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On Proper Polynomial Maps of $\mathbb{C}^2$

Cinzia Bisi · Francesco Polizzi

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Abstract Two proper polynomial maps $f_1, f_2 : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ are said to be *equivalent* if there exist $\Phi_1, \Phi_2 \in \mathrm{Aut}(\mathbb{C}^2)$ such that $f_2 = \Phi_2 \circ f_1 \circ \Phi_1$. We investigate proper polynomial maps of topological degree $d \geq 2$ up to equivalence. Under the further assumption that the maps are Galois coverings, we also provide the complete description of equivalence classes. This widely extends previous results obtained by Lamy in the case $d = 2$.

Keywords Proper polynomial maps · Galois coverings · Complex reflection groups

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1 Introduction

Let $f : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ be a dominant polynomial map. We say that f is *proper* if it is closed and for every point $y \in \mathbb{C}^2$ the set $f^{-1}(y)$ is compact. The *topological degree* d of f is defined as the number of preimages of a general point.

The semi-group of proper polynomial maps from $\mathbb{C}^2$ to $\mathbb{C}^2$ is not completely understood yet. It is known that these maps cannot provide any counterexample to the

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C. Bisi (✉) · F. Polizzi
Dipartimento di Matematica, Università della Calabria, Via P. Bucci Cubo 30B, Arcavacata di Rende, 87036 Cosenza, Italy
e-mail: bisi@mat.unical.it

F. Polizzi
e-mail: polizzi@mat.unical.it

Jacobian Conjecture, see [1, Theorem 2.1]. Nevertheless, it is worthwhile to study them from other points of view, for instance analyzing their dynamical behaviour; this investigation was recently started in [5, 6, 8] and [9]. In the present paper we do not consider any dynamical question but we try to generalize to arbitrary $d \geq 3$ the following theorem, proved in [15].

Theorem 1.1 (Lamy) *Let* $f: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ *be a proper polynomial map of topological degree* 2. *Then there exist* $\Phi_1, \Phi_2 \in \mathrm{Aut}(\mathbb{C}^2)$ *such that*

$$f = \Phi_2 \circ \tilde{f} \circ \Phi_1,$$

where $\tilde{f}(x, y) = (x, y^2)$.

We say that two proper polynomial maps $f_1, f_2: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ are *equivalent* if there exist $\Phi_1, \Phi_2 \in \mathrm{Aut}(\mathbb{C}^2)$ such that

$$f_2 = \Phi_2 \circ f_1 \circ \Phi_1.$$

One immediately check that equivalent maps have the same topological degree. Therefore Theorem 1.1 says that when $d = 2$ there is just one equivalence class, namely that of $\tilde{f}$.

The aim of our work is to answer some questions that naturally arise from Lamy's result. The first one, already stated in [15], is the following:

Question 1.2 Is every proper polynomial map $f: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ equivalent to some map of type $(x, y) \longrightarrow (x, P(y))$?

The answer is negative, and a counterexample is provided already in degree 3 by the proper map $f: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ given by

$$f(x, y) = (x, y^3 + xy).$$

This map was first considered by Whitney; clearly it is not equivalent to any map of the form $(x, y) \longrightarrow (x, P(y))$, since its branch locus is the cuspidal cubic of equation $4x^3 + 27y^2 = 0$ (see Remark 2.7). The particular form of this counterexample led us to the following very natural question:

Question 1.3 Is every polynomial map $f: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ equivalent to some map of type $(x, y) \longrightarrow (x, Q(x, y))$?

And, more generally:

Question 1.4 How many equivalence classes of proper polynomial maps of fixed topological degree $d \geq 3$ are there?

Answers to Questions 1.3 and 1.4 are the relevant results of Sect. 3, referred as Theorems A and B.

Theorem A *For every* $d \geq 3$ *there exists at least one proper polynomial map* $f\colon \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ *such that* f *is not equivalent to a map of type* $(x, y) \longrightarrow (x, Q(x, y))$.

Theorem B *For all positive integers* d, n, *with* $d \geq 3$ *and* $n \geq 2$, *consider the polynomial map* $f_{d,n}\colon \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ *given by*

$$f_{d,n}(x, y) := (x, y^d - dx^n y).$$

Then $f_{d,n}$ *and* $f_{d,m}$ *are equivalent if and only if* $m = n$. *It follows that there exist infinitely many different equivalence classes of proper polynomial maps* $f\colon \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ *of fixed topological degree* d.

