,i} = 1_A$$

and, for every $h, k = 1, \ldots n$, we have

$$\begin{aligned} e_{h,k} \cdot e &= \sum_{i=1}^{n} e_{h,k} \cdot e_{i,1} \otimes_R e_{1,i} = e_{h,1} \otimes_R e_{1,k} \\ e \cdot e_{h,k} &= \sum_{i=1}^{n} e_{i,1} \otimes_R e_{1,i} \cdot e_{h,k} = e_{h,1} \otimes_R e_{1.k} \end{aligned}$$

Therefore e is a separability element for $M_n(R)$ over R. □

Proposition 13.15. *Let R be a commutative ring and let G be a finite group whose order n is a invertible in R. Then the group algebra $A = RG$ is a separable R-algebra.*

Proof. Let
$$e = (n1_A)^{-1} \sum_{g \in G} g \otimes_R g^{-1}.$$
Then
$$m_R(e) = (n1_A)^{-1} \cdot (n1_A) = 1_A$$
and, for every $h \in G$, we have
$$h \cdot e = (n1_A)^{-1} \sum_{g \in G} hg \otimes_R g^{-1} = (n1_A)^{-1} \sum_{t \in G} t \otimes_R t^{-1} h = e \cdot h.$$
Therefore the element e is a separability element for RG over R. □

Proposition 13.16. *Let A be an algebra over a field k. Then*
$$A \text{ separable over } k \quad \Rightarrow \quad \dim_k(A) < \infty.$$

Proof. Let
$$e = \sum_{j=1}^{n} x_j \otimes y_j$$
be a separability element for A over k. For every $a \in A$ we have
$$\sum_{j=1}^{n} a x_j \otimes y_j = \sum_{j=1}^{n} x_j \otimes y_j a.$$

Let $(e_i)_{i \in I}$ be a basis of A over k and for every $i \in I$ let $e_i^* : A \to k$ be the k-linear map defined by
$$e_i^*(e_j) = \delta_{ij}$$
Then, for every $i \in I$, we have
$$\sum_{j=1}^{n} a x_j \otimes e_i^*(y_j) = \sum_{j=1}^{n} x_j \otimes e_i^*(y_j a). \tag{13.3}$$

Now any element $r \in A$ can be uniquely written as
$$r = \sum_{i \in F(r)} e_i^*(r) e_i$$
where $F(r)$ is a suitable finite subset of I.

Set
$$F = \bigcup_{j=1,\ldots n} F(y_j).$$

Thus, using (13.3) we obtain

$$\begin{aligned}
\sum_{j=1}^{n} ax_j \otimes y_j &= \sum_{j=1}^{n} ax_j \otimes \sum_{i\in F(y_j)} e_i^*(y_j)\, e_i \\
&= \sum_{j=1}^{n} ax_j \otimes \sum_{i\in F} e_i^*(y_j)\, e_i \\
&= \sum_{j=1}^{n} \sum_{i\in F} ax_j \otimes e_i^*(y_j)\, e_i \\
&= \sum_{j=1}^{n} \sum_{i\in F} x_j \otimes e_i^*(y_j a)\, e_i \\
&= \sum_{i\in F} \sum_{j=1}^{n} x_j \otimes e_i^*(y_j a)\, e_i
\end{aligned}$$

so that we obtain

$$\begin{aligned}
a &= m\sigma(a) = m\left(\sum_{j=1}^{n} ax_j \otimes y_j\right) \\
&= m\left(\sum_{i\in F} \sum_{j=1}^{n} x_j \otimes e_i^*(y_j a)\, e_i\right) \\
&= \sum_{i\in F} \sum_{j=1}^{n} x_j e_i^*(y_j a)\, e_i = \sum_{i\in F} \sum_{j=1}^{n} e_i^*(y_j a)\, x_j e_i.
\end{aligned}$$

It follows that the set $\{\, x_j e_i \mid j = 1, \ldots n,\ i \in F\}$ is a set of generators for A over k. □

Proposition 13.17. *Let A be a separable algebra over a field k. Then A is semisimple.*

Proof. Let σ be a section of the multiplication map $m_A : A \otimes_k A \to A$ which is an A-bimodule homomorphism and let

$$e = \sum_{i=1}^{n} a_i \otimes_k b_i$$

be a separability element of A over k.
We will prove that any epimorphism

$$f : M \to N$$

of left A-modules splits in A-Mod. Let $s : N \to M$ be a section of f in k-Mod and let us define a map

$$\sigma : N \to M$$

by setting

$$\sigma(x) = \sum_{i=1}^{n} a_i s(b_i x).$$

Clearly we have

$$f\sigma(x) = f\left[\sum_{i=1}^{n} a_i s(b_i x)\right] = \sum_{i=1}^{n} a_i f[s(b_i x)] = \sum_{i=1}^{n} a_i (b_i x) = x$$

so that σ is a section of f. Now in $A \otimes_k A$ we have, for every $a \in A$

$$\sum_{i=1}^{n} a a_i \otimes b_i = \sum_{i=1}^{n} a_i \otimes b_i a$$

so that in $A \otimes_k A \otimes_k N$ we have, for every $a \in A$ and $x \in N$

$$\sum_{i=1}^{n} a a_i \otimes b_i \otimes x = \sum_{i=1}^{n} a_i \otimes b_i a \otimes x$$

Let μ_M (resp. μ_N) be the multiplication map on M (resp. on N):

$$\mu_M : A \otimes_k M \to M.$$

Then, for every $a \in A$ and for every $x \in N$, we have

$$\begin{aligned} a\sigma(x) &= \sum_{i=1}^{n} a a_i s(b_i x) \\ &= \mu_N (A \otimes s)(A \otimes \mu_M)\left(\sum_{i=1}^{n} a a_i \otimes b_i \otimes x\right) \\ &= \left[\mu_N (A \otimes s)(A \otimes \mu_M)\left(\sum_{i=1}^{n} a_i \otimes b_i a \otimes x\right)\right] \\ &= \sum_{i=1}^{n} a_i s(b_i a x) = \sigma(ax) \end{aligned}$$

and hence σ is a morphism of left A-modules. □

Proposition 13.18. *Let R be a commutative ring, let A be an R-algebra and let S be a commutative R-algebra. Then*

A is a separable R-algebra $\Rightarrow$ $A_{(S)} = A \otimes_R S$ is a separable S-algebra.

Moreover if we assume that R is a subring of S and $\pi : S \to R$ is an R-bilinear retraction of the canonical inclusion $\iota : R \to S$, then

$A_{(S)} = A \otimes_R S$ is a separable S-algebra $\Rightarrow$ A is a separable R-algebra.

Proof. Let us remark that for any R-algebra B, the S-algebra structure (and hence the S-bimodule structure) of $B_{(S)} = B \otimes_R S$ is via the ring homomorphism

$$\lambda : S \to \ B \otimes_R S = B_{(S)}$$

defined by setting

$$\lambda (s) = 1 \otimes_R s$$

whose image lies in the center of $B_{(S)}$. This applies, in particular when $B = A$ or $B = A \otimes_R A$.

The map

$$\phi : (A \otimes_R S) \otimes_S (A \otimes_R S) = A_{(S)} \otimes_S A_{(S)} \longrightarrow (A \otimes_R A) \otimes_R S = (A \otimes_R A)_{(S)}$$

defined by setting

$$\phi ((a \otimes_R s) \otimes_S (b \otimes_R t)) = (a \otimes_R b) \otimes_S st$$

is well defined and an S-algebra isomorphism whose inverse is the map

$$\psi : A \otimes_R A \otimes_R S = (A \otimes_R A)_{(S)} \longrightarrow (A \otimes_R S) \otimes_S (A \otimes_R S) = A_{(S)} \otimes_S A_{(S)}$$

defined by setting

$$\psi (a \otimes_R b \otimes_R s) = (a \otimes_R 1_S) \otimes_S (b \otimes_R s) .$$

Let us note that ϕ is also an $A_{(S)} = (A \otimes_R S)$-bimodule homomorphism since the $A_{(S)}$-bimodule structure on $(A \otimes_R A)_{(S)}$ is given by

$$(c \otimes_R w) \cdot [(a \otimes_R b) \otimes_S s] = (ca \otimes_R b) \otimes_S ws$$

and

$$[(a \otimes_R b) \otimes_S s] \cdot (c \otimes_R w) = (a \otimes_R bc) \otimes_S sw$$

so that

$$\begin{aligned} \phi ((c \otimes_R w) \cdot [(a \otimes_R s) \otimes_S (b \otimes_R t)]) &= \phi ((ca \otimes_R ws) \otimes_S (b \otimes_R t)) \\ &= (ca \otimes_R b) \otimes_S wst = (c \otimes_R w) \cdot [(a \otimes_R b) \otimes_S st] \\ \phi ([(a \otimes_R s) \otimes_S (b \otimes_R t)] (c \otimes_R w)) &= \phi ((a \otimes_R s) \otimes_S (bc \otimes_R tw)) \\ &= (a \otimes_R bc) \otimes_S stw = [(a \otimes_R b) \otimes_S st] \cdot (c \otimes_R w) . \end{aligned}$$

Note that

$$(m_A \otimes_R S) \circ \phi = m_{A_{(S)}}$$

so that

$$m_{A_{(S)}} \circ \psi = m_A \otimes_R S \tag{13.4}$$

Let

$$\sigma : A \to A \otimes_R A$$

be an A-bimodule homomorphism which is a section of m_A. Then the map

$$\sigma \otimes_R S : A \otimes_R S = A_{(S)} \to (A \otimes_R A) \otimes_R S = (A \otimes_R A)_{(S)}$$

is clearly a section of $m_A \otimes_R S$ which is an $A_{(S)}$-bimodule homomorphism. In fact, for every $a \in A$ and $s \in S$, we have

$$\begin{aligned}(\sigma \otimes_R S)(a \otimes_R s) &= \sigma(a) \otimes_R s = a\sigma(1_R) \otimes_R s = (a \otimes s)(\sigma(1_R) \otimes_R 1_S)\\ (\sigma \otimes_R S)(a \otimes_R s) &= \sigma(a) \otimes_R s = \sigma(1_R)\, a \otimes_R s = (\sigma(1_R) \otimes_R 1_S)(a \otimes s).\end{aligned}$$

Then the map

$$\psi \circ (\sigma \otimes_R S) : A \otimes_R S = A_{(S)} \to (A \otimes_R S) \otimes_S (A \otimes_R S) = A_{(S)} \otimes_S A_{(S)}$$

is an $A_{(S)}$-bimodule homomorphism which is a section of $m_{A_{(S)}}$. In fact, in view of (13.4) we have

$$m_{A_{(S)}} \circ \psi \circ (\sigma \otimes_R S) = (m_A \otimes S) \circ (\sigma \otimes_R S) = A \otimes_R S.$$

Conversely, assume that $\theta : A_{(S)} \to (A \otimes_R S) \otimes_S (A \otimes_R S) = A_{(S)} \otimes_S A_{(S)}$ is $A_{(S)}$-bimodule homomorphism which is a section of $m_{A_{(S)}}$. Then we have

$$\begin{aligned}& m_A \circ r_{A \otimes_R A} \circ [(A \otimes_R A) \otimes_R \pi] \circ \phi \circ \theta \circ (A \otimes_R \iota)\, r_A^{-1}\\ =\; & r_A (m_A \otimes_R R)[(A \otimes_R A) \otimes_R \pi]\, \phi \circ \theta \circ (A \otimes_R \iota)\, r_A^{-1}\\ =\; & r_A [A \otimes_R \pi](m_A \otimes_R S)\, \phi \circ \theta \circ (A \otimes_R \iota)\, r_A^{-1}\\ =\; & r_A [A \otimes_R \pi] \circ m_{A_{(S)}} \circ \theta \circ (A \otimes_R \iota)\, r_A^{-1}\\ =\; & r_A [A \otimes_R \pi] \circ \mathrm{Id}_{A_{(S)}} (A \otimes_R \iota)\, r_A^{-1} = \mathrm{Id}_A\end{aligned}$$

where $r_A : A \otimes_R R \to A$ and $r_{A \otimes_R A} : A \otimes_R A \otimes_R R \to A \otimes_R A$ are the usual isomorphisms. Thus

$$\sigma = r_{A \otimes_R A} \circ [(A \otimes_R A) \otimes_R \pi] \circ \phi \circ \theta \circ (A \otimes \iota)\, r_A^{-1}$$

is a section of m_A. The proof that σ is an A-bimodule isomorphism is straightforward and is left as an exercise to the reader. □

Proposition 13.19. *Let A_1 and A_2 be algebras over a commutative ring R. Then*

A_1 and A_2 are separable R-algebras $\Leftrightarrow$ $A_1 \times A_2$ is a separable R-algebra.

Proof. " $\Rightarrow$ " Let $i_1 : A_1 \to A_1 \times A_2$ and let $i_2 : A_2 \to A_1 \times A_2$ the usual injective R-module homomorphisms and let us consider the codiagonal map

$$\theta = \nabla((i_1 \otimes_R i_1), (i_2 \otimes_R i_2)) : (A_1 \otimes_R A_1) \times (A_2 \otimes_R A_2) \to (A_1 \times A_2) \otimes_R (A_1 \times A_2).$$

We have

$$\theta((a_1 \otimes_R b_1), (a_2 \otimes_R b_2)) = [(a_1, 0_{A_2}) \otimes_R (b_1, 0_{A_2})] + [(0_{A_1}, a_2) \otimes_R (0_{A_1}, b_2)]$$

$$\begin{aligned}
& \left(m_{A_1\times A_2}\circ\theta\right)\left(\left(a_1\otimes_R b_1\right),\left(a_2\otimes_R b_2\right)\right) \\
= & \; m_{A_1\times A_2}\left(\left[\left(a_1,0_{A_2}\right)\otimes_R\left(b_1,0_{A_2}\right)\right]+\left[\left(0_{A_1},a_2\right)\otimes_R\left(0_{A_1},b_2\right)\right]\right) \\
= & \; \left(a_1b_1,0_{A_2}\right)+\left(0_{A_1},a_2b_2\right)=\left(a_1b_1,a_2b_2\right) \\
= & \; \left(m_{A_1}\times m_{A_2}\right)\left(\left(a_1\otimes_R b_1\right),\left(a_2\otimes_R b_2\right)\right)
\end{aligned}$$

so that

$$m_{A_1\times A_2}\circ\theta=m_{A_1}\times m_{A_2}$$

Let σ_1 be an A_1-bimodule sections of m_{A_1} and let σ_2 be an A_2-module section of m_{A_2}. It follows that

$$m_{A_1\times A_2}\circ\theta\circ\left(\sigma_1\times\sigma_2\right)=\left(m_{A_1}\times m_{A_2}\right)\circ\left(\sigma_1\times\sigma_2\right)=\mathrm{Id}_{A_1\times A_2}$$

and hence $\theta\circ\left(\sigma_1\times\sigma_2\right)$ is a section of $m_{A_1\times A_2}$. Let us prove that $\theta\circ\left(\sigma_1\times\sigma_2\right)$ is an $A_1\times A_2$-bimodule homomophism. From

$$\begin{aligned}
\theta\left(\left(\alpha_1a_1\otimes_R b_1,\alpha_2a_2\otimes_R b_2\right)\right) &= \left(\alpha_1a_1,0_{A_2}\right)\otimes_R\left(b_1,0_{A_2}\right)+\left(0_{A_1},\alpha_2a_2\right)\otimes_R\left(0_{A_1},b_2\right) \\
&= \left(\alpha_1,\alpha_2\right)\left(\left[\left(a_1,0_{A_2}\right)\otimes_R\left(b_1,0_{A_2}\right)\right]+\left[\left(0_{A_1},a_2\right)\otimes_R\left(0_{A_1},b_2\right)\right]\right) \\
&= \left(\alpha_1,\alpha_2\right)\theta\left(\left(a_1\otimes_R b_1\right),\left(a_2\otimes_R b_2\right)\right)
\end{aligned}$$

we deduce that θ is a left $A_1\times A_2$-module homomorphism. An analogous result on the right gives us that θ is in fact an $A_1\times A_2$-bimodule homorphism. Since we have

$$\begin{aligned}
\left(\sigma_1\times\sigma_2\right)\left(\left(a_1b_1,a_2b_2\right)\right) &= \left(\sigma_1\left(a_1b_1\right),\sigma_2\left(a_2b_2\right)\right)= \\
&= \left(a_1\sigma_1\left(b_1\right),a_2\sigma_2\left(b_2\right)\right)= \\
&= \left(a_1,a_2\right)\left(\sigma_1\left(b_1\right),\sigma_2\left(b_2\right)\right)
\end{aligned}$$

and similarly on the right side, we also conclude that $\sigma_1\times\sigma_2$ is an $A_1\times A_2$-bimodule homorphism.

