Localized intersection of currents and the Lefschetz coincidence point theorem

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Abstract : We introduce the notion of a Thom class of a current and define the localized intersection of currents. In particular we consider the situation where we have a C^∞ map of manifolds and study localized intersections of the source manifold and currents on the target manifold. We then obtain a residue theorem on the source manifold and give explicit formulas for the residues in some cases. These are applied to the problem of coincidence points of two maps. We define the global and local coincidence homology classes and indices. A representation of the Thom class of the graph as a Čech-de Rham cocycle immediately gives us an explicit expression of the index at an isolated coincidence point, which in turn gives explicit coincidence classes in some non-isolated components. Combining these, we have a general coincidence point theorem including the one by S. Lefschetz.

Keywords : Alexander duality, Thom class, localized intersections, residue theorem, coincidence classes and indices, Lefschetz coincidence point formula.

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Introduction

For two cycles in a manifold, the localized intersection product of their classes is defined, in the homology of their set theoretical intersection, via Alexander dualities and the cup product in the relative cohomology. Thus for a cycle C in a manifold W, the corresponding class in the relative cohomology carries the local information on C. For a submanifold M of W, this is the Thom class of M, which may be identified with the Thom class of the normal bundle of M in W by the tubular neighborhood theorem. In this paper we take up the localization problem of currents. We introduce the notion of a Thom class of a current and define the localized intersection of two currents. In particular we consider the intersections of a fixed submanifold M and currents on W, obtain a residue theorem on M and give explicit expressions of the residues in some cases (see Theorems 3.12 and 3.17 below for precise statements). As an application we study the coincidence point problem for two maps.

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The coincidence point formula discovered by S. Lefschetz (cf. [6], [7]) is formulated for a pair of continuous maps between compact oriented topological manifolds of the same dimension. Using the above, we define global and local homology classes of coincidence for a pair of C^∞ maps $M \to N$ and give a general coincidence point theorem, even in the case the dimensions m and n of M and N are different ($m \geq n$) and the coincidence points are non-isolated (Definitions 4.1 and 4.2). In the case $m = n$, the use of Thom class in the Čech-de Rham cohomology immediately gives us an explicit expression of the coincidence index at an isolated coincidence point (Propositions 4.5). This gives in turn an explicit coincidence homology class at a certain non-isolated coincidence component in the case $m > n$ (Proposition 4.8). We then have a general coincidence point formula, including the Lefschetz coincidence point formula.

The paper is organized as follows. In Section 2, we recall preliminary materials such as Poincaré and Alexander dualities, global and localized intersection products, Thom classes in various settings and an explicit expression, in the Čech-de Rham cohomology, of the Thom class of an oriented vector bundle (Proposition 2.16). In Section 3, we consider the localization problem of currents, introduce the notion of a Thom class of a current and give some examples. We then consider localized intersections of currents. In particular we study intersections of a fixed submanifold M in a manifold W and currents on W. In fact we consider a more general situation where we have a map $F : M \to W$ (Definition 3.10). For a closed current T on W, we have the intersection product $M \cdot_F T$ in the homology of M and, if T is localized at a compact set $\tilde{S}$ in W and if Ψ_T is a Thom class of T along $\tilde{S}$, we have the residue of $F^*\Psi_T$ in the homology of $S = F^{-1}\tilde{S}$ as a localized intersection product. If S has several connected components, we have a "residue theorem" (Theorem 3.12). We give an explicit formula for the residue at a non-isolated component of S in the case it is a submanifold of M (Theorem 3.17). These are conveniently used in Section 4, where we study the coincidence point problem for two maps.

Let M and N be manifolds of dimensions m and n and let $f, g : M \to N$ be two maps. We define the global coincidence class of the pair (f, g) in the $(m-n)$-th homology of M (Definition 4.1). If M is compact, we also define the local coincidence class of the pair in the $(m-n)$-th homology of the set of points in M where f and g coincide (Definition 4.2). We then apply Theorem 3.12 to get a general coincidence point theorem (Theorem 4.4). In the case $m = n$ we have a formula for the coincidence index at an isolated coincidence point as the local mapping degree of $g - f$ (Proposition 4.5). This is in fact a classical result, however we give a short direct proof using the aforementioned expression of the Thom class in the Čech-de Rham cohomology. This together with Theorem 3.17 gives an explicit expression of the coincidence homology class at a non-isolated component (Proposition 4.8). If M and N are compact manifolds of the same dimension, we have a general coincidence point formula (Theorem 4.11), which reduces to the Lefschetz coincidence point formula in the case the coincidence points are isolated (Corollary 4.12).

1 Notation and conventions

Let M be a C^∞ manifold. We denote by $H_*(M, \mathbb{C})$ and $H^*(M, \mathbb{C})$ its homology and cohomology of finite singular chains with $\mathbb{C}$ coefficients. Also we denote by $\breve{H}_*(M, \mathbb{C})$ the homology of locally finite chains (Borel-Moore homology). They can be computed taking a triangulation of M. Recall that any C^∞ manifold admits a C^∞ triangulation, which is essentially unique. In the sequel a locally finite C^∞ chain is simply called a chain unless otherwise stated. Thus a chain C is expressed as a locally finite sum $C = \sum a_i s_i$ with a_i in $\mathbb{C}$ and s_i oriented C^∞ simplices. We set $|C| = \bigcup s_i$ and call it the support of C. It is a closed set. For a cycle C, its class in the homology of the ambient space is denoted by $[C]$, while the class in the homology of its support is simply denoted by C.

For an open set U in M, we denote by $A^p(U)$ and $A^p_c(U)$, respectively, the spaces of complex valued C^∞ p-forms and p-forms with compact support on U. The cohomology of the complex $(A^*(M), d)$ is the de Rham cohomology $H^*_{\mathrm{dR}}(M)$ and that of $(A^*_c(M), d)$ is the cohomology $H^*_c(M)$ with compact support. A C^∞ form will be simply called a form unless otherwise stated.

2 Čech-de Rham cohomology and the Thom class

For the background on the Čech-de Rham cohomology, we refer to [2]. The integration theory on this cohomology is developed in [8]. See [9] also for these materials and for the description of the Thom class in the framework of relative Čech-de Rham cohomology. The relation with the combinatorial viewpoint, as given in [3], is discussed in [10].

In this section we let M denote a C^∞ manifold of dimension m.

2.1 Poincaré duality

We recall the Poincaré duality and global intersection products of homology classes.

Suppose that M is connected and oriented. Then the pairing

$$A^p(M) \times A^{m-p}_c(M) \longrightarrow \mathbb{C} \qquad \text{given by} \quad (\omega, \varphi) \mapsto \int_M \omega \wedge \varphi$$

induces the Poincaré duality for a possibly non-compact manifold:

$$P : H^p(M, \mathbb{C}) \simeq H^p_{\mathrm{dR}}(M) \xrightarrow{\sim} H^{m-p}_c(M)^* \simeq \breve{H}_{m-p}(M, \mathbb{C}). \tag{2.1}$$

In the sequel we sometimes omit the coefficient $\mathbb{C}$ in homology and cohomology. In fact the Poincaré duality holds with $\mathbb{Z}$ coefficient. Note that P is given by the left cap product with the fundamental class of M, the class of the sum of all m-simplices in M. We also denote P by P_M if we wish to make the manifold M under consideration explicit.

In the isomorphism (2.1), the class $[\omega]$ of a closed p-form ω corresponds to the functional on $H^{m-p}_c(M)$ given by

$$[\varphi] \mapsto \int_M \omega \wedge \varphi, \tag{2.2}$$

or to the class $[C]$ of an $(m-p)$-cycle C such that

$$\int_M \omega \wedge \varphi = \int_C \varphi \tag{2.3}$$

for any closed form φ in $A^{m-p}_c(M)$. We call ω a *de Rham representative* of C.

For two classes $[C_1] \in \breve{H}_{q_1}(M)$ and $[C_2] \in \breve{H}_{q_2}(M)$, the *intersection product* $[C_1] \cdot [C_2]$ is defined by

$$[C_1] \cdot [C_2] := P(P^{-1}[C_1] \smile P^{-1}[C_2]) \qquad \text{in } \breve{H}_{q_1+q_2-m}(M), \tag{2.4}$$

where $\smile$ denotes the cup product, which corresponds to the exterior product in the first isomorphism in (2.1).

If M is compact and connected, then $\breve{H}_0(M, \mathbb{C}) = H_0(M, \mathbb{C}) = \mathbb{C}$. Thus if $q_1 + q_2 = m$, $[C_1] \cdot [C_2]$ is a number given by

$$[C_1] \cdot [C_2] = \int_M \omega_1 \wedge \omega_2 = \int_{C_1} \omega_2 = (-1)^{q_1 q_2} \int_{C_2} \omega_1,$$

where ω_1 and ω_2 are de Rham representatives of C_1 and C_2, respectively.