Comparing Theorem 1.1 with Theorems A and B, one sees that the behaviour of proper polynomial maps $\mathbb{C}^2 \longrightarrow \mathbb{C}^2$ up to equivalence is completely different for $d = 2$ and for $d \geq 3$. It seems that a satisfactory description of all equivalence classes in the case $d \geq 3$ is at the moment out of reach; nevertheless, one could hope at least to classify those proper maps enjoying some additional property. For this reason, in Sect. 4 we restrict our attention to polynomial maps $f\colon \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ which are *Galois coverings* with finite Galois group G. All these maps are proper and their topological degree equals $|G|$; moreover $G \subset \mathrm{Aut}(\mathbb{C}^2)$ and f can be identified with the quotient map $\mathbb{C}^2 \longrightarrow \mathbb{C}^2/G$. Since G is a finite group, we may assume that $G \subset \mathrm{GL}(2, \mathbb{C})$ by a polynomial change of coordinates, see [13], and since $\mathbb{C}^2/G \cong \mathbb{C}^2$ it follows that G is a *finite complex reflection group*. These groups and their conjugacy classes in $\mathrm{GL}(2, \mathbb{C})$ were completely classified in [21] and [3]. Therefore we may exploit this classification in order to prove the main result of Sect. 4.

Theorem C (*See Theorem* 4.8) *Let* $f\colon \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ *be a polynomial map which is a Galois covering with finite Galois group* G. *Then, up to equivalence, we are in one of the cases in Table* 4 *of Sect.* 4.

Referring to Table 4, we observe that the case $d = 2$ corresponds to the map $\mathfrak{f}_2$ (and to the Galois group $\mathbb{Z}_2$); therefore our Theorem C widely extends Theorem 1.1.

We finally remark that the equivalence relation studied in the present paper is weaker than the conjugacy relation, in which we require $\Phi_2 = \Phi_1^{-1}$. For instance, the two maps

$$f_1(x, y) = (x, y^2) \quad \text{and} \quad f_2(x, y) = (x, y^2 + x)$$

are equivalent in our sense but they are not conjugate by any automorphism of $\mathbb{C}^2$, since their sets of fixed points are not biholomorphic. The study of conjugacy classes of proper maps of given topological degree is certainly an interesting problem, but we will not consider it here: some good references are [8] and [9].

Some of our computations were carried out by using the Computer Algebra Systems `GAP4` and `Singular`, see [11] and [20]. For the reader's convenience, we included the scripts in Appendix A.

2 Preliminaries

2.1 Proper Polynomial Maps

We recall the following

Definition 2.1 Let $f: \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a dominant polynomial map. We say that f is *proper* if it is closed and for every point $p \in \mathbb{C}^n$ the set $f^{-1}(p)$ is compact. Equivalently, f is proper if and only if for every compact set $K \subset \mathbb{C}^n$ the set $f^{-1}(K)$ is compact.

Notice that in the first part of the definition, the hypothesis f closed is necessary. For example, if one considers the map $f(x, y) = (x + x^2 y, y)$ on $\mathbb{C}^2$, $f^{-1}(p)$ is compact because it always consists of one or two points. However:

- f is not closed, since the image of the curve $xy + 1 = 0$ is the set of points $\{(0, y) \mid y \in \mathbb{C}^*\}$;
- f is not proper, since for any compact neighborhood K of $(0, 0)$ the set $f^{-1}(K)$ is never compact, see [15].

The map also provides an example of a surjective map which is not necessarily proper. On the other hand, every proper map must be surjective.

There is a purely algebraic condition for a polynomial map to be proper, see [12, Proposition 3]:

Proposition 2.2 *A dominant polynomial map $f: \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is proper if and only if the push-forward map $f_*: \mathbb{C}[s_1, \dots, s_n] \longrightarrow \mathbb{C}[x_1, \dots, x_n]$ is finite, i.e., $f_*\mathbb{C}[s_1, \dots, s_n] \subset \mathbb{C}[x_1, \dots, x_n]$ is an integral extension of rings.*

In the sequel we will focus on proper polynomial maps $f: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$. We write

$$f(x, y) = (f_1(x, y), f_2(x, y))$$

with $f_1, f_2 \in \mathbb{C}[x, y]$. Then the push-forward map will be given by

$$\begin{aligned} f_*: \mathbb{C}[s, t] &\longrightarrow \mathbb{C}[x, y] \\ s &\longrightarrow f_1(x, y) \\ t &\longrightarrow f_2(x, y). \end{aligned}$$

Given such a map f, its *Jacobian* J_f is the polynomial

$$J_f(x, y) = \begin{vmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{vmatrix}.$$

The *critical locus* $\mathrm{Crit}(f)$ of f is the affine variety $V(J_f) \subset \mathbb{C}^2$. The *branch locus* $B(f)$ of f is the image of the critical locus, that is $B(f) = f(\mathrm{Crit}(f))$. Since f is proper, the restriction

$$f: \mathbb{C}^2 \setminus f^{-1}(B(f)) \longrightarrow \mathbb{C}^2 \setminus B(f)$$

is an unramified covering of finite degree d; we will call d the *topological degree* of f.