" $\Leftarrow$ " It follows by applying Lemma 13.13. □

Lemma 13.20. *Let k be an algebraically closed field and assume that, for some $n\in\mathbb{N},n\geq1$, $k\subseteq Z\left(M_n\left(D\right)\right)$ where D is a ring with no zerodivisor.*
If $\dim_k M_n\left(D\right)<\infty$ then $k\simeq D$

Proof. Let $\sum_{i,j}a_{i,j}e_{i,j}\in Z\left(M_n\left(D\right)\right)$ and let $1\leq t,s\leq n$. Then from

$$e_{s,s}\left(\sum_{i,j}a_{i,j}e_{i,j}\right)e_{t,t}=a_{s,t}e_{ts}\qquad e_{t,t}\left(\sum_{i,j}a_{i,j}e_{i,j}\right)e_{s,s}=a_{t,s}e_{s,t}$$

we deduce that $a_{s,t}e_{ts}=a_{t,s}e_{s,t}$ for every t,s so that

$$a_{s,t}=0\text{ for }t\neq s\text{ and }a_{t,t}=a_{s,s}\text{ for }t=s.$$

so that $Z(M_n(D)) \subseteq D(\sum_t e_{t,t}) \cap Z(M_n(D)) = D1_{M_n(D)} \cap Z(M_n(D)) \subseteq Z\left(D1_{M_n(D)}\right)$. Therefore, via isomorphisms, we have that $k \subseteq Z(D)$ and $\dim_k D \leq \dim_k M_n(D) < \infty$ so that any element of D is algebraic over k. Thus let $a \in D$ and let $p(X) \in k[X]$ be a nonzero polynomial such that $p(a) = 0$. Since k is algebraically closed, there exists $\alpha_1, \ldots, \alpha_n \in k$ such that

$$p = \prod_{i=1}^{n} (X - \alpha_i).$$

Hence $0 = p(a) = \prod\limits_{i=1}^{n} (a - \alpha_i)$. Since D contains no zerodivisor, we get that there exists an i such that $a = \alpha_i \in k$. Thus we obtain that $k = D$. □

Lemma 13.21. *Let A be finite dimensional algebra over an algebraically closed field k. If $J(A) = \{0\}$ then A is separable over k.*

Proof. By Corollary 13.5, we get that A is semisimple. Then, by Wedderburn-Artin Theorem, we obtain that A is a direct product of rings, each isomorphic to a finite matrix ring $M_n(D)$ over a division ring D:

$$A \cong M_{n_1}(D_1) \times \ldots \times M_{n_t}(D_t)$$

The natural embedding of k in $Z(A)$ gives rise to the embeddings of k in $Z(M_{n_i}(D_i))$ for each $i = 1, \ldots, n$. Since $\dim_k A < \infty$ we have that $\dim_k M_n(D_i) < \infty$ and hence, by Lemma 13.20, we get that each D_i is isomorphic to k so that

$$A \cong M_{n_1}(k) \times \ldots \times M_{n_t}(k).$$

By Proposition 13.14 each $M_{n_i}(k)$ is separable over k. In view of Proposition 13.19, we conclude. □

Proposition 13.22. *Let A be an algebra over a field k. Then*

A separable over k $\Leftrightarrow$ $\dim_k(A) < \infty$ and A is classically separable over k.

Proof. ($\Rightarrow$) By Proposition 13.16, we already know that $\dim_k(A) < \infty$. Let now L be a field extension of k. Then, by Proposition 13.18, $A_{(L)}$ is a separable L-algebra and hence it is semisimple by Proposition 13.17. Then, in view of Proposition 13.7, A is classically separable.

($\Leftarrow$) Let L be an algebraic closure of k. Then $A_{(L)} = A \otimes_k L$ has finite dimension over L and hence it is left (and right) artinian. Moreover, since A is classically separable over k, we know that $J\left(A_{(L)}\right) = 0$. Hence, by Lemma 13.21, $A_{(L)}$ is separable over L. Thus, by Proposition 13.18 we conclude. □

Proposition 13.23. *Let k be a field and let H be a Hopf algebra over k. Then the following statements are equivalent:*

(*a*) *H is separable.*

(*b*) *H is semisimple.*

(*c*) *There exists a left integral t in H such that $\varepsilon_H(t) = 1$.*

Moreover, if one of these conditions hold, then $\dim_k(H) < \infty$.

Proof. $(a) \Rightarrow (b)$ is Proposition 13.17.
$(b) \Rightarrow (c)$. Since

$$\varepsilon_H : H \to k$$

is an algebra homomorphism, k becomes a left H-module via ε_H and it results that ε_H is a morphism of left H-modules. Since H is semisimple, the module ${}_H k$ is projective so that, as ε_H is surjective, there exists a left H-module homomorphism $\tau : k \to H$ which is a section of ε_H.

Let

$$t = \tau(1_k).$$

Then we have

$$\varepsilon_H(t) = \varepsilon_H(\tau(1_k)) = (\varepsilon_H \circ \tau)(1_k) = 1_k.$$

Also, for every $h \in H$ we have

$$h \cdot t = h \cdot \tau(1_k) = \tau(h \cdot 1_k) = \tau(\varepsilon_H(h) \cdot 1_k) = \varepsilon_H(h) \cdot \tau(1_k) = \varepsilon_H(h) \cdot t$$

and hence t is a left integral in H.
$(c) \Rightarrow (a)$. Let $t \in H$ be a left integral such that $\varepsilon_H(t) = 1$. Let us prove that

$$e = \sum t_{(1)} \otimes S\left(t_{(2)}\right)$$

is a separability element for H over k. We have

$$\sum t_{(1)} S\left(t_{(2)}\right) = \varepsilon_H(t) 1_H = 1_k 1_H = 1_H$$

so that e fulfills (13.1). Let $h \in H$. We have

$$\begin{aligned}
he &= \sum h t_{(1)} \otimes S\left(t_{(2)}\right) = \\
&= \sum h_{(1)} t_{(1)} \otimes S\left(t_{(2)}\right) \varepsilon_H\left(h_{(2)}\right) \\
&= \sum h_{(1)} t_{(1)} \otimes S\left(t_{(2)}\right) S\left(h_{(2)}\right) h_{(3)} \\
&= \sum h_{(1)} t_{(1)} \otimes S\left(h_{(2)} t_{(2)}\right) h_{(3)} \\
&= \sum \left[(\mathrm{Id}_H \otimes S) \circ \Delta\right]\left(h_{(1)} t\right) \cdot \left(1 \otimes h_{(2)}\right) \\
&= \left(\sum \left[(\mathrm{Id}_H \otimes S) \circ \Delta\right]\left(\varepsilon_H\left(h_{(1)}\right) t\right)\right) \cdot \left(1 \otimes h_{(2)}\right) \\
&= \left(\left[(\mathrm{Id}_H \otimes S) \circ \Delta\right](t)\right) \cdot \left(1 \otimes \sum \varepsilon_H\left(h_{(1)}\right) h_{(2)}\right) \\
&= \left[\sum t_{(1)} \otimes S\left(t_{(2)}\right)\right](1 \otimes h) \\
&= \sum t_{(1)} \otimes S\left(t_{(2)}\right) h = eh
\end{aligned}$$

so that e also fulfills (13.2).
The last assertion follows by Proposition 13.16. □

Theorem 13.24. *Let $\pi : E \to B$ be a surjective morphism of algebras over a field k, and let $f : A \to B$ be an algebra homomorphism. If A is separable and $\ker(\pi)^2 = \{0\}$, then*

$$h = \sigma f + m_E(\sigma f \otimes \sigma f)\nu - m_E(E \otimes m_E)(\sigma f \otimes \sigma f \otimes \sigma f)(\nu \otimes A)(u_A \otimes A)l_A^{-1} : A \to E$$

defines a morphism of algebras such that $\pi \circ h = f$. Here $\sigma : B \to E$ is a k-linear map such that $\pi \circ \sigma = \mathrm{Id}_B$ and $\sigma(1_B) = 1_E$ and $\nu : A \otimes A \to A$ is a morphism of A-bimodules such that $m_A \circ \nu = \mathrm{Id}_A$.

Proof. Let us set

$$\eta = m_E(\sigma f \otimes \sigma f)\nu - m_E(E \otimes m_E)(\sigma f \otimes \sigma f \otimes \sigma f)(\nu \otimes A)(u_A \otimes A)l_A^{-1}$$

and let

$$\nu(1_A) = \sum_{i=1}^{n} x_i \otimes y_i \text{ where } n \in \mathbb{N}, n \geq 1 \text{ and } x_i, y_i \in A \text{ for every } i = 1, \ldots, n$$

be a separability element of A over k. Then, for every $a \in A$, we have

$$\eta(a) = \sum \sigma f(x_i)\,\sigma f(y_i \cdot a) - \sum \sigma f(x_i)\,\sigma f(y_i)\,\sigma f(a)$$

and hence

$$\begin{aligned}\pi\eta(a) &= \sum \pi\sigma f(x_i)\,\pi\sigma f(y_i a) - \sum \pi\sigma f(x_i)\,\pi\sigma f(y_i)\,\pi\sigma f(a) = \\ &= \sum f(x_i)\,f(y_i a) - \sum f(x_i)\,f(y_i)\,f(a) = 0\end{aligned}$$

so that $\pi\eta = 0$. Now

$$h = \sigma f + \eta$$

and so

$$\pi h = \pi\sigma f + \pi\eta = f.$$

Let us prove that h is an algebra morphism. We have

$$h(1_A) = \sigma f(1_A) + \eta(1_A) = \sigma(1_B) + 0 = 1_E.$$

so that h is unital. Moreover we have

$$h(a) = \sigma f(a) + \eta(a)$$

so that, for every $a, b \in A$ we get,

$$\begin{gathered}h(a)\,h(b) = (\sigma f(a) + \eta(a)) \cdot_B (\sigma f(b) + \eta(b)) \\ = \sigma f(a) \cdot_B \sigma f(b) + \sigma f(a) \cdot_B \eta(b) + \eta(a) \cdot_B \sigma f(b) + \eta(a) \cdot_B \eta(b) \\ = \sigma f(a)\,\sigma f(b) + \sigma f(a)\left[\sum \sigma f(x_i)\,\sigma f(y_i \cdot b)\right] \\ -\sigma f(a)\left[\sum \sigma f(x_i)\,\sigma f(y_i)\,\sigma f(b)\right] \\ + \left[\sum \sigma f(x_i)\,\sigma f(y_i \cdot a)\right]\sigma f(b) - \left[\sum \sigma f(x_i)\,\sigma f(y_i)\,\sigma f(a)\right]\sigma f(b)\end{gathered}$$

and

$$h(ab) = \sigma f(ab) + \sum \sigma f(x_i) \sigma f(y_i \cdot ab) - \sum \sigma f(x_i) \sigma f(y_i) \sigma f(ab).$$

Since

$$\sum x_i \otimes y_i a = \sum a x_i \otimes y_i$$

we also get

$$\sum x_i \otimes y_i a \otimes b = \sum a x_i \otimes y_i \otimes b$$

and hence

$$\sum x_i \otimes y_i ab = \sum a x_i \otimes y_i b.$$

Therefore we obtain both

$$\begin{aligned} \sum \sigma f(x_i) \sigma f(y_i ab) &= \sum \sigma f(a x_i) \sigma f(y_i b) \text{ and} \\ \sum \sigma f(x_i) \sigma f(y_i a) &= \sum \sigma f(a x_i) \sigma f(y_i). \end{aligned}$$

Using these equalities and keeping in mind that $Ker(\pi)^2 = \{0\}$, we obtain

$$\begin{gathered} h(ab) - h(a) h(b) = \sigma f(ab) + \sum \sigma f(x_i) \sigma f(y_i ab) - \sum \sigma f(x_i) \sigma f(y_i) \sigma f(ab) \\ -\sigma f(a) \sigma f(b) - \sigma f(a) \left[\sum \sigma f(x_i) \sigma f(y_i \cdot b)\right] + \sigma f(a) \sum \sigma f(x_i) \sigma f(y_i) \sigma f(b) \\ - \sum \sigma f(x_i) \sigma f(y_i a) \sigma f(b) + \sum \sigma f(x_i) \sigma f(y_i) \sigma f(a) \sigma f(b) \\ = \sum [\sigma f(a x_i) - \sigma f(a) \sigma f(x_i)] \sigma f(y_i b) \\ - \sum [\sigma f(a x_i) - \sigma f(a) \sigma f(x_i)] [\sigma f(y_i) \sigma f(b)] \\ + \left[1 - \sum \sigma f(x_i) \sigma f(y_i)\right] \sigma f(ab) - \left[1 - \sum \sigma f(x_i) \sigma f(y_i)\right] \sigma f(a) \sigma f(b) \\ = \sum [\sigma f(a x_i) - \sigma f(a) \sigma f(x_i)] [\sigma f(y_i b) - \sigma f(y_i) \sigma f(b)] \\ + \left[1 - \sum \sigma f(x_i) \sigma f(y_i)\right] [\sigma f(ab) - \sigma f(a) \sigma f(b)] = 0. \end{gathered}$$

Thus h is an algebra homomorphism. □

Theorem 13.25 (Wedderburn Principal Theorem)**.** *Let T be a separable algebra over a field k and let*

$$f : R \longrightarrow T$$

be a surjective k-algebra morphism such that $\ker(f)$ is nilpotent. Then there exists a k-algebra homomorphism

$$\theta : T \longrightarrow R$$

such that $f \circ \theta = \mathbf{1}_T$ i.e. f has a section which is a k-algebra homomorphism.

Proof. Let $L = \ker(f)$. Assume that, for $n \in \mathbb{N}, n \geq 1$, we have that $L^n = \{0\}$. For every $i = 1, \ldots, n$ set $B_i = R/L^i$ and, for every $i = 1, \ldots, n-1$ let $\pi_i : R/L^{i+1} \to R/L^i$ be the canonical projection. Let $p : R \to R/L$ be the canonical projection and let $\overline{f} : R/L \to T$ be the unique algebra homomorphism such that $\overline{f} \circ p = f$. Since f is surjective, $\overline{f}$ is an isomorphism: let $g : T \to R/L$ be its inverse. Then $g \circ f = p$. By applying Theorem 13.24 to $A = T, E = R/L^2, B = R/L$, $\pi = \pi_1$ and $f = g$ we get that there exists an algebra morphism $h_1 : T \to R/L^2$ such that $\pi_1 \circ h_1 = g$.