2.2 Čech-de Rham cohomology

The Čech-de Rham cohomology is defined for an arbitrary open covering of M, however here we only consider coverings consisting of two open sets. Thus let $\mathcal{U} = \{U_0, U_1\}$ be an open covering of M. We set $U_{01} = U_0 \cap U_1$ and define the complex vector space $A^p(\mathcal{U})$ as

$$A^p(\mathcal{U}) := A^p(U_0) \oplus A^p(U_1) \oplus A^{p-1}(U_{01}).$$

An element σ in $A^p(\mathcal{U})$ is given by a triple $\sigma = (\sigma_0, \sigma_1, \sigma_{01})$ with σ_i a p-form on U_i, $i = 0, 1$, and σ_{01} a $(p-1)$-form on U_{01}. We define an operator $D : A^p(\mathcal{U}) \to A^{p+1}(\mathcal{U})$ by

$$D\sigma := (d\sigma_0, d\sigma_1, \sigma_1|_{U_{01}} - \sigma_0|_{U_{01}} - d\sigma_{01}).$$

Then we see that $D \circ D = 0$ so that we have a complex $(A^*(\mathcal{U}), D)$. The p-th *Čech-de Rham cohomology* of $\mathcal{U}$, denoted by $H^p_D(\mathcal{U})$, is the p-th cohomology of this complex. It is also abbreviated as ČdR cohomology. We denote the class of a cocycle σ by $[\sigma]$. It can be shown that the map $A^p(M) \to A^p(\mathcal{U})$ given by $\omega \mapsto (\omega|_{U_0}, \omega|_{U_1}, 0)$ induces an isomorphism

$$\alpha : H^p_{\mathrm{dR}}(M) \xrightarrow{\sim} H^p_D(\mathcal{U}). \tag{2.5}$$

Note that α^{-1} assigns to the class of a ČdR cocycle $(\sigma_0, \sigma_1, \sigma_{01})$ the class of the closed form $\rho_0\sigma_0 + \rho_1\sigma_1 - d\rho_0 \wedge \sigma_{01}$, where $\{\rho_0, \rho_1\}$ is a partition of unity subordinate to $\mathcal{U}$.

Now we could define the cup product for ČdR cochains and describe the Poincaré duality in terms of the ČdR cohomology as in [9] in the case M is compact. However here we proceed as follows. Let M and $\mathcal{U} = \{U_0, U_1\}$ be as above. A *system of honeycomb cells* adapted to $\mathcal{U}$ is a collection $\{R_0, R_1\}$ of two submanifolds of M of dimension m with C^∞ boundary having the following properties:

(1) $R_i \subset U_i$ for $i = 0, 1$,

(2) $\mathrm{Int}\, R_0 \cap \mathrm{Int}\, R_1 = \emptyset$ and

(3) $R_0 \cup R_1 = M$,

where Int denotes the interior. Suppose M is oriented. Then R_0 and R_1 are naturally oriented. Let $R_{01} = R_0 \cap R_1$ with the orientation as the boundary of R_0; $R_{01} = \partial R_0$, or equivalently, the orientation opposite to that of the boundary of R_1 ; $R_{01} = -\partial R_1$. We consider the pairing

$$A^p(\mathcal{U}) \times A_c^{m-p}(M) \longrightarrow \mathbb{C}$$

given by

$$(\sigma, \varphi) \mapsto \int_{R_0} \sigma_0 \wedge \varphi + \int_{R_1} \sigma_1 \wedge \varphi + \int_{R_{01}} \sigma_{01} \wedge \varphi. \tag{2.6}$$

Then it induces the Poincaré duality (2.1) through the isomorphism α in (2.5).

2.3 Relative Čech-de Rham cohomology and Alexander duality

We introduce the relative Čech-de Rham cohomology and describe the Alexander duality, which is used to define localized intersection products.

Let S be a closed set in M. Letting $U_0 = M \smallsetminus S$ and U_1 a neighborhood of S in M, we consider the covering $\mathcal{U} = \{U_0, U_1\}$ of M. If we set

$$A^p(\mathcal{U}, U_0) = \{\, \sigma \in A^p(\mathcal{U}) \mid \sigma_0 = 0 \,\},$$

we see that $(A^*(\mathcal{U}, U_0), D)$ is a subcomplex of $(A^*(\mathcal{U}), D)$. We denote by $H_D^p(\mathcal{U}, U_0)$ the p-th cohomology of this complex. From the short exact sequence

$$0 \longrightarrow A^*(\mathcal{U}, U_0) \xrightarrow{j^*} A^*(\mathcal{U}) \xrightarrow{\iota^*} A^*(U_0) \longrightarrow 0,$$

where j^* is the inclusion and ι^* is the homomorphism that assigns σ_0 to $\sigma = (\sigma_0, \sigma_1, \sigma_{01})$, we have the long exact sequence

$$\cdots \longrightarrow H_D^{p-1}(\mathcal{U}) \xrightarrow{\iota^*} H_{\mathrm{dR}}^{p-1}(U_0) \xrightarrow{\delta^*} H_D^p(\mathcal{U}, U_0) \xrightarrow{j^*} H_D^p(\mathcal{U}) \xrightarrow{\iota^*} H_{\mathrm{dR}}^p(U_0) \longrightarrow \cdots. \tag{2.7}$$

In the above, δ^* assigns the class $[(0, 0, -\theta)]$ to the class of a closed $(p-1)$-form θ on U_0. Comparing with the long cohomology exact sequence for the pair $(M, M \smallsetminus S)$, we have a natural isomorphism (see [10] for a precise proof):

$$H_D^p(\mathcal{U}, U_0) \simeq H^p(M, M \smallsetminus S; \mathbb{C}).$$

We describe the Alexander duality in terms of the relative ČdR cohomology in the case S is compact and admits a regular neighborhood (cf. [9]). Thus suppose M is oriented and let $\{R_0, R_1\}$ be a system of honeycomb cells adapted to $\mathcal{U}$. We assume that S is compact so that we may also assume that R_1 is compact. Consider the pairing

$$A^p(\mathcal{U}, U_0) \times A^{m-p}(U_1) \longrightarrow \mathbb{C} \tag{2.8}$$

given by

$$(\sigma, \varphi) \mapsto \int_{R_1} \sigma_1 \wedge \varphi + \int_{R_{01}} \sigma_{01} \wedge \varphi.$$

Then it induces the Alexander homomorphism:

$$A : H^p(M, M \smallsetminus S; \mathbb{C}) \simeq H_D^p(\mathcal{U}, U_0) \longrightarrow H_{\mathrm{dR}}^{m-p}(U_1)^* \simeq H_{m-p}(U_1, \mathbb{C}). \tag{2.9}$$

The homomorphism (2.9) depends on U_1 and is not an isomorphism in general. Here we consider the following hypothesis:

(*) there exists a triangulation of M such that S is (the polyhedron of) a subcomplex.

The hypothesis is satisfied if S is the support of a chain. Under this hypothesis, we may take as U_1 a regular neighborhood of S so that there is a deformation retract $U_1 \to S$. We then have $H_{m-p}(U_1) \simeq H_{m-p}(S)$ and (2.8) induces the Alexander duality:

$$A : H^p(M, M \smallsetminus S; \mathbb{C}) \simeq H^p_D(\mathcal{U}, U_0) \xrightarrow{\sim} H^{m-p}_{\mathrm{dR}}(U_1)^* \simeq H_{m-p}(S, \mathbb{C}). \tag{2.10}$$

Note that A is also given by the left cap product with the fundamental class M. We also denote A by $A_{M,S}$ if we wish to make the pair (M, S) explicit.

In the isomorphism (2.10), the class $[\sigma]$ of a p-cochain σ corresponds to the functional on $H^{m-p}_{\mathrm{dR}}(U_1)$ given by

$$[\varphi_1] \mapsto \int_{R_1} \sigma_1 \wedge \varphi_1 + \int_{R_{01}} \sigma_{01} \wedge \varphi_1, \tag{2.11}$$

or to the class $[C]$ of an $(m-p)$-cycle C in S such that

$$\int_{R_1} \sigma_1 \wedge \varphi_1 + \int_{R_{01}} \sigma_{01} \wedge \varphi_1 = \int_C \varphi_1 \tag{2.12}$$

for any closed form φ_1 in $A^{m-p}(U_1)$.

