(CO)MONADS AND DESCENT

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1. Introduction

The following is inspired to the Introductions of [MS] and [KLV] respectively.

Let R be an arbitrary ring and let us denote the category of right modules over R by Mod-R. If S is an extension of R, i.e. there is an arbitrary morphism of rings with unit $R \to S$, then the categories Mod-R and Mod-S are connected by a pair of adjoint functors (f^*, f_*) where $f^* : Mod$-$R \to Mod$-S, $f^*(N) = N \otimes_R S$ is the so called extension of scalars functor and $f^* : Mod$-$S \to Mod$-R, $f^*(M) = M$ regarded as an R-module via f, is the restriction of scalars functor. Roughly speaking, classical descent theory of modules and morphisms is concerned with the description of the image of f^*. To be more specific we list below three problems of classical descent theory.

(1) **(Descent of modules)** *Let M be a right S-module. Is there any right R-module N such that $M \simeq N \otimes_R S$ as right S-modules?*
(2) **(Descent of morphisms)** *Let N and N' be right R-modules and let $f : N \otimes_R S \to N' \otimes_R S$ be a morphism of right S-modules. Does there exist a morphism of right R-modules $g : N \to N'$ such that $f = g \otimes id_S$?*
(3) **(Classifications of S-forms)** *Given a right R-module N classify all right R-modules N' such that $N' \otimes_R S \simeq N \otimes_R S$.*

A well-known example, due to Grothendieck, is faithfully flat descent theory ($R \to S$ is now a faithfully flat extension of commutative rings), see [Gro] and [KO]. The existence of an $N \in Mod$-R as in the first problem is equivalent to the existence of a "descent datum" on M. Let us briefly recall the definition of descent datum in this setting. First let us note that we have an algebra morphism $i_S : S \to S \otimes_R S$, $i_S(x) = x \otimes 1$. Hence, for any $M \in Mod$-S, the S-modules $S \otimes_R M$ and $M \otimes_R S$ are modules over $S \otimes_R S$ via extension of scalars from S to $S \otimes_R S$. Let $g : S \otimes_R M \to M \otimes_R S$ be an arbitrary $S \otimes_R S$-linear map. We define $g_1 := S \otimes_R g$ and $g_3 := g \otimes_R S$ and let g_2 be the map from $S \otimes_R S \otimes_R M$ to $M \otimes_R S \otimes_R S$ given by

$$g_2(s \otimes t \otimes m) = \sum m_j \otimes t \otimes s_j,$$

where $g(s \otimes m) = \sum m_j \otimes s_j$. Then a descent datum on M is an $S \otimes_R S$-linear map $g : S \otimes_R M \to M \otimes_R S$ such that $g_2 = g_3 g_1$ and $\sum m_j s_j = m$ if $g(1 \otimes m) = \sum m_j \otimes s_j$.(See Theorem 6.17 and the considerations just above it).

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One can easily describe descent data in another equivalent way. Let $\sigma_M : M \to M \otimes_R S$ be the map $m \mapsto m \otimes_R 1$. Then any $S \otimes_R S$-linear map $g : S \otimes_R M \to M \otimes_R S$ is uniquely determined by the map $g\sigma_M : M \to M \otimes_R S$. Let us denote $g\sigma_M$ by ρ_g. Then g is a descent datum if and only if ρ_g is a morphism of right S-modules and satisfies the following properties (see Theorem 6.17)

$$(\rho_g \otimes_R S)\rho_g = (\sigma_M \otimes_R S)\rho_g,$$
$$\mu_M \rho_g = \mathrm{Id}_M,$$

This means that $(M, \rho_g) \in {}^{\mathbb{C}}(Mod\text{-}A)$ where $\mathbb{C}$ is the canonical comonad of the adjunction (f^*, f_*).

In the paper [MS], extending results by Nuss [Nu] on noncommutative rings, the situation (f^*, f_*) was replaced $({}_{\mathbb{A}}F, {}_{\mathbb{A}}U)$ where $\mathbb{A}$ is a monad over a category $\mathcal{A}$ and ${}_{\mathbb{A}}F : \mathcal{A} \to {}_{\mathbb{A}}\mathcal{A}$ is the free functor while ${}_{\mathbb{A}}U : {}_{\mathbb{A}}\mathcal{A} \to \mathcal{A}$ is the forgetful functor. Let $\mathbb{A}^* = ({}_{\mathbb{A}}F{}_{\mathbb{A}}U, {}_{\mathbb{A}}Fu_{\mathbb{A}}U, \lambda)$ be the comonad on the category ${}_{\mathbb{A}}\mathcal{A}$ associated to this adjunction. In this context, it was proved that, if the monad $\mathbb{A}$ is equipped with a "compatible flip" $\Phi : A^2 \to A^2$, then to give an $\mathbb{A}^*$-comodule structure on an $\mathbb{A}$-module (X, μ) is equivalent to giving a "symmetry" on X, that is an involution $AX \to AX$ satisfying some suitable conditions.

Unfortunately, the following natural example, which is a direct generalization of the classical case of commutative rings, does not fit into their general context: let $\mathcal{C}$ be a braided monoidal category and let (S, m_S, u_S) be an algebra in C, then the braiding

$$c_{S,S} : S \otimes S \to S \otimes S$$

induces a natural isomorphism $\Phi : A^2 \to A^2$ on the monad $\mathbb{A} = (- \otimes_R S, - \otimes_R m_S, (- \otimes_R u_S) \circ r_-)$, but this natural isomorphism is not a flip unless the braiding is a symmetry and the monoid is commutative. To encompass this example, in [KLV] the notion of BD-law on a monad $\mathbb{A}$ is introduced (see Definitions 6.1) and, given a BD-law Φ on the monad $\mathbb{A}$, the notion of "compatible flip" is substituted by Φ-braiding on an $\mathbb{A}$-module. In these notes we prefer to call this quasi Φ-symmetry (see Definitions 6.2) since we could not find meaningful relation with the usual meaning of a braiding (on the other hand a BD-law on a monad $\mathbb{A}$ could be called a braiding on the monad $\mathbb{A}$). We give a self-contained proof of [KLV, Theorem 3.7] (see Theorem 6.12) which shows that the category of quasi Φ-symmetries is isomorphic to the category of $\mathbb{A}^*$-comodules.

These are the notes of a short Ph.D. course I gave at the Department of Mathematics and Computer Science of the University of Ferrara during January and February 2016.

I would like to thank Leonardo Spinosa for a careful reading of them.

2. Monads

Definition 2.1. A *monad* on a category $\mathcal{A}$ is a triple $\mathbb{A} = (A, m_A, u_A)$, where $A : \mathcal{A} \to \mathcal{A}$ is a functor, $m_A : AA \to A$ and $u_A : \mathcal{A} \to A$ are functorial morphisms satisfying the associativity and the unitality conditions:

$$m_A \circ (m_A A) = m_A \circ (A m_A) \quad \text{and} \quad m_A \circ (A u_A) = A = m_A \circ (u_A A). \tag{1}$$

Definition 2.2. A *morphism between two monads* $\mathbb{A} = (A, m_A, u_A)$ and $\mathbb{B} = (B, m_B, u_B)$ on a category $\mathcal{A}$ is a functorial morphism $\varphi : A \to B$ such that

$$\varphi \circ m_A = m_B \circ (\varphi\varphi) \quad \text{and} \quad \varphi \circ u_A = u_B.$$

Here $\varphi\varphi = \varphi B \circ A\varphi = B\varphi \circ \varphi A$.

Example 2.3. Let $f : R \to S$ be a morphism of rings. Let ${}_RS_R$ denote the R-bimodule structure on S defined by

$$r \cdot s = f(r)\, s \qquad s \cdot r = s f(r) \qquad \text{for every } r \in R \text{ and } s \in S.$$

Since

$$(s \cdot r)\, s' = (s f(r))\, s' = s\, (f(r)\, s') = s\, (r \cdot s)$$

the multiplication $m : S \times S \to S$ on S factorizes through $S \otimes_R S$ i.e. there is a group morphism

$$m_S : S \otimes_R S \to S$$

such that $m_S = \tau \circ m$ where $\tau : S \times S \to S \otimes_R S$ is the canonical map. m_S is a morphism of S-S-bimodules. Clearly we get that

$$m_S \circ (S \otimes_R m_S) = m_S \circ (m_S \otimes_R S) \tag{2}$$

For any right R-module M let

$$r_M : M \to M \otimes_R R$$

denote the usual isomorphism defined by $r_M(x) = x \otimes_R 1_R$. It is easy to check that this defines a functorial isomorphism

$$r_- : Mod\text{-}R \to - \otimes_R R.$$

Set

$$u_S = - \otimes_R f : - \otimes_R R \to - \otimes_R S$$

and

$$u_A = (- \otimes_R u_S) \circ r_- : Mod\text{-}R \to - \otimes_R R \to - \otimes_R S$$

For every right R-module M

$$u_A M : M \to M \otimes_R S$$

is defined by

$$(u_A M)(x) = x \otimes_R 1_S \qquad \text{for every } x \in M.$$

For every $x \in M$ and $s \in S$ we compute

$$\begin{aligned} [(M \otimes_R m_S) \circ (u_A M \otimes_R S)](x \otimes_R s) &= (M \otimes_R m_S)(x \otimes_R 1_S \otimes_R s) \\ &= (x \otimes_R s) = (M \otimes_R S)(x \otimes_R s) \end{aligned}$$

so that we get

$$(M \otimes_R m_S) \circ (u_A M \otimes_R S) = M \otimes_R S. \tag{3}$$

A similar computation gives

$$(M \otimes_R m_S) \circ (u_A (M \otimes_R S)) = M \otimes_R S \tag{4}$$

Let us consider the triple $\mathbb{A} = (A, m_A, u_A)$ where

$$\begin{aligned} A &= - \otimes_R S : Mod\text{-}R \to Mod\text{-}R \\ m_A &= - \otimes_R m_S : - \otimes_R S \otimes_R S \to - \otimes_R S \\ u_A &= (- \otimes_R u_S) \circ r_- : Mod\text{-}R \to - \otimes_R S \end{aligned}$$

We prove that $\mathbb{A} = (A, m_A, u_A)$ is a monad on the category $Mod\text{-}R$. For every $M \in Mod\text{-}R$ we compute

$$\begin{aligned} [m_A \circ (m_A A)](M) &= (M \otimes_R m_S) \circ (M \otimes_R S \otimes_R m_S) = \\ & M \otimes_R [m_S \circ (S \otimes_R m_S)] \overset{(2)}{=} M \otimes_R [m_S \circ (m_S \otimes_R S)] \\ &= (M \otimes_R m_S) \circ (M \otimes_R m_S \otimes_R S) = [m_A \circ (A m_A)](M) \end{aligned}$$

$$\begin{aligned} [m_A \circ (A u_A)] M &= [(- \otimes_R m_S) \circ (u_A \otimes_R S)] M \\ &= (M \otimes_R m_S) \circ (u_A M \otimes_R S) \overset{3}{=} M \otimes_R S = AM \end{aligned}$$

and

$$\begin{aligned} [m_A \circ (u_A A)] M &= [(- \otimes_R m_S) \circ (u_A (- \otimes_R S))] M \\ &= (M \otimes_R m_S) \circ (u_A (M \otimes_R S)) \overset{4}{=} M \otimes_R S = AM. \end{aligned}$$

Proposition 2.4 ([H]). *Let (L, R) be an adjunction with unit η and counit ϵ where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$. Then $\mathbb{A} = (RL, R\epsilon L, \eta)$ is a monad on the category $\mathcal{B}$.*

Proof. We have to prove that

$$(R\epsilon L)\circ(RLR\epsilon L)=(R\epsilon L)\circ(R\epsilon LRL)\quad\text{and}$$
$$(R\epsilon L)\circ RL\eta=RL=(R\epsilon L)\circ(\eta RL)\,.$$

In fact we have

$$(R\epsilon L)\circ(RLR\epsilon L)\overset{\epsilon}{=}(R\epsilon L)\circ(R\epsilon LRL)$$

and

$$(R\epsilon L)\circ RL\eta\overset{(L,R)}{=}RL\overset{(L,R)}{=}(R\epsilon L)\circ(\eta RL)\,.$$

□

EXERCISE 2.5. Let A,B rings and let M be an B-A-bimodule. Consider the functors

$$\begin{aligned}L&=-\otimes_B M:Mod\text{-}B\to Mod\text{-}A\\ R&=\mathrm{Hom}_A(M,-):Mod\text{-}A\to Mod\text{-}B.\end{aligned}$$

Then $(L,R)=(-\otimes_B M,\mathrm{Hom}_A(M,-))$ is an adjunction. Compute the monad $\mathbb{RL}$ associated to this adjunction. Moreover, compute the monad $\mathbb{RL}$ in the particular case $B=R,A=S,f:R\to S$ is a ring morphism and $M=S$ endowed with the left B-module structure defined by f.

DEFINITION 2.6. A *module* for a monad $\mathbb{A}=(A,m_A,u_A)$ on a category $\mathcal{A}$ is a pair $\left(X,{}^A\mu_X\right)$ where $X\in\mathcal{A}$ and ${}^A\mu_X:AX\to X$ is a morphism in $\mathcal{A}$ such that

$$ {}^A\mu_X\circ\left(A^A\mu_X\right)={}^A\mu_X\circ(m_AX)\quad\text{and}\quad X={}^A\mu_X\circ(u_AX)\,. \tag{5}$$

A *morphism* f *between two* $\mathbb{A}$*-modules* $\left(X,{}^A\mu_X\right)$ and $\left(X',{}^A\mu_{X'}\right)$ is a morphism $f:X\to X'$ in $\mathcal{A}$ such that

$${}^A\mu_{X'}\circ(Af)=f\circ{}^A\mu_X.$$

We will denote by ${}_{\mathbb{A}}\mathcal{A}$ the category of $\mathbb{A}$-modules and their morphisms. This is the so-called *Eilenberg-Moore category* which is sometimes also denoted by $\mathcal{A}^{\mathbb{A}}$.

REMARK 2.7. Let $\mathbb{A}=(A,m_A,u_A)$ be a monad on a category $\mathcal{A}$ and let $\left(X,{}^A\mu_X\right)\in{}_{\mathbb{A}}\mathcal{A}$. From the unitality property of ${}^A\mu_X$ we deduce that ${}^A\mu_X$ is an epimorphism for every $\left(X,{}^A\mu_X\right)\in{}_{\mathbb{A}}\mathcal{A}$ and that u_AX is mono for every $\left(X,{}^A\mu_X\right)\in{}_{\mathbb{A}}\mathcal{A}$, i.e. u_A is a monomorphism.

EXAMPLE 2.8. Consider the monad $\mathbb{A}=(A,m_A,u_A)$ on Mod-R of Example 2.3. We want to understand the category of modules with respect to this monad. The underlying category is $\mathcal{A}=Mod\text{-}R$. Let $\left(X,{}^A\mu_X\right)\in{}_{\mathbb{A}}(Mod\text{-}R)$. This means that

$${}^A\mu_X:AX=X\otimes_R S\to X$$

is a morphism in Mod-R such that ${}^A\mu_X\circ\left(A^A\mu_X\right)={}^A\mu_X\circ(m_AX)$ and $X={}^A\mu_X\circ(u_AX)\,.$ For every $x\in X$ and $s\in S$ write $xs={}^A\mu_X(x\otimes_R s)$ .Then we get

$$\left({}^A\mu_X\circ\left(A^A\mu_X\right)\right)(x\otimes_R s\otimes_R s')={}^A\mu_X(xs)\otimes_R s'=(xs)s'$$

$$\left({}^A\mu_X\circ(m_AX)\right)(x\otimes_R s\otimes_R s')={}^A\mu_X(x\otimes_R ss')=x(ss')$$

$$\left({}^A\mu_X\circ(u_AX)\right)(x)={}^A\mu_X(x\otimes_R 1_S)=x1_S$$

Let $\tau:X\times S\to X\otimes_R S$ denote the canonical map. Then, in view of the equalities above we have that $\left(X,{}^A\mu_X\circ\tau\right)\in Mod\text{-}S$. It is easy to see that the assignment $\left(X,{}^A\mu_X\right)\mapsto\left(X,{}^A\mu_X\circ\tau\right)$ defines an isomorphism of categories from ${}_{\mathbb{A}}\mathcal{A}$ to Mod-S.

DEFINITION 2.9. Let $\mathbb{A}=(A,m_A,u_A)$ be a monad on $\mathcal{A}$. The functor

$$\begin{array}{rccc}{}_{\mathbb{A}}U: & {}_{\mathbb{A}}\mathcal{A} & \to & \mathcal{A}\\ & \left(X,{}^A\mu_X\right) & \to & X\\ & f & \to & f\end{array}$$

is called the *forgetful functor*.