Then, by applying Theorem 13.24 to $A = T, E = R/L^3, B = R/L^2$, $\pi = \pi_2$ and $f = h_1$ we get that there exists an algebra morphism $h_2 : T \to R/L^3$ such that $\pi_2 \circ h_2 = h_1$. Assume now that $h_i : T \to R/L^{i+1}$ is an algebra morphism such that $\pi_i \circ h_i = h_{i-1}$. Then we can apply again Theorem 13.24 to $A = T, E = R/L^{i+2}, B = R/L^{i+1}$, $\pi = \pi_{i+1}$ and $f = h_i$ we get that there exists an algebra morphism $h_{i+1} : T \to R/L^{i+1}$ such that $\pi_{i+1} \circ h_{i+1} = h_i$. Let $\chi : R \to R/L^n$ be the obvious isomorphism and let $\theta = \chi^{-1} \circ h_{n-1}$. Then we have

$$\begin{aligned} p \circ \theta &= (\pi_1 \pi_2 \cdots \pi_{n-1} \circ \chi) \circ \chi^{-1} \circ h_{n-1} = (\pi_1 \pi_2 \cdots \pi_{n-1}) \circ h_{n-1} \\ &= (\pi_1 \pi_2 \cdots \pi_{n-2}) \circ h_{n-2} = \ldots = \pi_1 \circ h_1 = g \end{aligned}$$

and hence

$$\overline{f} \circ p \circ \theta = \mathrm{Id}_T$$

which means that

$$f \circ \theta = \mathbf{1}_T.$$

□

Chapter 14

TAFT-WILSON Theorem

Definition 14.1. *A k-coalgebra C is said to have a* separable coradical *if, for every simple subcoalgebra D of C, D^* is a separable k-algebra.*

Lemma 14.2. *Let k be an algebraically closed field. Then any k-coalgebra C has a* separable coradical.

Proof. Let $D \subseteq C$ be a simple subcoalgebra. then, by Corollary 9.18, D^* is a finite dimensional simple algebra. Quindi, per la Proposizione 13.4, $Jac(D^*) = \{0\}$ so that, since k is algebraically closed, by Lemma 13.21, D^* is separable over k. □

Lemma 14.3. *Let C be a pointed k-coalgebra. Then C has separable coradical..*

Proof. Since C is pointed, every simple subcoalgebra of C is of the form kg where $g \in G(C)$ and hence $(kg)^*$ is a k-algebra isomorphic to k. □

Lemma 14.4. *Let C be a finite dimensional k-coalgebra. The following statements are equivalent*

(a) C has separable coradical.

(b) $(C_0)^*$ is a separable k-algebra.

Proof. By Proposition 9.28
$$C_0 = \bigoplus_{D \in \mathcal{D}} D$$
where $\mathcal{D}$ is the set of all simple subcoalgebras of C.Moreover, since $dim_k(C) < \infty$, we have that $\mathcal{D}$ is finite. Then we have a ring isomorphism
$$(C_0)^* \simeq \prod_{D \in \mathcal{D}} D^*.$$
By Proposition 13.19, $(C_0)^*$ is separable over k if and only if, for every $D \in \mathcal{D}$, each D^* is separable over k. □

Definition 14.5. *Let D be a subcoalgebra of a k-coalgebra C and let $i : D \to C$ be the canonical injection. A coalgebra morphism*

$$\pi : C \longrightarrow D$$

is called a (coalgebra) projection of C onto D *if* $\pi \circ i = \mathrm{Id}_D$.

Lemma 14.6. *Let C be a finite dimensional k-coalgebra with separable coradical and let D be a subcoalgebra of C. Then any projection π from D to D_0 can be extended to a projection of C onto C_0.*

Proof. Let $i_{C_0}^C : C_0 \longrightarrow C$ be the canonical injection and let $\pi' : C \longrightarrow C/\mathrm{Ker}(\pi) = E$ be the canonical projection. Set $\alpha = \pi' \circ i_{C_0}^C$. Since $\pi : D \longrightarrow D_0$ is a coalgebra morphism, $\mathrm{Ker}\,(\pi)$ is a coideal of D. Since

$$\begin{aligned} \Delta_C Ker(\pi) &= \Delta_D Ker(\pi) \subseteq Ker(\pi) \otimes D + D \otimes Ker(\pi) \\ &\subseteq Ker(\pi) \otimes C + C \otimes Ker(\pi) \end{aligned}$$

and

$$\varepsilon_C(Ker(\pi)) = \varepsilon_D(Ker(\pi)) = 0.$$

$\mathrm{Ker}\,(\pi)$ is a coideal also of C and hence π' and also α are coalgebra morphism. Now we have

$$Ker(\alpha) = C_0 \cap Ker(\pi') = C_0 \cap Ker(\pi)$$
$$= C_0 \cap D \cap Ker(\pi) \overset{\text{Lem9.40.}}{=} D_0 \cap Ker(\pi) = \{0\}$$

so that α is injective and hence the dual morphism

$$\alpha^* : E^* \longrightarrow (C_0)^*$$

is surjective. Let $(T_i)_{i \in I}$ be the family of simple subcoalgebras of C. Since α is injective, each $\alpha(T_i)$ is a simple subcoalgebra of E and hence

$$\alpha(C_0) = \sum_{i \in I} \alpha(T_i) \subseteq E_0.$$

On the other hand, by Corollary 11.7, we have that $E_0 \subseteq \pi'(C_0) = \left(\pi' \circ i_{C_0}^C\right)(C_0) = \alpha(C_0)$ and thus we deduce that $E_0 = \alpha(C_0) = \mathrm{Im}\,(\alpha)$. Therefore we have

$$\begin{aligned} Ker(\alpha^*) &= \{\eta \in E^* : | : \alpha^*(\eta) = 0\} \\ &= \{\eta \in E^* : | : \eta \circ \alpha = 0\} \\ &= \{\eta \in E^* : | : \eta(\mathrm{Im}(\alpha)) = 0\} \\ &= \{\eta \in E^* : | : \eta(E_0) = 0\} \\ &= E_0^{\perp}\ . \end{aligned}$$

and hence we have the exact sequence

$$0 \to E_0^{\perp} = Ker(\alpha^*) \longrightarrow E^* \xrightarrow{\alpha^*} (C_0)^* \to 0\ .$$

Since C is finite dimensional and has separable coradical, we deduce from Lemma 14.4 that$(C_0)^*$ is a separable k-algebra. On the other hand E is finite dimensional and hence $Jac(E^*)$ is a nilpotent two-sided ideal of E^*. Moreover, by Proposition 9.41 we know that $Jac(E^*) = E_0^{\perp}$. Therefore we get that $\alpha^* : E^* \longrightarrow C_0^*$ is a surjective algebra morphism and $Ker(\alpha^*) = E_0^{\perp} = Jac(E^*)$ is nilpotent. Thus we can apply Wedderburn Principal Theorem 13.25 and deduce that there exists an algebra morphism $\beta : C_0^* \longrightarrow E^*$ such that $\alpha^* \circ \beta = \mathrm{Id}_{C_0^*}$. Since C_0 and E are finite dimensional, there exists a coalgebra morphism $\pi^{''} : E \longrightarrow C_0$ such that $\beta = \pi^{''*}$ and we have

$$(\pi^{''} \circ \alpha)^* = \alpha^* \circ \pi^{''*} = \alpha^* \circ \beta = \mathrm{Id}_{C_0^*} = (\mathrm{Id}_{C_0})^* .$$

Therefore we obtain that

$$\mathrm{Id}_{C_0} = \pi^{''} \circ \alpha = \pi^{''} \circ \pi^{'} \circ i_{C_0}^{C}$$

i.e. the map

$$C \xrightarrow{\pi^{'}} E \xrightarrow{\pi^{''}} C_0 \to 0$$

is a projection of C onto C_0. Let us prove that $\tilde{\pi} = \pi^{''} \circ \pi^{'}$ extends $\pi : D \to D_0$. Let

$i_D^C : D \to C$, $\quad i_{D_0}^D : D_0 \longrightarrow D$ and $i_{D_0}^{C_0} : D_0 \longrightarrow C_0$ be the canonical injection

and let

$$\begin{array}{lcl} j & : & D/Ker(\pi) \hookrightarrow C/Ker(\pi) \quad \text{be the canonical injection and} \\ p & : & D \longrightarrow D/Ker(\pi) \text{ be the canonical projection.} \end{array}$$

Let $\tau : D/Ker(\pi) \longrightarrow D_0$ be the unique morphism such that $\tau \circ p = \pi$. As π is surjective, τ is an isomorphism. Since $j \circ p = \pi' \circ i_D^C$, we have that

$$\begin{aligned} \alpha \circ i_{D_0}^{C_0} &= \pi^{'} \circ i_{C_0}^{C} \circ i_{D_0}^{C_0} = \pi^{'} \circ i_D^C \circ i_{D_0}^D = j \circ p \circ i_{D_0}^D \\ &= j \circ \tau^{-1} \circ \tau \circ p \circ i_{D_0}^D = j \circ \tau^{-1} \circ \pi \circ i_{D_0}^D = j \circ \tau^{-1} \circ \mathrm{Id}_{D_0} = j \circ \tau^{-1}. \end{aligned}$$

and hence

$$\tilde{\pi} \circ i_D^C = \pi^{''} \circ \pi^{'} \circ i_D^C = \pi^{''} \circ j \circ p = \pi^{''} \circ j \circ \tau^{-1} \circ \tau \circ p = \pi^{''} \circ j \circ \tau^{-1} \circ \pi$$
$$= \pi^{''} \circ j \circ \tau^{-1} \circ \tau \circ p = \pi^{''} \circ j \circ \tau^{-1} \circ \pi = \pi^{''} \circ \alpha \circ i_{D_0}^{C_0} \circ \pi = \mathrm{Id}_{C_0} \circ i_{D_0}^{C_0} \circ \pi = i_{D_0}^{C_0} \circ \pi.$$

□

Theorem 14.7. *Let C be a k-coalgebra with separable coradical. Then there exists a coalgebra projection of C onto C_0.*

Proof. Let

$$\mathcal{F} = \{(F, \pi) \mid F \text{ is a subcoalgebra of } C \text{ and } \pi : F \longrightarrow F_0 \text{ is a coalgebra projection}\}.$$

Since $(C_0, \mathrm{Id}_{C_0}) \in \mathcal{F}$ we have that $\mathcal{F} \neq \emptyset$. Let us consider the partial order on $\mathcal{F}$ defined by setting

$$(F^{'}, \pi^{'}) \leq (F, \pi) \Longleftrightarrow F^{'} \subset F \text{ and } \pi_{|F'} = \pi \ .$$

It is easy to show that $(\mathcal{F}, \leq)$ is inductive. Hence, by applying Zorn's Lemma to $(\mathcal{F}, \leq)$ we obtain that there exists a maximal element (F, π) in $(\mathcal{F}, \leq)$. Let us assume that $F \subsetneqq C$ and let $c \in C$, $c \notin F$. Let L be the subcoalgebra of C generated by c and let

$$D = L + \pi(L \cap F).$$

Since L is finite dimensional also D is finite dimensional and since $\pi(L \cap F) \subseteq F_0 \subseteq F$, we get that

$$D \cap F = [L + \pi(L \cap F)] \cap F = (L \cap F) + \pi(L \cap F).$$

Now let $x \in \pi(F) = F_0$. Then we have that $x = \mathrm{Id}_{F_0}(x) = \left(\pi \circ i_{F_0}^{F}\right)(x) = \pi(x)$ where $i_{F_0}^{F} : F_0 \to F$ is the canonical inclusion. Therefore we have

$$X = \pi(X) \text{ for every subset } X \subseteq \pi(F) = F_0$$

In particular we have that

$$\pi(\pi(L \cap F)) = \pi(L \cap F)$$

and

$$D \cap F_0 = \pi(D \cap F_0) \subseteq \pi(D \cap F).$$

Therefore we obtain

$$\pi(D \cap F) = \pi(L \cap F) + \pi(\pi(L \cap F)) = \pi(L \cap F) \subseteq D \cap F_0 \overset{\text{Lem9.40}}{=} (D \cap F)_0 \subseteq \pi(D \cap F)$$

so that

$$\pi(D \cap F) \doteq (D \cap F)_0. \tag{14.1}$$

Let $\pi^{'}$ be the corestriction to $(D \cap F)_0$ of the restriction of π to $D \cap F$. Then, by (14.1), π' is a projection of $D \cap F$ onto $(D \cap F)_0$. Thus, being D finite dimensional, we can apply Lemma 14.6 and deduce that π' extends to a coalgebra projection

$$\pi_1 : D \longrightarrow D_0.$$

Let $\gamma : D + F \longrightarrow D_0 + F_0$ be the map defined by setting

$$\gamma(d + f) = \pi_1(d) + \pi(f) \qquad \text{for every } d \in D \text{ and } f \in F.$$

Note tha γ is well defined since $\pi_{1|D\cap F} = \pi' = \pi_{|D\cap F}$ and it is a coalgebra morphism. Let $d \in D_0$ and $f \in F_0$. We have

$$\gamma(d + f) = \pi_1(d) + \pi(f) = d + f$$

and hence $\gamma \circ i_{D_0+F_0}^{D+F} = \mathrm{Id}_{D_0+F_0}$ so that γ is a projection of $D + F$ onto $D_0 + F_0 = (F + D)_0$ in view of Proposition 9.29.Contradiction. □

Lemma 14.8. *Let $f : C \to D$ be a surjective morphism of k-coalgebras and let W_1,W_2 be subspaces of C such that $Ker(f) \subseteq W_1 \cap W_2$. Then*

$$f(W_1 \wedge_C W_2) = f(W_1) \wedge_D f(W_2).$$

Proof. For every $i = 1, 2$ we have that $Ker(f) \subseteq W_i$ and hence there exists an isomorphism

$$f_i : C/W_i \longrightarrow f(C)/f(W_i) = D/f(W_i)$$

such that the diagram

$$\begin{array}{ccc} C & \xrightarrow{f} & f(C) = D \\ \pi_{W_i}^C \downarrow & & \downarrow \pi_{f(W_i)}^D \\ C/W_i & \xrightarrow{f_i} & f(C)/f(W_i) \end{array}$$

where $\pi_{W_i}$and $\pi_{f(W_i)}$ are the canonical projections, is commutative. We compute

$$\begin{aligned} f^{\leftarrow}[f(W_1) \wedge_D f(W_2)] &= f^{\leftarrow}\left(\mathrm{Ker}\left[\left(\pi_{f(W_1)}^D \otimes \pi_{f(W_2)}^D\right) \circ \Delta_D\right]\right) = \\ &= \mathrm{Ker}\left[\left(\pi_{f(W_1)}^D \otimes \pi_{f(W_2)}^D\right) \circ \Delta_D \circ f\right] \end{aligned}$$

Now we have

$$\begin{aligned} \left(\pi_{f(W_1)}^D \otimes \pi_{f(W_2)}^D\right) \circ \Delta_D \circ f &= \left(\pi_{f(W_1)}^D \otimes \pi_{f(W_2)}^D\right) \circ (f \otimes f) = \\ &= \left[\left(\pi_{f(W_1)}^D \circ f\right) \otimes \left(\pi_{f(W_2)}^D \circ f\right)\right] \circ \Delta_C \\ = \left[\left(f_1 \circ \pi_{W_1}^C\right) \otimes \left(f_2 \circ \pi_{W_2}^C\right)\right] \circ \Delta_C &= (f_1 \otimes f_2) \circ \left(\pi_{W_1}^C \otimes \pi_{W_2}^C\right) \circ \Delta_C \end{aligned}$$

so that, since $f_1 \otimes f_2$ is bijective, we get

$$\begin{aligned} &\mathrm{Ker}\left[\left(\pi_{f(W_1)}^D \otimes \pi_{f(W_2)}^D\right) \circ \Delta_D \circ f\right] = \\ &= \mathrm{Ker}\left[(f_1 \otimes f_2) \circ \left(\pi_{W_1}^C \otimes \pi_{W_2}^C\right) \circ \Delta_C\right] \\ &= \mathrm{Ker}\left[\left(\pi_{W_1}^C \otimes \pi_{W_2}^C\right) \circ \Delta_C\right] = W_1 \wedge_C W_2. \end{aligned}$$

Thus we obtain

$$f^{\leftarrow}[f(W_1) \wedge_D f(W_2)] = W_1 \wedge_C W_2$$

from which, since f is surjective, we infer that

$$f(W_1) \wedge_D f(W_2) = f\left(f^{\leftarrow}[f(W_1) \wedge_D f(W_2)]\right) = f(W_1 \wedge_C W_2).$$

□

Definition 14.9. *Let C be a k-coalgebra and let $C^+ = Ker(\varepsilon)$. Then $R = R(C) = C/C_0^+$ is called the* associated connected coalgebra of C.