Denoting by $i : S \hookrightarrow M$ the inclusion, we have the following commutative diagram:

$$\begin{array}{ccc} H^p(M, M \smallsetminus S) & \xrightarrow[A]{\sim} & H^{m-p}_{\mathrm{dR}}(U_1)^* \simeq H_{m-p}(S) \\ \Big\downarrow {\scriptstyle j^*} & & \Big\downarrow {\scriptstyle i_*} \\ H^p(M) & \xrightarrow[P]{\sim} & H^{m-p}_c(M)^* \simeq \breve{H}_{m-p}(M). \end{array} \tag{2.13}$$

Remark 2.14 1. If we denote the homomorphism $H^{m-p}_{\mathrm{dR}}(U_1)^* \to H^{m-p}_c(M)^*$ corresponding to i_* in the homology also by i_*, it is described as follows. For any functional F_1 on $H^{m-p}_{\mathrm{dR}}(U_1)$, there is a corresponding cycle C in S and i_*F_1 is given by

$$i_*F_1[\varphi] = \int_C \varphi \qquad \text{for} \quad [\varphi] \in H^{m-p}_c(M).$$

Alternatively, if $(0, \sigma_1, \sigma_{01})$ is a ČdR representative of $A^{-1}F_1$, then

$$i_*F_1[\varphi] = \int_{R_1} \sigma_1 \wedge \varphi + \int_{R_{01}} \sigma_{01} \wedge \varphi.$$

2. For a closed set S (which may not be compact) in M satisfying (*), we may define the Alexander isomorphism

$$A : H^p(M, M \smallsetminus S) \xrightarrow{\sim} \breve{H}_{m-p}(S)$$

via combinatorial topology (cf. [3]).

Let S_1 and S_2 be compact sets in M satisfying (*) and set $S = S_1 \cap S_2$. Let A_1, A_2 and A denote the Alexander isomorphisms for (M, S_1), (M, S_2) and (M, S), respectively.

For two classes $c_1 \in H_{q_1}(S_1)$ and $c_2 \in H_{q_2}(S_2)$, the *localized intersection product* $(c_1 \cdot c_2)_S$ is defined by

$$(c_1 \cdot c_2)_S := A(A_1^{-1}c_1 \smile A_2^{-1}c_2) \qquad \text{in} \;\; H_{q_1+q_2-m}(S), \tag{2.15}$$

where $\smile$ denotes the cup product

$$H^{m-q_1}(M, M \smallsetminus S_1) \times H^{m-q_2}(M, M \smallsetminus S_2) \xrightarrow{\smile} H^{2m-q_1-q_2}(M, M \smallsetminus S).$$

Letting $i_1 : S_1 \hookrightarrow M$, $i_2 : S_2 \hookrightarrow M$ and $i : S \hookrightarrow M$ be the inclusions, from (2.13) we see that the definitions (2.4) and (2.15) are consistent in the sense that

$$i_*(c_1 \cdot c_2)_S = (i_1)_* c_1 \cdot (i_2)_* c_2.$$

2.4 Thom class

In this subsection, we sometimes omit the coefficient $\mathbb{C}$ in homology and cohomology. In fact the isomorphisms we consider below can be defined from combinatorial viewpoint in homology and cohomology with $\mathbb{Z}$ coefficient (cf. [3], also [10]).

(a) Thom class of an oriented real vector bundle : Let $\pi : E \to M$ be an oriented real vector bundle of rank k. We identify M with the image of the zero section. Then we have the Thom isomorphism

$$T_E : H^p(M, \mathbb{C}) \xrightarrow{\sim} H^{p+k}(E, E \smallsetminus M; \mathbb{C}),$$

whose inverse is given by the integration along the fiber of π (see [9, Ch.II, 5]).

The *Thom class* Ψ_E of E, which is in $H^k(E, E \smallsetminus M)$, is the image of the constant function 1 in $H^0(M, \mathbb{C})$ by T_E. Note that T_E is given by the cup product with Ψ_E. Let $W_0 = E \smallsetminus M$ and W_1 a neighborhood of M in E and consider the covering $\mathcal{W} = \{W_0, W_1\}$ of E. We refer to [9, Ch.II, Proposition 5.7] for an explicit expression of a ČdR cocycle representing Ψ_E in the isomorphism $H^k(E, E \smallsetminus M) \simeq H^k_D(\mathcal{W}, W_0)$. In particular suppose E is trivial on an open set U of M. Then, setting $A^k(\mathcal{W}, W_0)|_U = A^k(\mathcal{W}', W_0')$ with $W_i' = W_i \cap \pi^{-1}(U)$, we have (cf. [9, Ch.III, Lemma 1.4]) :

Proposition 2.16 *Suppose E is trivial on an open set U of M ; $E|_U \simeq \mathbb{R}^k \times U$ and let $\rho : E|_U \to \mathbb{R}^k$ denote the projection on to the fiber direction. Then $\Psi_{E|_U}$ is represented by a cocycle in $A^k(\mathcal{W}, W_0)|_U$ of the form*

$$(0, 0, -\rho^*\psi_k),$$

where ψ_k is an angular form on $\mathbb{R}^k \smallsetminus \{0\}$, i.e., a closed $(k-1)$-form with $\int_{S^{k-1}} \psi_k = 1$.

Suppose M is compact and oriented. We orient the total space E so that, if $\xi = (\xi_1, \dots, \xi_k)$ is a positive fiber coordinate system of E and if $x = (x_1, \dots, x_m)$ is a positive coordinate system on M, then (ξ, x) is a positive coordinate system on E. We then have the commutative diagram :

$$\begin{array}{ccc} H^p(M) & \xrightarrow[T_E]{\sim} & H^{p+k}(E, E \smallsetminus M) \\ \wr\big\downarrow P & & \wr\big\downarrow A \\ H_{m-p}(M) & \xrightarrow{=} & H_{m-p}(M). \end{array} \tag{2.17}$$

(b) Thom class of a submanifold : Let W be an oriented C^∞ manifold of dimension m' and M a compact and oriented submanifold of W of dimension m. Set $k = m' - m$. In view of (2.17), we define the Thom isomorphism T_M as the composition

$$T_M : H^p(M) \underset{P}{\xrightarrow{\sim}} H_{m-p}(M) \underset{A^{-1}}{\xrightarrow{\sim}} H^{p+k}(W, W \smallsetminus M)$$

and the Thom class Ψ_M of M, which is in $H^k(W, W \smallsetminus M)$, by

$$\Psi_M =: T_M(1) = A^{-1}(M). \tag{2.18}$$

Let $N_M \to M$ denote the normal bundle of M in W. Suppose the orientation of M is compatible with that of W in the sense that N_M is orientable. We orient N_M as follows. Namely, if $(x_1, \dots, x_k, \dots, x_{m'})$ is a positive coordinate system on W such that M is given by $x_1 = \cdots = x_k = 0$ and that $(x_{k+1}, \dots, x_{m'})$ is a positive coordinate system of M, then the vectors $(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k})$ determine a positive frame of N_M. By the tubular neighborhood theorem, there is a neighborhood W_1 of M in W and an orientation preserving diffeomorphism of W_1 onto a neighborhood of the zero section of N_M, which is identified with M. By excision we have

$$H^k(W, W \smallsetminus M) \simeq H^k(W_1, W_1 \smallsetminus M) \simeq H^k(N_M, N_M \smallsetminus M)$$

and, in the above isomorphisms, the Thom class Ψ_M of M corresponds to the Thom class Ψ_{N_M} of the vector bundle N_M.

Remark 2.19 More generally, for a pseudo-manifold M in W, we may define the Poincaré homomorphism $P : H^p(M) \longrightarrow \check{H}_{m-p}(M)$. Thus we have the Thom homomorphism and the Thom class Ψ_M of M (cf. [3]).

(c) Thom class of a cycle : Let C be a finite $(m' - p)$-cycle in W and $\tilde{S}$ its support. We may define the Thom class Ψ_C of C by

$$\Psi_C := A^{-1}(C),$$

where A is the Alexander isomorphism (2.10) for the pair $(W, \tilde{S})$.

3 Localized intersection of currents

3.1 Thom class of a current

Let W be an oriented C^∞ manifold of dimension m'. Recall that a *p-current* T on W is a continuous linear functional on the space $A_c^{m'-p}(W)$. We use the notation

$$T(\varphi) = \langle T, \varphi \rangle, \qquad \varphi \in A_c^{m'-p}(W).$$

Let $\mathcal{D}^p(W)$ denote the space of p-currents on W. The differential $d : \mathcal{D}^p(W) \to \mathcal{D}^{p+1}(W)$ is defined by

$$\langle dT, \varphi \rangle = (-1)^{p+1} \langle T, d\varphi \rangle, \qquad \varphi \in A_c^{m'-p-1}(W).$$

Then $(\mathcal{D}^*(W), d)$ forms a complex, whose p-th cohomology is denoted by $H^p(\mathcal{D}^*(W))$. For a closed p-current T, we denote by $[T]$ its cohomology class.