Definition 2.3 We say that two proper polynomial maps $f_1,\ f_2\colon \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ are *equivalent* if there exist $\Phi_1,\ \Phi_2 \in \mathrm{Aut}(\mathbb{C}^2)$ such that

$$f_2 = \Phi_2 \circ f_1 \circ \Phi_1. \tag{1}$$

Remark 2.4 This equivalence relation in the semi-group of proper polynomial maps is weaker than the conjugacy relation, in which we require $\Phi_2 = \Phi_1^{-1}$. For instance, the two maps $f_1(x, y) = (x, y^2)$ and $f_2(x, y) = (x, y^2 + x)$ are equivalent in our sense but they are not conjugate by any automorphism of $\mathbb{C}^2$, since their sets of fixed points are not biholomorphic. The study of conjugacy classes of proper polynomial maps of given topological degree is an interesting problem, but we will not consider it in this paper.

Proposition 2.5 *If f_1 and f_2 are equivalent then they have the same topological degree. Moreover* $\mathrm{Crit}(f_1)$ *is biholomorphic to* $\mathrm{Crit}(f_2)$ *and* $B(f_1)$ *is biholomorphic to* $B(f_2)$.

Proof Assume that (1) holds. Since Φ_1 and Φ_2 have topological degree 1, it follows that f_1 and f_2 have the same topological degree. By the chain rule we have

$$J_{f_2} = J_{\Phi_2} \cdot J_{f_1} \cdot J_{\Phi_1},$$

so we obtain

$$\mathrm{Crit}(f_2) = \Phi_1^{-1}(\mathrm{Crit}(f_1)), \qquad B(f_2) = \Phi_2(B(f_1))$$

and this completes the proof. □

Definition 2.6 We say that a polynomial map $f\colon \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ is *semi-separate* if it is of the form

$$f(x, y) = (x, Q(x, y)),$$

where $Q(x, y) \in \mathbb{C}[x, y]$. In particular we say that it is *separate* if it is of the form

$$f(x, y) = (x, P(y)),$$

where $P(y) \in \mathbb{C}[y]$.

Recall that a polynomial $Q(x, y) \in \mathbb{C}[x, y]$ is called *monic with respect to* y if

$$Q(x, y) = ay^n + \text{terms of lower degree in } y, \quad a \in \mathbb{C}^*.$$

By Proposition 2.2 it follows that a semi-separate polynomial map f is proper if and only if $Q(x, y)$ is monic with respect to y; in this case, up to a dilation we may assume that f has the form

$$f(x, y) = (x, y^d + q_{d-1}(x)y^{d-1} + \cdots + q_0(x)), \tag{2}$$

where d is the topological degree. Notice that the Jacobian of (2) is

$$J_f(x, y) = dy^{d-1} + (d-1)q_{d-1}(x)y^{d-2} + \cdots + q_1(x). \tag{3}$$

For example, let us consider the case of a general semi-separate map $f\colon \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ of topological degree 3. By using a linear transformation we can get rid of the term in y^2; therefore, up to equivalence, f has the form

$$f(x, y) = (x, y^3 + p(x)y + q(x)).$$

Then $\mathrm{Crit}(f)$ has equation

$$3y^2 + p(x) = 0,$$

whereas $B(f)$ has equation

$$y^2 - 2q(x)y + \frac{\Delta(x)}{27} = 0,$$

where $\Delta(x) := 27q(x)^2 + 4p(x)^3$ is the discriminant of $y^3 + p(x)y + q(x)$. In particular, taking $p(x) = x$ and $q(x) = 0$, we obtain the *Whitney map*

$$f(x, y) = (x, y^3 + xy),$$

whose branch locus is the cuspidal cubic curve of equation $4x^3 + 27y^2 = 0$.

Remark 2.7 By (3) it follows that the branch locus of a separate map is a disjoint union of lines. Therefore the previous computations together with Proposition 2.5 show that the Whitney map is not equivalent to a separate one.

The following lemma will be used in the proof of Theorem A, see Sect. 3.

Lemma 2.8 *Let $f\colon \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ be a semi-separate map as in* (2), *with $d \geq 3$. If there exist two polynomials $H_1(x,\ y)$, $H_2(x,\ y)$ such that*

$$J_f(x, y) = H_1(x, y)^{d-2}H_2(x, y),$$

then both affine curves $V(H_1)$ and $V(H_2)$ are biholomorphic to $\mathbb{C}$.