Proposition 2.10. *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad on a category $\mathcal{A}$. Let $f, g : \left(X, {}^A\mu_X\right) \to \left(Y, {}^A\mu_Y\right)$ be morphisms in ${}_{\mathbb{A}}\mathcal{A}$. Then*

$$f = g \Leftrightarrow {}_{\mathbb{A}}Uf = {}_{\mathbb{A}}Ug$$

i.e. the functor ${}_{\mathbb{A}}U : {}_{\mathbb{A}}\mathcal{A} \to \mathcal{A}$ is faithful

Proposition 2.11. *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad on a category $\mathcal{A}$. Then ${}_{\mathbb{A}}U$ reflects isomorphisms.*

Proof. Let $f : \left(X, {}^A\mu_X\right) \to \left(Y, {}^A\mu_Y\right)$ be a morphism in ${}_{\mathbb{A}}\mathcal{A}$ such that ${}_{\mathbb{A}}Uf$ is an isomorphism in $\mathcal{A}$. Since

$${}^A\mu_Y \circ (A {}_{\mathbb{A}}Uf) = {}_{\mathbb{A}}Uf \circ {}^A\mu_X$$

we get that

$$({}_{\mathbb{A}}Uf)^{-1} \circ {}^A\mu_Y = {}^A\mu_X \circ \left(A\left({}_{\mathbb{A}}Uf\right)^{-1}\right).$$

which entails that $({}_{\mathbb{A}}Uf)^{-1}$ gives rise to a morphism $g : \left(Y, {}^A\mu_Y\right) \to \left(X, {}^A\mu_X\right)$ such that ${}_{\mathbb{A}}Ug = ({}_{\mathbb{A}}Uf)^{-1}$. Hence

$${}_{\mathbb{A}}U\left(f \circ g\right) = \mathrm{Id}_Y \qquad \text{and} \qquad {}_{\mathbb{A}}U\left(g \circ f\right) = \mathrm{Id}_X$$

so that

$$f \circ g = \mathrm{Id}_{(Y, {}^A\mu_Y)} \qquad \text{and} \qquad g \circ f = \mathrm{Id}_{(X, {}^A\mu_X)}.$$

□

Definition 2.12. Let $\mathbb{A} = (A, m_A, u_A)$ be a monad on $\mathcal{A}$. The functor

$$\begin{array}{rccc} {}_{\mathbb{A}}F : & \mathcal{A} & \to & {}_{\mathbb{A}}\mathcal{A} \\ & X & \to & (AX, m_AX) \\ & f & \to & Af. \end{array}$$

is called the *free functor*.

Proposition 2.13. *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad on $\mathcal{A}$. Then $({}_{\mathbb{A}}F, {}_{\mathbb{A}}U)$ is an adjunction with unit the unit u_A of the monad $\mathbb{A}$*

$$u_A : \mathcal{A} \to {}_{\mathbb{A}}U {}_{\mathbb{A}}F = A.$$

The counit $\lambda_A : {}_{\mathbb{A}}F {}_{\mathbb{A}}U \to {}_{\mathbb{A}}\mathcal{A}$ is uniquely determined by setting

$${}_{\mathbb{A}}U\left(\lambda_A\left(X, {}^A\mu_X\right)\right) = {}^A\mu_X \text{ for every } \left(X, {}^A\mu_X\right) \in {}_{\mathbb{A}}\mathcal{A}.$$

Moreover we have

$$ {}_{\mathbb{A}}U\lambda_{A\mathbb{A}}F = m_A \tag{6}$$

Proof. Let $\left(X, {}^A\mu_X\right) \in {}_{\mathbb{A}}\mathcal{A}$. In view of (5) we have

$${}^A\mu_X \circ \left(A {}^A\mu_X\right) = {}^A\mu_X \circ (m_AX).$$

This means that there exists a morphism

$$\lambda_A\left(X, {}^A\mu_X\right) : (AX, m_AX) = {}_{\mathbb{A}}F {}_{\mathbb{A}}U\left(X, {}^A\mu_X\right) \to \left(X, {}^A\mu_X\right)$$

such that

$${}_{\mathbb{A}}U\lambda_A\left(X, {}^A\mu_X\right) = {}^A\mu_X.$$

It is easy to show that in this way we get a functorial morphism $\lambda_A : {}_{\mathbb{A}}F {}_{\mathbb{A}}U \to {}_{\mathbb{A}}\mathcal{A}$.

Let $\left(X, {}^A\mu_X\right) \in {}_{\mathbb{A}}\mathcal{A}$. We compute

$$\begin{aligned}[({}_{\mathbb{A}}U\lambda_A) \circ (u_{A\mathbb{A}}U)]\left(\left(X, {}^A\mu_X\right)\right) &= ({}_{\mathbb{A}}U\lambda_A)\left(\left(X, {}^A\mu_X\right)\right) \circ (u_{A\mathbb{A}}U)\left(\left(X, {}^A\mu_X\right)\right) \\ &= {}^A\mu_X \circ u_AX \overset{5}{=} X.\end{aligned}$$

From this we deduce that $({}_{\mathbb{A}}U\lambda_A) \circ (u_{A\mathbb{A}}U) = {}_{\mathbb{A}}U$.

Let $X \in \mathcal{A}$. We compute

$${}_{\mathbb{A}}U\left[(\lambda_{A\mathbb{A}}F) \circ ({}_{\mathbb{A}}Fu_A)\right](X) = \left[{}_{\mathbb{A}}U\left(\lambda_{A\mathbb{A}}F\right) \circ \left({}_{\mathbb{A}}U {}_{\mathbb{A}}Fu_A\right)\right](X)$$

$$= {}_{\mathbb{A}}U\left(\lambda_{A\mathbb{A}}F\right)(X) \circ \left({}_{\mathbb{A}}U_{\mathbb{A}}Fu_A\right)(X) = m_A X \circ A u_A X \overset{1}{=} X.$$

From this we deduce that

$${}_{\mathbb{A}}U\left[\left(\lambda_{A\mathbb{A}}F\right) \circ \left({}_{\mathbb{A}}Fu_A\right)\right] = {}_{\mathbb{A}}U\left({}_{\mathbb{A}}F\right)$$

and hence, by Proposition 2.10, that $\left(\lambda_{A\mathbb{A}}F\right) \circ \left({}_{\mathbb{A}}Fu_A\right) = {}_{\mathbb{A}}F$.

Fore every $\left(X, {}^{A}\mu_X\right) \in {}_{\mathbb{A}}\mathcal{A}$ we compute

$$\left({}_{\mathbb{A}}U\lambda_{A\mathbb{A}}F\right) X = {}_{\mathbb{A}}U\lambda_A\left(X, m_A X\right) = m_A X.$$

□

EXERCISE 2.14. Prove that ${}_{\mathbb{A}}FX = (AX, m_A X) \in {}_{\mathbb{A}}\mathcal{A}$.

PROPOSITION 2.15. *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad on a category $\mathcal{A}$. Then for every $Z, W \in {}_{\mathbb{A}}\mathcal{A}$ we have that*

$$Z = W \text{ if and only if } {}_{\mathbb{A}}U(Z) = {}_{\mathbb{A}}U(W) \text{ and } {}_{\mathbb{A}}U(\lambda_A Z) = {}_{\mathbb{A}}U(\lambda_A W).$$

In particular, if $F, G : \mathcal{X} \to {}_{\mathbb{A}}\mathcal{A}$ are functors, we have

$$F = G \text{ if and only if } {}_{\mathbb{A}}UF = {}_{\mathbb{A}}UG \text{ and } {}_{\mathbb{A}}U(\lambda_A F) = {}_{\mathbb{A}}U(\lambda_A G)$$

LEMMA 2.16. *Let (L, R) be an adjunction where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$. and let $\mathbb{A} = (A = RL, m_A = R\epsilon L, u_A = \eta)$ be the associated monad on the category $\mathcal{B}$. Then*

- *for every $X \in \mathcal{A}$ we have that $(RX, R\epsilon X) \in {}_{\mathbb{A}}\mathcal{B}$,*
- *for every morphism $f : X \to X'$ in $\mathcal{A}$ there is a unique morphism $\overline{R(f)} : (RX, R\epsilon X) \to (RX', R\epsilon X')$ in ${}_{\mathbb{A}}\mathcal{B}$ such that ${}_{\mathbb{A}}U\left(\overline{R(f)}\right) = R(f)$*

Proof. For every $X \in \mathcal{A}$ we compute

$$R\epsilon X \circ RLR\epsilon X \overset{\epsilon}{=} R\epsilon X \circ R\epsilon LRX$$

and

$$R\epsilon X \circ \eta RX = RX.$$

Thus we deduce that $(RX, R\epsilon X) \in {}_{\mathbb{A}}\mathcal{B}$. Let $f : X \to X'$ be a morphism in $\mathcal{A}$. We compute

$$R\epsilon X' \circ RLRf \overset{\epsilon}{=} Rf \circ R\epsilon X.$$

Thus we deduce that there is a morphism $\overline{R(f)} : (RX, R\epsilon X) \to (RX', R\epsilon X')$ in ${}_{\mathbb{A}}\mathcal{B}$ such that ${}_{\mathbb{A}}U\left(\overline{R(f)}\right) = R(f)$. This morphism is unique in view of Proposition 2.10. □

DEFINITIONS 2.17. Let (L, R) be an adjunction where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$. and let $\mathbb{A} = (A = RL, m_A = R\epsilon L, u_A = \eta)$ be the associated monad on the category $\mathcal{B}$. In view of Lemma 2.16, we can consider the functor

$$K = {}_R K : \mathcal{A} \to {}_{\mathbb{A}}\mathcal{B}$$

defined by setting

$$K(X) = (RX, R\epsilon X) \qquad \text{and} \qquad K(f) = \overline{R(f)}.$$

This is called the *comparison functor* of the adjunction (L, R). Note that ${}_{\mathbb{A}}U \circ K = R$.

A functor $R : \mathcal{A} \to \mathcal{B}$ which has a left adjoint $L : \mathcal{B} \to \mathcal{A}$ for which the corresponding comparison functor $K : \mathcal{A} \to {}_{\mathbb{A}}\mathcal{B}$ is an equivalence of categories is called *monadic* (*tripleable* in Beck's terminology [[Be2, Definition 3, page 8]]).

PROPOSITION 2.18. *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad on a category $\mathcal{A}$. Then the monad associate to the adjunction $({}_{\mathbb{A}}F, {}_{\mathbb{A}}U)$ is the monad $\mathbb{A}$ and the corresponding comparison functor is the identity on the category ${}_{\mathbb{A}}\mathcal{A}$. In particular the functor ${}_{\mathbb{A}}U$ is monadic.*

Proof. We already observed that ${}_{\mathbb{A}}U \circ {}_{\mathbb{A}}F = A$ and that the unit of this adjunction is u_A. For every $X \in \mathcal{A}$ we compute

$$ {}_{\mathbb{A}}U\lambda_{A\mathbb{A}}FX = {}_{\mathbb{A}}U\lambda_A\left(AX, m_AX\right) = m_AX. $$

We deduce that ${}_{\mathbb{A}}U\lambda_{A\mathbb{A}}F = m_A$ and hence we get that the monad associate to the adjunction $({}_{\mathbb{A}}F, {}_{\mathbb{A}}U)$ is the monad $\mathbb{A}$. Let now $(X,\mu) \in {}_{\mathbb{A}}\mathcal{A}$ and we compute

$$ K\left((X,\mu)\right) = \left({}_{\mathbb{A}}U\left(X,\mu\right), {}_{\mathbb{A}}U\lambda\left(X,\mu\right)\right) = \left(X,\mu\right). $$

Let $f : (X,\mu) \to (X',\mu')$ be a morphism in ${}_{\mathbb{A}}\mathcal{A}$. Then $K\left(f\right) = \overline{{}_{\mathbb{A}}U\left(f\right)}$ where

$$ \overline{{}_{\mathbb{A}}U\left(f\right)} : K\left((X,\mu)\right) = (X,\mu) \to K\left((X',\mu')\right) = (X',\mu') $$

is the unique morphism such that ${}_{\mathbb{A}}U\left(\overline{{}_{\mathbb{A}}U\left(f\right)}\right) = {}_{\mathbb{A}}U\left(f\right)$. Since${}_{\mathbb{A}}U$ is faithful, this entails $K\left(f\right) = \overline{{}_{\mathbb{A}}U\left(f\right)} = f$ and we deduce that $K = {}_{\mathbb{A}}\mathcal{A}$. □

3. Comonads

Definition 3.1. A *comonad* on a category $\mathcal{A}$ is a triple $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$, where $C : \mathcal{A} \to \mathcal{A}$ is a functor, $\Delta^C : C \to CC$ and $\varepsilon^C : C \to \mathcal{A}$ are functorial morphisms satisfying the coassociativity and the counitality conditions

$$ \left(\Delta^C C\right) \circ \Delta^C = \left(C\Delta^C\right) \circ \Delta^C \quad \text{and} \quad \left(C\varepsilon^C\right) \circ \Delta^C = C = \left(\varepsilon^C C\right) \circ \Delta^C. $$

Definition 3.2. A *morphism between two comonads* $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ and $\mathbb{D} = \left(D, \Delta^D, \varepsilon^D\right)$ on a category $\mathcal{A}$ is a functorial morphism $\varphi : C \to D$ such that

$$ \Delta^D \circ \varphi = (\varphi\varphi) \circ \Delta^C \quad \text{and} \quad \varepsilon^D \circ \varphi = \varepsilon^C. $$

Example 3.3. Let $\left(\mathcal{C}, \Delta^{\mathcal{C}}, \varepsilon^{\mathcal{C}}\right)$ an A-coring where A is a ring. This means that

- $\mathcal{C}$ is an A-A-bimodule
- $\Delta^{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \otimes_A \mathcal{C}$ is a morphism of A-A-bimodules
- $\varepsilon^{\mathcal{C}} : \mathcal{C} \to A$ is a morphism of A-A-bimodules satisfying the following

$$ \left(\Delta^{\mathcal{C}} \otimes_A \mathcal{C}\right) \circ \Delta^{\mathcal{C}} = \left(\mathcal{C} \otimes_A \Delta^{\mathcal{C}}\right) \circ \Delta^{\mathcal{C}}, \left(\mathcal{C} \otimes_A \varepsilon^{\mathcal{C}}\right) \circ \Delta^{\mathcal{C}} = r_{\mathcal{C}}^{-1} \quad \text{and} \quad \left(\varepsilon^{\mathcal{C}} \otimes_A \mathcal{C}\right) \circ \Delta^{\mathcal{C}} = l_{\mathcal{C}}^{-1} $$

where $r_{\mathcal{C}} : \mathcal{C} \otimes_A A \to \mathcal{C}$ and $l_{\mathcal{C}} : A \otimes_A \mathcal{C} \to \mathcal{C}$ are the right and left constraints. Let

$$ \begin{aligned} C &= - \otimes_A \mathcal{C} : Mod\text{-}A \to Mod\text{-}A \\ \Delta^C &= - \otimes_A \Delta^{\mathcal{C}} : - \otimes_A \mathcal{C} \to - \otimes_A \mathcal{C} \otimes_A \mathcal{C} \\ \varepsilon^C &= r_- \circ \left(- \otimes_A \varepsilon^{\mathcal{C}}\right) : - \otimes_A \mathcal{C} \to - \otimes_A A \to - \end{aligned} $$

We prove that $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ is a comonad on the category Mod-A. For every $M \in Mod$-A we compute

$$ \begin{gathered} \left[\left(\Delta^C C\right) \circ \Delta^C\right](M) = \left(\Delta^C CM\right) \circ \left(\Delta^C M\right) \\ = \left(M \otimes_A \mathcal{C} \otimes_A \Delta^{\mathcal{C}}\right) \circ \left(M \otimes_A \Delta^{\mathcal{C}}\right) = M \otimes_A \left[\left(\mathcal{C} \otimes_A \Delta^{\mathcal{C}}\right) \circ \Delta^{\mathcal{C}}\right] \\ \overset{\mathcal{C} coring}{=} M \otimes_A \left[\left(\Delta^{\mathcal{C}} \otimes_A \mathcal{C}\right) \circ \Delta^{\mathcal{C}}\right] = \left(M \otimes_A \Delta^{\mathcal{C}} \otimes_A \mathcal{C}\right) \circ \left(M \otimes_A \Delta^{\mathcal{C}}\right) \\ = \left(C\Delta^C M\right) \circ \left(\Delta^C M\right) = \left[\left(C\Delta^C\right) \circ \Delta^C\right](M) \end{gathered} $$

and

$$ \begin{gathered} \left[\left(\varepsilon^C C\right) \circ \Delta^C\right](M) = \left(\varepsilon^C CM\right) \circ \left(\Delta^C M\right) \\ = r_{CM} \circ \left(M \otimes_A \mathcal{C} \otimes_A \varepsilon^{\mathcal{C}}\right) \circ \left(M \otimes_A \Delta^{\mathcal{C}}\right) = r_{M \otimes_A \mathcal{C}} \circ \left(M \otimes_A \left[\left(\mathcal{C} \otimes_A \varepsilon^{\mathcal{C}}\right) \circ \Delta^{\mathcal{C}}\right]\right) \\ \overset{\mathcal{C} coring}{=} r_{M \otimes_A \mathcal{C}} \circ \left(M \otimes_A r_{\mathcal{C}}^{-1}\right) = M \otimes_A \mathcal{C} = CM \end{gathered} $$

$$ \begin{gathered} \left[\left(C\varepsilon^C\right) \circ \Delta^C\right](M) = \left(C\varepsilon^C M\right) \circ \left(\Delta^C M\right) \\ = \left(\left[r_M \circ \left(M \otimes_A \varepsilon^{\mathcal{C}}\right)\right] \otimes_A \mathcal{C}\right) \circ \left(M \otimes_A \Delta^{\mathcal{C}}\right) \\ = \left(r_M \otimes_A \mathcal{C}\right) \circ \left(M \otimes_A \varepsilon^{\mathcal{C}} \otimes_A \mathcal{C}\right) \circ \left(M \otimes_A \Delta^{\mathcal{C}}\right) \end{gathered} $$

$$= (r_M \otimes_A \mathcal{C}) \circ \left[M \otimes_A \left(\left(\varepsilon^{\mathcal{C}} \otimes_A \mathcal{C}\right) \circ \Delta^{\mathcal{C}}\right)\right]$$
$$= (r_M \otimes_A \mathcal{C}) \circ \left(M \otimes_A l_{\mathcal{C}}^{-1}\right) = M \otimes_A \mathcal{C} = CM.$$

PROPOSITION 3.4. *Let (L, R) be an adjunction with unit η and counit ϵ where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$. Then $\mathbb{C} = (LR, L\eta R, \epsilon)$ is a comonad on the category $\mathcal{A}$.*

Proof. Dual to the proof of Proposition 2.4. □

DEFINITION 3.5. A *comodule* for a comonad $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ on a category $\mathcal{A}$ is a pair $\left(X, {}^C\rho_X\right)$ where $X \in \mathcal{A}$ and ${}^C\rho_X : X \to CX$ is a morphism in $\mathcal{A}$ such that

$$\left(C\,{}^C\rho_X\right) \circ {}^C\rho_X = \left(\Delta^C X\right) \circ {}^C\rho_X \quad \text{and} \quad X = \left(\varepsilon^C X\right) \circ {}^C\rho_X.$$

A *morphism between two $\mathbb{C}$-comodules* $\left(X, {}^C\rho_X\right)$ and $\left(X', {}^C\rho_{X'}\right)$ is a morphism $f : X \to X'$ in $\mathcal{A}$ such that

$${}^C\rho_{X'} \circ f = (Cf) \circ {}^C\rho_X.$$

We denote by ${}^{\mathbb{C}}\mathcal{A}$ the category of $\mathbb{C}$-comodule and their morphisms.