A form ω in $A^p(W)$ may be naturally thought of as a p-current T_ω by

$$\langle T_\omega, \varphi\rangle = \int_W \omega\wedge\varphi, \qquad \varphi\in A_c^{m'-p}(W).$$

If ω is closed, then T_ω is closed and the assignment $\omega\mapsto T_\omega$ induces an isomorphism

$$\beta : H^p_{\mathrm{dR}}(W) \xrightarrow{\ \sim\ } H^p(\mathcal{D}^*(W)). \tag{3.1}$$

A Čech-de Rham cochain σ on a covering $\mathcal{W}$ of W may be also thought of as a current T_σ via integration given as (2.6). If $D\sigma = 0$, then T_σ is closed and the assignment $\sigma\mapsto T_\sigma$ induces the isomorphism $\beta\circ\alpha^{-1} : H^p_D(\mathcal{W}) \xrightarrow{\sim} H^p(\mathcal{D}^*(W))$.

Also an $(m'-p)$-chain C may be thought of as a p-current T_C by

$$\langle T_C, \varphi\rangle = \int_C \varphi, \qquad \varphi\in A_c^{m'-p}(W).$$

If C is a cycle, T_C is closed and the assignment $C\mapsto T_C$ induces an isomorphism

$$\gamma : \breve{H}_{m-p}(W) \xrightarrow{\ \sim\ } H^p(\mathcal{D}^*(W)).$$

By (2.3), we have $\gamma\circ P = \beta$. Thus for any closed current T on W, there exist a closed p-form a ČdR cocycle σ and an $(m'-p)$-cycle C such that

$$[T] = [T_\omega] = [T_\sigma] = [T_C].$$

We call ω, σ and C, respectively, de Rham, ČdR and cycle representatives of T.

If U is an open set of W, there is a natural inclusion $A_c^*(U)\hookrightarrow A_c^*(W)$, given by extension by zero, so that we may consider the restriction $T|_U$ to U of a current T on W. The *support* $\operatorname{supp}(T)$ of T is the smallest closed subset of W such that $T|_{W\smallsetminus \operatorname{supp}(T)} = 0$.

Now we consider the localization problem of currents. Thus let $\tilde{S}$ be a closed set in W. Let $W_0 = W\smallsetminus\tilde{S}$ and W_1 a neighborhood of $\tilde{S}$ in W and consider the covering $\mathcal{W} = \{W_0, W_1\}$ of W. We have the commutative diagram with exact row (cf. (2.7)):

$$\begin{array}{ccccc} H^p_D(\mathcal{W}, W_0) & \xrightarrow{j^*} & H^p_D(\mathcal{W}) & \xrightarrow{\iota^*} & H^p_{\mathrm{dR}}(W_0) \\ & & \wr\downarrow \beta\circ\alpha^{-1} & & \wr\downarrow\beta \\ & & H^p(\mathcal{D}^*(W)) & \xrightarrow{\iota^*} & H^p(\mathcal{D}^*(W_0)). \end{array} \tag{3.2}$$

Suppose T is a closed p-current on W such that $\iota^*[T] = 0$, i.e., $[T|_{W_0}] = 0$. Then there is a class Ψ_T in $H^p_D(\mathcal{W}, W_0)$ such that $[T] = j^*\Psi_T$. We then say that T is *localized at* $\tilde{S}$ and call Ψ_T a *Thom class of* T *along* $\tilde{S}$. Here some comments are in order:

(1) Any closed current T is localized at $\operatorname{supp}(T)$ in the above sense. It is also localized at the support of a cycle representive of T. Thus the set $\tilde{S}$ as above may be different from $\operatorname{supp}(T)$, see Example 3.6 below.

(2) The class Ψ_T is not uniquely determined, as j^* is not injective in general, however in some cases, there is a natural choice of Ψ_T, see Examples 3.4 and 3.8 below.

In the above situation, suppose $\tilde{S}$ is a compact set satisfying (*). Let $(0, \psi_1, \psi_{01})$ be a ČdR representative of Ψ_T and $\{\tilde{R}_0, \tilde{R}_1\}$ a system of honeycomb cells adapted to $\mathcal{W}$. Then, from the commutativity of the diagram obtained by replacing M and $\mathcal{U}$ by W and $\mathcal{W}$ in (2.13) (see also Remark 2.14), for any closed form φ in $A_c^{m'-p}(W)$ we have:

$$\langle T, \varphi \rangle = \int_{\tilde{R}_1} \psi_1 \wedge \varphi + \int_{\tilde{R}_{01}} \psi_{01} \wedge \varphi. \tag{3.3}$$

Thus the value of T is "concentrated" near $\tilde{S}$ and is explicitly given by the above. Also note that the right hand side does not depend on the choice of Ψ_T.

Example 3.4 Let C be a finite $(m'-p)$-cycle in W. Then T_C is localized at $\tilde{S} = |C|$ and Ψ_C is a natural choice for Ψ_{T_C}. In particular, if $C = M$ is a compact oriented submanifold of codimension p, then the Thom class Ψ_M of M is a natural choice for Ψ_{T_M}.

We show that, starting from a closed p-form ω such that $[T_\omega] = [T]$, there is a natural way of constructing such a class. Thus let $\tilde{S}$ be a closed set of W and let $\mathcal{W} = \{W_0, W_1\}$ be as before.

Proposition 3.5 *Let T be a closed p-current on W such that $[T|_{W_0}] = 0$ and ω a de Rham representative of T. Then there exists a Thom class Ψ_T which is represented by a ČdR cocycle of the form $(0, \omega, -\psi)$ with ψ a $(p-1)$-form on W_{01} satisfying $\omega = -d\psi$ on W_{01}.*

PROOF: Just to make sure we denote the restrictions of forms explicitly. From the assumption, there exists a $(p-1)$-form ψ on W_0 such that $d\psi = -\omega|_{W_0}$. Hence the cocycle $(\omega|_{W_0}, \omega|_{W_1}, 0)$ is cohomologous as a ČdR cocycle to $(0, \omega|_{W_1}, -\psi|_{W_{01}})$ since

$$(0, \omega|_{W_1}, -\psi|_{W_{01}}) - (\omega|_{W_0}, \omega|_{W_1}, 0) = (d\psi, 0, -\psi|_{W_{01}}) = D(\psi, 0, 0).$$

Thus the class $\Psi_T = [(0, \omega|_{W_1}, -\psi|_{W_{01}})]$ satisfies $[T] = j^*\Psi_T$. □

In this case, if $\tilde{S}$ is a compact set satisfying (*), (3.3) is written as

$$\int_W \omega \wedge \varphi = \int_{\tilde{R}_1} \omega \wedge \varphi - \int_{\tilde{R}_{01}} \psi \wedge \varphi.$$

Thus the value of the integral away from $\tilde{R}_1$ is cut off and is compensated by an integral on $\tilde{R}_{01}$.

Example 3.6 Let C be an $(m'-p)$-cycle in W and ω a de Rham representative of C. Let $\tilde{S}$ be the support of C and set $W_0 = W \smallsetminus \tilde{S}$. Then T_ω is localized at $\tilde{S}$ as $[T_\omega] = [T_C]$, although we do not have any precise information about $\mathrm{supp}(T_\omega)$. Its Thom class Ψ_{T_ω} along $\tilde{S}$ is represented by a ČdR cocycle of the form $(0, \omega, -\psi)$.

Remark 3.7 Let C, ω and $\tilde{S}$ be as in Example 3.6. Then there is a $(p-1)$-current R such that

$$T_C - T_\omega = dR.$$

We may think of R as the current defined by a $(p-1)$-form ψ on $W \smallsetminus \tilde{S}$ that can be extended as a locally integrable L^1 form on W and with $d\psi = -\omega$ on $W \smallsetminus \tilde{S}$. The equation above becomes then

$$dT_\psi - T_{d\psi} = T_C,$$

which is a residue formula (cf. [5, Ch.3,1]), and the identitiy

$$D(\psi, 0, 0) + (\omega, \omega, 0) = (0, \omega, -\psi)$$

may be thought of as the corresponding expression in terms of ČdR cochains.

Example 3.8 Let $\pi : E \to W$ be a C^∞ complex vector bundle of rank r and ∇ a connection for E. For $q = 0, \dots, r$, we have the q-th Chern form $c_q(\nabla)$, which is a closed $2q$-form defining the q-th Chern class $c_q(E)$ in $H^{2q}_{\mathrm{dR}}(W)$. We call $T_{c_q(\nabla)}$ the q-th *Chern current associated with* ∇. Suppose E admits ℓ sections $\boldsymbol{s} = (s_1, \dots, s_\ell)$ that are linearly independent on the complement of a closed set $\tilde{S} \subset W$. Then we see that $T_{c_q(\nabla)}$ is localized at $\tilde{S}$ and there is a natural way of choosing a Thom class along $\tilde{S}$ for $q = r - \ell + 1, \dots, r$.