Lemma 14.10. *Let D be a subcoalgebra of a k-coalgebra C and let I be a coideal of C. Then $I \cap D$ is a coideal of D and hence of C.*

Proof. Let $p : C \to C/I$ be the canonical projection and let $i_D : D \to C$ be the canonical injection. We have that $I \cap D = \mathrm{Ker}(p \circ i_D)$ is a coideal of D since $p \circ i_D$ is a coalgebra morphism. □

Lemma 14.11. *Let C be a k-coalgebra and let $\pi = \pi^C_{C_0^+} : C \to R(C) = C/C_0^+$ be the canonical projection. Then C_0^+ is a coideal of C so that $R(C)$ is a coalgebra. Moreover, for every $n \in \mathbb{N}$, we have that $R(C)_n = \pi(C_n)$. In particular $R(C)$ is connected.*

Proof. Since $\varepsilon : C \to k$ is a morphism of k-coalgebras (see 1.26), and since, by Proposition 9.23, C_0 is a subcoalgebra of C, in view of Lemma 14.10 $C_0^+ = C_0 \cap \mathrm{Ker}(\varepsilon_C) = \mathrm{Ker}(\varepsilon_{C_0})$ is a coideal of C and hence $R = R(C)$ is a coalgebra and π is a coalgebra morphism.

Since π is surjective, we can apply Proposition 11.7 to infer that $R_0 \subseteq \pi(C_0) = C_0/C_0^+ \simeq k$ and hence (note that $R \neq \{0\}$. Why?) $R_0 = \pi(C_0) \simeq k$ so that R is connected.

Now let us assume that $R_n = \pi(C_n)$ for some $n \in \mathbb{N}$ and let us prove it for $n+1$. Since π is surjective, we can apply Lemma 14.8 to get that

$$\pi(C_{n+1}) = \pi(C_0 \wedge_C C_n) = \pi(C_0) \wedge_R \pi(C_n) \overset{\text{Indhypo}}{=} R_0 \wedge R_n = R_{n+1}.$$

□

Theorem 14.12. *Let $f : C \to D$ be a morphism of k-coalgebras. If $f_{|C_1}$ is injective, then f is injective.*

Proof. Since $Ker(f) \cap C_1^+ = \{0\}$ it is enough to show that $N = \{0\}$ whenever N is a coideal of C such that $N \cap C_1^+ = \{0\}$. Let R be the associated connected coalgebra and let $\pi : C \to R = C/C_0^+$ be the canonical projection. We compute

$$\begin{aligned} R_1^+ &= R_1 \cap (Ker(\varepsilon_R)) = \pi(C_1) \cap (Ker(\varepsilon)/C_0^+) \\ &= (C_1/C_0^+) \cap (Ker(\varepsilon)/C_0^+) = (C_1 \cap Ker(\varepsilon))/C_0^+ \\ &= C_1^+/C_0^+ = \pi(C_1^+). \end{aligned}$$

Therefore we have

$$\begin{aligned} \pi(N) \cap R_1^+ &= \pi(N) \cap \pi(C_1^+) = \pi\left[\pi^{\leftarrow}\left(\pi(N) \cap \pi(C_1^+)\right)\right] \\ &= \pi\left[\pi^{\leftarrow}(\pi(N)) \cap \pi^{\leftarrow}\left(\pi(C_1^+)\right)\right] = \pi\left[\left(N + C_0^+\right) \cap \left(C_1^+ + C_0^+\right)\right] \\ &= \pi\left[\left(N + C_0^+\right) \cap C_1^+\right] \subseteq \pi\left(C_0^+\right) = \{0\} \end{aligned}$$

where the inclusion follows by the following: let $n \in N$, $x \in C_0^+$ such that $n + x = y \in C_1^+$ is an element of the intersection $\left(N + C_0^+\right) \cap C_1^+$. Then $n = y - x \in N \cap C_1^+ = \{0\}$, therefore $n = 0$ and thus $\left(N + C_0^+\right) \cap C_1^+ \subseteq C_0^+$. Now let

$$p : R \to R/\pi(N) = \pi(C)/\pi(N).$$

be the canonical projection. By Proposition 12.2, we know that $P(R) \subseteq R_1^+$ so that

$$\mathrm{Ker}\,(p) \cap P(R) \subseteq \mathrm{Ker}\,(p) \cap R_1^+ = \pi(N) \cap R_1^+ = \{0\}.$$

Since R is connected we can apply Lemma 12.5 and get that p is injective i.e. $\pi(N) = \{0\}$. This means that $N \subseteq Ker(\pi) = C_0^+ \subseteq C_1^+$ so that $N = N \cap C_1^+ = \{0\}$. □

Definition 14.13. *Let C be a k-coalgebra and let $g, h \in G(C)$ be grouplike elements. The set of g, h-*primitive elements *of C is the set*

$$P_{g,h}(C) = \{c \in C \mid \Delta(c) = c \otimes g + h \otimes c\}.$$

Lemma 14.14. *Let C be a k-coalgebra and let $g, h \in G(C)$ be grouplike elements. Then*

1) $\varepsilon(x) = 0$ *for every $x \in P_{g,h}(C)$.*

2) *We have*

$$k(g - h) \subseteq P_{g,h}(C) \cap P_{h,g}(C) \cap C_0.$$

3) *If C is pointed and $g \neq h$ we have*

$$P_{g,h}(C) \cap P_{h,g}(C) \cap C_0 \subseteq k(g - h).$$

Proof. 1) Let $x \in P_{g,h}(C)$. Then from $\Delta(x) = x \otimes g + h \otimes x$, we deduce that $x = \varepsilon(x)\, g + x$ and hence $\varepsilon(x) = 0$.

2) Since,

$$\Delta(g - h) = g \otimes g - h \otimes h = (g - h) \otimes g + h \otimes (g - h) = (g - h) \otimes h + g \otimes (g - h),$$

it is clear that $g - h \in P_{g,h}(C) \cap P_{h,g}(C) \cap C_0$.

3) Let $x \in P_{g,h}(C) \cap P_{h,g}(C) \cap C_0$. By Proposition 9.30 we have that $C_0 = kG(C)$ so that we can write

$$x = \lambda g + \mu h + v \qquad \text{where } \lambda, \mu \in k \text{ and } v \in \sum_{g_i \neq g, g_i \neq h} k g_i.$$

Since $x \in P_{g,h}$, we have that

$$\begin{aligned} \Delta(x) &= x \otimes g + h \otimes x = \lambda g \otimes g + \mu h \otimes g + v \otimes g + h \otimes \lambda g + h \otimes \mu h + h \otimes v. \\ &= (\mu + \lambda)(h \otimes g) + v \otimes g + h \otimes v + \lambda g \otimes g + h \otimes \mu h. \end{aligned}$$

Since $x \in P_{h,g}$, we also have

$$\begin{aligned}\Delta(x) &= x \otimes h + g \otimes x = \lambda g \otimes h + \mu h \otimes h + v \otimes h + g \otimes \lambda g + g \otimes \mu h + g \otimes v \\ &= (\mu + \lambda)(g \otimes h) + v \otimes h + g \otimes v + g \otimes \lambda g + \mu h \otimes h.\end{aligned}$$

and hence we obtain

$$(\mu + \lambda)(h \otimes g) + v \otimes g + h \otimes v = (\mu + \lambda)(g \otimes h) + v \otimes h + g \otimes v.$$

From this, we infer that

$$\mu + \lambda = 0 \quad \text{and} \quad v = 0.$$

Thus we obtain $x = \lambda(g - h)$. □

14.15. *Let C be a pointed k-coalgebra. In view of Lemma 14.3, we know that C has separable coradical. By Theorem 14.7, there exist a projection π of C onto C_0. Let $I = \mathrm{Ker}(\pi)$. Then $I \cap C_0 = \{0\}$ and $C = I + C_0$ so that*

$$C = I \oplus C_0.$$

For every $x \in G = G(C)$, we define $e_x \in C^*$ by setting:

$$e_x(I) = 0 \quad \text{and} \quad e_x(y) = \delta_{x,y} \quad \text{for every} \quad y \in G. \tag{14.2}$$

The family $(e_x)_{x \in G}$ is a family of pairwise orthogonal idempotents of C^*. Since I is a coideal of C we have that $I \subseteq \mathrm{Ker}(\varepsilon)$ and hence

$$\sum_{x \in G} e_x = \varepsilon.$$

For every $c \in C$ and $x, y \in G$ we set

$${}^x c = c \cdot e_x, \quad c^y = e_y \cdot c \quad \text{and} \quad {}^x c^y = ({}^x c)^y = {}^x(c^y),$$

and

$${}^x C^y = \{{}^x c^y | c \in C\}.$$

Note that I (and hence the ${}^x C^y$) are not unique since they are related to the projection that appears in Wedderburn Principal Theorem 13.25 which is not unique.

For every $g \in G$ we denote by L_g the left multiplication by e_g on C, and by R_g the right multiplication by e_g on C i.e.

$$L_g(c) = e_g \cdot c = \sum c_1 e_g(c_2) \qquad \text{and} \qquad R_g(c) = c \cdot e_g = \sum e_g(c_1) c_2 \text{ for every } c \in C.$$

For every $c \in C$, we have

$$\begin{aligned}(\Delta \circ L_g)(c) &= \Delta(e_g \cdot c) = \Delta\left(\sum c_1 e_g(c_2)\right) = \sum c_1 \otimes c_2 e_g(c_3) \\ &= \sum c_1 \otimes (e_g \cdot c_2) = [(C \otimes L_g) \circ \Delta](c)\end{aligned}$$

and

$$\begin{aligned}(\Delta \circ R_g)(c) &= \Delta(c \cdot e_g) = \Delta\left(\sum e_g(c_1)c_2\right) = \sum e_g(c_1)c_2 \otimes c_3 \\ &= \sum (c_1 \cdot e_g) \otimes c_2 = [(R_g \otimes C) \circ \Delta](c)\end{aligned}$$

so that we deduce that

$$(\Delta \circ L_g) = (C \otimes L_g) \circ \Delta \text{ and } \Delta \circ R_g = (R_g \otimes C) \circ \Delta \tag{14.3}$$

Now we have

$$\begin{aligned}[(L_g \otimes R_h) \circ \Delta](c) &= \sum (e_g \cdot c_1) \otimes (c_2 \cdot e_h) = \sum c_1 e_g(c_2) \otimes e_h(c_3) c_4 = \\ &= \sum c_1 \otimes (e_g(c_2) e_h(c_3) c_4) = \sum c_1 \otimes R_h(e_g(c_2) c_3) \\ &= \sum c_1 \otimes (R_h \circ R_g)(c_2) = \{[C \otimes (R_h \circ R_g)] \circ \Delta\}(c)\end{aligned}$$

so that we get

$$(L_g \otimes R_h) \circ \Delta = [C \otimes (R_h \circ R_g)] \circ \Delta \tag{14.4}$$

Now we compute

$$\begin{aligned}\sum_{z \in G} [(L_z \otimes R_z) \circ \Delta] &\overset{(14.4)}{=} \sum_{z \in G} \{[C \otimes (R_z \circ R_z)] \circ \Delta\} = \left(\sum_{z \in G} [C \otimes (R_z \circ R_z)]\right) \circ \Delta = \\ &= \left[\sum_{z \in G} C \otimes R_z\right] \circ \Delta = \left(C \otimes \sum_{z \in G} R_z\right) \circ \Delta = \Delta\end{aligned}$$

hence we get

$$\sum_{z \in G} [(L_z \otimes R_z) \circ \Delta] = \Delta. \tag{14.5}$$

Lemma 14.16. *Let C be a pointed k-coalgebra with $C_0 = kG$ and let us write $C = I \oplus C_0$ as in 14.15. By using the notations introduced thereby, we have that*

$$\varepsilon = \sum_{x \in G} e_x$$

so that

$$c = \varepsilon \cdot c \cdot \varepsilon = \sum_{x,y \in G} (e_y \cdot c \cdot e_x) = \sum_{x,y \in G} ({}^x c^y). \tag{14.6}$$

Hence we obtain

$$C = \sum_{x,y \in G} {}^x C^y = \bigoplus_{x,y \in G} {}^x C^y \tag{14.7}$$

where the second equality depends on the fact that the elements e_x are pairwise orthogonal.

Proof. Let $c \in C = I \oplus C_0$.and let us write

$c = w + \sum_{g\in G} \lambda_g g$ where $w \in I,\ \lambda_g \in k$ for every $g \in G$ and $\lambda_g = 0$ for almost every g.

Then

$$e_y(c) = e_y(w) + \lambda_g e_y(g) = \lambda_y$$

and hence $e_y(c) = 0$ for almost every $y \in G$. It follows that

$$e_y \cdot c = \sum c_1 e_y(c_2) = 0 \text{ for almost every } y \in G$$

and also

$$c \cdot e_y = \sum e_y(c_1)\, c_2 = 0 \text{ for almost every } y \in G.$$

Now

$$\sum_{y\in G} e_y(c) = e_y(w) + \sum_{y\in G}\sum_{g\in G} \lambda_g e_y(g) = \sum_{g\in G} \lambda_g = \varepsilon(c)$$

and since this holds for every $c \in C$ we deduce that

$$\sum_{y\in G} e_y = \varepsilon.$$

Therefore, for every $c \in C$ we have

$$c = \varepsilon \cdot c \cdot \varepsilon = \sum_{y\in G} e_y \cdot c \cdot \sum_{x\in G} e_x = \sum_{x,y\in G} e_y \cdot c \cdot e_x = \sum_{x,y\in G} ({}^x c^y).$$

We note that this sums make sense since $e_y \cdot c = \sum c_1 e_y(c_2) = 0$ for almost every $y \in G$ and $c \cdot e_y = \sum e_y(c_1)\, c_2 = 0$ for almost every $y \in G$. □

Lemma 14.17. *Let C be a pointed k-coalgebra with $C_0 = kG$ and let us write $C = I \oplus C_0$ as in 14.15. By using the notations introduced thereby, we have that*

0) $e_x \cdot I \subseteq I$ and $I \cdot e_x \subseteq I$ for every $x \in G$.