Proof By using (3) we can write

$$dy^{d-1} + (d-1)q_{d-1}(x)y^{d-2} + \cdots + q_1(x) = H_1(x, y)^{d-2}H_2(x, y). \tag{4}$$

The left-hand side of (4) is monic with respect to y, so it cannot be divided by a polynomial in x. It follows that both H_1 and H_2 contain y. Therefore, by comparing the degrees, it follows that both H_1 and H_2 are monic of degree 1 in y, that is we may assume

$$H_1(x, y) = y + h_1(x), \qquad H_2(x, y) = dy + h_2(x),$$

for some $h_1(x), h_2(x) \in \mathbb{C}[x]$. This completes the proof. □

2.2 Milnor Number of a Plane Curve Singularity

In this subsection we summarize without proofs the definition and the properties of the Milnor number of a plane curve singularity. For further details we refer the reader to [16, Chap. 1] and [7, Chap. 3] and [4].

Let $\mathbb{C}\{x, y\}$ be the ring of convergent power series in two variables; it is a local ring whose maximal ideal $\mathfrak{m}$ consists of series with zero constant term, that is of series vanishing at the point $o = (0, 0)$.

Definition 2.9 A *plane curve singularity* X is a germ of an analytic space $(V(F), o)$, where $F \in \mathfrak{m} \subset \mathbb{C}\{x, y\}$.

Definition 2.10 Let $X = (V(F), o)$ be a plane curve singularity. We define the *Milnor number* $\mu(X, o)$ by

$$\mu(X, o) := \dim_{\mathbb{C}} \frac{\mathbb{C}\{x, y\}}{\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right)}.$$

Theorem 2.11 *The Milnor number is well defined and it is an invariant of the singularity. Moreover $\mu(X, o) < +\infty$ if and only if X is a germ of an* isolated *plane curve singularity.*

Example 2.12 Set $F_{d,n}(x, y) = y^d - x^n$ with $d, n \geq 2$. The point $o = (0, 0)$ is the only singularity of the affine curve $C_{d,n} = V(F_{d,n})$, and the corresponding Milnor number is given by

$$\mu_{d,n} := \mu(C_{d,n}, o) = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x, y\}}{(x^{n-1}, y^{d-1})} = (d-1)(n-1).$$

3 Proofs of Theorems A and B

We start by proving Theorem A.

Theorem A *For every $d \geq 3$ there exists at least one proper polynomial map $f\colon \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ such that f is not equivalent to a map of type $(x, y) \longrightarrow (x, Q(x, y))$.*

Proof Consider the polynomial map $f_d\colon \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ defined as follows:

$$f_d(x, y) := (x + y + xy, x^{d-1}y).$$

Claim 3.1 *The map f_d is proper and has topological degree d for all $d \geq 2$.*

Indeed, look at the push-forward map

$$\begin{aligned} f_{d*}\colon \mathbb{C}[s, t] &\longrightarrow \mathbb{C}[x, y] \\ s &\longrightarrow x + y + xy \\ t &\longrightarrow x^{d-1}y. \end{aligned}$$

The element $x \in \mathbb{C}[x,\ y]$ satisfies the monic equation of degree d

$$X^d - sX^{d-1} + tX + t = 0.$$

Analogously, the element y satisfies the monic equation

$$Y(s-Y)^{d-1} - t(1+Y)^{d-1} = 0.$$

This shows that $f_{d*}\mathbb{C}[s,t] \subset \mathbb{C}[x,y]$ is a integral extension of rings of degree d, hence Proposition 2.2 implies that f_d is a proper map of degree d. This proves our claim.

Now we want to show that f_d is not equivalent to any semi-separate map for all $d \geq 3$.

The Jacobian $J_{f_d}(x,y)$ splits as

$$J_{f_d}(x,y) = H_1(x,y)^{d-2} H_2(x,y), \tag{5}$$

where

$$H_1(x,y) = x, \quad H_2(x,y) = (2-d)xy + x - (d-1)y.$$

For all $d \geq 3$, the conic $V(H_2)$ is biholomorphic to $\mathbb{C}^*$. Since $\mathbb{C}$ and $\mathbb{C}^*$ are obviously not biholomorphic, it follows by Proposition 2.5 and Lemma 2.8 that there exists no semi-separate map equivalent to f_d, for all $d \geq 3$. This concludes the proof of Theorem A. □

Remark 3.2 For $d = 2$ the map f_d is equivalent to a semi-separate one. Indeed, consider $\Phi_1, \Phi_2 \in \mathrm{Aut}(\mathbb{C}^2)$ defined by

$$\Phi_1(x,y) = \left(\frac{x+y}{2}, \frac{x-y}{2}\right), \qquad \Phi_2(x,y) = (x^2 + 2x - y, x^2 - y).$$

Then we have

$$f_2(x,y) = \Phi_2 \circ \tilde{f} \circ \Phi_1(x,y),$$

where $\tilde{f}(x,y) = (x, y^2)$, in accordance with Theorem 1.1.

Now let us prove Theorem B.