For this, we take an $\boldsymbol{s}$-trivial connection ∇_0 for E on $W_0 = W \smallsetminus \tilde{S}$, i.e., a connection satisfying $\nabla_0 s_i = 0$ for $i = 1, \dots, \ell$. Denoting by $c_q(\nabla_0, \nabla)$ the Bott difference form (cf. [1], [9]), we have $c_q(\nabla)|_{W_0} - c_q(\nabla_0) = d\, c_q(\nabla_0, \nabla)$. Since the connection ∇_0 is $\boldsymbol{s}$-trivial, it follows that $c_q(\nabla_0) = 0$ so that $c_q(\nabla)|_{W_0}$ is exact for $q = r - \ell + 1, \dots, r$. Hence the Chern current $T_{c_q(\nabla)}$ localizes at $\tilde{S}$. As its Thom class along $\tilde{S}$, we may take the class $c_q(E, \boldsymbol{s})$ in $H^{2q}(W, W \smallsetminus \tilde{S}) \simeq H^{2q}_D(\mathcal{W}, W_0)$ represented by the cocycle (cf. Proposition 3.5):

$$(0, c_q(\nabla)|_{W_1}, c_q(\nabla_0, \nabla)).$$

This class does not depend on the choice of ∇ or ∇_0 (cf. [9, Ch.III, Lemma 3.1]) and is a natural choice of Thom class for $T_{c_q(\nabla)}$. It is the *localization of* $c_q(E)$ *at* $\tilde{S}$ *by* $\boldsymbol{s}$.

The Thom class of a complex vector bundle as a real oriented bundle may be expressed in this manner (cf. [9, Ch.III, Theorem 4.4]).

3.2 Localized intersection of currents

Let W be an oriented C^∞ manifold of dimension m' as before. For closed currents T_1 and T_2, it is possible to define the intersection product $T_1 \cdot T_2$ in the homology of W using the isomorphism β and the Poincaré duality (cf. (3.1), (2.4)). Also if T_1 and T_2 are localized at compact sets $\tilde{S}_1$ and $\tilde{S}_2$ satisfying (*), we may define the localized intersection $(T_1 \cdot T_2)_{\tilde{S}}$ in the homology of $\tilde{S} = \tilde{S}_1 \cap \tilde{S}_2$ using Thom classes Ψ_{T_1} and Ψ_{T_2} and the Alexander duality (cf. (2.15)).

Here we consider the case $T_1 = T_M$ with M a compact oriented submanifold of dimension m in W and obtain a residue theorem on M. In the sequel we take the Thom class Ψ_M of M (cf. Subsection 2.4 (b)) as Ψ_{T_M} and set $k = m' - m$. Recall that by the Alexander isomorphism

$$A_{W,M} : H^k(W, W \smallsetminus M) \xrightarrow{\sim} H_m(M),$$

the class Ψ_M corresponds to the fundamental class M. Let $i : M \hookrightarrow W$ denote the inclusion.

First localization : Let c be a class in $\breve{H}_{m'-p}(W)$. Recall that we have the intersection product $(M\cdot c)_M$ localized at M (cf. (2.15)), which is a class in $H_{m-p}(M)$ defined as $A_{W,M}(\Psi_M\smile P_W^{-1}c)$. We denote it by $M\cdot c$:

$$M\cdot c:=(M\cdot c)_M.$$

It is sent to $[M]\cdot c$ by $i_*:H_{m-p}(M)\to\breve{H}_{m-p}(W)$.

Second localization : Let $\tilde{S}$ be a compact set of W satisfying (*). We have the Alexander isomorphism

$$A_{W,\tilde{S}}:H^p(W,W\setminus\tilde{S})\xrightarrow{\sim}H_{m'-p}(\tilde{S}).$$

We set $S=\tilde{S}\cap M$ and suppose it also satisfies (*). For a class c in $H_{m'-p}(\tilde{S})$, we have the class $(M\cdot c)_S$ in $H_{m-p}(S)$ (cf. (2.15)).

Proposition 3.9 *The following diagrams are commutative :*

$$\begin{array}{ccc} H^p(W) & \xrightarrow[P_W]{\sim} & \breve{H}_{m'-p}(W) \\ \Big\downarrow{\scriptstyle i^*} & & \Big\downarrow{\scriptstyle M\cdot} \\ H^p(M) & \xrightarrow[P_M]{\sim} & H_{m-p}(M), \end{array} \qquad \begin{array}{ccc} H^p(W,W\setminus\tilde{S}) & \xrightarrow[A_{W,\tilde{S}}]{\sim} & H_{m'-p}(\tilde{S}) \\ \Big\downarrow{\scriptstyle i^*} & & \Big\downarrow{\scriptstyle (M\cdot\ )_S} \\ H^p(M,M\setminus S) & \xrightarrow[A_{M,S}]{\sim} & H_{m-p}(S). \end{array}$$

PROOF: We prove the commutativity of the second diagram, the proof for the first one being similar. We have the cup product followed by the Alexander isomorphism :

$$H^k(W,W\setminus M)\times H^p(W,W\setminus\tilde{S})\xrightarrow{\smile}H^{k+p}(W,W\setminus S)\xrightarrow{A_{W,S}}H_{m-p}(S).$$

Nothing that the Alexander isomorphism is given by the left cap product with the fundamental class and using properties of cap and cup products, we have, for a class u in $H^p(W,W\setminus\tilde{S})$,

$$A_{M,S}(i^*u)=A_{W,S}(\Psi_M\smile u)=(M\cdot A_{W,\tilde{S}}u)_S.$$

□

In view of the above, we define intersection products in a more general situation where M is not necessarily a submanifold of W :

Definition 3.10 Let W and M be oriented C^∞ manifolds of dimensions m' and m, respectively, and $F:M\to W$ a C^∞ map. We define the intersection product $M\cdot_F$ so that the first diagram below is commutative. Also, for a compact set $\tilde{S}$ satisfying (*) in W, we set $S=F^{-1}(\tilde{S})$ and suppose S is compact and satisfy (*). We then define the localized intersection product $(M\cdot_F\ )_S$ so that the second diagram is commutative :

$$\begin{array}{ccc} H^p(W) & \xrightarrow[P_W]{\sim} & \breve{H}_{m'-p}(W) \\ \Big\downarrow{\scriptstyle F^*} & & \Big\downarrow{\scriptstyle M\cdot_F} \\ H^p(M) & \xrightarrow[P_M]{\sim} & \breve{H}_{m-p}(M), \end{array} \qquad \begin{array}{ccc} H^p(W,W\setminus\tilde{S}) & \xrightarrow[A_{W,\tilde{S}}]{\sim} & H_{m'-p}(\tilde{S}) \\ \Big\downarrow{\scriptstyle F^*} & & \Big\downarrow{\scriptstyle (M\cdot_F\ )_S} \\ H^p(M,M\setminus S) & \xrightarrow[A_{M,S}]{\sim} & H_{m-p}(S). \end{array}$$

Remark 3.11 1. Let M be a submanifold of W and $i : M \hookrightarrow W$ the inclusion. If M is compact, $M \cdot_i$ is the product $M \cdot$ defined before. We may also define the product $M \cdot$ as $M \cdot_i$ in the case M is not compact.

2. The products as above are defined in the algebraic category in [4].

For a closed p-current T on W, we define

$$M \cdot_F T := M \cdot_F P\beta^{-1}[T].$$

Suppose T is localized at $\tilde{S}$. Then taking a Thom class Ψ_T of T along $\tilde{S}$, we define the *residue* of Ψ_T on M at S by

$$\mathrm{Res}(F^*\Psi_T, S) := (M \cdot_F A(\Psi_T))_S.$$

Suppose S has a finite number of connected components $(S_\lambda)_\lambda$. Then we have a decomposition $H_{m-p}(S) = \bigoplus_\lambda H_{m-p}(S_\lambda)$ and accordingly $\mathrm{Res}(F^*\Psi_T, S)$ determines a class in $H_{m-p}(S_\lambda)$, which is denoted by $\mathrm{Res}(F^*\Psi_T, S_\lambda)$. We can state the following general residue theorem, which follows from the commutativity of the diagram (2.13):

Theorem 3.12 *Let W and M be oriented C^∞ manifolds of dimensions m' and m, respectively, and $F : M \to W$ a C^∞ map. Let T be a closed p-current on W such that $[T|_{W \smallsetminus \tilde{S}}] = 0$ for some compact subset $\tilde{S}$ satisfying (*) in W. Suppose that $S = F^{-1}\tilde{S}$ is compact, satisfies (*) and has a finite number of connected components $(S_\lambda)_\lambda$. Then*

(1) *For each λ we have a class $\mathrm{Res}(F^*\Psi_T,\, S_\lambda)$ in $H_{m-p}(S_\lambda)$.*

(2) *We have the "residue formula":*

$$M \cdot_F T = \sum_\lambda (i_\lambda)_* \,\mathrm{Res}(F^*\Psi_T,\, S_\lambda) \qquad \text{in } \breve{H}_{m-p}(M),$$

where $i_\lambda : S_\lambda \hookrightarrow M$ denotes the inclusion.