DEFINITION 3.6. Let $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad on a category$\mathcal{A}$. The functor

$$\begin{array}{cccc} {}^{\mathbb{C}}U : & {}^{\mathbb{C}}\mathcal{A} & \to & \mathcal{A} \\ & \left(X, {}^C\rho_X\right) & \to & X \\ & f & \to & f \end{array}$$

is called the *forgetful functor*.

PROPOSITION 3.7. *Let $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad on a category $\mathcal{A}$. Let $f, g : \left(X, {}^C\rho_X\right) \to \left(Y, {}^C\rho_Y\right)$ be morphisms in ${}^{\mathbb{C}}\mathcal{A}$. Then*

$$f = g \Leftrightarrow {}_{\mathbb{A}}Uf = {}_{\mathbb{A}}Ug$$

i.e. the functor ${}^{\mathbb{C}}U : {}^{\mathbb{C}}\mathcal{A} \to \mathcal{A}$ is faithful

PROPOSITION 3.8. *Let $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad on a category $\mathcal{A}$. Then ${}^{\mathbb{C}}U$ reflects isomorphisms.*

Proof. Analogous to the proof of Proposition 2.11. □

DEFINITION 3.9. Let $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad on a category $\mathcal{A}$. The functor

$$\begin{array}{cccc} {}^{\mathbb{C}}F : & \mathcal{A} & \to & {}^{\mathbb{C}}\mathcal{A} \\ & X & \to & \left(CX, \Delta^C X\right) \\ & f & \to & Cf \end{array}$$

is called the *free functor*.

PROPOSITION 3.10. *Let $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad on a category $\mathcal{A}$. Then $\left({}^{\mathbb{C}}U, {}^{\mathbb{C}}F\right)$ is an adjunction with counit the counit ε^C of the comonad $\mathbb{C}$*

$$\varepsilon^C : C = {}^{\mathbb{C}}U^{\mathbb{C}}F \to \mathcal{A}.$$

The unit $\gamma^C : {}^{\mathbb{C}}\mathcal{A} \to {}^{\mathbb{C}}F^{\mathbb{C}}U$ is defined by setting

$${}^{\mathbb{C}}U\left(\gamma^C\left(X, {}^C\rho_X\right)\right) = {}^C\rho_X \text{ for every } \left(X, {}^C\rho_X\right) \in {}^{\mathbb{C}}\mathcal{A}.$$

Moreover we have

$${}^{\mathbb{C}}U\gamma^{C\mathbb{C}}F = \Delta^C.$$

LEMMA 3.11. *Let (L, R) be an adjunction where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$. and let $\mathbb{C} = \left(C = LR, \Delta^C = L\eta R, \varepsilon^C = \epsilon\right)$ be the associated comonad on the category $\mathcal{A}$. Then*

- *for every $Y \in \mathcal{B}$ we have that $(LY, L\eta Y) \in {}^{\mathbb{C}}\mathcal{A}$,*
- *for every morphism $f : Y \to Y'$in $\mathcal{B}$ there is a unique morphism $\overline{L(f)} : (LY, L\eta Y) \to (LY', L\eta Y')$ in ${}^{\mathbb{C}}\mathcal{A}$ such that ${}_{\mathbb{A}}U\left(\overline{L(f)}\right) = L(f)$.*

Proof. Dual to the proof of Lemma 2.16. □

DEFINITIONS 3.12. Let (L, R) be an adjunction where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$ and let $\mathbb{C} = \left(C = LR, \Delta^C = L\eta R, \varepsilon^C = \epsilon\right)$ be the associated comonad on the category $\mathcal{A}$. In view of Lemma 3.11, we can consider the functor

$$K^{co} = K_L^{co} : \mathcal{B} \to {}^{\mathbb{C}}\mathcal{A}$$

defined by setting

$$K^{co}(Y) = (LY, L\eta Y) \qquad \text{and} \qquad K^{co}(f) = \overline{L(f)}.$$

This is called the *cocomparison functor* of the adjunction (L, R). Note that ${}^{\mathbb{C}}U \circ K^{co} = L$.

A functor $L : \mathcal{B} \to \mathcal{A}$ which has a right adjoint $R : \mathcal{A} \to \mathcal{B}$ for which the corresponding cocomparison functor $K_L^{co} : \mathcal{B} \to {}^{\mathbb{C}}\mathcal{A}$ is an equivalence of categories is called *comonadic*.

4. Johnstone for Monads

PROPOSITION 4.1 ([Appel] and [J]). *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad on a category $\mathcal{A}$ and let $\mathbb{B} = (B, m_B, u_B)$ be a monad on a category $\mathcal{B}$ and let $Q : \mathcal{A} \to \mathcal{B}$ be a functor. Then there is a bijection between the following collections of data*

$\mathcal{F}$ *functors $\widetilde{Q} : {}_{\mathbb{A}}\mathcal{A} \to {}_{\mathbb{B}}\mathcal{B}$ that are liftings of Q (i.e. ${}_{\mathbb{B}}U\widetilde{Q} = Q{}_{\mathbb{A}}U$)*

$\mathcal{M}$ *functorial morphisms $\Phi : BQ \to QA$ such that*

$$\Phi \circ (m_B Q) = (Qm_A) \circ (\Phi A) \circ (B\Phi) \qquad \textit{and} \qquad \Phi \circ (u_B Q) = Qu_A$$

given by

$$a : \mathcal{F} \to \mathcal{M} \textit{ where } a\left(\widetilde{Q}\right) = \left({}_{\mathbb{B}}U\lambda_B\widetilde{Q}{}_{\mathbb{A}}F\right) \circ (BQu_A)$$

$$b : \mathcal{M} \to \mathcal{F} \textit{ where } b(\Phi)\left(\left(X, {}^{A}\mu_X\right)\right) = \left(QX, \left(Q^{A}\mu_X\right) \circ (\Phi X)\right)$$

$$\textit{and } {}_{\mathbb{B}}U\left[b(\Phi)(f)\right] = Q\left({}_{\mathbb{A}}Uf\right).$$

Proof. First of all let us note that,

$$\lambda_A \circ {}_{\mathbb{A}}F{}_{\mathbb{A}}U\lambda_A \overset{\lambda_A}{=} \lambda_A \circ \lambda_{A\mathbb{A}}F{}_{\mathbb{A}}U$$

so that we get

$${}_{\mathbb{A}}U\lambda_A \circ {}_{\mathbb{A}}U{}_{\mathbb{A}}F{}_{\mathbb{A}}U\lambda_A = {}_{\mathbb{A}}U\lambda_A \circ {}_{\mathbb{A}}U\lambda_{A\mathbb{A}}F{}_{\mathbb{A}}U \overset{(6)}{=} {}_{\mathbb{A}}U\lambda_A \circ m_{A\mathbb{A}}U$$

and hence

$$\text{(7)} \qquad {}_{\mathbb{A}}U\lambda_A \circ A{}_{\mathbb{A}}U\lambda_A = {}_{\mathbb{A}}U\lambda_A \circ m_{A\mathbb{A}}U$$

Let $\widetilde{Q} : {}_{\mathbb{A}}\mathcal{A} \to {}_{\mathbb{B}}\mathcal{B}$ be a lifting of the functor $Q : \mathcal{A} \to \mathcal{B}$ (i.e. ${}_{\mathbb{B}}U\widetilde{Q} = Q{}_{\mathbb{A}}U$).

Define a functorial morphism Φ by setting:

$$\Phi = \left({}_{\mathbb{B}}U\lambda_B\widetilde{Q}{}_{\mathbb{A}}F\right) \circ (BQu_A) : BQ \to {}_{\mathbb{B}}U\widetilde{Q}{}_{\mathbb{A}}F = Q{}_{\mathbb{A}}U{}_{\mathbb{A}}F = QA$$

where $u_A : \mathcal{A} \to {}_{\mathbb{A}}U{}_{\mathbb{A}}F = A$ is also the unit of the adjunction $({}_{\mathbb{A}}F, {}_{\mathbb{A}}U)$ and $\lambda_B : {}_{\mathbb{B}}F{}_{\mathbb{B}}U \to {}_{\mathbb{B}}\mathcal{B}$ is the counit of the adjunction. We have to prove that such a Φ satisfies $\Phi \circ (m_B Q) = (Qm_A) \circ (\Phi A) \circ (B\Phi)$ and $\Phi \circ (u_B Q) = Qu_A$. First, let us note that

$$\text{(8)} \qquad Qm_A = Q{}_{\mathbb{A}}U\lambda_{A\mathbb{A}}F = {}_{\mathbb{B}}U\widetilde{Q}\lambda_{A\mathbb{A}}F$$

Now let us compute

$$\begin{aligned}(Qm_A) \circ (\Phi A) \circ (B\Phi) &= (Qm_A) \circ \left({}_{\mathbb{B}}U\lambda_B\widetilde{Q}{}_{\mathbb{A}}FA\right) \circ (BQu_AA) \\ &\quad \circ \left(B{}_{\mathbb{B}}U\lambda_B\widetilde{Q}{}_{\mathbb{A}}F\right) \circ (BBQu_A) \\ &\overset{(8)}{=} \left({}_{\mathbb{B}}U\widetilde{Q}\lambda_{A\mathbb{A}}F\right) \circ \left({}_{\mathbb{B}}U\lambda_B\widetilde{Q}{}_{\mathbb{A}}FA\right) \circ (BQu_AA) \\ &\quad \circ \left(B{}_{\mathbb{B}}U\lambda_B\widetilde{Q}{}_{\mathbb{A}}F\right) \circ (BBQu_A) \\ &= {}_{\mathbb{B}}U\left[\left(\widetilde{Q}\lambda_{A\mathbb{A}}F\right) \circ \left(\lambda_B\widetilde{Q}{}_{\mathbb{A}}FA\right) \circ \left({}_{\mathbb{B}}FQu_AA\right)\right]\end{aligned}$$

$$\circ\left(B_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F\right)\circ(BBQu_A)$$

$$\overset{\lambda_B}{=}{}_{\mathbb{B}}U\left[\left(\lambda_B\widetilde{Q}_{\mathbb{A}}F\right)\circ\left({}_{\mathbb{B}}F_{\mathbb{B}}U\widetilde{Q}\lambda_{A\mathbb{A}}F\right)\circ({}_{\mathbb{B}}FQu_AA)\right]$$

$$\circ\left(B_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F\right)\circ(BBQu_A)$$

$$\overset{\widetilde{Q}\text{lifting}}{=}{}_{\mathbb{B}}U\left[\left(\lambda_B\widetilde{Q}_{\mathbb{A}}F\right)\circ({}_{\mathbb{B}}FQ_{\mathbb{A}}U\lambda_{A\mathbb{A}}F)\circ({}_{\mathbb{B}}FQu_AA)\right]$$

$$\circ\left(B_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F\right)\circ(BBQu_A)$$

$$\overset{(8)}{=}{}_{\mathbb{B}}U\left[\left(\lambda_B\widetilde{Q}_{\mathbb{A}}F\right)\circ({}_{\mathbb{B}}FQm_A)\circ({}_{\mathbb{B}}FQu_AA)\right]$$

$$\circ\left(B_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F\right)\circ(BBQu_A)$$

$$\overset{A\text{monad}}{=}\left({}_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F\right)\circ\left(B_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F\right)\circ(BBQu_A)$$

$$\overset{(7)}{=}\left[({}_{\mathbb{B}}U\lambda_B\circ m_{B\mathbb{B}}U)\,\widetilde{Q}_{\mathbb{A}}F\right]\circ(BBQu_A)$$

$$=\left({}_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F\right)\circ\left(m_{B\mathbb{B}}U\widetilde{Q}_{\mathbb{A}}F\right)\circ(BBQu_A)$$

$$\overset{m_B}{=}\left({}_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F\right)\circ(BQu_A)\circ(m_BQ)$$

$$=\left({}_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F\right)\circ({}_{\mathbb{B}}U_{\mathbb{B}}FQu_A)\circ(m_BQ)$$

$$=\Phi\circ(m_BQ)\,.$$

Moreover we have

$$\Phi\circ(u_BQ)=\left({}_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F\right)\circ(BQu_A)\circ(u_BQ)$$

$$\overset{u_B}{=}\left({}_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F\right)\circ(u_BQA)\circ(Qu_A)$$

$$=\left({}_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F\right)\circ(u_BQ_{\mathbb{A}}U_{\mathbb{A}}F)\circ(Qu_A)$$

$$\overset{\widetilde{Q}\text{lifting}}{=}\left({}_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F\right)\circ\left(u_{B\mathbb{B}}U\widetilde{Q}_{\mathbb{A}}F\right)\circ(Qu_A)$$

$$\overset{({}_{\mathbb{B}}F,{}_{\mathbb{B}}U)\text{adj}}{=}Qu_A.$$

Conversely, let $\Phi : BQ\to QA$ be a functorial morphism satisfying $\Phi\circ(m_BQ)=(Qm_A)\circ(\Phi A)\circ(B\Phi)$ and $\Phi\circ(u_BQ)=Qu_A$. We define $\widetilde{Q}:{}_{\mathbb{A}}\mathcal{A}\to{}_{\mathbb{B}}\mathcal{B}$ by setting, for every $(X,\mu)\in{}_{\mathbb{A}}\mathcal{A}$,

$$\widetilde{Q}\left((X,\mu)\right)=(QX,(Q\mu)\circ(\Phi X))=\left(Q_{\mathbb{B}}U\left(X,\mu\right),\left[Q_{\mathbb{A}}U\lambda_A\circ\Phi_{\mathbb{A}}U\right](X,\mu)\right).$$

Note that, a posteriori, we will have

$$ {}_{\mathbb{B}}U\lambda_B\widetilde{Q}=Q_{\mathbb{A}}U\lambda_A\circ\Phi_{\mathbb{A}}U \tag{9}$$

We have to check that $(Q\left(X\right),(Q\mu)\circ(\Phi X))\in{}_{\mathbb{B}}\mathcal{B}$, that is

$$\widetilde{\mu}\circ B\widetilde{\mu}=\widetilde{\mu}\circ(m_BQX)\quad\text{ and }\quad\widetilde{\mu}\circ(u_BQX)=QX$$

where $\widetilde{\mu}=(Q\mu)\circ(\Phi X)$. We compute

$$\widetilde{\mu}\circ(B\widetilde{\mu})=(Q\mu)\circ(\Phi X)\circ(BQ\mu)\circ(B\Phi X)$$

$$\overset{\Phi}{=}(Q\mu)\circ(QA\mu)\circ(\Phi AX)\circ(B\Phi X)$$

$$\overset{5}{=}(Q\mu)\circ(Qm_AX)\circ(\Phi AX)\circ(B\Phi X)$$

$$\overset{\text{propertyof}\Phi}{=}(Q\mu)\circ(\Phi X)\circ(m_BQX)$$

$$=\widetilde{\mu}\circ(m_BQX)\,.$$

Moreover we have

$$\begin{aligned}\widetilde{\mu}\circ(u_BQX) &= (Q\mu)\circ(\Phi X)\circ(u_BQX)\\ &\overset{\text{propertyof}\Phi}{=} (Q\mu)\circ(Qu_AX)\\ &\overset{5}{=} QX.\end{aligned}$$

Now, let $f:(X,\mu)\to(X',\mu')$ be a morphism of $\mathbb{A}$-modules, that is a morphism $f:X\to X'$ in $\mathcal{A}$ such that

$$\mu'\circ(Af)=f\circ\mu.$$

We want to prove that $Q(f)$ lifts to a morphism $\widetilde{Q}(f):\widetilde{Q}(X,\mu)=(QX,(Q\mu)\circ(\Phi X))\to\widetilde{Q}(X',\mu')=(QX',(Q\mu')\circ(\Phi X'))$ of $\mathbb{B}$-modules i.e.

$$[(Q\mu')\circ(\Phi X')]\circ(BQf)\overset{?}{=}(Qf)\circ[(Q\mu)\circ(\Phi X)].$$

We compute

$$\begin{aligned}{}[(Q\mu')\circ(\Phi X')]\circ(BQf) &\overset{\Phi}{=} (Q\mu')\circ(QAf)\circ(\Phi X)\\ &\overset{f\,\text{morph}A\text{-mod}}{=} (Qf)\circ(Q\mu)\circ(\Phi X).\end{aligned}$$

Let now check that $\widetilde{Q}$ is a lifting of Q. Let $(X,\mu)\in{}_{\mathbb{A}}\mathcal{A}$ and let us compute

$${}_{\mathbb{B}}U\widetilde{Q}((X,\mu))={}_{\mathbb{B}}U(QX,(Q\mu)\circ(\Phi X))=QX=Q_{\mathbb{A}}U((X,\mu)).$$

Let $f:(X,\mu)\to(X',\mu')$ be a morphism in ${}_{\mathbb{A}}\mathcal{A}$. By construction we have

$${}_{\mathbb{B}}U\widetilde{Q}(f)=Q_{\mathbb{A}}U(f):QX\to QX'.$$

Therefore $\widetilde{Q}$ is a lifting of the functor Q.