Theorem B *For all positive integers d, n, with $d \geq 3$ and $n \geq 2$, consider the polynomial map $f_{d,n}: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ given by*

$$f_{d,n}(x,y) := (x, y^d - dx^n y).$$

Then $f_{d,n}$ and $f_{d,m}$ are equivalent if and only if $m = n$. It follows that there exist infinitely many different equivalence classes of proper polynomial maps $f: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ of fixed topological degree d.

Proof The critical locus of the map $f_{d,n}$ is the affine curve $C_{d-1,n}$ of equation $y^{d-1} - x^n = 0$, whose unique singularity is $o = (0, 0)$. The Milnor number of $C_{d-1,n}$ in o is given by

$$\mu_{d-1,n} = \mu(C_{d-1,n}, o) = (d-2)(n-1),$$

see Example 2.12. Hence $\mu_{d-1,n} = \mu_{d-1,m}$ if and only if $m = n$. It follows by Theorem 2.11 that the curves $C_{d-1,n}$ and $C_{d-1,m}$ are not biholomorphic if $m \neq n$, therefore Proposition 2.5 implies that $f_{d,n}$ and $f_{d,m}$ are not equivalent if $m \neq n$.

This concludes the proof of Theorem B. □

4 The Case of Galois Coverings

Let $f : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ be a polynomial map which is a Galois covering with finite Galois group G. By Proposition 2.2, f is proper and its topological degree equals $|G|$; moreover $G \subset \mathrm{Aut}(\mathbb{C}^2)$, and f can be identified with the quotient map $\mathbb{C}^2 \longrightarrow \mathbb{C}^2/G$. Since G is a finite group, we may assume $G \subset \mathrm{GL}(2, \mathbb{C})$ by a polynomial change of coordinates [13, Corollary 4.4] and, since $\mathbb{C}^2/G \cong \mathbb{C}^2$, it follows that G is a *finite complex reflection group*. Let us denote by $\mathbb{C}[x, y]^G$ the subalgebra of G-invariant polynomials; then the following two conditions are equivalent, see [3, p. 380]:

(i) there are two algebraically independent homogeneous polynomials $\phi_1, \phi_2 \in \mathbb{C}[x, y]^G$ which satisfy $|G| = \deg(\phi_1) \cdot \deg(\phi_2)$;
(ii) there are two algebraically independent homogeneous polynomials $\phi_1, \phi_2 \in \mathbb{C}[x, y]^G$ such that $1, \phi_1, \phi_2$ generate $\mathbb{C}[x, y]^G$ as an algebra over $\mathbb{C}$.

We say that ϕ_1, ϕ_2 are a *basic set of invariants* for G. Furthermore, putting $d_1 := \deg(\phi_1)$, $d_2 := \deg(\phi_2)$, the set $\{d_1, d_2\}$ is independent of the particular choice of ϕ_1, ϕ_2. We call d_1, d_2 the *degrees* of G.

Proposition 4.1 *Let ϕ_1, ϕ_2 and ψ_1, ψ_2 be two basic sets of invariants for G. Then the two polynomial maps $\phi, \psi : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ defined by*

$$\phi(x, y) = (\phi_1(x, y), \phi_2(x, y)),$$
$$\psi(x, y) = (\psi_1(x, y), \psi_2(x, y))$$

are equivalent.

Proof Set

$$d_1 = \deg(\phi_1) = \deg(\psi_1), \qquad d_2 = \deg(\phi_2) = \deg(\psi_2),$$

with $d_1 \leq d_2$. Since both $\{1, \phi_1, \phi_2\}$ and $\{1, \psi_1, \psi_2\}$ generate $\mathbb{C}[x, y]^G$, we may express both ϕ_1 and ϕ_2 as polynomials in ψ_1, ψ_2. Looking at the degrees, one sees that there are three cases.

• If $d_1 \nmid d_2$, then there exist $a, b \in \mathbb{C}^*$ such that

$$\phi_1 = a\psi_1, \qquad \phi_2 = b\psi_2.$$

Set $\Phi(x, y) = (ax, by)$.
• If $d_1|d_2$ and $d_1 \neq d_2$, set $s = d_2/d_1$. Then there exist $a, c, d \in \mathbb{C}$, $ad \neq 0$, such that

$$\phi_1 = a\psi_1, \qquad \phi_2 = c\psi_1^s + d\psi_2.$$

Set $\Phi(x, y) = (ax, cx^s + dy)$.
• If $d_1 = d_2$, then there exist $a, b, c, d \in \mathbb{C}$, $(ad - bc) \neq 0$, such that

$$\phi_1 = a\psi_1 + b\psi_2, \qquad \phi_2 = c\psi_1 + d\psi_2.$$

Set $\Phi(x, y) = (ax + by, cx + dy)$.