1) ${}^xC^x = ({}^xC^x)^+ + kx = ({}^xC^x)^+ \oplus kx$ for every $x \in G$.

2) ${}^xC^y = ({}^xC^y) \cap I = ({}^xC^y)^+$ for every $x, y \in G$ with $x \neq y$.

3) $I = \bigcap_{x\in G} Ker(e_x)$.

4) $I = \oplus_{x,y\in G} ({}^xC^y)^+$.

5) For every $c \in C$ and $x, y \in G$ we have

$$\Delta({}^x c^y) = \sum_{z\in G} {}^x(c_1)^z \otimes {}^z(c_2)^y \tag{14.8}$$

Proof. 0) Let $x \in G$ and $a \in I$. Since I is a coideal of C we can write

$$\Delta(a) = \sum_{i=1}^{m} a_i \otimes c_i + \sum_{j=1}^{n} d_j \otimes b_j \text{ where } a_i, b_j \in I \text{ and } c_i, d_j \in C$$

so that, since $e_x(b_j) \in e_x(I) = \{0\}$ we have

$$e_x \cdot a = \sum_{i=1}^{m} a_i e_x(c_i) + \sum_{j=1}^{n} d_j e_x(b_j) = \sum_{i=1}^{m} a_i e_x(c_i) \in I.$$

In a similar way one proves that $I \cdot e_x \subseteq I$.

1) and 2) Let $x \in G$. First of all note that, since $\varepsilon(x) = 1$, we have that $({}^xC^x)^+ \cap kx = \{0\}$ and hence $({}^xC^x)^+ + kx = ({}^xC^x)^+ \oplus kx$. Moreover, since $x \in {}^xC^x$, it is clear that $({}^xC^x)^+ + kx \subseteq {}^xC^x$. Let $c \in C = I \oplus C_0$.and let us write

$$c = w + \sum_{g \in G} \lambda_g g \text{ where } w \in I,\ \lambda_g \in k \text{ for every } g \in G \text{ and } \lambda_g = 0 \text{ for almost every } g.$$

Then,

$$e_x c\, e_x = e_x w\, e_x + \lambda_x x.$$

Now, by 0), $e_x w\, e_x \in {}^xC^x \cap I$ and since $I \subseteq Ker(\varepsilon)$ we get that $e_x w\, e_x \in ({}^xC^x)^+$ whence

$$e_x c\, e_x \in ({}^xC^x)^+ + kx$$

which implies that

$${}^xC^x \subseteq ({}^xC^x)^+ \oplus kx.$$

Let now $y \in G$ such that $x \neq y$. Then

$$e_y c\, e_x = e_y w\, e_x \in I$$

so that

$${}^xC^y = ({}^xC^y) \cap I$$

and since $I \subseteq Ker(\varepsilon)$, we get that $e_y c\, e_x \in Ker(\varepsilon) \cap {}^xC^y = ({}^xC^y)^+$. Thus we get that

$${}^xC^y = ({}^xC^y) \cap I = ({}^xC^y)^+.$$

3) Since $e_x(I) = \{0\}$ for every $x \in G$, it is clear that $I \subseteq \bigcap_{x \in G} Ker(e_x)$. Conversely, let $c \in \bigcap_{x \in G} Ker(e_x)$. Since $c \in C$, we may write $c = w + \sum_{g \in F} \lambda_g g$ where $w \in I$, $\lambda_g \in k$ for every $g \in G$ and $\lambda_g = 0$ for almost every g. Now, for every $g \in G$ we have

$$\begin{aligned} 0 &= e_g(x) = e_g(w + \sum_{h \in F} \lambda_h h) = e_g(w) + \lambda_g e_g(g) \\ &= 0 + \lambda_g = 0 \end{aligned}$$

so that we deduce that $\lambda_g = 0$ for every $g \in G$ and hence $c = w \in I$.

4) First of all, let us prove that

$$\sum ({}^{x}C^{y})^{+} \subseteq I.$$

If $x \neq y$, this is clear in view of 3). Let us assume that $x = y$. As before, let $c \in C$ and let us write $c = w + \sum_{g\in G} \lambda_g g$ where $w \in I$, $\lambda_g \in k$ for every $g \in G$ and $\lambda_g = 0$ for almost every g. Then

$$e_x c\, e_x = e_x w\, e_x + \lambda_x x.$$

Assume now that $e_x c\, e_x \in Ker(\varepsilon)$ and let $t \in G$. Since $e_x w\, e_x \in I = \bigcap_{g\in G} Ker(e_g)$, we have that $e_t\,(e_x w\, e_x) = 0$. Since $\varepsilon = \sum_{t\in G} e_t$ we deduce that

$$0 = \varepsilon\,(e_x c\, e_x) = \sum_{t\in G} e_t\,(e_x c\, e_x) = \sum_{t\in G} e_t\,(e_x w\, e_x) + \sum_{t\in G} e_t\,(\lambda_x x) = \sum_{t\in G} \delta_{t,x}\lambda_x = \lambda_x$$

so that we get $e_x c\, e_x = e_x w\, e_x \in I$.

Now let $w \in I$. Then, by (14.6) we have

$$w = \sum_{x,y\in G} {}^{x}w^{y}$$

where ${}^{x}w^{y} = e_y w e_x \in I$ since I is a coideal. Thus ${}^{x}w^{y} \in ({}^{x}C^{y}) \cap I \subseteq ({}^{x}C^{y}) \cap \mathrm{Ker}\,(\varepsilon) = ({}^{x}C^{y})^{+}$ and we deduce that

$$w \in \sum_{x,y\in G} ({}^{x}C^{y})^{+}.$$

Therefore we get that

$$I = \sum_{x,y\in G} ({}^{x}C^{y})^{+}.$$

In view of (14.7) , this sum is direct.

5) By applying to ${}^{x}c^{y}$ formula (14.5)we have

$$\Delta\,({}^{x}c^{y}) = \Delta\,(e_y \cdot c \cdot e_x) = \sum_{z\in G} [(L_z \otimes R_z) \circ \Delta]\,(e_y \cdot c \cdot e_x) = \sum_{z\in G} (L_z \otimes R_z)\,(c_1 \cdot e_x \otimes e_y \cdot c_2) =$$

$$= \sum_{z\in G} e_z \cdot c_1 \cdot e_x \otimes e_y \cdot c_2 \cdot e_z = \sum_{z\in G} {}^{x}\,(c_1)\,{}^{z} \otimes {}^{z}\,(c_2)\,{}^{y}.$$

□

Notation 14.18. *Let C be a pointed k-coalgebra and let $g \neq h \in G\,(C)$ be grouplike elements.Then, by Lemma 14.14 $k(g-h) = P_{g,h}(C) \cap P_{h,g}(C) \cap C_0$. In the following we fix a subspace $P^{'}_{g,h}(C)$ of $P_{g,h}(C)$ such that $P_{g,h}(C) = k(g-h) \oplus P^{'}_{g,h}(C)$.*

Theorem 14.19 (Taft-Wilson)**.** *Let C be a pointed k-coalgebra with $G = G(C)$. Then*

1) For every $n \in \mathbb{N}$, $n \geq 1$ and $c \in C_n \cap ({}^{x}C^{y})^{+}$ we have that

$$\Delta(c) = c \otimes y + x \otimes c + t \text{ where } t \in C_{n-1} \otimes C_{n-1}$$

2) For every $n \in \mathbb{N}$, $n \geq 1$ and $c \in C_n$, we have that

$$c = \sum_{g,h \in G} c_{g,h} \text{ where } \Delta(c_{g,h}) = c_{g,h} \otimes g + h \otimes c_{g,h} + w \text{ and } w \in C_{n-1} \otimes C_{n-1}.$$

3) $C_1 = kG \oplus (\bigoplus_{g,h \in G} P'_{g,h}(C))$.

Proof. We will use the notations introduced in 14.15.

1) For every every $n \in \mathbb{N}$, $n \geq 1$, let $I_n = I \cap C_n$. Since $C = I \oplus C_0$ and, by 3) inTheorem 10.10 $C_0 \subseteq C_n$ we have that

$$C_n = I_n \oplus C_0. \tag{14.9}$$

Now, since, by Lemma 14.17, every $({}^xC^y)^+ \subseteq I$ we have

$$C_n \cap ({}^xC^y)^+ = C_n \cap (I \cap ({}^xC^y)^+) = I_n \cap ({}^xC^y)^+ \tag{14.10}$$

and hence

$$\bigoplus_{x,y \in G} (C_n \cap ({}^xC^y)^+) = \bigoplus_{x,y \in G} (I_n \cap ({}^xC^y)^+) \subseteq I_n$$

Let $c \in I_n = I \cap C_n$. Since by 2) inTheorem 10.10 C_n is a subcoalgebra of C and by 0) of Lemma 14.17, we have

$${}^xc^y = e_x c\, e_y \in ({}^xC^y) \cap I_n \subseteq [({}^xC^y) \cap \mathrm{Ker}\,(\varepsilon)] \cap I_n = ({}^xC^y)^+ \cap I_n$$

and hence, by form 14.6

$$c = \sum_{x,y \in G} ({}^xc^y) \in \bigoplus_{x,y \in G} (I_n \cap ({}^xC^y)^+)$$

so that

$$I_n \subseteq \bigoplus_{x,y \in G} (I_n \cap ({}^xC^y)^+).$$

Therefore we get

$$I_n = \bigoplus_{x,y \in G} (I_n \cap ({}^xC^y)^+) \overset{(14.10)}{=} \bigoplus_{x,y \in G} (C_n \cap ({}^xC^y)^+) \tag{14.11}$$

Thus we can assume that $c \in I_n \cap ({}^xC^y)^+$. Now, since $C_n = I_n \oplus C_0$ and, by Theorem 10.10, we have $C_i \subseteq C_{n-1}$ for every $i = 0, \ldots, n-1$, we get that

$$\begin{aligned} \Delta(c) &\in C_n \otimes C_0 + C_0 \otimes C_n + \sum_{i=1}^{n-1} C_i \otimes C_{n-i} \\ &= I_n \otimes C_0 + C_0 \otimes C_0 + C_0 \otimes I_n + C_0 \otimes C_0 + \sum_{i=1}^{n-1} C_i \otimes C_{n-i} \\ &\subseteq I_n \otimes C_0 + C_0 \otimes I_n + C_{n-1} \otimes C_{n-1}. \end{aligned}$$

Therefore we can write

$$\Delta(c) = \sum_{g\in G} c_g \otimes g + \sum_{h\in G} h \otimes d_h + \sum_{i=1}^{t} v_i \otimes w_i \tag{14.12}$$

where $c_g, d_g \in I_g$ for every $g \in G$ and $v_i, w_i \in C_{n-1}$ for every $i = 1, \dots t$.

Let us apply formula (14.8) to (14.12) and get

$$\Delta(c) = \Delta\left({}^{x}c^{y}\right) = \sum_{z\in G} {}^{x}\left(c_1\right)^{z} \otimes {}^{z}\left(c_2\right)^{y} = \tag{14.13}$$

(14.14)

$$= \sum_{g,z\in G} {}^{x}\left(c_g\right)^{z} \otimes {}^{z}\left(g\right)^{y} + \sum_{h,z\in G} {}^{x}\left(h\right)^{z} \otimes {}^{z}\left(d_h\right)^{y} + \sum_{z\in G}\sum_{i=1}^{t} {}^{x}\left(v_i\right)^{z} \otimes {}^{z}\left(w_i\right)^{y}.$$

Now

$$\begin{aligned} {}^{z}\left(g\right)^{y} &= e_y \cdot g \cdot e_z = \left(g e_y\left(g\right)\right) \cdot e_z = e_y\left(g\right) e_z\left(g\right) g \\ &= \delta_{y,g}\delta_{z,g} g = \delta_{z,g,y} y \end{aligned}$$

and

$$\begin{aligned} {}^{x}\left(h\right)^{z} &= e_z \cdot h \cdot e_x = \left(h e_z\left(h\right)\right) \cdot e_y = e_z\left(h\right) e_x\left(h\right) h \\ &= \delta_{z,h}\delta_{x,h} h = \delta_{x,h,z} x. \end{aligned}$$

Then we can rewrite (14.12) as

$$\Delta(c) = {}^{x}\left(c_y\right)^{y} \otimes y + x \otimes {}^{x}\left(d_x\right)^{y} + \sum_{z\in G}\sum_{i=1}^{t} {}^{x}\left(v_i\right)^{z} \otimes {}^{z}\left(w_i\right)^{y} \tag{14.15}$$

Let us apply $l_C \circ (\varepsilon \otimes C)$ to (14.15) and we get

$$\begin{aligned} c &= \varepsilon({}^{x}\left(c_y\right)^{y})y + \varepsilon(x){}^{x}\left(d_x\right)^{y} + \sum_{z\in G}\sum_{i=1}^{t} \varepsilon\left[{}^{x}\left(v_i\right)^{z}\right] {}^{z}\left(w_i\right)^{y} \\ &= 0 + {}^{x}\left(d_x\right)^{y} + \sum_{z\in G}\sum_{i=1}^{t} \varepsilon\left[{}^{x}\left(v_i\right)^{z}\right] {}^{z}\left(w_i\right)^{y} = {}^{x}\left(c_x\right)^{y} + v, \end{aligned}$$

where $\varepsilon({}^{x}\left(c_y\right)^{y}) = 0$ since ${}^{x}\left(c_y\right)^{y} \in I_n \subseteq \mathrm{Ker}\left(\varepsilon\right)$ and $v = \sum_{z\in G}\sum_{i=1}^{t} \varepsilon\left[{}^{x}\left(v_i\right)^{z}\right] {}^{z}\left(w_i\right)^{y} \in C_{n-1}$. In fact C_n is a C^*-sub-bimodule of C^x so that ${}^{x}\left(v_i\right)^{z}$ and ${}^{z}\left(w_i\right)^{y} \in C_{n-1}$.

In a similar way, by applying $r_C \circ (C \otimes \varepsilon)$, way one gets

$$c = {}^{x}(c_y)^{y} + u \text{ where } u \in C_{n-1}.$$

Substituting in (14.15)we get

$$\begin{aligned}
\Delta(c) &= (c-u) \otimes y + x \otimes (c-v) + \sum_{z\in G}\sum_{i=1}^{t} {}^{x}(v_i)^{z} \otimes {}^{z}(w_i)^{y} \\
&= c \otimes y + x \otimes c - u \otimes y - x \otimes v + \sum_{z\in G}\sum_{i=1}^{t} {}^{x}(v_i)^{z} \otimes {}^{z}(w_i)^{y} \\
&= c \otimes y + x \otimes c + t
\end{aligned}$$

where $t \in C_{n-1} \otimes C_{n-1}$. In fact, as noted before, each ${}^{x}(v_i)^{z} \otimes {}^{z}(w_i)^{y} \in C_{n-1} \otimes C_{n-1}$ and also $u \otimes y \in C_{n-1} \otimes C_0 \subseteq C_{n-1} \otimes C_{n-1}$, $x \otimes v \in C_0 \otimes C_{n-1} \subseteq C_{n-1} \otimes C_{n-1}$.Thus 1) is proved.