We have to prove that we have a bijection. Let us start with $\widetilde{Q}:{}_{\mathbb{A}}\mathcal{A}\to{}_{\mathbb{B}}\mathcal{B}$ a lifting of the functor $Q:\mathcal{A}\to\mathcal{B}$. Then we construct $\Phi:BQ\to QA$ given by

$$\Phi=\left({}_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F\right)\circ(BQu_A)$$

and using this functorial morphism we define a functor $\overline{Q}:{}_{\mathbb{A}}\mathcal{A}\to{}_{\mathbb{B}}\mathcal{B}$ as follows: for every $(X,\mu)\in{}_{\mathbb{A}}\mathcal{A}$

$$\overline{Q}((X,\mu))=(QX,(Q\mu)\circ(\Phi X)).$$

Since both $\widetilde{Q}$ and $\overline{Q}$ are liftings of Q, we have that ${}_{\mathbb{B}}U\widetilde{Q}=Q_{\mathbb{A}}U={}_{\mathbb{B}}U\overline{Q}$. In view of Proposition 2.15, it remains to prove that ${}_{\mathbb{B}}U\left(\lambda_B\overline{Q}\right)={}_{\mathbb{B}}U\left(\lambda_B\widetilde{Q}\right)$. Since $\overline{Q}(X,\mu)=(Q_{\mathbb{B}}U(X,\mu),[Q_{\mathbb{A}}U\lambda_A\circ\Phi_{\mathbb{A}}U](X,\mu))$ for every $((X,\mu))\in{}_{\mathbb{A}}\mathcal{A}$ we have that

$${}_{\mathbb{B}}U\lambda_B\overline{Q}=Q_{\mathbb{A}}U\lambda_A\circ\Phi_{\mathbb{A}}U$$

We compute

$$\begin{aligned}{}_{\mathbb{B}}U\left(\lambda_B\overline{Q}\right) &= Q_{\mathbb{A}}U\lambda_A\circ\Phi_{\mathbb{A}}U\\ &=(Q_{\mathbb{A}}U\lambda_A)\circ\left({}_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F_{\mathbb{A}}U\right)\circ(BQu_{A\mathbb{A}}U)\\ &\overset{\widetilde{Q}\text{lifting}Q}{=}\left({}_{\mathbb{B}}U\widetilde{Q}\lambda_A\right)\circ\left({}_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F_{\mathbb{A}}U\right)\circ(BQu_{A\mathbb{A}}U)\\ &\overset{\lambda_B}{=}\left({}_{\mathbb{B}}U\lambda_B\widetilde{Q}\right)\circ\left({}_{\mathbb{B}}U_{\mathbb{B}}F_{\mathbb{B}}U\widetilde{Q}\lambda_A\right)\circ(BQu_{A\mathbb{A}}U)\\ &=\left({}_{\mathbb{B}}U\lambda_B\widetilde{Q}\right)\circ\left(B\left[{}_{\mathbb{B}}U\widetilde{Q}\lambda_A\circ Qu_{A\mathbb{A}}U\right]\right)\\ &=\left({}_{\mathbb{B}}U\lambda_B\widetilde{Q}\right)\circ\left(B\left[Q_{\mathbb{A}}U\lambda_A\circ Qu_{A\mathbb{A}}U\right]\right)\\ &\overset{({}_{\mathbb{A}}F,{}_{\mathbb{A}}U)\text{adj}}{=}{}_{\mathbb{B}}U\lambda_B\widetilde{Q}.\end{aligned}$$

Conversely, let us start with a functorial morphism $\Phi : BQ \to QA$ satisfying $\Phi \circ (m_B Q) = (Q m_A) \circ (\Phi A) \circ (B\Phi)$ and $\Phi \circ (u_B Q) = Q u_A$. Then we construct a functor $\widetilde{Q} : {}_{\mathbb{A}}\mathcal{A} \to {}_{\mathbb{B}}\mathcal{B}$ by setting, for every $(X, \mu) \in {}_{\mathbb{A}}\mathcal{A}$,

$$\widetilde{Q}\left((X,\mu)\right) = (QX, (Q\mu) \circ (\Phi X))$$

which lifts $Q : \mathcal{A} \to \mathcal{B}$. Now, we define a functorial morphism $\Psi : BQ \to QA$ given by

$$\Psi = \left({}_{\mathbb{B}}U \lambda_B \widetilde{Q}_{\mathbb{A}} F\right) \circ (BQu_A).$$

Then we have

$$\begin{aligned}\Psi &= \left({}_{\mathbb{B}}U \lambda_B \widetilde{Q}_{\mathbb{A}} F\right) \circ (BQu_A) \\ &\overset{(9)}{=} (Q_{\mathbb{A}} U \lambda_{A\mathbb{A}} F) \circ (\Phi_{\mathbb{A}} U_{\mathbb{A}} F) \circ (BQu_A) \\ &= (Qm_A) \circ (\Phi A) \circ (BQu_A) \\ &\overset{\Phi}{=} (Qm_A) \circ (QAu_A) \circ \Phi \\ &\overset{A\text{monad}}{=} \Phi.\end{aligned}$$

□

Definition 4.2. A *left module functor* for a monad $\mathbb{A} = (A, m_A, u_A)$ on a category $\mathcal{A}$ is a pair $\left(Q, {}^{A}\mu_Q\right)$ where $Q : \mathcal{B} \to \mathcal{A}$ is a functor and ${}^{A}\mu_Q : AQ \to Q$ is a functorial morphism satisfying:

$${}^{A}\mu_Q \circ \left(A\, {}^{A}\mu_Q\right) = {}^{A}\mu_Q \circ (m_A Q) \quad \text{and} \quad Q = {}^{A}\mu_Q \circ (u_A Q).$$

Example 4.3. In the setting of Example 2.3, $\mathbb{A} = (A, m_A, u_A)$ where

$$\begin{aligned} A &= -\otimes_R S : Mod\text{-}R \to Mod\text{-}R \\ m_A &= -\otimes_R m_S : -\otimes_R S \otimes_R S \to -\otimes_R S \\ u_A &= \ : Mod\text{-}R \to -\otimes_R S \end{aligned}$$

Let M be an R-S-bimodule and let $Q =: Mod\text{-}R \to Mod\text{-}R$. Then Q is a left module functor for the monad $\mathbb{A}$ via the map via the map

$${}^{A}\mu_Q = -\otimes_R \mu_M^{A} : AQ = -\otimes_R M \otimes_R S \longrightarrow Q = -\otimes_R M$$

where we denote by $\mu_M^S : M \otimes_R S \longrightarrow M$ the map induced by the multiplication by S on M.

Corollary 4.4. *Let $\mathcal{X}, \mathcal{A}$ be categories, let $\mathbb{A} = (A, m_A, u_A)$ be a monad on a category $\mathcal{A}$ and let $F : \mathcal{X} \to \mathcal{A}$ be a functor. Then there exists a bijective correspondence between the following collections of data:*

- $\mathcal{H}$ *Left $\mathbb{A}$-module actions ${}^{A}\mu_F : AF \to F$*
- $\mathcal{G}$ *Functors ${}_AF : \mathcal{X} \to {}_{\mathbb{A}}\mathcal{A}$ such that ${}_{\mathbb{A}}U {}_AF = F$,*

given by

$$\begin{gathered} \widetilde{a} : \mathcal{H} \to \mathcal{G} \text{ where } {}_{\mathbb{A}}U \widetilde{a}\left({}^{A}\mu_F\right) = F \text{ and } {}_{\mathbb{A}}U \lambda_A \widetilde{a}\left({}^{A}\mu_F\right) = {}^{A}\mu_F \text{ i.e.} \\ \widetilde{a}\left({}^{A}\mu_F\right)(X) = \left(FX, {}^{A}\mu_F X\right) \text{ and } \widetilde{a}\left({}^{A}\mu_F\right)(f) = F(f) \\ \widetilde{b} : \mathcal{G} \to \mathcal{H} \text{ where } \widetilde{b}\left({}_AF\right) = {}_{\mathbb{A}}U \lambda_{A\,A} F : AF \to F. \end{gathered}$$

Proof. Apply Proposition 4.1 to the case $\mathcal{A} = \mathcal{X}, \mathcal{B} = \mathcal{A}$, $\mathbb{A} = \mathrm{Id}_{\mathcal{X}}$ and $\mathbb{B} = \mathbb{A}$. Then $\widetilde{Q} = {}_AF$ is the lifting of F and $\Phi = {}^{A}\mu_F$ satisfies ${}^{A}\mu_F \circ (m_A F) = {}^{A}\mu_F \circ \left(A\, {}^{A}\mu_F\right)$ and ${}^{A}\mu_F \circ (u_A F) = F$ that is $\left(F, {}^{A}\mu_F\right)$ is a left $\mathbb{A}$-module functor. □

Corollary 4.5. *Let (L, R) be an adjunction with $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$ and let $\mathbb{A} = (A, m_A, u_A)$ be a monad on $\mathcal{B}$. Then there is a bijective correspondence between the following collections of data*

- $\mathfrak{K}$ *Functors $K : \mathcal{A} \to {}_{\mathbb{A}}\mathcal{B}$ such that ${}_{\mathbb{A}}U \circ K = R$,*

$\mathfrak{L}$ *functorial morphism* $\alpha : AR \to R$ *such that* (R, α) *is a left module functor for the monad* $\mathbb{A}$

given by

$$\Phi \ : \ \mathfrak{K} \to \mathfrak{L} \text{ where } \Phi(K) = {}_{\mathbb{A}}U\lambda_A K : AR \to R$$
$$\Omega \ : \ \mathfrak{L} \to \mathfrak{K} \text{ where } \Omega(\alpha)(X) = (RX, \alpha X) \text{ and } {}_{\mathbb{A}}U\Omega(\alpha)(f) = R(f).$$

Proof. Apply Corollary 4.4 to the case "F" $= R : \mathcal{A} \to \mathcal{B}$ where (L, R) is an adjunction with $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$ and $\mathbb{A} = (A, m_A, u_A)$ a monad on $\mathcal{B}$. □

5. Distributive laws and lifting of monads

From 4.1 we get

Proposition 5.1. *Let* $\mathbb{A} = (A, m_A, u_A)$ *be a monad on a category* $\mathcal{A}$ *and let* $B : \mathcal{A} \to \mathcal{A}$ *be a functor. Then there is a bijection between the following collections of data*

$\mathcal{F}$ *functors* $\widetilde{B} : {}_{\mathbb{A}}\mathcal{A} \to {}_{\mathbb{A}}\mathcal{A}$ *that are liftings of* B *(i.e.* ${}_{\mathbb{A}}U\widetilde{B} = B{}_{\mathbb{A}}U$*)*

$\mathcal{M}$ *functorial morphisms* $\Phi : AB \to BA$ *such that*

$$\Phi \circ (m_A B) = (B m_A) \circ (\Phi A) \circ (A\Phi) \qquad \text{and} \qquad \Phi \circ (u_A B) = B u_A$$

given by

$$a : \mathcal{F} \to \mathcal{M} \text{ where } a\left(\widetilde{B}\right) = \left({}_{\mathbb{A}}U\lambda_A \widetilde{B}{}_{\mathbb{A}}F\right) \circ (ABu_A)$$
$$b : \mathcal{M} \to \mathcal{F} \text{ where } b(\Phi)\left((X, {}^A\mu_X)\right) = \left(BX, \left(B^A\mu_X\right) \circ (\Phi X)\right)$$
$$\text{and } {}_{\mathbb{A}}U\left[b(\Phi)(f)\right] = B{}_{\mathbb{A}}U(f).$$

Definition 5.2. [Be1] Let $\mathbb{A} = (A, m_A, u_A)$ and $\mathbb{B} = (B, m_B, u_B)$ be monads on a category $\mathcal{A}$. A functorial morphisms $\Phi : AB \to BA$ such that

$$\Phi \circ (m_A B) = (B m_A) \circ (\Phi A) \circ (A\Phi) \qquad \text{and} \qquad \Phi \circ (u_A B) = B u_A \tag{10}$$

and

$$\Phi \circ (A m_B) = (m_B A) \circ (B\Phi) \circ (\Phi B) \qquad \text{and} \qquad \Phi \circ (A u_B) = u_B A \tag{11}$$

is said to be *a distributive law of* $\mathbb{A}$ *over* $\mathbb{B}$.

Theorem 5.3. *Let* $\mathbb{A} = (A, m_A, u_A)$ *and* $\mathbb{B} = (B, m_B, u_B)$ *be monads on a category* $\mathcal{A}$. *Then there is a bijection between the following collections of data*

$\mathcal{D}$ *distributive laws of* $\mathbb{A}$ *over* $\mathbb{B}$

$\mathcal{M}$ *monads* $\widehat{\mathbb{B}} = \left(\widehat{B}, m_{\widehat{B}}, u_{\widehat{B}}\right)$ *on* ${}_{\mathbb{A}}\mathcal{A}$ *that are lifting of* $\mathbb{B}$ *(i.e.* ${}_{\mathbb{A}}U\widehat{B} = B{}_{\mathbb{A}}U, {}_{\mathbb{A}}Um_{\widehat{B}} = m_B{}_{\mathbb{A}}U, {}_{\mathbb{A}}Uu_{\widehat{B}} = u_B{}_{\mathbb{A}}U$*)*

given by

$$a \ : \ \mathcal{D} \to \mathcal{M} \text{ where } a(\Phi) = \widehat{\mathbb{B}} \text{ where } \widehat{\mathbb{B}} = \left(\widehat{B}, m_{\widehat{B}}, u_{\widehat{B}}\right) \text{ and}$$
$$\widehat{B}\left((X, {}^A\mu_X)\right) \ = \ \left(BX, \left(B^A\mu_X\right) \circ (\Phi X)\right), \ {}_{\mathbb{A}}U\widehat{B}(f) = B{}_{\mathbb{A}}U(f)$$
$$b \ : \ \mathcal{M} \to \mathcal{D} \text{ where } b\left(\left(\widehat{B}, m_{\widehat{B}}, u_{\widehat{B}}\right)\right) = \left({}_{\mathbb{A}}U\lambda_A \widehat{B}{}_{\mathbb{A}}F\right) \circ (ABu_A).$$

Proof. Let $\Phi : AB \to BA$ be a distributive law of $\mathbb{A}$ over $\mathbb{B}$. By Proposition 5.1 we know that $\widehat{B} : {}_{\mathbb{A}}\mathcal{A} \to {}_{\mathbb{A}}\mathcal{A}$ defined by setting $\widehat{B}\left((X, {}^A\mu_X)\right) = \left(BX, \left(B^A\mu_X\right) \circ (\Phi X)\right)$, ${}_{\mathbb{A}}U\widehat{B}(f) = B{}_{\mathbb{A}}U(f)$ is a functor.