In all cases $\Phi \in \mathrm{Aut}(\mathbb{C}^2)$, see [10], and $\phi = \Phi \circ \psi$. This completes the proof. □

Corollary 4.2 *Let $f : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ be a Galois covering with finite Galois group G. Then f is equivalent to the map $\phi(x, y) = (\phi_1(x, y), \phi_2(x, y))$, where ϕ_1, ϕ_2 is any basic set of invariants for G.*

It is well known that there exists a unitary inner product on $\mathbb{C}^2$ invariant under G, hence we may assume that G is a subgroup of the unitary group $\mathrm{U}(2)$, see [3, p. 382]. There are two cases, according whether the representation $G \subset \mathrm{U}(2)$ is reducible or not.

4.1 The Reducible Case

Assume that there exists a 1-dimensional linear subspace $V \subset \mathbb{C}^2$ which is invariant under G; then its orthogonal complement $V^\perp$ is also invariant [19, Chap. 1], and up to a linear change of coordinates we may assume $V = \langle e_1 \rangle$, $V^\perp = \langle e_2 \rangle$, where $\{e_1, e_2\}$ is the canonical basis of $\mathbb{C}^2$. This means that G is generated by

$$g_1(x, y) = (\theta_m x, y), \qquad g_2(x, y) = (x, \theta_n y),$$

where θ_m is a primitive m-th root of unity and θ_n is a primitive n-th root of unity, respectively. Therefore we obtain the following

Proposition 4.3 *Let $G \subset \mathrm{U}(2)$ be a reducible finite complex reflection group acting on $\mathbb{C}^2$. Then, up to a change of coordinates, we are in one of the following cases*:

(1) *$G = \mathbb{Z}_m$, generated by*

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \exp(2\pi i/m) \end{pmatrix};$$

(2) *$G = \mathbb{Z}_m \times \mathbb{Z}_n$, generated by*

$$g_1 = \begin{pmatrix} \exp(2\pi i/m) & 0 \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad g_2 = \begin{pmatrix} 1 & 0 \\ 0 & \exp(2\pi i/n) \end{pmatrix}.$$

4.2 The Irreducible Case

The finite irreducible complex reflection groups were classified by Shephard and Todd in [21]. They found an infinite family $G(m, p, 2)$, depending on two positive integer parameters m, p, with $p|m$, and 19 exceptional cases, that they numbered from 4 to 22. We start by describing the groups belonging to the infinite family. One has

$$G(m, p, 2) = \mathbb{Z}_2 \ltimes A(m, p, 2),$$

where $A(m, p, 2)$ is the Abelian group of order m^2/p whose elements are the matrices $\left(\begin{smallmatrix} \theta^{\alpha_1} & 0 \\ 0 & \theta^{\alpha_2} \end{smallmatrix}\right)$, with $\theta = \exp(2\pi i/m)$ and $\alpha_1 + \alpha_2 \equiv 0 \pmod p$, whereas $\mathbb{Z}_2$ is generated by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. In particular, $G(m, m, 2)$ is the dihedral group of order $2m$.

Proposition 4.4 (1) *$G(m, p, 2)$ acts irreducibly on $\mathbb{C}^2$, except in the case $G(2, 2, 2)$. In particular, $G(m, p, 2)$ is non-Abelian provided that $(m, p) \neq (2, 2)$.*

(2) *The only groups in the family $G(m, p, 2)$ which are isomorphic as abstract groups are $G(2, 1, 2)$ and $G(4, 4, 2)$.*

Proof (1) Suppose that $G = G(m, p, 2)$ leaves invariant a nontrivial proper linear subspace $V \subset \mathbb{C}^2$. In particular, V must be invariant under the linear transformation $(x, y) \longrightarrow (y, x)$, hence we may assume, up to an interchanging of V and $V^{\perp}$, that V is the line $x - y = 0$. As $A(m, p, 2)$ stabilizes V, all diagonal coefficients of an element of $A(m, p, 2)$ must be equal. From this, one easily deduces that $m = p = 2$. On the other hand, it is obvious that $G(2, 2, 2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ acts reducibly on $\mathbb{C}^2$.

(2) Assume that $G(m, p, 2)$ and $G(m', p', 2)$ are isomorphic as abstract groups. In particular $|G(m, p, 2)| = |G(m', p', 2)|$ and $|Z(G(m, p, 2))| = |Z(G(m', p', 2))|$. Setting $q = m/p$, $q' = m'/p'$, by [3, p. 387] we obtain

$$mq = m'q', \qquad q \cdot \gcd(p, 2) = q' \cdot \gcd(p', 2).$$

If $\gcd(p, 2) = \gcd(p', 2)$ we have $q = q'$, hence $m = m'$ and $p = p'$. Therefore, we may suppose that p is odd and p' is even. Hence $q = 2q'$, that is $m' = 2m$ and $p' = 4p$. Since $p'|m'$, it follows that m must be even. Summing up, we are left to understand when $G(m, p, 2)$ and $G(2m, 4p, 2)$, m even, p odd are isomorphic as abstract groups. If m is even and p is odd, there are exactly $m + 3$ elements of order 2 in $G(m, p, 2)$, namely