2) Let $c \in C_n$. Then $c = \sum_{g,h\in G} \left({}^{g}c^{h}\right)$ where ${}^{g}c^{h} = e_h c e_g \in C_n \cap {}^{g}C^{h}$. Now, if $g = h$ by we can write

$${}^{g}c^{g} = c_g + \lambda g \text{ where } c_g \in ({}^{g}C^{g})^{+} \text{ and } \lambda \in k.$$

If $g \neq h$ we have

$${}^{g}c^{h} \in \left({}^{g}C^{h}\right) \overset{2) of Lemma 14.17}{=} ({}^{g}C^{h})^{+}.$$

Let F be a finite subset of G such that

$$c = \sum_{g,h\in F} {}^{g}c^{h}.$$

Then, by 1) we have

$$\Delta\left({}^{g}c^{h}\right) = {}^{g}c^{h} \otimes g + h \otimes {}^{g}c^{h} + w \text{ where } w \in C_{n-1} \otimes C_{n-1} \text{ if } g \neq h$$

and

$$\Delta\left({}^{g}c^{g}\right) = \Delta(c_g) + \Delta(\lambda g) = c_g \otimes g + g \otimes c_g + u + \lambda(g \otimes g) \text{ where } u \in C_{n-1} \otimes C_{n-1}$$

so that

$$\Delta\left({}^{g}c^{g}\right) = \Delta(c_g) + \Delta(\lambda g) = c_g \otimes g + g \otimes c_g + w \text{ where } w = u + \lambda(g \otimes g) \in C_{n-1} \otimes C_{n-1}.$$

3) From formula 14.9, we know that $C_1 = C_0 \oplus I_1$ and from formula 14.11 that $I_1 = \oplus_{x,y}({}^{x}C_1^{y})^{+}$.
Let us prove that

$$P_{y,x}(C) = k(y-x) \oplus {}^{x}(C_1)^{y+}. \tag{14.16}$$

First of all, let us prove that the sum $k(y-x) + {}^x(C_1)^{y+}$ is direct i.e. that $k(y-x) \cap {}^x(C_1)^{y+} = \{0\}$. Let $x \neq y$ and let $\lambda(y-x) \in^x (C_1)^{y+}$. Then we have

$$\lambda(y-x) = e_y \lambda(y-x) e_x = \lambda(e_y y e_x - e_y x e_x) = 0.$$

" $\subseteq$ " Let $c \in P_{y,x} = P_{y,x}(C)$, then $\Delta(c) = c \otimes y + x \otimes c$. Let us apply formula 14.8. Then, for every $g, h \in G$, we have that

$$\begin{aligned} \Delta\left({}^g c^h\right) &= \sum_{z \in G} {}^g\left(c_1\right)^z \otimes^z \left(c_2\right)^h = \\ &= \sum_{z \in G} {}^g\left(c\right)^z \otimes^z \left(y\right)^h + \sum_{z \in G} {}^g\left(x\right)^z \otimes^z \left(c\right)^h \\ &= \delta_{h,y}\, {}^g c^y \otimes y + \delta_{g,x} x \otimes {}^x c^h \end{aligned}$$

If $h \neq y$ and $g \neq x$, then $\Delta\, {}^g c^h = 0$ so that ${}^g c^h = 0$.
If $h = y$ and $g \neq x$, then $\Delta\, {}^g c^h = {}^g c^y \otimes y$ which yields, by applying $l_C(\varepsilon \otimes I)$, ${}^g c^h = \varepsilon({}^g c^h) y \in ky$.
If $h \neq y$ and $g = x$, then $\Delta {}^g c^h = x \otimes {}^x c^h$ which yields, by applying $r_C(I \otimes \varepsilon)$, ${}^g c^h = x\varepsilon({}^x c^h) \in kx$.
Finally if $h = y$ and $g = x$, then $\Delta\, {}^g c^h = {}^x c^y \otimes y + x \otimes {}^x c^y$ so that ${}^x c^y \in P_{y,x}$.
Thus we obtain

$$\begin{aligned} c &= \sum_{g,h \in G} {}^g c^h = \sum_{\substack{g,h \in G \\ h \neq y \\ g \neq x}} {}^g c^h + \sum_{\substack{g,h \in G \\ h = y \\ g \neq x}} {}^g c^h + \sum_{\substack{g,h \in G \\ h \neq y \\ g = x}} {}^g c^h + {}^x c^y \\ &= 0 + \sum_{\substack{g,h \in G \\ h = y \\ g \neq x}} \varepsilon({}^g c^h) y + \sum_{\substack{g,h \in G \\ h \neq y \\ g = x}} \varepsilon({}^x c^h) x + {}^x c^y \end{aligned}$$

so that we get

$$c = {}^x c^y + \alpha x + \beta y \text{ where } {}^x c^y \in P_{y,x} \text{ and } \alpha, \beta \in k.$$

Since both c and ${}^x c^y \in P_{y,x}$, we deduce that also $\alpha x + \beta y \in P_{y,x}$. Since, by Lemma 14.14, even $\alpha(x-y) \in P_{y,x}$ we deduce that

$$(\alpha + \beta) y = (\alpha x + \beta y) - \alpha(x - y) \in P_{y,x}$$

and hence, by 1) of Lemma 14.14, we get

$$0 = \varepsilon\left((\alpha + \beta) y\right) = \alpha + \beta$$

which implies that

$$c = \alpha(x - y) + {}^x c^y \text{ where } {}^x c^y \in P_{y,x} \text{ and } \alpha \in k.$$

If $x = y$ we get

$$c = {}^x c^x \in P_{x,x}$$

and hence, by 1) of Lemma 14.14 we know that $\varepsilon(c) = 0$. If $x \neq y$, by 2) of Lemma 14.17 we know that ${}^x C^y = ({}^x C^y)^+$. Thus, in both case we have that ${}^x c^y \in ({}^x C^y)^+$.
" $\supseteq$ " By Lemma 14.14, we have that $k(y - x) \in P_{y,x}$.
Let now $c \in C_1 \cap ({}^x C^y)^+$. Then, in view of 1),

$$\Delta(c) = c \otimes y + x \otimes c + t \text{ where } t \in C_0 \otimes C_0$$

i.e.

$$\Delta(c) = c \otimes y + x \otimes c + \sum_{g,h \in G} \alpha_{g,h} g \otimes h \text{ where } \alpha_{g,h} \in k \text{ and they are almost all zero.}$$

Since, by formula (14.6) , $c = \sum_{g,h \in G} \left({}^g c^h\right)$ and since $c \in {}^x C^y$, we get that

$$c = e_y \cdot c \cdot e_x = {}^x c^y.$$

Therefore

$$\begin{aligned}
\Delta(c) = \Delta({}^x c^y) &\overset{\text{form}(14.8)}{=} \sum_{z \in G} {}^x(c_1)^z \otimes {}^z(c_2)^y \\
= \sum_{z \in G} {}^x(c)^z \otimes {}^z(y)^y + {}^x(x)^z \otimes {}^z(c)^y &+ \sum_{g,h \in G} \alpha_{g,h} {}^x(g)^z \otimes {}^z(h)^y \\
= {}^x(c)^y \otimes {}^y(y)^y + {}^x(x)^x \otimes {}^x(c)^y &+ \sum_{g,h \in G} \alpha_{g,h} \sum_{z \in G} \delta_{g,x,z} g \otimes \delta_{h,z,y} h \\
&= c \otimes y + x \otimes c + \delta_{x,y} \alpha_{x,y}\, x \otimes\, y.
\end{aligned}$$

Since $\varepsilon(c) = 0$, by applying $(I \otimes \varepsilon)$ we obtain

$$c = c + \delta_{x,y} \alpha_{x,y} y$$

and hence $\delta_{x,y} \alpha_{x,y} = 0$. Thus we get

$$\Delta(c) = c \otimes y + x \otimes c$$

i.e. $c \in P_{y,x}$.
Thus $P_{y,x}(C) = k(y - x) + {}^x(C_1)^{y+}$ and hence formula (14.16) is proved.

Since $P_{y,x}(C) \overset{(14.16)}{=} k(y - x) \oplus {}^x(C_1)^{y+}$, we have

$$C_0 \oplus ({}^x C_1^y)^+ = C_0 + P_{y,x}(C) \tag{14.17}$$

$$\begin{aligned}
C_1 &\overset{(14.9)}{=} C_0 \oplus (C_1 \cap I) = C_0 \oplus I_1 = \\
&\overset{(14.11)}{=} C_0 \oplus [\bigoplus_{x,y \in G} (C_1 \cap ({}^x C^y)^+)] = C_0 \oplus [\bigoplus_{x,y \in G} ({}^x C_1^y)^+)] \\
&\overset{(14.17)}{=} C_0 + \sum_{x,y \in G} P_{y,x}(C).
\end{aligned}$$

Since
$$P_{g,h}(C) = k(g-h) \oplus P'_{g,h}(C)$$
and since $k(g-h) \in C_0$, we get that
$$C_1 = C_0 + \sum_{g,h\in G} [k(g-h) + P'_{g,h}(C)] = C_0 + \sum_{g,h\in G} P'_{g,h}(C).$$
Let us prove that the sum
$$C_0 + \sum_{g,h\in G} P'_{g,h}(C)$$
is direct. Assume that
$$c + \sum_{g,h\in G} d_{g,h} = 0 \text{ where } c \in C_0 \text{ and, for every } g,h \in G,\ d_{g,h} \in P'_{g,h}(C).$$
Now, for every $g,h \in G$, we have
$$P'_{g,h}(C) \subseteq P_{g,h}(C) \overset{(14.16)}{=} k(g-h) \oplus (C_1 \cap ({}^gC^h)^+)$$
and hence we can write
$$d_{g,h} = \alpha_{g,h}(g-h) + b_{g,h} \text{ where } \alpha_{g,h} \in k \text{ and } b_{g,h} \in C_1 \cap ({}^gC^h)^+.$$
Therefore we get that
$$c + \sum_{g,h\in G} \alpha_{g,h}(g-h) + \sum_{g,h\in G} b_{g,h} = 0$$
i.e.
$$c + \sum_{g,h\in G} \alpha_{g,h}(g-h) = -\sum_{g,h\in G} b_{g,h} \in C_0 \cap (\sum_{g,h\in G} (C_1 \cap ({}^gC^h)^+) \overset{(14.10)}{\subseteq} C_0 \cap I_1 \subseteq C_0 \cap I = \{0\}$$
and hence
$$c + \sum_{g,h\in G} \alpha_{g,h}(g-h) = -\sum_{g,h\in G} b_{g,h} = 0$$
Since $\sum_{g,h\in G} b_{g,h} \in \sum_{g,h\in G}(C_1 \cap ({}^gC^h)^+ = \bigoplus_{x,y\in G}(C_1 \cap ({}^xC^y)^+)$ we deduce that $b_{g,h} = 0$ for every $g,h \in G$ so that
$$d_{g,h} = \alpha_{g,h}(g-h) \in k(g-h) \cap P'_{g,h} = \{0\} \text{ for every } g,h \in G$$
and hence
$$c = -\sum_{g,h\in G} d_{g,h} = 0.$$
□

Corollary 14.20. *Let $f : C \to D$ be a k-coalgebra morphism. If C is pointed and $f_{|P_{g,h}(C)}$ is injective for every $g, h \in G$, then f is injective.*

Proof. We can assume w.l.o.g. that f is surjective. Then, by Corollary 11.8 also D is pointed. Now, in view of Theorem 14.12, it is enough to show that f is injective on C_1. Let $g, h \in G = G(C)$, $g \neq h$. Then, by Lemma 14.14, we have that $0 \neq g - h \in P_{g,h}(C)$ and hence, in view of our assumptions, we get $0 \neq f(g-h) = f(g) - f(h)$, which implies that f is injective on G so that the family

$$(f(g))_{g \in G}$$

is a family of distinct grouplike elements of $G(D)$ and hence, by Theorem 1.54, these elements are linearly independent. Let $w \in C_0 = kG$, $w = \sum_{g \in G} \lambda_g g$ and assume that $f(w) = 0$. Then from $\sum_{g \in G} \lambda_g f(g) = 0$ we deduce that $\lambda_g = 0$ for every $g \in G$ and hence $w = 0$. Thus f is injective on $C_0 = kG$. Let $c \in P_{g,h}(C)$, i.e. $\Delta(c) = c \otimes g + h \otimes c$. Then we get

$$\begin{aligned} \Delta(f(c)) &= (f \otimes f)\Delta(c) = (f \otimes f)(c \otimes g + h \otimes c) \\ &= f(c) \otimes f(g) + f(h) \otimes f(c) \in P_{f(g),f(h)}(D) \end{aligned}$$

and hence we obtain that $f(P_{g,h}(C)) \subseteq P_{f(g),f(h)}(D)$. Let $P'_{g,h}(C)$ be a complement subspace of $k(g-h)$ in $P_{g,h}(C)$. Then

$$P'_{g,h}(C) \cap k(g-h) = \{0\}$$

and since f is injective on C_1 we get

$$f\left(P'_{g,h}(C)\right) \cap f(k(g-h)) = \{0\}$$

Hence we can choose a complement subspace $P'_{f(g),f(h)}(D)$ of $k(f(g) - f(h))$ in $P_{f(g),f(h)}(D)$ containing $f\left(P'_{g,h}(C)\right)$. Since both C and D are pointed, by Taft-Wilson Theorem 14.19, we have that $C_1 = kG \oplus (\bigoplus_{g,h \in G} P'_{g,h}(C))$ and $D_1 = kG(D) \oplus (\bigoplus_{a,b \in G(D)} P'_{a,b}(D))$. In particular we get that

$$\begin{aligned} f(kG) \cap \left[\left(\sum_{g,h \in G} f\left[P'_{g,h}(C)\right] \right) \right] &\subseteq kG(D) \cap \sum_{a,b \in G(D)} P'_{a,b}(D) = \\ &= kG(D) \cap \bigoplus_{a,b \in G(D)} P'_{a,b}(D) = \{0\}. \end{aligned}$$

Let $c \in C_1$ and let us write

$$c = w + \sum t_{g,h} \text{ where } w \in kG \text{ and } t_{g,h} \in P'_{g,h}(C) \text{ for every } g, h \in G.$$

Assume that $f(c) = 0$. Then we obtain

$$f(w) = -\sum f(t_{g,h}) \in f(kG) \cap \sum_{g,h \in G} f\left(P'_{g,h}(C)\right) = \{0\}$$

Since $w \in kG$ and f is injective on $C_0 = kG$ we deduce that $w = 0$. Moreover since $\sum_{a,b\in G(D)} P'_{a,b}(D) = \bigoplus_{a,b\in G(D)} P'_{a,b}(D)$ and, , $f\left(P'_{g,h}(C)\right) \subseteq P'_{f(g),f(h)}(D)$ where $f(g), f(h) \in G(D)$, we get that $\sum_{g,h\in G} f\left(P'_{g,h}(C)\right) = \bigoplus_{g,h\in G} f\left(P'_{g,h}(C)\right)$ so that, from $\sum f(t_{g,h}) = 0$ we infer that $f(t_{g,h}) = 0$ for every $g, h \in G$ and hence, since $t_{g,h} \in P'_{g,h}(C) \subseteq C_1$ and f is injective on C_1, that $t_{g,h} = 0$. Therefore we obtain that $c = 0$. □