Let $\left(X, {}^A\mu_X\right) \in {}_{\mathbb{A}}\mathcal{A}$ and let us prove that $m_B X : B^2X \to BX$ lifts to a morphism $m_{\widehat{B}}\left(X, {}^A\mu_X\right)$ in ${}_{\mathbb{A}}\mathcal{A}$ from $\left(\widehat{B}\right)^2\left((X, {}^A\mu_X)\right)$ to $\widehat{B}\left((X, {}^A\mu_X)\right)$. Note that

$$\left(\widehat{B}\right)^2\left((X, {}^A\mu_X)\right) \ = \ \widehat{B}\left(\widehat{B}\left((X, {}^A\mu_X)\right)\right) = \widehat{B}\left(BX, \left(B^A\mu_X\right) \circ (\Phi X)\right)$$
$$= \ \left(B^2(X), \left(B^{2A}\mu_X\right) \circ (B\Phi X) \circ (\Phi BX)\right).$$

We compute

$$
\begin{aligned}
&(m_B X) \circ \left(B^{2A}\mu_X\right) \circ (B\Phi X) \circ \Phi B X \overset{m_B}{=} \left(B^A \mu_X\right) \circ (m_B A X) \circ (B\Phi X) \circ (\Phi B X) \\
&\overset{(11)}{=} \left(B^A \mu_X\right) \circ (\Phi X) \circ (A m_B X) .
\end{aligned}
$$

We have to check that in this way we get a functorial morphism $m_{\widehat{B}} : \left(\widehat{B}\right)^2 \to \widehat{B}$. Let $f : (X, \mu) \to (X', \mu')$ be a morphism in ${}_{\mathbb{A}}\mathcal{A}$. We have to prove that

$$
m_{\widehat{B}}(X', \mu') \circ \left(\widehat{B}\right)^2 f = \left(\widehat{B}\right) f \circ m_{\widehat{B}}(X, \mu)
$$

which amounts, in view of Proposition 2.10, to

$$
{}_{\mathbb{A}}U\left[m_{\widehat{B}}(X', \mu') \circ \left(\widehat{B}\right)^2 f\right] = {}_{\mathbb{A}}U\left[\left(\widehat{B}\right) f \circ m_{\widehat{B}}(X, \mu)\right].
$$

We compute

$$
\begin{aligned}
{}_{\mathbb{A}}U\left[m_{\widehat{B}}(X', \mu') \circ \left(\widehat{B}\right)^2 f\right] &= {}_{\mathbb{A}}U m_{\widehat{B}}(X', \mu') \circ {}_{\mathbb{A}}U \left(\widehat{B}\right)^2 f = m_B X' \circ B^2 {}_{\mathbb{A}}U f \\
\overset{m_B}{=} B {}_{\mathbb{A}}U f \circ m_B X &= {}_{\mathbb{A}}U \widehat{B} f \circ {}_{\mathbb{A}}U m_{\widehat{B}}(X, \mu) = {}_{\mathbb{A}}U\left[\left(\widehat{B}\right) f \circ m_{\widehat{B}}(X, \mu)\right].
\end{aligned}
$$

Let us prove that $u_B X : X \to BX$ lifts to a morphism $u_{\widehat{B}}(X, \mu)$ in ${}_{\mathbb{A}}\mathcal{A}$ from $\left((X, {}^A\mu_X)\right)$ to $\widehat{B}\left((X, {}^A\mu_X)\right)$. We compute

$$
\left(B^A \mu_X\right) \circ (\Phi X) \circ (A u_B X) \overset{(11)}{=} \left(B^A \mu_X\right) \circ (u_B A X) \overset{u_B}{=} (u_B X) \circ {}^A \mu_X .
$$

We have to check that in this way we get a functorial morphism $u_{\widehat{B}} : {}_{\mathbb{A}}\mathcal{A} \to \widehat{B}$. Let $f : (X, \mu) \to (X', \mu')$ be a morphism in ${}_{\mathbb{A}}\mathcal{A}$. We have to prove that

$$
u_{\widehat{B}}(X', \mu') \circ f = \left(\widehat{B}\right) f \circ u_{\widehat{B}}(X, \mu)
$$

which amounts, in view of Proposition 2.10, to

$$
{}_{\mathbb{A}}U\left[u_{\widehat{B}}(X', \mu') \circ f\right] = {}_{\mathbb{A}}U\left[\left(\widehat{B}\right) f \circ u_{\widehat{B}}(X, \mu)\right].
$$

We compute

$$
\begin{aligned}
{}_{\mathbb{A}}U\left[u_{\widehat{B}}(X', \mu') \circ f\right] &= {}_{\mathbb{A}}U u_{\widehat{B}}(X', \mu') \circ {}_{\mathbb{A}}U f = u_B X' \circ {}_{\mathbb{A}}U f \overset{u_B}{=} B {}_{\mathbb{A}}U f \circ u_B X \\
&= {}_{\mathbb{A}}U\left[\left(\widehat{B}\right) f \circ u_{\widehat{B}}(X, \mu)\right].
\end{aligned}
$$

Now we have to check that $\widehat{\mathbb{B}} = \left(\widehat{B}, m_{\widehat{B}}, u_{\widehat{B}}\right)$ is a monad on ${}_{\mathbb{A}}\mathcal{A}$. We compute

$$
\begin{aligned}
{}_{\mathbb{A}}U\left[m_{\widehat{B}} \circ \left(m_{\widehat{B}} \widehat{B}\right)\right] &= m_{B \mathbb{A}}U \circ m_B B_{\mathbb{A}} U \\
\overset{(B, m_B, u_B) \text{ is a monad}}{=} & m_{B\mathbb{A}}U \circ (B m_{B\mathbb{A}} U) = {}_{\mathbb{A}}U\left[m_{\widehat{B}} \circ \widehat{B} m_{\widehat{B}}\right]
\end{aligned}
$$

so that, in view of Proposition 2.10, we conclude that

$$
m_{\widehat{B}} \circ \left(m_{\widehat{B}} \widehat{B}\right) = m_{\widehat{B}} \circ \widehat{B} m_{\widehat{B}}.
$$

We compute

$$
\begin{aligned}
{}_{\mathbb{A}}U\left[m_{\widehat{B}} \circ \left(\widehat{B} u_{\widehat{B}}\right)\right] &= m_{B\mathbb{A}}U \circ B u_{B\mathbb{A}} U \\
\overset{(B, m_B, u_B) \text{ is a monad}}{=} & m_{B\mathbb{A}}U \circ u_B B_{\mathbb{A}} U = {}_{\mathbb{A}}U\left[m_{\widehat{B}} \circ \left(\widehat{B} u_{\widehat{B}}\right)\right]
\end{aligned}
$$

so that, in view of Proposition 2.10, we conclude that

$$
m_{\widehat{B}} \circ \left(\widehat{B} u_{\widehat{B}}\right) = \widehat{B} = m_{\widehat{B}} \circ \left(\widehat{B} u_{\widehat{B}}\right).
$$

Let now $\widehat{\mathbb{B}} = \left(\widehat{B}, m_{\widehat{B}}, u_{\widehat{B}}\right)$ be a monad on ${}_{\mathbb{A}}\mathcal{A}$ that is a lifting of $\mathbb{B}$. By Proposition 5.1 we already know that $\Phi = \left({}_{\mathbb{A}}U\lambda_A\widehat{B}{}_{\mathbb{A}}F\right) \circ (ABu_A)$ is a functorial morphism from AB to BA which satisfies (10). Let us prove it satisfies also (11). We compute

$$\begin{aligned}
&(m_BA) \circ (B\Phi) \circ (\Phi B)\\
&= (m_BA) \circ \left(B{}_{\mathbb{A}}U\lambda_A\widehat{B}{}_{\mathbb{A}}F\right) \circ (BABu_A) \circ \left({}_{\mathbb{A}}U\lambda_A\widehat{B}{}_{\mathbb{A}}FB\right) \circ (ABu_AB)\\
&= (m_B{}_{\mathbb{A}}U{}_{\mathbb{A}}F) \circ \left(B{}_{\mathbb{A}}U\lambda_A\widehat{B}{}_{\mathbb{A}}F\right) \circ (B{}_{\mathbb{A}}U{}_{\mathbb{A}}FBu_A) \circ \left({}_{\mathbb{A}}U\lambda_A\widehat{B}{}_{\mathbb{A}}FB\right) \circ ({}_{\mathbb{A}}U{}_{\mathbb{A}}FBu_AB)\\
&= \left({}_{\mathbb{A}}Um_{\widehat{B}}{}_{\mathbb{A}}F\right) \circ \left({}_{\mathbb{A}}U\widehat{B}\lambda_A\widehat{B}{}_{\mathbb{A}}F\right) \circ \left({}_{\mathbb{A}}U\widehat{B}{}_{\mathbb{A}}FBu_A\right) \circ \left({}_{\mathbb{A}}U\lambda_A\widehat{B}{}_{\mathbb{A}}FB\right) \circ ({}_{\mathbb{A}}U{}_{\mathbb{A}}FBu_AB)\\
&= {}_{\mathbb{A}}U\left[\left(m_{\widehat{B}}{}_{\mathbb{A}}F\right) \circ \left(\widehat{B}\lambda_A\widehat{B}{}_{\mathbb{A}}F\right) \circ \left(\widehat{B}{}_{\mathbb{A}}FBu_A\right) \circ \left(\lambda_A\widehat{B}{}_{\mathbb{A}}FB\right) \circ ({}_{\mathbb{A}}FBu_AB)\right]\\
&= {}_{\mathbb{A}}U\left[\left(m_{\widehat{B}}{}_{\mathbb{A}}F\right) \circ \left[\left(\widehat{B}\lambda_A\widehat{B}{}_{\mathbb{A}}F\right) \circ \left(\widehat{B}{}_{\mathbb{A}}FBu_A\right)\right] \circ \left(\lambda_A\widehat{B}{}_{\mathbb{A}}FB\right) \circ ({}_{\mathbb{A}}FBu_AB)\right]\\
&\overset{\lambda_A}{=} {}_{\mathbb{A}}U\left[\left(m_{\widehat{B}}{}_{\mathbb{A}}F\right) \circ \left(\lambda_A\widehat{B}\widehat{B}{}_{\mathbb{A}}F\right) \circ \left({}_{\mathbb{A}}F{}_{\mathbb{A}}U\widehat{B}\lambda_A\widehat{B}{}_{\mathbb{A}}F\right) \circ \left({}_{\mathbb{A}}F{}_{\mathbb{A}}U\widehat{B}{}_{\mathbb{A}}FBu_A\right) \circ ({}_{\mathbb{A}}FBu_AB)\right]\\
&= {}_{\mathbb{A}}U\left[\left(m_{\widehat{B}}{}_{\mathbb{A}}F\right) \circ \left(\lambda_A\widehat{B}\widehat{B}{}_{\mathbb{A}}F\right) \circ \left({}_{\mathbb{A}}F{}_{\mathbb{A}}U\widehat{B}\lambda_A\widehat{B}{}_{\mathbb{A}}F\right) \circ ({}_{\mathbb{A}}FB{}_{\mathbb{A}}U{}_{\mathbb{A}}FBu_A) \circ ({}_{\mathbb{A}}FBu_AB)\right]\\
&= {}_{\mathbb{A}}U\left[\left(m_{\widehat{B}}{}_{\mathbb{A}}F\right) \circ \left(\lambda_A\widehat{B}\widehat{B}{}_{\mathbb{A}}F\right) \circ \left({}_{\mathbb{A}}F{}_{\mathbb{A}}U\widehat{B}\lambda_A\widehat{B}{}_{\mathbb{A}}F\right) \circ {}_{\mathbb{A}}FB\,(ABu_A \circ u_AB)\right]\\
&\overset{u_A}{=} {}_{\mathbb{A}}U\left[\left(m_{\widehat{B}}{}_{\mathbb{A}}F\right) \circ \left(\lambda_A\widehat{B}\widehat{B}{}_{\mathbb{A}}F\right) \circ \left({}_{\mathbb{A}}F{}_{\mathbb{A}}U\widehat{B}\lambda_A\widehat{B}{}_{\mathbb{A}}F\right) \circ {}_{\mathbb{A}}FB\,(u_ABA \circ Bu_A)\right]\\
&= {}_{\mathbb{A}}U\left[\left(m_{\widehat{B}}{}_{\mathbb{A}}F\right) \circ \left(\lambda_A\widehat{B}\widehat{B}{}_{\mathbb{A}}F\right) \circ \left({}_{\mathbb{A}}FB{}_{\mathbb{A}}U\lambda_A\widehat{B}{}_{\mathbb{A}}F\right) \circ {}_{\mathbb{A}}FB\,(u_ABA \circ Bu_A)\right]\\
&= {}_{\mathbb{A}}U\left[\left(m_{\widehat{B}}{}_{\mathbb{A}}F\right) \circ \left(\lambda_A\widehat{B}\widehat{B}{}_{\mathbb{A}}F\right) \circ \left({}_{\mathbb{A}}FB\left({}_{\mathbb{A}}U\lambda_A\widehat{B}{}_{\mathbb{A}}F \circ (u_AB{}_{\mathbb{A}}U{}_{\mathbb{A}}F \circ Bu_A)\right)\right)\right]\\
&= {}_{\mathbb{A}}U\left[\left(m_{\widehat{B}}{}_{\mathbb{A}}F\right) \circ \left(\lambda_A\widehat{B}\widehat{B}{}_{\mathbb{A}}F\right) \circ \left({}_{\mathbb{A}}FB\left(\left({}_{\mathbb{A}}U\lambda_A\widehat{B}{}_{\mathbb{A}}F\right) \circ \left(u_A{}_{\mathbb{A}}U\widehat{B}{}_{\mathbb{A}}F\right) \circ Bu_A\right)\right)\right]\\
&\overset{({}_{\mathbb{A}}U\lambda_A)\circ(u_A{}_{\mathbb{A}}U)={}_{\mathbb{A}}U}{=} {}_{\mathbb{A}}U\left[\left(m_{\widehat{B}}{}_{\mathbb{A}}F\right) \circ \left(\lambda_A\widehat{B}\widehat{B}{}_{\mathbb{A}}F\right) \circ ({}_{\mathbb{A}}FBBu_A)\right]\\
&\overset{\lambda_A}{=} {}_{\mathbb{A}}U\left[\left(\lambda_A\widehat{B}{}_{\mathbb{A}}F\right) \circ \left({}_{\mathbb{A}}F{}_{\mathbb{A}}Um_{\widehat{B}}{}_{\mathbb{A}}F\right) \circ ({}_{\mathbb{A}}FBBu_A)\right]\\
&= {}_{\mathbb{A}}U\left[\left(\lambda_A\widehat{B}{}_{\mathbb{A}}F\right) \circ ({}_{\mathbb{A}}Fm_B{}_{\mathbb{A}}U{}_{\mathbb{A}}F) \circ ({}_{\mathbb{A}}FBBu_A)\right]\\
&= {}_{\mathbb{A}}U\left[\left(\lambda_A\widehat{B}{}_{\mathbb{A}}F\right) \circ ({}_{\mathbb{A}}Fm_BA) \circ ({}_{\mathbb{A}}FBBu_A)\right]\\
&\overset{m_B}{=} {}_{\mathbb{A}}U\left[\left(\lambda_A\widehat{B}{}_{\mathbb{A}}F\right) \circ ({}_{\mathbb{A}}FBu_A) \circ ({}_{\mathbb{A}}Fm_B)\right]\\
&= \left({}_{\mathbb{A}}U\lambda_A\widehat{B}{}_{\mathbb{A}}F\right) \circ (ABu_A) \circ (Am_B)\\
&= \Phi \circ (Am_B)\,.
\end{aligned}$$

We also compute

$$\begin{aligned}
\Phi \circ (Au_B) &= \left({}_{\mathbb{A}}U\lambda_A\widehat{B}{}_{\mathbb{A}}F\right) \circ (ABu_A) \circ (Au_B)\\
&= \left({}_{\mathbb{A}}U\lambda_A\widehat{B}{}_{\mathbb{A}}F\right) \circ ({}_{\mathbb{A}}U{}_{\mathbb{A}}FBu_A) \circ ({}_{\mathbb{A}}U{}_{\mathbb{A}}Fu_B)\\
&= {}_{\mathbb{A}}U\left[\left(\lambda_A\widehat{B}{}_{\mathbb{A}}F\right) \circ {}_{\mathbb{A}}F\,(Bu_A \circ u_B)\right]\\
&\overset{u_B}{=} {}_{\mathbb{A}}U\left[\left(\lambda_A\widehat{B}{}_{\mathbb{A}}F\right) \circ {}_{\mathbb{A}}F\,(u_BA \circ u_A)\right]\\
&= {}_{\mathbb{A}}U\left[\left(\lambda_A\widehat{B}{}_{\mathbb{A}}F\right) \circ {}_{\mathbb{A}}F\,(u_B{}_{\mathbb{A}}U{}_{\mathbb{A}}F \circ u_A)\right]\\
&= {}_{\mathbb{A}}U\left[\left(\lambda_A\widehat{B}{}_{\mathbb{A}}F\right) \circ {}_{\mathbb{A}}F\,\left({}_{\mathbb{A}}Uu_{\widehat{B}}{}_{\mathbb{A}}F \circ u_A\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= {}_{\mathbb{A}}U\left[\left(\lambda_A\widehat{B}_{\mathbb{A}}F\right)\circ\left({}_{\mathbb{A}}F_{\mathbb{A}}Uu_{\widehat{B}\mathbb{A}}F\right)\circ\left({}_{\mathbb{A}}Fu_A\right)\right] \\
&\overset{\lambda_A}{=} {}_{\mathbb{A}}U\left[\left(u_{\widehat{B}\mathbb{A}}F\right)\circ\left(\lambda_{A\mathbb{A}}F\right)\circ\left({}_{\mathbb{A}}Fu_A\right)\right] \\
&\overset{(\lambda_{A\mathbb{A}}F)\circ({}_{\mathbb{A}}Fu_A)={}_{\mathbb{A}}F}{=} {}_{\mathbb{A}}Uu_{\widehat{B}\mathbb{A}}F = u_{B\mathbb{A}}U_{\mathbb{A}}F = u_BA.
\end{aligned}
$$

□

6. Descent data and quasi-symmetries associated to a monad

Definitions 6.1. Let $\mathbb{A} = (A, m, u)$ be a monad on a category $\mathcal{A}$. Let $\Phi : A^2 \to A^2$ be a functorial morphism.

We will say that Φ *satisfies the Yang-Baxter equation* if

$$A\Phi \circ \Phi A \circ A\Phi = \Phi A \circ A\Phi \circ \Phi A \tag{12}$$

holds true.

We will say that Φ *is a BD-law on* $\mathbb{A}$ [KLV, Definition 2.2] provided it is a distributive law of $\mathbb{A}$ over itself i.e. it satisfies

$$\Phi \circ (m_A A) = (Am_A) \circ (\Phi A) \circ (A\Phi) \qquad \text{and} \qquad \Phi \circ (u_A A) = Au_A \tag{13}$$

and

$$\Phi \circ (Am_A) = (m_A A) \circ (A\Phi) \circ (\Phi A) \qquad \text{and} \qquad \Phi \circ (Au_A) = u_A A \tag{14}$$

and it satisfies the Yang-Baxter equation.