We may express $M \cdot_F T$ and $\mathrm{Res}(F^*\Psi_T,\, S_\lambda)$ as follows. Let T be a closed p-current on W and ω a de Rham representative of T. From (2.2) and (2.3), we have:

Proposition 3.13 (1) *The intersection product $M \cdot_F T$ in $\breve{H}_{m-p}(M)$ is represented by a cycle C such that*

$$\int_M F^*\omega \wedge \varphi = \int_C \varphi$$

for any closed form φ in $A_c^{m-p}(M)$.

(2) *In the isomorphism $\breve{H}_{m-p}(M) \simeq H_c^{m-p}(M)^*$, $M \cdot_F T$ corresponds to the functional on $H_c^p(M)$ that assigns to $[\varphi]$ the left hand side above.*

(3) *In particular, if $p = m$ and if M is compact, $M \cdot_F T$ is a number given by*

$$M \cdot_F T = \int_M F^*\omega.$$

Suppose T satisfies the conditions in Theorem 3.12. Let $W_0 = W \smallsetminus \tilde{S}$ and W_1 a neighborhood of $\tilde{S}$ and consider the covering $\mathcal{W} = \{W_0, W_1\}$. Let $\varPsi_T$ be represented by a ČdR cocycle $(0, \psi_1, \psi_{01})$ in $A^p(\mathcal{W}, W_0)$. For each λ we take a regular neighborhood U_λ of S_λ in M such that $F(U_\lambda) \subset W_1$ and that $U_\lambda \cap U_\mu = \emptyset$ if $\lambda \neq \mu$. For each λ, we take a compact submanifold R_λ of dimension m with C^∞ boundary in U_λ, containing S_λ in its interior. From (2.11) and (2.12), we have:

Proposition 3.14 (1) *The residue* $\mathrm{Res}(F^*\varPsi_T, S_\lambda)$ *in* $H_{m-p}(S_\lambda)$ *is represented by a cycle* C *such that*

$$\int_{R_\lambda} F^*\psi_1 \wedge \varphi + \int_{R_{0\lambda}} F^*\psi_{01} \wedge \varphi = \int_C \varphi$$

for any closed form φ *in* $A^{m-p}(U_\lambda)$.

(2) *In the isomorphism* $H_{m-p}(S_\lambda) \simeq H^{m-p}_{\mathrm{dR}}(U_\lambda)^*$, $\mathrm{Res}(F^*\varPsi_T, S_\lambda)$ *corresponds to the functional on* $H^{m-p}_{\mathrm{dR}}(U_\lambda)$ *that assigns to* $[\varphi]$ *the left hand side above.*

(3) *In particular, if* $p = m$, *the residue is a number given by*

$$\mathrm{Res}(F^*\varPsi_T, S_\lambda) = \int_{R_\lambda} F^*\psi_1 + \int_{R_{0\lambda}} F^*\psi_{01}.$$

Example 3.15 Let C be a finite $(m'-p)$-cycle on W, $\tilde{S} = |C|$ and $S = F^{-1}\tilde{S}$. We take $\varPsi_C$ as $\varPsi_{T_C}$. Then

$$M \cdot_F T_C = M \cdot_F [C], \qquad \mathrm{Res}(F^*\varPsi_C, S_\lambda) = (M \cdot_F C)_{S_\lambda}$$

and the residue formula becomes

$$M \cdot_F [C] = \sum_\lambda (i_\lambda)_*(M \cdot_F C)_{S_\lambda} \qquad \text{in} \quad \breve{H}_{m-p}(M).$$

In particular, if M is compact and $p = m$,

$$M \cdot_F [C] = \sum_\lambda (M \cdot_F C)_{S_\lambda}.$$

Let ω be a de Rham representative of C. Then T_ω is localized at $\tilde{S}$. As $\varPsi_{T_\omega}$ we may take the class represented by a cocycle of the form $(0, \omega, -\psi)$ (cf. Example 3.6). As a homology class, $M \cdot_F T_\omega = M \cdot_F [C]$. As a functional, it is given as in Proposition 3.13. Also $\mathrm{Res}(F^*\varPsi_{T_\omega}, S_\lambda)$ is a functional given as in Proposition 3.14 with $\psi_1 = \omega$ and $\psi_{01} = -\psi$.

See Propositions 4.5 and 4.8 below for explicit expressions of $(M \cdot_F C)_{S_\lambda}$ in some special cases.

Example 3.16 Let W be a complex manifold of dimension n' and M a complex submanifold of dimension n. Also let V be an analytic subvariety of W of dimension k. Recall that there exists a subanalytic triangulation of W compatible with M, V and $\mathrm{Sing}(V)$, the singular set of V. Thus V may be thought of as a chain, which is not C^∞ but still has the associated current T_V of integration. Moreover it is a cycle, as the real codimension of Sing(V) in V is greater than or equal to two. If $n+k=n'$ and if p is an isolated point of $M\cap V$, we have

$$(M\cdot V)_p \geq \mathrm{mult}_p(V),$$

the multiplicity of V at p. The equality holds, by definition, if M is general with respect to V, i.e., the intersection of the tangent space of M at p and the tangent cone of V at p consists only of p. Note that $\mathrm{mult}_p(V)$ coincides with the Lelong number of T_V at p (e.g., [5, Ch.3, 2]).

We finish this section by giving a formula for the residue at a non-isolated component. Thus, in the situation of Theorem 3.12, suppose that S_λ is an oriented submanifold of M of dimension $m-p$ with orientation compatible with that of M in the sense described in Subsection 2.4 (b). Let p_λ be a point in S_λ and B_λ a small open ball of dimension p in M transverse to S_λ at p_λ. We orient B_λ so that the orientation of B_λ followed by that of S_λ gives the orientation of M. Setting $F_\lambda = F|_{B_\lambda}$, we have the commutative diagram

$$\begin{array}{ccc} H^p(W, W\smallsetminus \tilde{S}) & \xrightarrow[A_{W,\tilde{S}}]{\sim} & H_{m'-p}(\tilde{S}) \\ \downarrow{\scriptstyle F_\lambda^*} & & \downarrow{\scriptstyle (B_\lambda\cdot_{F_\lambda}\ )_{p_\lambda}} \\ H^p(B_\lambda, B_\lambda\smallsetminus p_\lambda) & \xrightarrow[A_{B_\lambda,p_\lambda}]{\sim} & H_0(p_\lambda). \end{array}$$

We have the residue $\mathrm{Res}(F_\lambda^*\Psi_T, p_\lambda) = (B_\lambda \cdot_{F_\lambda} A(\Psi_T))_{p_\lambda}$ in $H_0(p_\lambda)\simeq\mathbb{C}$ so that it is a number.

Theorem 3.17 *In the situation of Theorem 3.12, suppose that S_λ is an oriented submanifold of M of dimension $m-p$ and let p_λ and F_λ be as above. Then we have:*

$$\mathrm{Res}(F^*\Psi_T, S_\lambda) = \mathrm{Res}(F_\lambda^*\Psi_T, p_\lambda)\cdot S_\lambda \qquad \text{in} \ \ H_{m-p}(S_\lambda).$$

PROOF: We try to find $\mathrm{Res}(F^*\Psi_T, S_\lambda)$ by Proposition 3.14. As U_λ, we take a tubular neighborhood of S_λ with a C^∞ projection $\pi : U_\lambda \to S_\lambda$, which gives U_λ the structure of a bundle of open balls of dimension p. Setting $U_0 = U_\lambda \smallsetminus S_\lambda$, we consider the covering $\mathcal{U}_\lambda = \{U_0, U_\lambda\}$ of U_λ. As R_λ, we take a bundle on S_λ of closed balls of dimension p in U_λ. Then $R_{0\lambda}$ is a bundle on S_λ of $(p-1)$-spheres. We denote the restrictions of π to R_λ and $R_{0\lambda}$ by π_λ and $\pi_{0\lambda}$, respectively. For a closed $(m-p)$-form φ on U_λ, we compute the integral

$$I := \int_{R_\lambda} F^*\psi_1\wedge\varphi + \int_{R_{0\lambda}} F^*\psi_{01}\wedge\varphi.$$

Since π induces an isomorphism $\pi^* : H^{m-p}_{\mathrm{dR}}(S_\lambda) \xrightarrow{\sim} H^{m-p}_{\mathrm{dR}}(U_\lambda)$, there exist a closed $(m-p)$-form θ on S_λ and an $(m-p-1)$-form τ on U_λ such that

$$\varphi = \pi^*\theta + d\tau.$$

Using the projection formula, the fact that $dF^*\psi_1 = 0$ and the Stokes formula, we have

$$\int_{R_\lambda} F^*\psi_1\wedge\varphi = \int_{S_\lambda} (\pi_\lambda)_* F^*\psi_1\cdot\theta + (-1)^{p+1}\int_{R_{0\lambda}} F^*\psi_1\wedge\tau,$$

where $(\pi_\lambda)_*$ denotes the integration along the fiber of π_λ. Note that $(\pi_\lambda)_* F^*\psi_1$ is a C^∞ function on S_λ. Noting that $dF^*\psi_{01} = F^*\psi_1$ on $U_{0\lambda}$ and $\partial R_{0\lambda} = \emptyset$, we also compute to get

$$\int_{R_{0\lambda}} F^*\psi_{01} \wedge \varphi = \int_{S_\lambda} (\pi_{0\lambda})_* F^*\psi_{01} \cdot \theta + (-1)^p \int_{R_{0\lambda}} F^*\psi_1 \wedge \tau.$$