Remark 14.21. *Let $n \in \mathbb{N}, n \geq 2$ and let $U = U_n$ be the k-algebra of the $n \times n$ upper triangular matrices over the field k. Then a basis of U over k is given by $\{e_{i,j} \mid 1 \leq i \leq j \leq n\}$ where $e_{i,j}$ is defined by setting $(e_{ij})_{a,b} = \delta_{i,a}\delta_{j,b}$. Fix an $i, 1 \leq i \leq n$ and let*

$$P_i = \sum_{\substack{1\leq a\leq b\leq n \\ (a,b)\neq(i,i)}} ke_{a,b}.$$

P_i is a left ideal of U. In fact let $1 \leq s \leq t \leq n$ and $1 \leq a \leq b \leq n$ with $(a, b) \neq (i, i)$. Then

$$e_{s,t}e_{a,b} = \delta_{t,a}e_{s,b}$$

Assume $t = a$ and $(s, b) = (i, i)$. Then we would get $i = s \leq t = a \leq b = i$ and hence $(a, b) = (i, i)$. Contradiction. Clearly we have

$$U/P_i \simeq ke_{i,i}$$

and hence P_i is a left maximal ideal of U. Conversely let P be a left maximal ideal of U. Since $1_U = \sum_{a=1}^{n} e_{a,a}$ there exists an i such that $e_{i,i} \notin P$. Since

$$P + Ue_{i,i} = U$$

for every $(a, b) \neq (i, i)$ there exists a $p \in P$ and an $u \in U$ such that

$$p + ue_{i,i} = e_{a,b}.$$

Write

$$p = \sum_{s\leq t} p_{s,t}e_{s,t} \text{ and } u = \sum_{s\leq t} u_{s,t}e_{s,t}, \text{ where } p_{s,t}, u_{s,t} \in k.$$

Then we have

$$ue_{i,i} = \sum_{s\leq t} u_{s,t}e_{s,t}e_{i,i} = \sum_{s\leq i} u_{s,i}e_{s,i}$$

and

$$P \ni e_{a,a}p = \sum_{s\leq t} p_{s,t}e_{a,a}e_{s,t} = \sum_{a\leq t} p_{a,t}e_{a,t} \text{ and}$$

$$e_{a,a}ue_{i,i} = \sum_{s\leq i} u_{s,i}e_{a,a}e_{s,i} = 0 \text{ unless } a \leq i \text{ in which case we get } e_{a,a}ue_{i,i} = u_{a,i}e_{a,i}.$$

Thus we obtain

$$e_{a,b} = e_{a,a}e_{a,b} = e_{a,a}(p + ue_{i,i}) = \sum_{a\leq t} p_{a,t}e_{a,t} + e_{a,a}ue_{i,i}$$

In the case $i < a$ this means

$$P \ni e_{a,a}p = \sum_{a \leq t} p_{a,t}e_{a,t} = e_{a,b}$$

In the case $a \leq i$ we get

$$\sum_{a \leq t} p_{a,t}e_{a,t} + u_{a,i}e_{a,i} = e_{a,b.}$$

Now if* $b \neq i$ *this implies

$$P \ni e_{a,a}p = \sum_{a \leq t} p_{a,t}e_{a,t} = e_{a,b.}$$

Let us consider the case $b = i$. Then, since $(a, b) \neq (i, i)$, we have $a < b = i$ and hence

$$\sum_{a \leq t} p_{a,t}e_{a,t} + u_{a,i}e_{a,i} = e_{a,i}.$$

If* $e_{a,i} \notin P$ *we have

$$P + Ue_{a,i} = U$$

Hence there exists a $q \in P$ and a $w \in U$ such that

$$q + we_{a,i} = e_{i,i}$$

Write

$$q = \sum_{s \leq t} q_{s,t}e_{s,t} \text{ and } w = \sum_{s \leq t} w_{s,t}e_{s,t} \text{ where } q_{s,t}, w_{s,t} \in k.$$

Then we have

$$we_{a,i} = \sum_{s \leq t} w_{s,t}e_{s,t}e_{a,i} = \sum_{s \leq a} w_{s,a}e_{s,i}$$

and

$$P \ni e_{i,i}q = \sum_{s \leq t} q_{s,t}e_{i,i}e_{s,t} = \sum_{i \leq t} q_{i,t}e_{i,t} \ .$$

Now we have that

$$0 = e_{i,i}we_{a,i} = \sum_{s \leq a} w_{s,a}e_{i,i}e_{s,i}$$

In fact

$$e_{i,i}e_{s,i} \neq 0 \text{ and } s \leq a \text{ implies } i = s \leq a$$

and since $a < i$, this cannot happen. Therefore we obtain

$$P \ni e_{i,i}q = e_{i,i}q + e_{i,i}we_{a,i} = e_{i,i}e_{i,i} = e_{i,i}.$$

Contradiction. We deduce that* P *contains all* $e_{s,t}$ *with *$(s, t) \neq (i, i)$ and hence $P = P_i$. Therefore for the Jacobson radical $J(U)$ of U we have*

$$J(U) = P_1 \cap \ldots \cap P_n = \sum_{\substack{1 \leq a \leq b \leq n \\ (a,b) \neq (1,1)}} ke_{a,b} \cap \ldots \cap \sum_{\substack{1 \leq a \leq b \leq n \\ (a,b) \neq (n,n)}} ke_{a,b} = \sum_{1 \leq a < b \leq n} ke_{a,b}$$

i.e. $J(U)$ *is the set of strictly upper triangular matrices. For every* $s \in \mathbb{N}, 1 \leq s \leq n$ *we have*

$$(J(U))^s = \sum_{\substack{1\leq a\leq b\leq n \\ s\leq b-a}} ke_{a,b}.$$

In particular

$$(J(U))^n = \{0\}.$$

Example 14.22. *Let* $C = M^C(n,k)$ *the* $n \times n$ matrix k-coalgebra.*introduced in 1.12 2).* C *has a basis of* n^2 *elements* X_{ij}, $1 \leq i,j \leq n$, *and its coalgebra structure is defined by setting*

$$\Delta(X_{ij}) = \sum_{h=1}^{n} X_{ih} \otimes X_{hj} \quad \text{and} \quad \varepsilon(X_{ij}) = \delta_{ij} \text{ for every } 1 \leq i,j \leq n.$$

Recall that $C^* \cong M_n(k)$, *the* $n \times n$ *matrix* k*-algebra which is a simple algebra. Thus* C *is a simple coalgebra i.e.* $C = C_0$. *Let* I *be the subspace of* C *spanned by* $\{X_{ij} : |1 \leq i,j \leq n, i > j\}$. *For every* $1 \leq i,j \leq n$ *with* $i > j$, *we have that*

$$\Delta(X_{ij}) = \sum_{h=1}^{n} X_{ih} \otimes X_{hj} =$$

$$= \sum_{i>h} X_{ih} \otimes X_{hj} + \sum_{h>i} X_{ih} \otimes X_{hj} + X_{ii} \otimes X_{ij} \in I \otimes C + C \otimes I$$

and also that

$$\varepsilon(X_{ij}) = 0 \text{ for every } i > j.$$

Thus I *is a coideal of* C *so that* $D = C/I$ *is a coalgebra and* $\{\bar{X}_{i,j} = X_{ij} + I | i \leq j\}$ *is a basis for* D. *Note that* D^* *is the subalgebra of* C^* *consisting of upper triangular matrices. Now, for every* $1 \leq i \leq j \leq n$ *and* $1 \leq h \leq n$, *if* $h < i$ *then* $X_{ih} \in I$ *while if* $j < h$ *we have that* $X_{hj} \in I$ *so that*

$$\Delta(\bar{X}_{i,j}) = \bar{X}_{i,i} \otimes \bar{X}_{i,j} + \bar{X}_{i,i+1} \otimes \bar{X}_{i+1,j} + ... + \bar{X}_{i,j} \otimes \bar{X}_{jj}.$$

Let $J = Jac(D^*)$ *be the Jacobson radical of* D^*. *By Remark 14.21, for every* $s \in \mathbb{N}, 1 \leq s \leq n$ *we have*

$$(J(U))^s = \sum_{\substack{1\leq a\leq b\leq n \\ s\leq b-a}} ke_{a,b}.$$

where for each (a,b) *with* $1 \leq a \leq b \leq n, e_{a,b} = \left(\bar{X}_{a,b}\right)^*$ *i.e.*

$$e_{a,b}\left(\bar{X}_{i,j}\right) = \delta_{(a,b)(i,j)} = 0 \text{ unless } (a,b) = (i,j) \text{ in which case it is } 1.$$

Now, by Proposition 11.9 for every $s \in \mathbb{N}, 0 \leq s \leq n-1$ we have

$$D_s = (J^{s+1})^{\perp} = \left(\sum_{\substack{1 \leq a \leq b \leq n \\ s+1 \leq b-a}} k e_{a,b} \right)^{\perp} = \bigcap_{\substack{1 \leq a \leq b \leq n \\ s+1 \leq b-a}} (k e_{a,b})^{\perp} = \sum_{\substack{1 \leq i \leq j \leq n \\ j-i \leq s}} k \bar{X}_{i,j}.$$

Thus

$$D_0 = J^{\perp} = \sum_{\substack{1 \leq i \leq j \leq n \\ j-i \leq 1}} k \bar{X}_{i,j} = \sum_{i=1}^{n} k \bar{X}_{i,i} = kG(D),$$

and in general for every $s \in \mathbb{N}, 0 \leq s \leq n-1$

$$D_s = \sum_{\substack{1 \leq i \leq j \leq n \\ j-i \leq s}} k \bar{X}_{i,j} = \sum_{\substack{1 \leq i \leq j \leq n \\ j-i \leq s-1}} k \bar{X}_{i,j} + \sum_{\substack{1 \leq i \leq j \leq n \\ j-i = s}} k \bar{X}_{i,j} = D_{s-1} + \sum_{i=1}^{n-s} k \bar{X}_{i,i+s}$$

and

$$D_n = D.$$

Now, for every $1 \leq i, s \leq n$ and $0 \leq s \leq n$ with $1 \leq i+s \leq n$ we have

$$\begin{aligned} \Delta(\bar{X}_{i,i+s}) &= \bar{X}_{i,i} \otimes \bar{X}_{i,i+s} + \bar{X}_{i,i+1} \otimes \bar{X}_{i+1,i+s} + \ldots + \bar{X}_{i,i+s} \otimes \bar{X}_{i+s,i+s} \\ &= \bar{X}_{i,i} \otimes \bar{X}_{i,i+s} + \bar{X}_{i,i+s} \otimes \bar{X}_{i+s,i+s} + w \end{aligned}$$

where $\bar{X}_{i,i}$ and $\bar{X}_{i,i+s}$ are in $G(D)$ and

$$w = \bar{X}_{i,i+1} \otimes \bar{X}_{i+1,i+s} + \ldots + \bar{X}_{i,i+s-1} \otimes \bar{X}_{i+s-1,i+s-1} \in D_{s-1} \otimes D_{s-1}.$$

Thus the elements $\bar{X}_{i,i+s} \in D_s$ have the form described in Taft-Wilson Theorem 14.19. Let $\pi : C \to D$ be the canonical projection. Then we have

$$D_0 \subsetneqq \pi(C_0) = \pi(C) = D.$$

Therefore Corollary 11.7, in general, cannot be improved and the coradical filtration is not preserved in homorphic images.

Chapter 15

Some Useful Results

Lemma 15.1. *Let k be a field and let $f : V \to W$ and $f' : V' \to W'$ be k-linear maps. Then*
$$\mathrm{Ker}\,(f \otimes f') = \mathrm{Ker}\,(f) \otimes V' + V \otimes \mathrm{Ker}\,(f')\,.$$

Proof. Let X be a basis of $\mathrm{Ker}\,(f)$ which we complete to a basis Y of V. Let X' be a basis of $\mathrm{Ker}\,(f')$ which we complete to a basis Y' of V'. Let $a \in \mathrm{Ker}\,(f \otimes f')$ and write

$$a = \sum_{x \in X, x' \in X'} \lambda_{x,x'} x \otimes x' + \sum_{y \in Y \setminus X, x' \in X'} \lambda_{y,x'} y \otimes x' + \sum_{x \in X, y' \in Y' | X'} \lambda_{x,y'} x \otimes y' + \sum_{y \in Y \setminus X, y' \in Y' | X'} \lambda_{y,y'} y \otimes y'.$$

Then we get

$$\begin{aligned} 0 = \sum_{x \in X, x' \in X'} \lambda_{x,x'} f(x) \otimes f'(x') &+ \sum_{y \in Y \setminus X, x' \in X'} \lambda_{y,x'} f(y) \otimes f'(x') + \sum_{x \in X, y' \in Y' \setminus X'} \lambda_{x,y'} f(x) \otimes f'(y') \\ &+ \sum_{y \in Y \setminus X, y' \in Y' \setminus X'} \lambda_{y,y'} f(y) \otimes f'(y') \\ &= \sum_{y \in Y \setminus X, y' \in Y' \setminus X'} \lambda_{y,y'} f(y) \otimes f'(y') \end{aligned}$$

so that, we get

$$\sum_{y \in Y \setminus X, y' \in Y' \setminus X'} \lambda_{y,y'} f(y) \otimes f'(y') = 0 \tag{15.1}$$

Now $f(Y \setminus X)$ is a linar indipendent subset of W. In fact, from

$$\sum_{y \in Y \setminus X,} \lambda_y f(y) = 0$$

we get, for Z the subspace spanned by $Y \setminus X$

$$\sum_{y \in Y \setminus X,} \lambda_y y \in \mathrm{Ker}\,(f) \cap Z = \{0\}$$

and hence, $\lambda_y = 0$ for every $y \in Y \backslash X$. The same holds for $f'(Y' \backslash X')$. Hence, from (15.1) we deduce that $\lambda_{y,y'} = 0$ for every $y \in Y \backslash X, y' \in Y' \backslash X'$. Hence, we obtain that $a \in \operatorname{Ker}(f) \otimes V' + V \otimes \operatorname{Ker}(f')$. The other inclusion is trivial. □

15.2. *Let V be a vector space over a field k and let $V^* = \operatorname{Hom}_k(V, k)$ be its dual. Given a subvector space W of V we set:*

$$W^{\perp} = \{f \in V^* \mid f(W) = 0\} ;$$

and for every subspace X of V^* we set

$$X^{\perp} = \{v \in V \mid \xi(v) = 0 \text{ for every } \xi \in X\} = \bigcap_{\xi \in X} \operatorname{Ker}(\xi).$$

Note that $W^{\perp\perp} = W$ while $X = X^{\perp\perp}$ whenever V is finite dimensional.