Definitions 6.2. Let $\mathbb{A} = (A, m, u)$ be a monad on a category $\mathcal{A}$ and let $\Phi : A^2 \to A^2$ be a BD-law on $\mathbb{A}$. Let $(X, \mu) \in {}_{\mathbb{A}}\mathcal{A}$. A *quasi* Φ*-symmetry on* (X, μ) is a morphism $c : AX \to AX$ such that

$$\mu \circ c \circ uX = X \tag{15}$$

$$Ac \circ \Phi X \circ Ac = \Phi X \circ Ac \circ \Phi X \tag{16}$$

$$c \circ A\mu = mX \circ Ac \circ \Phi X \tag{17}$$

We denote by Φ-QSymm(X, μ) the set of quasi Φ-symmetries on (X, μ). Moreover we write QSymm$(\mathbb{A}, \Phi)$ for the category having as objects pairs

$$((X, \mu), c) \text{ where } (X, \mu) \in {}_{\mathbb{A}}\mathcal{A} \text{ and } c \in \Phi\text{-QSymm}(X, \mu).$$

A morphism $f : ((X, \mu), c) \to ((X', \mu'), c')$ is a morphism $f : (X, \mu) \to (X', \mu')$ in ${}_{\mathbb{A}}\mathcal{A}$ such that $c' \circ Af = Af \circ c$.

A quasi Φ-symmetry c on (X, μ) is called a Φ*-symmetry* if $c^2 = AX$. We denote by Φ-Symm(X, μ) the subset of Φ-QSymm(X, μ) consisting of Φ-symmetries and by Symm$(\mathbb{A}, \Phi)$ the full subcategory of QSymm$(\mathbb{A}, \Phi)$ whose objects are pairs $((X, \mu), c)$ where $(X, \mu) \in {}_{\mathbb{A}}\mathcal{A}$ and $c \in \Phi$-Symm(X, μ).

Remark 6.3. $(X, \mu) \in {}_{\mathbb{A}}\mathcal{A}$. In [KLV, Definition 3.3] a *quasi* Φ*-symmetry on* (X, μ) is called Φ*-braiding on* (X, μ).

Remark 6.4. Let $f : B \to A$ be a morphism of rings. Every $M \in Mod\text{-}A$ has a natural structure of right B-module defined by setting

$$m \cdot b = mf(b) \qquad \text{for every } m \in M \text{ and } b \in B.$$

We will denote by M endowed with this $f_*(M)$ right B-module structure. It is easy to check that every morphism of right A-modules $g : M \to M'$ becomes automatically a morphism $f_*(g) : f_*(M) \to f_*(M')$ in $Mod\text{-}B$ and in this way we get a functor $f_* : Mod\text{-}A \to Mod\text{-}B$. On the other hand, A has a left B-module structure defined by

$$b \cdot a = f(b)\,a \qquad \text{for every } b \in B \text{ and } a \in A.$$

In this way A becomes a B-A-bimodule. Let $L := (-) \otimes_B A : Mod\text{-}B \to Mod\text{-}A$ be the extension of scalars functor and $R := \mathrm{Hom}_A({}_BA, -) : Mod\text{-}A \to Mod\text{-}B$ be the restriction of scalars functor (see 2.5). In the following we will identify R with f_* through the natural isomorphism of right B-modules:

$$\nu_M : \mathrm{Hom}_A({}_BA_A, M) \to f_*(M), \qquad h \mapsto h(1_A).$$

EXAMPLE 6.5. Let $f : B \to A$ be a morphism of rings. Let $L := (-) \otimes_B A = f^* : Mod\text{-}B \to Mod\text{-}A$ be the extension of scalars functor and $R := \mathrm{Hom}_A({}_BA, -) = f_* : Mod\text{-}A \to Mod\text{-}B$ be the restriction of scalars functor (see 2.5). Let $\mathbb{A} = (RL, m = R\epsilon L, u = \eta)$ be the associated monad on $Mod\text{-}B$ (see Proposition 2.4). For any $E \in Mod\text{-}B$ we have

$$RLE = E \otimes_B A \text{ regarded as a right } B\text{-module}$$

$$\begin{aligned} mE &: E \otimes_B A \otimes_B A \to E \otimes_B A \\ & x \otimes a \otimes a' \mapsto x \otimes aa' \end{aligned}$$

$$\begin{aligned} uE &: E_B \to E \otimes_B A \\ & x \mapsto x \otimes 1. \end{aligned}$$

Assume now that $\mathrm{Im}(f)$ is contained in the center of A. Let $\Phi : (RL)^2 \to (RL)^2$, be the functorial morphism defined by

$$\Phi E = E \otimes_B \tau : E \otimes_B A \otimes_B A \to E \otimes_B A \otimes_B A \text{ for any } E \in Mod\text{-}B$$

where $\tau : A \otimes_B A \to A \otimes_B A$ is the usual flip $\tau(x \otimes y) = y \otimes x$. Note for $\Phi E = E \otimes_B \tau$ to be a morphism in $Mod\text{-}B$ we need that

$$\begin{aligned} x \otimes a' \otimes ab &= (x \otimes a' \otimes a)\, b = [\Phi E(x \otimes a \otimes a')]\, b = \Phi E((x \otimes a \otimes a')\, b) \\ &= \Phi E(x \otimes a \otimes a'b) = x \otimes a'b \otimes a = x \otimes a' \otimes ba \end{aligned}$$

which is satisfied in view of our assumption. We compute

$$\begin{aligned} RLuE &: E \otimes_B A \to E \otimes_B A \otimes_B A \\ & x \otimes a \mapsto x \otimes 1 \otimes a \end{aligned}$$

$$\begin{aligned} uRLE &: E \otimes_B A \to E \otimes_B A \otimes_B A \\ & x \otimes a \mapsto x \otimes a \otimes 1 \end{aligned}$$

$$\begin{aligned} RLmE &: [E \otimes_B A \otimes_B A] \otimes_B A \to [E \otimes_B A] \otimes_B A \\ x \otimes a \otimes a' \otimes a'' &\mapsto x \otimes aa' \otimes a'' \end{aligned}$$

i.e.

$$RLm = - \otimes_B m \otimes_B A$$

$$\begin{aligned} mRLE &: [E \otimes_B A] \otimes_B A \otimes_B A \to [E \otimes_B A] \otimes_B A \\ x \otimes a \otimes a' \otimes a'' &\mapsto x \otimes a \otimes a'a'' \end{aligned}$$

i.e.

$$mRL = - \otimes_B A \otimes_B m$$

$$\begin{aligned} \Phi RLE &: [E \otimes_B A] \otimes_B A \otimes_B A \to [E \otimes_B A] \otimes_B A \otimes_B A \\ & x \otimes a \otimes a' \otimes a'' \mapsto x \otimes a \otimes a'' \otimes a' \end{aligned}$$

so that

$$\Phi RL = - \otimes_B A \otimes_B \tau$$

$$\begin{aligned} RL(\Phi E) &: [E \otimes_B A \otimes_B A] \otimes_B A \to [E \otimes_B A \otimes_B A] \otimes_B A \\ & x \otimes a \otimes a' \otimes a'' \mapsto x \otimes a' \otimes a \otimes a'' \end{aligned}$$

so that

$$RL\Phi = - \otimes_B \tau \otimes_B A$$

Let us check that Φ satisfies (13). For every $x \in E, a, a', a'' \in A$ we have:

$$(\Phi E \circ mRLE)(x \otimes a \otimes a' \otimes a'') = \Phi E(x \otimes a \otimes a'a'') = x \otimes a'a'' \otimes a$$

$$\begin{aligned}&[(RLmE) \circ (\Phi RLE) \circ (RL\Phi E)](x \otimes a \otimes a' \otimes a'')\\&= [(RLmE) \circ (\Phi RLE)](x \otimes a' \otimes a \otimes a'')\\&= (RLmE)(x \otimes a' \otimes a'' \otimes a) = x \otimes a'a'' \otimes a\end{aligned}$$

and

$$[\Phi E \circ (uRLE)](x \otimes a) = \Phi E(x \otimes a \otimes 1) = x \otimes 1 \otimes a = (RLuE)(x \otimes a).$$

Let us check that Φ satisfies (14). For every $x \in E, a, a', a'' \in A$ we have:

$$[\Phi E \circ (RLmE)](x \otimes a \otimes a' \otimes a'') = \Phi E(x \otimes aa' \otimes a'') = x \otimes a'' \otimes aa'$$

$$\begin{aligned}&[(mRLE) \circ (RL\Phi E) \circ (\Phi RLE)](x \otimes a \otimes a' \otimes a'')\\&= [(mRLE) \circ (RL\Phi E)](x \otimes a \otimes a'' \otimes a')\\&= (mRLE)(x \otimes a'' \otimes a \otimes a') = x \otimes a'' \otimes aa'\end{aligned}$$

so that we get

$$\Phi E \circ (RLmE) = (mRLE) \circ (RL\Phi E) \circ (\Phi RLE).$$

We compute

$$[\Phi E \circ (RLuE)] = (\Phi E)(x \otimes 1 \otimes a) = x \otimes a \otimes 1 = (uRLE)(x \otimes a).$$

Thus we obtain

$$\Phi \circ (RLm) = (mRL) \circ (RL\Phi) \circ (\Phi RL) \qquad \text{and} \qquad \Phi \circ (RLu) = uRL$$

Let us check that Φ satisfies (12). We have

$$\begin{aligned}RL\Phi \circ \Phi RL \circ RL\Phi &= - \otimes_B [(\tau \otimes_B A) \circ (A \otimes_B \tau) \circ (\tau \otimes_B A)]\\&= - \otimes_B [(A \otimes_B \tau) \circ (\tau \otimes_B A) \circ (A \otimes_B \tau)] = \Phi RL \circ RL\Phi \circ \Phi RL.\end{aligned}$$

Thus Φ is a BD-law on $Mod\text{-}B$.

Remark 6.6. Let $\mathbb{A} = (A, m, u)$ be a monad on a category $\mathcal{A}$ and let $\Phi : A^2 \to A^2$ be a BD-law on $\mathbb{A}$. For every $X \in \mathcal{A}$, $\Phi X : A^2X \to A^2X$ is a quasi Φ-symmetry on ${}_{\mathbb{A}}F(X) = (AX, mX)$. In fact we have

$$mX \circ \Phi X \circ uAX \overset{(13)}{=} mX \circ AuX = AX$$

$$A\Phi X \circ \Phi AX \circ A\Phi X \overset{(12)}{=} \Phi AX \circ A\Phi X \circ \Phi AX$$

$$\Phi X \circ AmX \overset{(14)}{=} mAX \circ A\Phi X \circ \Phi AX$$

Note that if $f : X \to X'$ is a morphism in $\mathcal{A}$, then

$$Af : ((AX, mX), \Phi AX) \to ((AX', mX'), \Phi AX')$$

is a morphism in $\mathrm{QSymm}(\mathbb{A}, \Phi)$. Then it is easy to show that in this way we obtain a functor

$$\begin{array}{rccc} J: & \mathcal{A} & \to & \mathrm{QSymm}(\mathbb{A}, \Phi) \\ & X & \mapsto & ((AX, mX), \Phi X) \end{array}.$$

Definition 6.7. Let $\mathbb{A} = (A, m, u)$ be a monad on a category $\mathcal{A}$. and let $({}_{\mathbb{A}}F, {}_{\mathbb{A}}U)$ the corresponding adjunction with unit u and counit λ. Let $\mathbb{A}^* = ({}_{\mathbb{A}}F_{\mathbb{A}}U, {}_{\mathbb{A}}Fu_{\mathbb{A}}U, \lambda)$ be the comonad on the category ${}_{\mathbb{A}}\mathcal{A}$ associated to this adjunction (Proposition 3.4). Let $(X, \mu) \in {}_{\mathbb{A}}\mathcal{A}$. A *descent datum on* (X, μ) is a morphism

$$\rho : (X, \mu) \to {}_{\mathbb{A}}F_{\mathbb{A}}U(X, \mu) = (AX, mX)$$

in ${}_{\mathbb{A}}\mathcal{A}$ such that $((X, \mu), \rho) \in {}^{\mathbb{A}^*}({}_{\mathbb{A}}\mathcal{A})$ i.e. the following equalities are satisfied

(18) $$mX \circ A\rho = \rho \circ \mu \text{ i.e. } \rho \text{ is a morphism in } {}_{\mathbb{A}}\mathcal{A}$$

(19) $$A\rho \circ \rho = AuX \circ \rho$$

$$\mu \circ \rho = \mathrm{Id}_X \tag{20}$$

The set of all descent data on (X,μ) will be denoted by $\mathrm{Des}\,(X,\mu)$.

REMARK 6.8. Let $\mathbb{A} = (A, m, u)$ be a monad on a category $\mathcal{A}$. For every $X \in \mathcal{A}$, $AuX : {}_{\mathbb{A}}FX = (AX, mX) \to (AAX, mAX)$ is a descent datum on (AX, mX). In fact we have:

$$mAX \circ AAuX \overset{m}{=} AuX \circ mX$$

$$AAuX \circ AuX \overset{u}{=} AuAX \circ AuX$$

$$mX \circ AuX = AX.$$

This is the canonical comparison $K : \mathcal{A} \to {}^{\mathbb{A}^*}({}_{\mathbb{A}}\mathcal{A})$ of the adjoint pair $({}_{\mathbb{A}}F, {}_{\mathbb{A}}U)$ sending $X \in \mathcal{A}$ to

$$({}_{\mathbb{A}}FX, {}_{\mathbb{A}}FuX) = ((AX, mX), AuX).$$

NOTATIONS 6.9. Let $\mathbb{A} = (A, m, u)$ be a monad on a category $\mathcal{A}$ and let $\Phi : A^2 \to A^2$ be a BD-law on $\mathbb{A}$. We denote by $V : \mathrm{QSymm}\,(\mathbb{A}, \Phi) \to {}_{\mathbb{A}}\mathcal{A}$ the forgetful functor and with $J : \mathcal{A} \to \mathrm{QSymm}\,(\mathbb{A}, \Phi)$ the functor defined by (see Remark 6.6)

$$J(X) = ((AX, mX), \Phi X).$$

PROPOSITION 6.10. [KLV, Theorem 3.7]*Let $\mathbb{A} = (A, m, u)$ be a monad on a category $\mathcal{A}$ and let $\Phi : A^2 \to A^2$ be a BD-law on $\mathbb{A}$. Then*

$$\left(\mathcal{A} \overset{{}_{\mathbb{A}}F}{\to} {}_{\mathbb{A}}\mathcal{A}\right) = \left(\mathcal{A} \overset{J}{\to} \mathrm{QSymm}\,(\mathbb{A}, \Phi) \overset{V}{\to} {}_{\mathbb{A}}\mathcal{A}\right)$$

$${}_{\mathbb{A}}F = V \circ J, \qquad {}_{\mathbb{A}}F \circ {}_{\mathbb{A}}U = V \circ J \circ {}_{\mathbb{A}}U$$

and $(V, J \circ {}_{\mathbb{A}}U)$ is an adjunction with counit $\lambda_A : {}_{\mathbb{A}}F \circ {}_{\mathbb{A}}U = V \circ J \circ {}_{\mathbb{A}}U \to {}_{\mathbb{A}}\mathcal{A}$ and unit β : $\mathrm{QSymm}\,(\mathbb{A}, \Phi) \to J \circ {}_{\mathbb{A}}U \circ V$
defined by

$$\beta((X,\mu), c) = c \circ uX \qquad \textit{for every } ((X,\mu), c) \in \mathrm{QSymm}\,(\mathbb{A}, \Phi).$$

Moreover the comonad corresponding to the adjunction $(V, J \circ {}_{\mathbb{A}}U)$ coincides with the comonad $\mathbb{A}^* = ({}_{\mathbb{A}}F{}_{\mathbb{A}}U, {}_{\mathbb{A}}Fu{}_{\mathbb{A}}U, \lambda)$ corresponding to the adjunction $({}_{\mathbb{A}}F, {}_{\mathbb{A}}U)$.

Proof. Let $X \in \mathcal{A}$. Then

$$(V \circ J)(X) = V((AX, mX), \Phi X) = (AX, mX) = {}_{\mathbb{A}}F(X).$$

Thus ${}_{\mathbb{A}}F \circ {}_{\mathbb{A}}U = V \circ J \circ {}_{\mathbb{A}}U$. Let now $((X,\mu), c) \in \mathrm{QSymm}\,(\mathbb{A}, \Phi)$ and let us check that $\beta((X,\mu), c) = c \circ uX$ is a morphism

$$\beta((X,\mu), c) : ((X,\mu), c) \to (J \circ {}_{\mathbb{A}}U \circ V)((X,\mu), c) = ((AX, mX), \Phi X)$$

in $\mathrm{QSymm}\,(\mathbb{A}, \Phi)$. We compute

$$c \circ uX \circ \mu \overset{u}{=} c \circ A\mu \circ uAX \overset{(17)}{=} mX \circ Ac \circ \Phi X \circ uAX \overset{(13)}{=} mX \circ Ac \circ AuX$$

and

$$Ac \circ AuX \circ c \overset{(13)}{=} Ac \circ \Phi X \circ uAX \circ c \overset{u}{=} Ac \circ \Phi X \circ Ac \circ uAX$$

$$\overset{(16)}{=} \Phi X \circ Ac \circ \Phi X \circ uAX \overset{(13)}{=} \Phi X \circ Ac \circ AuX.$$

Let us check that in this way we get a functorial morphism $\beta : \mathrm{QSymm}\,(\mathbb{A}, \Phi) \to J \circ {}_{\mathbb{A}}U \circ V$. Let

$$f : ((X,\mu), c) \to ((X', \mu'), c')$$

be a morphism in $\mathrm{QSymm}\,(\mathbb{A}, \Phi)$. We have

$$(J \circ {}_{\mathbb{A}}U \circ V)(f)\,\beta((X,\mu), c) = Af \circ c \circ uX$$

$$= c' \circ Af \circ uX \overset{u}{=} c' \circ uX' \circ f = \beta((X', \mu'), c') \circ f.$$

Let us know show $(V, J \circ {}_{\mathbb{A}}U)$ is an adjunction with counit $\lambda = \lambda_A$ and unit β.