$$\begin{pmatrix} \theta^{m/2} & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & \theta^{m/2} \end{pmatrix}, \quad \begin{pmatrix} \theta^{m/2} & 0 \\ 0 & \theta^{m/2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \theta^{\alpha_1} \\ \theta^{\alpha_2} & 0 \end{pmatrix},$$

where $\alpha_1 + \alpha_2 = m$. On the other hand, the two matrices

$$\begin{pmatrix} \theta^{m} & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & \theta^{m} \end{pmatrix}$$

belong to $G(2m, 4p, 2)$ if and only if $4|m$. So $G(2m, 4p, 2)$ contains $2m + 3$ elements of order 2 if $4|m$, and $2m + 1$ elements of order 2 if $4 \nmid m$. Consequently, if

$G(m, p, 2)$ and $G(2m, 4p, 2)$, m even, p odd are isomorphic as abstract groups the only possibility is $4 \nmid m$ and $m+3 = 2m+1$, that is $m = 2$, $p = 1$. Finally, it is not difficult to check that $G(2, 1, 2)$ and $G(4, 4, 2)$ are conjugate in U(2), hence they are isomorphic not only as abstract groups, but actually as complex reflection groups, see [3, p. 388]. □

Now let us consider the exceptional groups in the Shephard-Todd's list. We closely follow the treatment given in [21]. For $p = 3, 4, 5$, the abstract group

$$\langle s, t \mid s^2 = t^3 = (st)^p = 1\rangle$$

is isomorphic to $\mathcal{A}_4$, $\mathcal{S}_4$ and $\mathcal{A}_5$, respectively. These are the well-known groups of symmetries of regular polyhedra: $\mathcal{A}_4$ is the symmetry group of the tetrahedron, $\mathcal{S}_4$ is the symmetry group of the cube (and of the octahedron) and $\mathcal{A}_5$ is the symmetry group of the dodecahedron (and of the icosahedron). We take Klein's representation of these groups by complex matrices [14], and we call S_1, T_1 the matrices corresponding to the generators s and t, respectively. Therefore the exceptional finite complex reflection groups are generated by matrices

$$S = \lambda S_1, \qquad T = \mu T_1, \qquad Z = \exp(2\pi i/k)I,$$

where λ, μ are suitably chosen roots of unity and k is a suitable integer. The corresponding abstract presentations are of the form

$$\big\langle S, T, Z \mid S^2 = Z^{k_1}, T^3 = Z^{k_2}, (ST)^p = Z^{k_3}, [S, Z] = 1, [T, Z] = 1, Z^k = 1\big\rangle \quad (6)$$

where $p = 1, 2, 3$ and k_1, k_2, k_3, k are suitably chosen integers. We shall arrange the possible values of λ, μ, k_1, k_2, k_3, k in tabular form, according to Shephard-Todd's list [21, p. 280–286].

Exceptional groups derived from $\mathcal{A}_4$. Set $\omega = \exp(2\pi i/3)$, $\varepsilon = \exp(2\pi i/8)$. We have

$$S_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad T_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} \varepsilon & \varepsilon^3 \\ \varepsilon & \varepsilon^7 \end{pmatrix}.$$

The four corresponding groups are shown in Table 1. Here `IdSmallGroup`(G) denotes the label of G in the `GAP4` database of small groups, which includes all groups of order less than 2000, with the exception of 1024 [11]. For instance, one has `[24,3]` $= \mathrm{SL}_2(\mathbb{F}_3)$ and this means that $\mathrm{SL}_2(\mathbb{F}_3)$ is the third in the list of groups of order 24 (see the `GAP4` script 1 in Appendix A).

Exceptional groups derived from $\mathcal{S}_4$. We have

$$S_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ -1 & -i \end{pmatrix}, \qquad T_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} \varepsilon & \varepsilon \\ \varepsilon^3 & \varepsilon^7 \end{pmatrix}.$$

The eight corresponding groups are shown in Table 2.