Lemma 15.3. *Let V_1 and V_2 be vector spaces over a field k and let $X_1 \leq V_1^*$ and $X_2 \leq V_2^*$. Then we have*

$$(X_1 \otimes X_2)^{\perp} = V_1 \otimes (X_2)^{\perp} + (X_1)^{\perp} \otimes V_2 \text{ in } V_1 \otimes V_2.$$

Proof. Clearly we have

$$\bigcap_{\xi \in X_1 \otimes X_2} \operatorname{Ker}(\xi) \subseteq \bigcap_{\xi_1 \in X_1, \xi_2 \in X_2} \operatorname{Ker}(\xi_1 \otimes \xi_2)$$

Let $\xi \in X_1 \otimes X_2$. Then $\xi = \sum_{i=1}^{n} \xi_1^i \otimes \xi_2^i$ where $n \in \mathbb{N}, n \geq 1, \xi_1^i \in X_1$ and $\xi_2^i \in X_2$ for every $i = 1, \ldots, n$. Then

$$\bigcap_{i=1}^{n} \operatorname{Ker}\left(\xi_1^i \otimes \xi_2^i\right) \subseteq \operatorname{Ker}(\xi)$$

so that we have

$$\bigcap_{\xi_1 \in X_1, \xi_2 \in X_2} \operatorname{Ker}(\xi_1 \otimes \xi_2) \subseteq \bigcap_{\xi \in X_1 \otimes X_2} \operatorname{Ker}(\xi)$$

and we deduce that

$$(X_1 \otimes X_2)^{\perp} = \bigcap_{\xi \in X_1 \otimes X_2} \operatorname{Ker}(\xi) = \bigcap_{\xi_1 \in X_1, \xi_2 \in X_2} \operatorname{Ker}(\xi_1 \otimes \xi_2) \tag{15.2}$$

Recall that, by Proposition 1.38,for every $\xi_1^* \in X_1^*, \xi_2^* \in X_2^*$, the assignment $\xi_1 \otimes \xi_2 \mapsto \xi_1^*(\xi_1)\, \xi_2^*(\xi_2)$ defines a k-linear map $\Lambda_{\xi_1^*, \xi_2^*} : X_1 \otimes X_2 \to k$. Moreover the assignment $\xi_1^* \otimes \xi_2^* \mapsto \Lambda_{\xi_1^*, \xi_2^*}$ defines an injective k-linear map

$$\Lambda = \Lambda_{X_1, X_2} : X_1^* \otimes X_2^* \to (X_1 \otimes X_2)^*.$$

For every $i = 1, 2$, let $\Gamma_i : V_i \to X_i^*$ be the map defined by setting

$$\Gamma_i(v_i) = \widetilde{v}_{i|X_i} \text{ where } \widetilde{v}_i : V_i \to k \text{ is the evaluation map.}$$

Let $\pi_i : V_i \to V_i \otimes V_i/X_i^{\perp}$ be the canonical projection. Since $\mathrm{Ker}\,(\Gamma_i) = X_i^{\perp}$, there exists an injective map $\overline{\Gamma}_i : V_i/X_i^{\perp} \to X_i^*$ such that $\overline{\Gamma}_i \circ \pi_i = \Gamma_i$. Let

$$T = \Lambda \circ \left(\overline{\Gamma}_1 \otimes \overline{\Gamma}_2\right) \circ (\pi_1 \otimes \pi_2) : V_1 \otimes V_2 \to (X_1 \otimes X_2)^*$$

For every $i = 1, 2$, let $\xi_i \in X_i$ and $v_i \in V_i$. We compute

$$\begin{gathered}[T (v_1 \otimes v_2)] (\xi_1 \otimes \xi_2) \left\{ \left[\Lambda \circ \left(\overline{\Gamma}_1 \otimes \overline{\Gamma}_2\right) \circ (\pi_1 \otimes \pi_2)\right] (v_1 \otimes v_2) \right\} (\xi_1 \otimes \xi_2) = \\ = [(\Lambda \circ \Gamma_i) (v_1 \otimes v_2)] (\xi_1 \otimes \xi_2) = \Lambda \left(\widetilde{v}_{1|X_1} \otimes \widetilde{v}_{2|X_2}\right) (\xi_1 \otimes \xi_2) = \xi_1 (v_1) \otimes \xi_2 (v_2) .\end{gathered}$$

Then, by using (15.2), we deudce that

$$\mathrm{Ker}\,(T) = \bigcap_{\xi \in X_1 \otimes X_2} \mathrm{Ker}\,(\xi) = \bigcap_{\xi_1 \in X_1, \xi_2 \in X_2} \mathrm{Ker}\,(\xi_1 \otimes \xi_2) = (X_1 \otimes X_2)^{\perp}$$

On the other hand, since $\Lambda \circ \left(\overline{\Gamma}_1 \otimes \overline{\Gamma}_2\right)$ is injective, we get

$$\mathrm{Ker}\,(T) = \mathrm{Ker}\,(\pi_1 \otimes \pi_2) = V_1 \otimes X_2^{\perp} + X_1^{\perp} \otimes V_2$$

□

Lemma 15.4. *Let V be a vector space over a field k and let*

$$\{0\} = V_0 \subseteq V_1 \subseteq V_2 \subseteq ...$$

be an ascending chain of subspaces of V. Then

$$\bigcap_{i=0}^{n} (V \otimes V_{n-i} + V_i \otimes V) = \sum_{i=1}^{n} V_i \otimes V_{n+1-i}.$$

Proof. We have

$$\begin{aligned}(V \otimes V_n + V_0 \otimes V) \cap (V \otimes V_0 + V_n \otimes V) &= \\ = (V \otimes V_n + \{0\} \otimes V) \cap (V \otimes \{0\} + V_n \otimes V) \\ = (V \otimes V_n) \cap (V_n \otimes V) \\ = V_n \otimes V_n.\end{aligned}$$

Therefore we get

$$\begin{aligned}\bigcap_{i=0}^{n} (V \otimes V_{n-i} + V_i \otimes V) &= (V \otimes V_n + V_0 \otimes V) \cap (\bigcap_{i=1}^{n-1} V \otimes V_{n-i} + V_i \otimes V) \cap (V \otimes V_0 + V_n \otimes V) \\ &= (V_n \otimes V_n) \cap (\bigcap_{i=1}^{n-1} V \otimes V_{n-i} + V_i \otimes V) \\ &= \bigcap_{i=1}^{n-1} (V_n \otimes V_{n-i} + V_i \otimes V_n).\end{aligned}$$

Thus we may assume

$$V = V_n = \bigcup_{i \leqslant n} V_i.$$

and we have to prove that

$$\bigcap_{i=1}^{n-1} (V_n \otimes V_{n-i} + V_i \otimes V_n) = \sum_{i=1}^{n} V_i \otimes V_{n+1-i}.$$

Now for $i = 1, \ldots, n$, let $W_i \subseteq V_i$ be such that

$$V_i = V_{i-1} \oplus W_i.$$

Then

$$V_i = \bigoplus_{a=1}^{i} W_a$$

so that

$$\begin{aligned}
V_n \otimes V_{n-i} + V_i \otimes V_n &= \left(\bigoplus_{a=1}^{n} W_a\right) \otimes \left(\bigoplus_{b=1}^{n-i} W_b\right) + \left(\bigoplus_{a=1}^{i} W_a\right) \otimes \left(\bigoplus_{b=1}^{n} W_b\right) \\
&= \bigoplus_{a=1}^{n} \bigoplus_{b=1}^{n-i} (W_a \otimes W_b) + \bigoplus_{a=1}^{i} \bigoplus_{b=1}^{n} (W_a \otimes W_b) = \\
&= \bigoplus_{a=1}^{i} \bigoplus_{b=1}^{n-i} (W_a \otimes W_b) + \bigoplus_{a=i}^{n} \bigoplus_{b=1}^{n-i} (W_a \otimes W_b) \\
&\quad + \bigoplus_{a=1}^{i} \bigoplus_{b=1}^{n-i} (W_a \otimes W_b) + \bigoplus_{a=1}^{i} \bigoplus_{b=i}^{n} (W_a \otimes W_b) \\
&= \bigoplus_{a=1}^{i} \bigoplus_{b=1}^{n-i} (W_a \otimes W_b) + \bigoplus_{a=i}^{n} \bigoplus_{b=1}^{n-i} (W_a \otimes W_b) + \bigoplus_{a=1}^{i} \bigoplus_{b=i}^{n} (W_a \otimes W_b) \\
&= \bigoplus_{a=1}^{n} \bigoplus_{b=1}^{n-i} (W_a \otimes W_b) + \bigoplus_{a=1}^{i} \bigoplus_{b=i}^{n} (W_a \otimes W_b) \\
&= \bigoplus_{\substack{a \le i \\ \text{or} \\ b \le n-i}} W_a \otimes W_b
\end{aligned}$$

i.e.

$$V_n \otimes V_{n-i} + V_i \otimes V_n = \bigoplus_{\substack{a \le i \\ \text{or} \\ b \le n-i}} W_a \otimes W_b \tag{15.3}$$

Now

$$\bigcap_{i=0}^{n} \bigoplus_{\substack{a\leq i \\ \text{or} \\ b\leq n-i}} (W_a \otimes W_b) = \bigoplus_{a+b\leq n+1} (W_a \otimes W_b) \tag{15.4}$$

In fact it is clear that

$$\bigoplus_{a+b\leq n+1} (W_a \otimes W_b) \subseteq \bigoplus_{\substack{a\leq i \\ \text{or} \\ b\leq n-i}} (W_a \otimes W_b) \text{ for every } i = 0, \dots, n$$

Conversely let $x \in \bigcap_{i=0}^{n} \bigoplus_{\substack{a\leq i \\ \text{or} \\ b\leq n-i}} (W_a \otimes W_b)$. Since $x \in V_n \otimes V_n$ and $V_n = \bigoplus_{t=1}^{n} W_t$ we can write

$$x = \sum_{t,s=1}^{n} x_t \otimes y_s \text{ where } x_t \in W_t \text{ and } y_s \in W_s.$$

Assume that $x \notin \bigoplus_{a+b\leq n+1} (W_a \otimes W_b)$. Then there exist t and s such that $1 \leq t, s \leq n$, $t + s > n + 1$ and $x_t \otimes y_s \neq 0$.

Let $k \in \mathbb{N}$ such that either $n = 2k$ or $n = 2k + 1$.

Assume that $t \leq k$. Then if $s \leq n - k$ we would get $t + s \leq n$. Therefore $n - k < s$. If $k = n - (n - k) \leq s$ then we would get $n = n - k + k < s$. Therefore $n - k < s$ implies $s < k$ and hence $t + s < 2k \leq n$.Contradiction.

Assume that $k < t$. Since

$$x \in \bigoplus_{\substack{a\leq k \\ \text{or} \\ b\leq n-k}} (W_a \otimes W_b)$$

we deduce that $s \leq n - k$. Now if $n - k \leq t$ we would get $n = n - k + k < t$. Thus $t < n - k$ and hence $t + s < n - k + n - k = n + n - 2k \leq n + 1$. Contradiction. Therefore (15.4) is proved. Let us show that

$$\bigoplus_{a+b\leq n+1} (W_a \otimes W_b) = \sum_{i=1}^{n} V_i \otimes V_{n+1-i}. \tag{15.5}$$

In fact if $a + b \leq n + 1$, then $W_b \subseteq V_{n+1-a}$ and hence

$$W_a \otimes W_b \subseteq V_a \otimes V_{n+1-a}.$$

On the other hand

$$V_i \otimes V_{n+1-i} = \left(\bigoplus_{a=1}^{i} W_a\right) \otimes \left(\bigoplus_{b=1}^{n+1-i} W_b\right) = \bigoplus_{a=1}^{i} \bigoplus_{b=1}^{n+1-i} (W_a \otimes W_b) \subseteq \bigoplus_{a+b\leq n+1} (W_a \otimes W_b).$$

Therefore we get

$$\bigcap_{i=0}^{n}(V\otimes V_{n-i}+V_i\otimes V)\overset{(15.3)}{=}\bigcap_{i=0}^{n}\bigoplus_{\substack{a\leq i\\ \text{or}\\ b\leq n-i}}(W_a\otimes W_b)\overset{(15.4)}{=}\bigoplus_{a+b\leq n+1}(W_a\otimes W_b)\overset{(15.5)}{=}\sum_{i=1}^{n}V_i\otimes V_{n+1-i}.$$

□

Lemma 15.5. *Let D be a subspace of a vector space C over a field k and let I, J, X and Y be subspaces of D. Then we have:*

$$(I\otimes D+D\otimes J)\cap(X\otimes Y)=(I\cap X)\otimes Y+X\otimes(J\cap Y)$$

In particular for $D=C$ and $X=Y=E$ we get

$$(I\otimes C+C\otimes J)\cap(E\otimes E)=(I\cap E)\otimes E+E\otimes(J\cap E).$$

Proof. Let $p_I : D\to D/I$ and $p_J : D\to D/J$ be the canonical projections. Then we have

$$\begin{aligned}(I\otimes D+D\otimes J)\cap(X\otimes Y)&=\mathrm{Ker}\,(p_I\otimes p_J)\cap(X\otimes Y)=\mathrm{Ker}\,(p_I\otimes p_J)_{|X\otimes Y}=\\ &=\mathrm{Ker}\left(p_{I|X}\otimes p_{J|Y}\right)=\mathrm{Ker}\left(p_{I|X}\right)\otimes Y+X\otimes\mathrm{Ker}\left(p_{J|Y}\right)=\\ &=(I\cap X)\otimes Y+X\otimes(J\cap Y).\end{aligned}$$

□

Lemma 15.6. *Let $(W_i^1)_{i\in I}$ be a finite family of subspaces of a vector space V_1 and let $(W_i^1)_{i\in I}$ be a finite family of subspaces of a vector space V_2. Then*

$$\bigcap_{j\in J,i\in I}\left(V_1\otimes W_j^2+W_i^1\otimes V_2\right)=V_1\otimes\left(\bigcap_{j\in J}W_j^2\right)+\left(\bigcap_{i\in I}W_i^1\right)\otimes V_2.$$

Proof. For every $i\in I$ and $j\in J$, let $p_i^1 : V_1\to V_1/W_i^1$ and $p_j^2 : V_2\to V_2/W_j^2$ be the canonical projection. Then

$$V_1\otimes W_j^2+W_i^1\otimes V_2=\mathrm{Ker}\left(p_i^1\otimes p_j^2\right)$$

so that

$$\bigcap_{j\in J,i\in I}\left(V_1\otimes W_j^2+W_i^1\otimes V_2\right)=\bigcap_{i\in I,j\in J}\mathrm{Ker}\left(p_i^1\otimes p_j^2\right)=\mathrm{Ker}\,(\Delta)$$

where

$$\Delta : V_1\otimes V_2\to\prod_{i\in I,j\in J}V_1/W_i^1\otimes V_2/W_j^2$$

is the diagonal morphism of the family $\left(p_i^1 \otimes p_j^2\right)_{i\in I, j\in J}$. Let $\Delta_1 : V_1 \to \prod_{i\in I} V_1/W_i^1$ be the diagonal morphism of the family $(p_i^1)_{i\in I}$ and let $\Delta_2 : V_2 \to \prod_{j\in J} V_2/W_j^2$ be the diagonal morphism of the family $\left(p_j^2\right)_{j\in I}$. Let

$$\Phi : \prod_{i\in I, j\in J} V_1/W_i^1 \otimes V_2/W_j^2 \to \prod_{i\in I} V_1/W_i^1 \otimes \prod_{j\in J} V_2/W_j^2$$

be the canonical isomorphism. Then

$$\mathrm{Ker}\,(\Delta) = \mathrm{Ker}\,(\Phi \circ \Delta) = \mathrm{Ker}\,(\Delta_1 \otimes \Delta_2)$$

Therefore we obtain

$$\begin{aligned} \bigcap_{j\in J, i\in I} \left(V_1 \otimes W_j^2 + W_i^1 \otimes V_2\right) &= \mathrm{Ker}\,(\Delta_1 \otimes \Delta_2) = V_1 \otimes \mathrm{Ker}\,(\Delta_2) + \mathrm{Ker}\,(\Delta_1) \otimes V_2 = \\ &= V_1 \otimes \left(\bigcap_{j\in J} W_j^2\right) + \left(\bigcap_{i\in I} W_i^1\right) \otimes V_2. \end{aligned}$$

□

Bibliography

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