For every $((X,\mu),c)\in \mathrm{QSymm}(\mathbb{A},\Phi)$, we compute

$$\begin{aligned}{}_{\mathbb{A}}U\left[\lambda V\left(((X,\mu),c)\right)\circ V\beta\left(((X,\mu),c)\right)\right] &= {}_{\mathbb{A}}U\lambda\left((X,\mu)\right)\circ c\circ uX\\ &= \mu\circ c\circ uX \overset{(15)}{=} X = {}_{\mathbb{A}}U\left[V\left(((X,\mu),c)\right)\right]\end{aligned}$$

and for every $(X,\mu)\in {}_{\mathbb{A}}\mathcal{A}$, we compute

$$\begin{aligned}&({}_{\mathbb{A}}U\circ V)\left([(J\circ {}_{\mathbb{A}}U)\,\lambda\,(X,\mu)]\circ[\beta\,(J\circ {}_{\mathbb{A}}U)\,(X,\mu)]\right)\\ &= J\mu\circ(V\circ {}_{\mathbb{A}}U)\left[\beta\left((AX,mX),\Phi X\right)\right] = A\mu\circ\Phi X\circ uAX\\ &\overset{(13)}{=} A\mu\circ AuX \overset{5}{=} AX = ({}_{\mathbb{A}}U\circ V)\,(J\circ {}_{\mathbb{A}}U)\,(X,\mu)\,.\end{aligned}$$

Since both the functors ${}_{\mathbb{A}}U$ and $V\circ {}_{\mathbb{A}}U$ are faithful, we conclude.

In view of the foregoing, to prove the last statement it remains to prove that

$$V\beta J_{\mathbb{A}}U = {}_{\mathbb{A}}Fu_{\mathbb{A}}U.$$

Let $(X,\mu)\in {}_{\mathbb{A}}\mathcal{A}$. We compute

$$\begin{aligned}(V\beta J_{\mathbb{A}}U)\,(X,\mu) &= V\beta J\,(X) = V\beta\,(AX,mX,\Phi X)\\ &= \Phi X\circ uAX \overset{(13)}{=} AuX = ({}_{\mathbb{A}}Fu_{\mathbb{A}}U)\,(X,\mu)\,.\end{aligned}$$

□

Proposition 6.11. *Let $\mathbb{A}=(A,m,u)$ be a monad on a category $\mathcal{A}$ and let $\Phi: A^2\to A^2$ be a BD-law on $\mathbb{A}$. For every $(X,\mu)\in {}_{\mathbb{A}}\mathcal{A}$ the assignment*

$$c\mapsto c\circ uX$$

defines a bijection

$$\Gamma\,(X,\mu):\Phi\text{-QSymm}\,(X,\mu)\to \mathrm{Des}\,(X,\mu)$$

whose inverse $\Gamma'(X,\mu)$ is defined by setting

$$\Gamma'\,(X,\mu)\,(\rho) = A\mu\circ\Phi X\circ A\rho.$$

Moreover if $\Phi X\circ\Phi X = A^2X$, then Φ-QSymm$\,(X,\mu) = \Phi$-Symm$\,(X,\mu)$.

Proof. Let $c\in\Phi$-QSymm$\,(X,\mu)$ and let us check that $c\circ uX\in\mathrm{Des}\,(X,\mu)$. Let us check (18).

$$\begin{aligned}mX\circ A\,(c\circ uX) &= mX\circ Ac\circ AuX \overset{(13)}{=} mX\circ Ac\circ\Phi X\circ uAX \overset{(17)}{=} c\circ A\mu\circ uAX\\ &\overset{u}{=} (c\circ uX)\circ\mu.\end{aligned}$$

Let us check (19).

$$\begin{aligned}AuX\circ(c\circ uX) &\overset{(13)}{=} \Phi X\circ(uAX\circ c)\circ uX \overset{u}{=} \Phi X\circ Ac\circ uAX\circ uX\\ &\overset{(14)}{=} (\Phi X\circ Ac\circ\Phi X)\circ AuX\circ uX \overset{(16)}{=} Ac\circ\Phi X\circ(Ac\circ AuX)\circ uX\\ &\overset{u}{=} Ac\circ\Phi X\circ uAX\circ c\circ uX \overset{(13)}{=} Ac\circ AuX\circ c\circ uX = A\,(c\circ uX)\circ c\circ uX.\end{aligned}$$

Let us check (20).

$$\mu\circ(c\circ uX) \overset{(15)}{=} X$$

Let $\rho\in\mathrm{Des}\,(X,\mu)$. Let us check that $A\mu\circ\Phi X\circ A\rho\in\Phi$-QSymm$\,(X,\mu)$. Let us check (15)

$$\begin{aligned}\mu\circ A\mu\circ\Phi X\circ A\rho\circ uX &\overset{(5)}{=} \mu\circ mX\circ\Phi X\circ(A\rho\circ uX) \overset{u}{=} \mu\circ mX\circ(\Phi X\circ uAX)\circ\rho\\ &\overset{(13)}{=} \mu\circ mX\circ(AuX\circ\rho) \overset{(19)}{=} \mu\circ(mX\circ A\rho)\circ\rho \overset{(18)}{=} \mu\circ\rho\circ\mu\circ\rho = X.\end{aligned}$$

Let us check (16)

$$\begin{aligned}&A^2\mu\circ A\Phi X\circ A^2\rho\circ\left(\Phi X\circ A^2\mu\right)\circ A\Phi X\circ A^2\rho\\ &\overset{\Phi}{=} A^2\mu\circ A\Phi X\circ\left(A^2\rho\circ A^2\mu\right)\circ\Phi AX\circ A\Phi X\circ A^2\rho\end{aligned}$$

$$
\begin{aligned}
&\overset{(18)}{=} A^2\mu \circ A\Phi X \circ \left(A^2mX \circ A^3\rho \circ \Phi AX\right) \circ A\Phi X \circ A^2\rho \\
&\overset{\Phi}{=} A^2\mu \circ A\Phi X \circ \Phi AX \circ A^2mX \circ \left(A^3\rho \circ A\Phi X\right) \circ A^2\rho \\
&\overset{\Phi}{=} A^2\mu \circ A\Phi X \circ \Phi AX \circ A^2mX \circ A\Phi AX \circ \left(A^3\rho \circ A^2\rho\right) \\
&\overset{(19)}{=} A^2\mu \circ A\Phi X \circ \Phi AX \circ A^2mX \circ \left(A\Phi AX \circ A^3uX\right) \circ A^2\rho \\
&\overset{\Phi}{=} A^2\mu \circ A\Phi X \circ \Phi AX \circ \left(A^2mX \circ A^3uX\right) \circ A\Phi X \circ A^2\rho \\
&= A^2\mu \circ (A\Phi X \circ \Phi AX \circ A\Phi X) \circ A^2\rho \\
&\overset{(12)}{=} \left(A^2\mu \circ \Phi AX\right) \circ A\Phi X \circ \left(\Phi AX \circ A^2\rho\right) \overset{\Phi}{=} \Phi X \circ A^2\mu \circ A\Phi X \circ A^2\rho \circ \Phi X
\end{aligned}
$$

Let us check (17)

$$
\begin{aligned}
& mX \circ A\left(A\mu \circ \Phi X \circ A\rho\right) \circ \Phi X \\
= {} & \left(mX \circ A^2\mu\right) \circ A\Phi X \circ A^2\rho \circ \Phi X \\
& \overset{m}{=} A\mu \circ mAx \circ A\Phi X \circ \left(A^2\rho \circ \Phi X\right) \\
& \overset{\Phi}{=} A\mu \circ \left(mAX \circ A\Phi X \circ \Phi AX\right) \circ A^2\rho \\
& \overset{(14)}{=} A\mu \circ \Phi X \circ \left(AmX \circ A^2\rho\right) \\
& \overset{(18)}{=} A\mu \circ \Phi X \circ A\rho \circ A\mu \\
= {} & \left(A\mu \circ \Phi X \circ A\rho\right) \circ A\mu.
\end{aligned}
$$

Let $c \in \Phi\text{-QSymm}\,(X,\mu)$. Since, by Proposition 6.10, $\beta\left((X,\mu), c\right) = c \circ uX$ is a morphism in $\mathrm{QSymm}\,(\mathbb{A},\Phi)$, we have that

$$
Ac \circ AuX \circ c = \Phi X \circ Ac \circ AuX. \tag{21}
$$

We deduce that

$$
\left(\Gamma'(X,\mu) \circ \Gamma(X,\mu)\right)(c) = A\mu \circ \left(\Phi X \circ Ac \circ AuX\right) \overset{(21)}{=} \left(A\mu \circ Ac \circ AuX\right) \circ c \overset{(15)}{=} c.
$$

Let $\rho \in \mathrm{Des}\,(X,\mu)$.

$$
\begin{aligned}
\left(\Gamma(X,\mu) \circ \Gamma'(X,\mu)\right)(\rho) &= A\mu \circ \Phi X \circ \left(A\rho \circ uX\right) \overset{u}{=} A\mu \circ \Phi X \circ uAX \circ \rho \\
&\overset{(13)}{=} A\mu \circ AuX \circ \rho \overset{(5)}{=} \rho
\end{aligned}
$$

Assume now that $\Phi X \circ \Phi X = A^2X$ and let $\rho \in \mathrm{Des}\,(X,\mu)$. We compute

$$
\begin{aligned}
A\mu \circ \Phi X \circ \left(A\rho \circ A\mu\right) \circ \Phi X \circ A\rho &\overset{(18)}{=} A\mu \circ \Phi X \circ AmX \circ \left(A^2\rho \circ \Phi X\right) \circ A\rho \\
&\overset{\Phi}{=} A\mu \circ \Phi X \circ AmX \circ \Phi AX \circ \left(A^2\rho \circ A\rho\right) \\
&\overset{(19)}{=} A\mu \circ \Phi X \circ AmX \circ \left(\Phi AX \circ A^2uX\right) \circ A\rho \\
&\overset{\Phi}{=} A\mu \circ \Phi X \circ AmX \circ A^2uX \circ \Phi X \circ A\rho \\
&= A\mu \circ \Phi X \circ \Phi X \circ A\rho = A\mu \circ A\rho \overset{(20)}{=} AX.
\end{aligned}
$$

Since any $c \in \Phi\text{-QSymm}\,(X,\mu)$ is of the form $\Gamma'(X,\mu)(\rho)$ for $\rho = \Gamma(X,\mu)(c)$, we conclude. □

We now give a new and self-contained proof of the following Theorem.

Theorem 6.12. [KLV, Theorem 3.7] *Let $\mathbb{A} = (A,m,u)$ be a monad on a category $\mathcal{A}$ and let $\Phi : A^2 \to A^2$ be a BD-law on $\mathbb{A}$. Let K^{co} be the cocomparison functor K^{co} of the adjunction $(V, J_{\mathbb{A}}U)$*

$$
K^{co} : \mathrm{QSymm}\,(\mathbb{A},\Phi) \to^{VJ_{\mathbb{A}}U} ({}_{\mathbb{A}}\mathcal{A}) =^{{}_{\mathbb{A}}F_{\mathbb{A}}U} ({}_{\mathbb{A}}\mathcal{A}) =^{\mathbb{A}^*} ({}_{\mathbb{A}}\mathcal{A})
$$

defined by

$$K^{co}\left(\left(\left(X,\mu\right),c\right)\right)=\left(V\left(\left(\left(X,\mu\right),c\right)\right),V\beta\left(\left(X,\mu\right),c\right)\right)=\left(\left(X,\mu\right),c\circ uX\right)$$

is an isomorphism of categories whose inverse is the functor Λ *defined by setting*

$$\Lambda\left(\left(\left(X,\mu\right),\rho\right)\right)=\left(\left(X,\mu\right),A\mu\circ\Phi X\circ A\rho\right).$$

In particular the functor V *is comonadic.*

Proof. In view of Proposition 6.11, we know that $\left(\left(X,\mu\right),A\mu\circ\Phi X\circ A\rho\right)\in\mathrm{QSymm}\left(\mathbb{A},\Phi\right)$ for every $\left(\left(X,\mu\right),\rho\right)\in^{\mathbb{A}^*}\left({}_{\mathbb{A}}\mathcal{A}\right)$. Let

$$f:\left(\left(X,\mu\right),\rho\right)\to\left(\left(X',\mu'\right),\rho'\right)$$

be a morphism in ${}^{\mathbb{A}^*}\left({}_{\mathbb{A}}\mathcal{A}\right)$. We have

$$Af\circ\left(A\mu\circ\Phi X\circ A\rho\right)=A\mu'\circ A^2f\circ\Phi X\circ A\rho\overset{\Phi}{=}A\mu'\circ\Phi X'\circ A^2f\circ A\rho=\left(A\mu'\circ\Phi X'\circ A\rho'\right)\circ Af$$

so that

$$f:\left(\left(X,\mu\right),A\mu\circ\Phi X\circ A\rho\right)\to\left(\left(X',\mu'\right),A\mu'\circ\Phi X'\circ A\rho'\right)$$

is a morphism in $\mathrm{QSymm}\left(\mathbb{A},\Phi\right)$. We deduce that Λ is a functor. In view of Proposition 6.11, we get that K^{co} is an isomorphism of categories with inverse Λ. □

Example 6.13. Let $f:B\to A$ be a morphism of rings. Let $L:=(-)\otimes_B A=f^*:Mod\text{-}B\to Mod\text{-}A$ be the extension of scalars functor and $R:=\mathrm{Hom}_A\left({}_BA,-\right)=f_*:Mod\text{-}A\to Mod\text{-}B$ be the restriction of scalars functor . Let $\mathbb{A}=(RL,m=R\epsilon L,u=\eta)$ be the associated monad on $Mod\text{-}B$. Recall from Example (6.5) that for any $E\in Mod\text{-}B$ we have

$$RLE=E\otimes_B A\text{ regarded as a right }B\text{-module}$$

$$\begin{gathered}mE:E\otimes_B A\otimes_B A\to E\otimes_B A\\ x\otimes a\otimes a'\mapsto x\otimes aa'\end{gathered}$$

$$\begin{gathered}uE:E_B\to E\otimes_B A\\ x\mapsto x\otimes 1.\end{gathered}$$

Let $(E,\mu)\in{}_{\mathbb{A}}\left(Mod\text{-}B\right)$. Then $\mu:RLE=E\otimes_B A\to E$ is a morphism in $Mod\text{-}B$ satisfying

$$\mu\circ\left(\mu\otimes_B A\right)=\mu\circ RL\mu=\mu\circ mE\text{ and }E=\mu\circ uE$$

i.e.

$$\left(xa\right)a'=\left[\mu\circ\left(\mu\otimes_B A\right)\right]\left(x\otimes a\otimes a'\right)=\left(\mu\circ mE\right)\left(x\otimes a\otimes a'\right)=x\left(aa'\right)$$

where, for any $x\in E$ and $a\in A$ we write $xa=\mu\left(x\otimes a\right)$ and

$$x=x1$$

Let

$$t:E\times A\to E\otimes_B A$$

the canonical projection. Then $(E,\mu\circ t)\in Mod\text{-}A$. Let $f:(E,\mu)\to(E',\mu')$ be a morphism in ${}_{\mathbb{A}}\left(Mod\text{-}B\right)$. This means that $f:E\to E'$ is a morphism in $(Mod\text{-}B)$ and $f\circ\mu=\mu'\circ\left(f\otimes_B A\right)$ i.e.

$$f\left(xa\right)=f\left(x\right)a$$

i.e. $f:(E,\mu\circ t)\to(E',\mu'\circ t)$ is a morphism in $Mod\text{-}A$.