Thus we have

$$I = \int_{S_\lambda} ((\pi_\lambda)_* F^*\psi_1 + (\pi_{0\lambda})_* F^*\psi_{01}) \cdot \theta$$

Now recall that we have the integration along the fiber on the ČdR cochains:

$$\pi_* : A^q(\mathcal{U}_\lambda, U_0) \longrightarrow A^{q-p}(S_\lambda),$$

which assigns to $\sigma = (0, \sigma_\lambda, \sigma_{0\lambda})$ the form $(\pi_\lambda)_*\sigma_\lambda + (\pi_{0\lambda})_*\sigma_{0\lambda}$ on S_λ. Moreover it is compatible with the differentials D and d (cf. [9, Ch.II, 5]). Since $(0, F^*\psi_1|_{U_\lambda}, F^*\psi_{01}|_{U_{0\lambda}})$ is a ČdR cocycle in $A^p(\mathcal{U}_\lambda, U_0)$, the function $(\pi_\lambda)_* F^*\psi_1 + (\pi_{0\lambda})_* F^*\psi_{01}$ is d-closed so that it is a constant. By definition the constant is exactly $\mathrm{Res}(F_\lambda^*\Psi_T, p_\lambda)$ above. Finally from

$$\int_{S_\lambda} \theta = \int_{S_\lambda} \varphi,$$

we have the theorem. □

Remark 3.18 1. In the above situation, $H_{m-p}(S_\lambda, \mathbb{Z}) \simeq \mathbb{Z}$ and is generated by the fundamental class S_λ. Thus if $\mathrm{Res}(F_\lambda^*\Psi_T, p_\lambda)$ is an integer, $\mathrm{Res}(F^*\Psi_T, S_\lambda)$ is an integral class.
2. The above theorem can also be proved topologically as [10, Theorem 4.1.1, see also Theorem 7.3.2]. In fact, using techniques and results in [10], we may compute residues in various settings.

4 Coincidence point theorem

4.1 Coincidence homology classes and indices

Let M and N be connected and oriented C^∞ manifolds of dimensions m and n, respectively, with $m \geq n$. We set $W = M \times N$ and orient W so that the orientation of M followed by that of N gives the orientation of W. Let f, $g : M \to N$ be C^∞ maps and denote by Γ_f and Γ_g the graphs of f and g in W. We consider the map

$$\tilde{f} : M \longrightarrow \Gamma_f \subset W \qquad \text{defined by} \quad \tilde{f}(x) = (x, f(x)),$$

which is a diffeomorphism onto Γ_f. We orient Γ_f so that $\tilde{f}$ is orientation preserving. Similarly we define $\tilde{g}$ for g. Recall the diagram (cf. Definition 3.10):

$$\begin{array}{ccc} H^n(W) & \xrightarrow[P]{\sim} & \breve{H}_m(W) \\ \big\downarrow \tilde{f}^* & & \big\downarrow M\cdot_{\tilde{f}} \\ H^n(M) & \xrightarrow[P]{\sim} & \breve{H}_{m-n}(M). \end{array}$$

Definition 4.1 The *global coincidence class* $\Lambda(f,g)$ *of the pair* (f,g) is defined by

$$\Lambda(f,g) = M \cdot_{\tilde{f}} [\Gamma_g] \qquad \text{in} \ \ \check{H}_{m-n}(M).$$

Note that $\tilde{f}$ induces an isomorphism $\tilde{f}_* : \check{H}_{m-n}(M) \xrightarrow{\sim} \check{H}_{m-n}(\Gamma_f)$ and $\Lambda(f,g)$ corresponds to $\Gamma_f \cdot [\Gamma_g]$ in $\check{H}_{m-n}(\Gamma_f)$, which is sent to $[\Gamma_f] \cdot [\Gamma_g]$ in $\check{H}_{m-n}(W)$ by the canonical homomorphism $\check{H}_{m-n}(\Gamma_f) \to \check{H}_{m-n}(W)$.

We define the *coincidence point set* of the pair (f,g) by

$$\operatorname{Coin}(f,g) = \{\, p \in M \mid f(p) = g(p) \,\}.$$

Note that $\operatorname{Coin}(f,g) = \tilde{f}^{-1}(\Gamma_g)$. For shortness, we set $S = \operatorname{Coin}(f,g)$.

From now on we assume that M is compact so that Γ_g and S are compact. Recall the diagram (cf. Definition 3.10) :

$$\begin{array}{ccc} H^n(W, W \smallsetminus \Gamma_g) & \xrightarrow[A]{\sim} & H_m(\Gamma_g) \\ \Big\downarrow \tilde{f}^* & & \Big\downarrow (M \cdot_{\tilde{f}}\ )_S \\ H^n(M, M \smallsetminus S) & \xrightarrow[A]{\sim} & H_{m-n}(S). \end{array}$$

Definition 4.2 The *local coincidence class* $\Lambda(f,g;S)$ *of the pair* (f,g) *at* S is defined to be the localized intersection class :

$$\Lambda(f,g;S) = (M \cdot_{\tilde{f}} \Gamma_g)_S \qquad \text{in} \ \ H_{m-n}(S).$$

Note that $\tilde{f}$ induces a homomorphism $\tilde{f}_* : H_{m-n}(S) \to H_{m-n}(\Gamma_f)$ and $\Lambda(f,g;S)$ is sent to $\Gamma_f \cdot [\Gamma_g]$.

Remark 4.3 1. The classes $\Lambda(f,g)$ and $\Lambda(f,g;S)$ are in fact in homology with $\mathbb{Z}$ coefficients.
2. We have

$$\Lambda(g,f) = (-1)^m \Lambda(f,g), \qquad \Lambda(g,f;S) = (-1)^m \Lambda(f,g;S).$$

Suppose $S = \operatorname{Coin}(f,g)$ has a finite number of connected components $(S_\lambda)_\lambda$. Then we have $H_{m-n}(S) = \bigoplus_\lambda H_{m-n}(S_\lambda)$ and accordingly we have the local coincidence class $\Lambda(f,g;S_\lambda)$ in $H_{m-n}(S_\lambda)$. From Theorem 3.12, we have a general coincidence point theorem :

Theorem 4.4 *In the above situation*

$$\Lambda(f,g) = \sum_\lambda (i_\lambda)_* \Lambda(f,g;S_\lambda) \qquad \text{in} \ \ H_{m-n}(M).$$

In general, $\Lambda(f,g)$ and $\Lambda(f,g;S_\lambda)$ are given as in Propositions 3.13 and 3.14. The theorem becomes more meaningful if we have explicit descriptions of them.

In the case $m = n$, $\Lambda(f,g;S_\lambda)$ is in $H_0(S_\lambda) = \mathbb{C}$ so that it is a number (in fact an integer), which we call the *coincidence index* of (f,g) at S_λ. If S_λ consists of a point p, we have the following explicit formula. In fact it is already known, however we give an alternative short proof using the Thom class in the Čech-de Rham cohomology. Let U be a coordinate neighborhood around p with coordinates $x = (x_1, \dots, x_m)$ in M and V a coordinate neighborhood around $f(p) = g(p)$ with coordinates $y = (y_1, \dots, y_m)$ in N. Also let D be a closed ball around p in U such that $f(D) \subset V$ and $g(D) \subset V$. Thus we may consider the map $g - f : D \to \mathbb{R}^m$ whose image is the origin 0 in $\mathbb{R}^m$ only at p. The boundary ∂D is homeomorphic to the unit sphere S^{m-1} and we have the map

$$\gamma : \partial D \longrightarrow S^{m-1} \qquad \text{defined by } \; \gamma(x) = \frac{g(x) - f(x)}{\|g(x) - f(x)\|}.$$

We denote the degree of this map by $\deg(g - f, p)$.