Conversely let $(M,\nu)\in Mod\text{-}A$. Since ν is B-balanced, there is a unique morphism $\mu:M\otimes_B A\to M$ such that $\nu=\mu\circ t$. Hence the assignment $(E,\mu)\mapsto(E,\mu\circ t)$ yields a category isomorphism

$$H:{}_{\mathbb{A}}\left(Mod\text{-}B\right)\to Mod\text{-}A.$$

Let $\left({}_{\mathbb{A}}F,{}_{\mathbb{A}}U\right)$ be the adjunction corresponding to our monad $\mathbb{A}$. Then it is easy to check that

$$Mod\text{-}A\overset{H^{-1}}{\to}{}_{\mathbb{A}}\left(Mod\text{-}B\right)\overset{{}_{\mathbb{A}}U}{\to}Mod\text{-}B$$

is just the restriction of scalars functor $R = \mathrm{Hom}_A(A, -) = f_* : Mod\text{-}A \to Mod\text{-}B$ while

$$Mod\text{-}B \xrightarrow{{}_{\mathbb{A}}F} {}_{\mathbb{A}}(Mod\text{-}B) \xrightarrow{H} Mod\text{-}A$$

coincides with the extension of scalars functor $L := (-) \otimes_B A = f^* : Mod\text{-}B \to Mod\text{-}A$. Therefore the category isomorphism H induces a category isomorphism

$${}^{{}_{\mathbb{A}}F_{\mathbb{A}}U}\left({}_{\mathbb{A}}(Mod\text{-}B)\right) \to {}^{\mathbb{C}}(Mod\text{-}A)$$

where $\mathbb{C}$ is the canonical comonad of the adjunction (L, R) i.e. $\mathbb{C} = (LR, \Delta = L\eta R = LuR, \varepsilon)$. For any $M \in Mod\text{-}A$ we have

$$LRM = M \otimes_B A \text{ regarded as a right } A\text{-module}$$

$$\begin{aligned} \Delta M : M \otimes_B A &\to M \otimes_B A \otimes_B A \\ x \otimes a &\mapsto x \otimes 1 \otimes a \end{aligned}$$

$$\begin{aligned} \varepsilon M : LRM = M \otimes_B A &\to M \\ x \otimes a &\mapsto xa. \end{aligned}$$

Let $(M, \rho) \in {}^{\mathbb{C}}(Mod\text{-}A)$ and for every $x \in M$ we write

$$\rho(x) = \sum x_i \otimes \alpha_i \text{ where } x_i \in M \text{ and } \alpha_i \in A \text{ for every } i.$$

(18) means that

$$\sum x_i \otimes \alpha_i a = \rho(xa) \qquad \text{for every } x \in M \text{ and } a \in A. \tag{22}$$

(19) means that

$$\sum \rho(x_i) \otimes \alpha_i = \sum x_i \otimes 1 \otimes \alpha_i \qquad \text{for every } x \in M \tag{23}$$

(20) means that

$$\sum x_i \alpha_i = x \qquad \text{for every } x \in M. \tag{24}$$

Now let us consider the cocomparison functor of the adjunction (L, R)

$$K^{co} : Mod\text{-}B \to {}^{\mathbb{C}}(Mod\text{-}A).$$

For every $E \in B\text{-}Mod$ we have

$$K^{co}(E) := (L(E), L\eta(E))$$

where $\rho_{L(E)} = L\eta(E) : E \otimes_B A \to E \otimes_B A \otimes_B A$ and

$$L\eta(E)(x \otimes a) = x \otimes 1 \otimes a.$$

Let $e : RM \to RLRM = M \otimes_B A$ be the map defined by $e(x) = x \otimes 1$. Note that e is a map in $Mod\text{-}B$. Let $E = \mathrm{Ker}(\rho - e)$. We have the exact sequence in $Mod\text{-}B$

$$0 \to E \xrightarrow{i} RM \xrightarrow{\rho - e} RLRM$$

and

$$M^{cov} = E = \{x \in M \mid \rho(x) = x \otimes 1\}.$$

It is easy to show that the assignment $M \mapsto M^{cov}$ defines a functor

$$()^{cov} : {}^{\mathbb{C}}(Mod\text{-}A) \to Mod\text{-}B.$$

Theorem 6.14. [CIP, Teorema page 45] *Using the assumptions and notations of Example 6.13, assume also that A is a faithfully flat left B-module. Then the cocomparison functor $K^{co} : Mod\text{-}B \to {}^{\mathbb{C}}(Mod\text{-}A)$ is an equivalence of categories with inverse functor*

$$()^{cov} : {}^{\mathbb{C}}(Mod\text{-}A) \to Mod - B.$$

Proof. Let $(M,\rho) \in {}^{\mathbb{C}}(Mod\text{-}A)$. Since A is a flat left B-module we get the exact sequence

$$0 \to M^{cov} \otimes_B A \to RM \otimes_B A \overset{(\rho - e)\otimes_B A}{\longrightarrow} RLRM \otimes_B A.$$

Let us show that $\mathrm{Im}\,(\rho) \subseteq M^{cov} \otimes_B A$ i.e. that $[(\rho - e) \otimes_B A]\,(\mathrm{Im}\,(\rho)) = 0$. Let $x \in M$ and let

$$\rho\,(x) = \sum x_i \otimes \alpha_i \text{ where } x_i \in M \text{ and } \alpha_i \in A \text{ for every } i.$$

We compute

$$[(\rho - e) \otimes_B A]\,(\rho)\,(x) = [(\rho - e) \otimes_B A]\left(\sum x_i \otimes \alpha_i\right)$$
$$= \sum \rho\,(x_i) \otimes \alpha_i - \sum x_i \otimes 1 \otimes \alpha_i \overset{(23)}{=} 0.$$

Hence we can consider the corestriction $\overline{\rho} : RM \to RLM^{cov} = M^{cov} \otimes_B A$ of ρ to $M^{cov} \otimes_B A$ so that $\rho = (i \otimes A) \circ \overline{\rho}$. Clearly $\overline{\rho}$ is a morphism in $Mod\text{-}A$. Let us show that it is a morphism in ${}^{\mathbb{C}}(Mod\text{-}A)$ from (M,ρ) to $K^{co}\,(M^{cov})$. For every $x \in M$, let

$$\rho\,(x) = \sum x_i \otimes \alpha_i \text{ where } x_i \in M \text{ and } \alpha_i \in A \text{ for every } i.$$

We compute

$$\begin{aligned}[(i \otimes_B A \otimes_B A) \circ (\overline{\rho} \otimes_B A) \circ \rho]\,(x) &= [(\rho \otimes_B A) \circ \rho]\,(x) = \sum \rho\,(x_i) \otimes \alpha_i \\ \overset{(23)}{=} \sum x_i \otimes 1 \otimes \alpha_i &= \left((i \otimes A \otimes A) \circ \rho_{L(M^{cov})} \circ \overline{\rho}\right)(x)\,.\end{aligned}$$

Since ${}_BA$ is flat, $i \otimes_B A \otimes_B A$ is a monomorphism so that we deduce that

$$(\overline{\rho} \otimes_B A) \circ \rho = \rho_{L(M^{cov})} \circ \overline{\rho}$$

and hence $\overline{\rho}$ is a morphism in ${}^{\mathbb{C}}(Mod\text{-}A)$.

Let $h : M^{cov} \otimes_B A \to M$ be defined by

$$h\,(x \otimes a) = xa.$$

For every $x \in M$, let

$$\rho\,(x) = \sum x_i \otimes \alpha_i \text{ where } x_i \in M \text{ and } \alpha_i \in A \text{ for every } i.$$

We compute

$$(h \circ \overline{\rho})\,(x) = \sum x_i \alpha_i \overset{(24)}{=} x$$

and for every $x \in M^{cov}$ and $a \in A$

$$(\overline{\rho} \circ h)\,(x \otimes a) = \rho\,(xa) \overset{(22)}{=} \sum x_i \otimes \alpha_i a$$
$$= \left(\sum x_i \otimes \alpha_i\right) a \overset{x \in M^{cov}}{=} (x \otimes 1)\,a = x \otimes a.$$

This proves that $\overline{\rho}$ is an isomorphism in ${}^{\mathbb{C}}(Mod\text{-}A)$ with inverse h.

Let now $E \in Mod\text{-}B$ and let $x \in E$. Then

$$\rho_{L(E)}\,(x \otimes 1) = x \otimes 1 \otimes 1$$

so that $x \otimes 1 \in (E \otimes_B A)^{cov}$ and hence we can consider the morphism of right B-modules $v : E \to (E \otimes_B A)^{cov}$ defined by $v\,(x) = x \otimes 1$. We want to prove that v is an isomorphism in $Mod\text{-}B$. Since A is a faithfully flat left B-module, in view of [Bou, Proposition 2 page 47], this is equivalent to show that $v \otimes_B A$ is bijective. For every $x \in E$ and $a \in A$ we have

$$(v \otimes_B A)\,(x \otimes a) = x \otimes 1 \otimes a = \overline{\rho_{L(E)}}\,(x \otimes a)$$

so that we deduce that $v \otimes_B A = \overline{\rho_{L(E)}}$. By the foregoing we know that $\overline{\rho_{L(E)}}$ is an isomorphism in $Mod\text{-}A$. □

NOTATION 6.15. *Let $R : \mathcal{A} \to \mathcal{B}$. We will denote by* $\mathrm{Im}R$ *the full subcategory of $\mathcal{B}$ consisting of those objects $B \in \mathcal{B}$ such that there is an object $A \in \mathcal{A}$ and an isomorphism $B \cong RA$ in $\mathcal{B}$.*

PROBLEM 1. (Descent problem for modules) Let $M \in A\text{-}Mod$. Is there any $E \in B\text{-}Mod$ such that $M \cong L(E) = E \otimes_B A$ in $A\text{-}Mod$? Such an E will be called a **form** of M over B.

THEOREM 6.16. *Using the assumptions and notations of Example 6.13, let $f : B \to A$ be a morphism of rings and assume that A is a faithfully flat left B-module. Then*

$$\mathrm{Obj}\,(\mathrm{Im}\,(L)) = \mathrm{Obj}\left(U^{\mathbb{C}}\left[{}^{\mathbb{C}}(Mod\text{-}A)\right]\right).$$

Proof. In view of Theorem 6.14, $K^{co} : Mod\text{-}B \to {}^{\mathbb{C}}(Mod\text{-}A)$ is an equivalence of categories so that $\mathrm{Obj}\,(\mathrm{Im}\,(K^{co})) = {}^{\mathbb{C}}(Mod\text{-}A)$. Therefore

$$\mathrm{Obj}\,(\mathrm{Im}\,(L)) = \mathrm{Obj}\left(\mathrm{Im}\left(U^{\mathbb{C}} \circ K^{co}\right)\right) = \mathrm{Obj}\left(U^{\mathbb{C}}\left[{}^{\mathbb{C}}(Mod\text{-}A)\right]\right)$$

□

Assume now A and B commutative. All modules over a commutative ring S are considered as symmetrical S-S-bimodules.

Let M be an A-module and let $g : A \otimes_B M \to M \otimes_B A$ be a morphism of A-A-bimodules. Let $g_1 = A \otimes_B g, g_3 = g \otimes_B A$ and define $g_2 : A \otimes_B A \otimes_B M \to M \otimes_B A \otimes_B A$ and $\overline{g} : M \to M$ by setting

$$\begin{aligned} g_2\,(a \otimes a' \otimes x) &= \sum x_i \otimes a' \otimes \alpha_i \qquad \text{where } g\,(a \otimes x) = \sum x_i \otimes \alpha_i. \\ \overline{g}\,(x) &= \sum x_i \alpha_i \qquad \text{where } g\,(1 \otimes x) = \sum x_i \otimes \alpha_i. \end{aligned}$$

$$\begin{gathered} (g_3 \circ g_1)\,(a \otimes a' \otimes x) = g_3\,(a \otimes g\,(a' \otimes x)) = g_3\,(a \otimes a' g\,(1 \otimes x)) \\ = g_3\left(\sum a \otimes a' x_i \otimes \alpha_i\right) = \sum a g\,(1 \otimes a' x_i) \otimes \alpha_i \\ \text{where } g\,(1 \otimes x) = \sum x_i \otimes \alpha_i \\ g_2\,(a \otimes a' \otimes x) = \sum a x_i \otimes a' \otimes \alpha_i \\ \text{where } g\,(a \otimes x) = a g\,(1 \otimes x) = \sum a x_i \otimes \alpha_i \text{ where } g\,(1 \otimes x) = \sum x_i \otimes \alpha_i \end{gathered}$$

Hence $g_2 = g_3 \circ g_1$ means

$$\sum a g\,(1 \otimes a' x_i) \otimes \alpha_i = \sum a x_i \otimes a' \otimes \alpha_i \text{ where } g\,(1 \otimes x) = \sum x_i \otimes \alpha_i \tag{25}$$

while $\overline{g} = \mathrm{Id}_M$ means

$$\sum x_i \alpha_i = x \qquad \text{where } g\,(1 \otimes x) = \sum x_i \otimes \alpha_i. \tag{26}$$

Let

$$\Gamma = \left\{ \begin{array}{c} g : A \otimes_B M \to M \otimes_B A \\ \mid g \text{ is a morphism of } A\text{-}A\text{-bimodules } g_2 = g_3 \circ g_1 \text{ and } \overline{g} = \mathrm{Id}_M \end{array} \right\}$$

For every $g \in \Gamma$ consider the map

$$\rho_g : M \to M \otimes_B A$$

defined by

$$\rho_g\,(x) = g\,(1 \otimes x).$$

For every $\rho \in \mathrm{Des}\,(X, \mu)$, where $\mu : A \otimes_B M \to M$ denotes the map induced by the multiplication by A on M, consider the map

$$g_\rho : A \otimes_B M \to M \otimes_B A$$

defined by

$$g_\rho\,(a \otimes x) = a\rho\,(x) = \sum a x_i \otimes \alpha_i \text{ where } \rho\,(x) = \sum x_i \otimes \alpha_i$$

THEOREM 6.17. *The assignment $g \mapsto \rho_g$ defines a bijection $\Lambda : \Gamma \to \mathrm{Des}\,(M, \mu)$ whose inverse is defined by the assignment $\rho \mapsto g_\rho$.*

Proof. Let us check that $\rho_g \in \mathrm{Des}(M,\mu)$. Let $x \in M$. We write

$$\rho_g(x) = g(1 \otimes x) = \sum x_i \otimes \alpha_i.$$

For every $a \in A$, we compute

$$\rho_g(xa) = g(1 \otimes xa) = g((1 \otimes x)a) = g(1 \otimes x)a = \rho_g(x)a$$
$$= \left(\sum x_i \otimes \alpha_i\right)a = \left(\sum x_i \otimes \alpha_i a\right)$$

so that ρ_g satisfies (22).

We compute

$$\sum \rho_g(x_i) \otimes \alpha_i = \sum g(1 \otimes x_i) \otimes \alpha_i \overset{(25)}{=} \sum x_i \otimes 1 \otimes \alpha_i$$

so that ρ_g fulfils (23). Moreover, in view of (26), for every $x \in X$ we have

$$\sum x_i a_i = x \qquad \text{where } \rho_g(x) = g(1 \otimes x) = \sum x_i \otimes a_i.$$

so that ρ_g fulfils (24).

Conversely let $\rho \in \mathrm{Des}(M,\mu)$. Let $x \in M$. We write

$$\rho(x) = \sum x_i \otimes \alpha_i.$$

Then, for every $a \in A$ we have

$$g_\rho(a \otimes x) = a\rho(x) = \sum a x_i \otimes a_i$$

For every $a, a' \in A$, we compute

$$\sum a g_\rho(1 \otimes a' x_i) \otimes \alpha_i = \sum a g_\rho(1 \otimes a' x_i) \otimes \alpha_i = \sum a g_\rho(1 \otimes x_i a') \otimes \alpha_i$$
$$= \sum a\rho(x_i a') \otimes \alpha_i = \sum a\rho(x_i)a' \otimes \alpha_i \overset{(23)}{=} \sum a x_i \otimes a' \otimes \alpha_i$$

so that g_ρ fulfils (25). Moreover in view of (24) we have that

$$\sum x_i \alpha_i = x$$

so that g_ρ fulfils (26).

Let now $g \in \Gamma$ and, for every $a \in A$ and $x \in M$ let us compute

$$g_{\rho_g}(a \otimes x) = a\rho_g(1 \otimes x) = ag(1 \otimes x) = g(a \otimes x).$$

Therefore we deduce that $g_{\rho_g} = g$. Let now $\rho \in \mathrm{Des}(M,\mu)$ and, for every $x \in M$ let us compute

$$\rho_{g_\rho}(x) = g_\rho(1 \otimes x) = \rho(x).$$

Therefore we deduce that $\rho = \rho_{g_\rho}$. □

References

[Appel] H. Appelgate, *Acyclic models and resolvent functors*, Ph.D. thesis (Columbia University, 1965).

[Be1] J.M. Beck, *Distributive laws*, Lect. Notes Math.,**80** (1969), 119–140.

[Be2] J.M. Beck, *Triples, algebras and cohomology*, Reprints in Theory and Applications of Categories **2** (2003), 1–59.

[Bou] N. Bourbaki, *Éléments de mathématique. Algèbre commutative.* Chapitres **1** a **4**, Springer Verlag Berlin (2006). (Réimpression inchangée de l'édition originale de 1985 Masson, Paris 1985).

[1] J. Cuntz and D. Quillen, *Algebra extensions and nonsingularity*, Journal of the Amer. Math. Soc., **8** (1995), 251-289.

[CIP] M. Cipolla, *Discesa fedelmente piatta dei moduli*, Rend. Circ. Mat. Palermo **25** (1976), 43–46

[Gro] A. Grothendieck, *Technique de descente et théorèmes d'existence en géometrie algébrique. I. Généralités. Descente par morphismes fidèlement plats*, Séminaire Bourbaki, Vol. **5**, Exp. No. 190, 299–327, (1995).

[H] P. Huber, *Homotopy theory in general categories*, Math. Ann. **144** (1961), 361-385.

[J] P.T. Johnstone, *Adjoint lifting theorems for categories of algebras*, Bull. London Math. Soc., **7** (1975), 294-297.

[KLV] S. Kasangian, S. Lack, E. Vitale, *Coalgebras, braidings, and distributive laws*, Theory Appl. Categ. **13** (2004), 129–146.

[KO] M. Knus, M. Ojanguren, *Théorie de la descente et algèbres d'Azumaya*, Lect. Notes Math., **389** (1974)

[MS] C. Menini, D. Ştefan, *Descent theory and Amitsur cohomology of triples*, J. Algebra **266** (2003), 261–304.
[Nu] Ph. Nuss, *Noncommutative descent and nonabelian cohomologie*, *K*-Theory, **12** (1997), 23-74.