Proposition 4.5 *In the above situation*

$$\varLambda(f, g; p) = \deg(g - f, p).$$

Proof: Let $\varPsi_{\varGamma_g}$ be the Thom class of $\varGamma_g$ and $(0, \psi_1, \psi_{01})$ its ČdR representative. We may take D as R_λ in Proposition 3.14. Since $R_{0\lambda} = -\partial D$, we have

$$\varLambda(f, g; p) = \int_D \tilde{f}^* \psi_1 - \int_{\partial D} \tilde{f}^* \psi_{01}. \tag{4.6}$$

Recall that $\varPsi_{\varGamma_g}$ may be naturally identified with the Thom class of the normal bundle $N_{\varGamma_g}$, which is trivial over $\tilde{g}(U)$; $N_{\varGamma_g}|_{\tilde{g}(U)} \simeq \mathbb{R}^m \times \tilde{g}(U)$ (cf. Subsection 2.4 (b)). Let $\rho_g : N_{\varGamma_g}|_{\tilde{g}(U)} \to \mathbb{R}^m$ denote the projection onto the fiber direction. Also let ψ_m be an angular form on $\mathbb{R}^m \smallsetminus \{0\}$. Then on $\tilde{g}(U)$ the Thom class of $N_{\varGamma_g}$ is represented by the cocycle (cf. Proposition 2.16)

$$(0, 0, -\rho_g^* \psi_m).$$

Let π_1 and π_2 denote the projections of $W = M \times N$ onto the first and second factors, respectively. We set $x = \pi_1^* x$ and $y = \pi_2^* y$ on $U \times V$. By our orientation convention, in a neighborhood of $\tilde{g}(p)$ in $N_{\varGamma_g}$, we may take $g(x) - y$ as fiber coordinates and x as base coordinates of the bundle $N_{\varGamma_g}$ so that we may write $\rho_g(x, y) = g(x) - y$. Then (4.6) becomes

$$\varLambda(f, g; p) = \int_{\partial D} (\rho_g \circ \tilde{f})^* \psi_m = \int_{\partial D} (g - f)^* \psi_m,$$

which is nothing but $\deg(g - f, p)$. □

In the above situation, let $J_f(p)$ and $J_g(p)$ denote the Jacobian matrices of f and g at p. A coincidence point p of the pair (f, g) is said to be *non-degenerate* if

$$\det(J_g(p) - J_f(p)) \neq 0.$$

Corollary 4.7 *If p is a non-degenerate coincidence point,*

$$\varLambda(f, g; p) = \operatorname{sgn} \det(J_g(p) - J_f(p)).$$

Now consider the case $m > n$. Suppose S_λ is an oriented submanifold of M of dimension $m - n$. Let p_λ be a point in S_λ and B_λ a small open ball of dimension n in M transverse to S_λ at p_λ. Setting $f_\lambda = f|_{B_\lambda}$ and $g_\lambda = g|_{B_\lambda}$, we have $\deg(g_\lambda - f_\lambda, p_\lambda)$. From Theorem 3.17, we have:

Proposition 4.8 *Let S_λ be a connected component of* $\mathrm{Coin}(f,g)$*. If S_λ is an oriented submanifold of M of dimension $m-n$,*

$$\Lambda(f,g;S_\lambda) = \deg(g_\lambda - f_\lambda, p_\lambda)\cdot S_\lambda \qquad \textit{in} \;\; H_{m-n}(S_\lambda).$$

4.2 Lefschetz coincidence point formula

Let M and N be compact, connected and oriented C^∞ manifolds of the same dimension m and let $f,g\colon M\to N$ be C^∞ maps. In this situation, $\check{H}_0(M)=H_0(M)=\mathbb{C}$ and $\Lambda(f,g)$ is a number (in fact an integer), which has an explicit description. Let

$$H^p(f): H^p(N) \longrightarrow H^p(M)$$

be the homomorphism induced by f on the p-th cohomology group and similarly for $H^p(g)$. We set $q=m-p$. The Poincaré duality allows us to define the composition

$$H^q(M)\simeq H^p(M)^* \xrightarrow{H^p(g)^*} H^p(N)^* \simeq H^q(N) \xrightarrow{H^q(f)} H^q(M).$$

We define the *Lefschetz coincidence number $L(f,g)$ of the pair (f,g)* as

$$L(f,g) := \sum_{q=0}^{m} (-1)^q\cdot \mathrm{tr}(H^q(f)\circ H^{m-q}(g)^*).$$

Although the following is already known, we include a proof for the sake of completeness. It is a modification of the presentation as given in [5] for the fixed point case, i.e., the case $M=N$ and $g=1_M$, the identity map of M.

Proposition 4.9 *In the above situation we have:*

$$\Lambda(f,g)=L(f,g).$$

Proof: Let $\{\mu_i^p\}_i$ be a set of closed forms representing a basis of $H^p_{\mathrm{dR}}(M)$. We set $q=m-p$ and let $\{\check{\mu}_j^q\}_j$ be a set of forms representing a basis of $H^q_{\mathrm{dR}}(M)$ dual to $\{[\mu_i^p]\}_i$:

$$\int_M \mu_i^p\wedge\check{\mu}_j^q = \delta_{ij}.$$

We also take a set of forms $\{\nu_k^p\}_k$ representing a basis of $H^p_{\mathrm{dR}}(N)$ and a set of forms $\{\check{\nu}_\ell^q\}_\ell$ representing a basis of $H^q_{\mathrm{dR}}(N)$ dual to $\{[\nu_k^p]\}_k$. By the Künneth formula, a basis of $H^m_{\mathrm{dR}}(W)$, $W=M\times N$, is represented by

$$\left\{\, \xi_{i,\ell}^{p,q} = \pi_1^*\mu_i^p \wedge \pi_2^*\check{\nu}_\ell^q \,\right\}_{p+q=m},$$

where π_1 and π_2 are projections onto the first and second factors.

Note that in general, for a p-form ω on M and a q-form θ on N, we have

$$\int_{\Gamma_f} \pi_1^*\omega\wedge\pi_2^*\theta = \int_M \omega\wedge f^*\theta \tag{4.10}$$

and similarly for the integration on Γ_g.

Let $G^p = (g^p_{ki})$ be the matrix representing $H^p(g)$ in the bases $\{[\nu^p_k]\}_k$ and $\{[\mu^p_i]\}_i$:

$$H^p(g)[\nu^p_k] = \sum_i g^p_{ik}[\mu^p_i].$$

Thus the dual map $H^p(g)^*$ is represented by the transposed ${}^tG^p$ in the bases $\{[\check{\mu}^q_j]\}_j$ and $\{[\check{\nu}^q_\ell]\}_\ell$. Also let $\check{F}^q = (\check{f}^q_{\ell j})$ be the matrix representing $H^q(f)$ in the bases $\{[\check{\nu}^q_\ell]\}_\ell$ and $\{[\check{\mu}^q_j\}_j$:

$$H^q(f)[\check{\nu}^q_\ell] = \sum_j \check{f}^q_{j\ell}[\check{\mu}^q_j].$$

Let η_g be an m-form representing the Poincaré dual of $[\varGamma_g]$ in W. Using (2.3) and (4.10) for $\varGamma_g$, we compute to get

$$[\eta_g] = \sum_{q,i,\ell}(-1)^q g^p_{i\ell}\,[\xi^{p,q}_{i,\ell}].$$

Thus we have

$$\varLambda(f,g) = \int_M \tilde{f}^*\eta_g = \sum_{q,i,\ell}(-1)^q g^p_{i\ell}\int_M \mu^p_i \wedge f^*\check{\nu}^q_\ell = \sum_{q,i,\ell}(-1)^q g^p_{i\ell}\,\check{f}^q_{i\ell}.$$

Since $\check{f}^q_{i\ell}$ is the ℓi entry of $\check{F}^q$ and $g^p_{i\ell}$ the $i\ell$ entry of ${}^tG^p$, we have the proposition. □

From Theorem 4.4 and Propositions 4.5 and 4.9, we have :

Theorem 4.11 *Let M and N be compact oriented C^∞ manifolds of same dimension and let $f, g\colon M \to N$ be C^∞ maps. Suppose* $\mathrm{Coin}(f,g)$ *has a finite number of connected components $(S_\lambda)_\lambda$. Then*

$$L(f,g) = \sum_\lambda \varLambda(f,g;S_\lambda).$$

In the case the set of coincidence points consists only of isolated points, we have :

Corollary 4.12 (Lefschetz coincidence point formula) *Let M and N be compact oriented C^∞ manifolds of the same dimension and let $f, g\colon M \to N$ be C^∞ maps. Suppose* $\mathrm{Coin}(f,g)$ *consists of a finite number of isolated points. Then*

$$L(f,g) = \sum_{p\in\mathrm{Coin}(f,g)} \deg(g-f,p).$$

Moreover, if all coincidence points are isolated and non-degenerate then

$$L(f,g) = \sum_{p\in\mathrm{Coin}(f,g)} \mathrm{sgn}\ \det(J_g(p) - J_f(p)).$$

Remark 4.13 1. The above theory applied to the case $N = M$ and $g = 1_M$, the identity map of M, gives a general fixed point theorem for f and the Lefschetz fixed point formula, which is effective also in the study of periodic points.

2. Let $f, g : M \to N$ be C^∞ maps. If g is a diffeomorphism, the coincidence theory for the pair (f,g) is equivalent to the fixed point theory for the map $g^{-1}\circ f$ of M.

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