Exceptional groups derived from $\mathcal{A}_5$. Set $\eta = \exp(2\pi i/5)$. We have

$$S_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} \eta^4 - \eta & \eta^2 - \eta^3 \\ \eta^2 - \eta^3 & \eta - \eta^4 \end{pmatrix}, \qquad T_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} \eta^2 - \eta^4 & \eta^4 - 1 \\ 1 - \eta & \eta^3 - \eta \end{pmatrix}.$$

Table 1 Exceptional groups derived from $\mathcal{A}_4$

No.	`IdSmallGroup(G)`	λ	μ	k_1	k_2	k_3	k	Degrees
4	`[24,3]`	-1	$-\omega$	1	2	2	2	4, 6
5	`[72,25]`	$-\omega$	$-\omega$	1	6	6	6	6, 12
6	`[48,33]`	i	$-\omega$	4	4	1	4	4, 12
7	`[144,157]`	$i\omega$	$-\omega$	8	12	3	12	12, 12

Table 2 Exceptional groups derived from $\mathcal{S}_4$

No.	`IdSmallGroup(G)`	λ	μ	k_1	k_2	k_3	k	Degrees
8	`[96,67]`	ε^3	1	1	2	4	4	8, 12
9	`[192,963]`	i	ε	8	7	8	8	8, 24
10	`[288,400]`	$\varepsilon^7\omega^2$	$-\omega$	7	12	12	12	12, 24
11	`[576,5472]`	i	$\varepsilon\omega$	24	21	8	24	24, 24
12	`[48,29]`	i	1	2	1	1	2	6, 8
13	`[96,192]`	i	i	4	1	2	4	8, 12
14	`[144,122]`	i	$-\omega$	6	6	5	6	6, 24
15	`[288,903]`	i	$i\omega$	12	3	10	12	12, 24

Table 3 Exceptional groups derived from $\mathcal{A}_5$

No.	`IdSmallGroup(G)`	λ	μ	k_1	k_2	k_3	k	Degrees
16	`[600,54]`	$-\eta^3$	1	7	10	10	10	20, 30
17	`[1200,483]`	i	$i\eta^3$	20	11	20	20	20, 60
18	`[1800,328]`	$-\omega\eta^3$	ω^2	11	30	30	30	30, 60
19	`[3600, ]`	$i\omega$	$i\eta^3$	40	33	40	60	60, 60
20	`[360,51]`	1	ω^2	3	6	5	6	12, 30
21	`[720,420]`	i	ω^2	12	12	1	12	12, 60
22	`[240, 93]`	i	1	4	4	3	4	12, 20

The seven corresponding groups are shown in Table 3.

Proposition 4.5 *None of the groups in Tables 1, 2, 3 is isomorphic as an abstract group to some* $G(m, p, 2)$.

Proof Let G be one of the groups in the tables. Looking at the presentation (6), one easily sees that the center of G is $\langle Z\rangle \cong \mathbb{Z}_k$ and that this is the maximal normal Abelian subgroup of G. Since in every case $2k < |G|$, this implies that G contains no normal Abelian subgroups of index 2, hence it cannot be isomorphic to $G(m, p, 2) = \mathbb{Z}_2 \ltimes A(m, p, 2)$. □

Definition 4.6 A finite group G of unitary automorphisms of $\mathbb{C}^2$ is called *imprimitive* if $\mathbb{C}^2 = V_1 \oplus V_2$, where V_1 and V_2 are nontrivial proper linear subspaces such that the set $\{V_1, V_2\}$ is invariant under G. If such a direct splitting of $\mathbb{C}^2$ does not exist, G is called *primitive*.

Notice that every group $G(m, p, 2)$ is imprimitive, since we can take $V_1 = \langle e_1 \rangle$, $V_2 = \langle e_2 \rangle$. By Proposition 4.4 and [3, p. 386 and p. 394] one obtains

Proposition 4.7 *Let G be an irreducible finite complex reflection group acting on $\mathbb{C}^2$.*

(1) *If G is imprimitive, then G is conjugate in* $\mathrm{U}(2)$ *to $G(m, p, 2)$ for some $m, p \in \mathbb{N}$, $p|m$, $(m, p) \neq (2, 2)$. The pair (m, p) is uniquely determined, with the exception of $G(2, 1, 2)$ which is conjugate to $G(4, 4, 2)$.*
(2) *If G is primitive, then G is conjugate in* $\mathrm{U}(2)$ *to exactly one of the groups* $4, \ldots, 22$ *in Tables* 1, 2, 3.

4.3 The Classification

Now we can give the classification, up to equivalence, of finite Galois coverings $f\colon \mathbb{C}^2 \longrightarrow \mathbb{C}^2$. Set

$$\begin{aligned}
\mathsf{a}_4(x, y) &= x^4 + (4\xi - 2)x^2y^2 + y^4, \quad \xi = \exp(2\pi i/6),\\
\mathsf{b}_6(x, y) &= x^5y - xy^5,\\
\mathsf{c}_8(x, y) &= x^8 + 14x^4y^4 + y^8,\\
\mathsf{d}_{12}(x, y) &= x^{12} - 33x^8y^4 - 33x^4y^8 + y^{12},\\
\mathsf{e}_{12}(x, y) &= x^{11}y + 11x^6y^6 - xy^{11},\\
\mathsf{f}_{20}(x, y) &= x^{20} - 228x^{15}y^5 + 494x^{10}y^{10} + 228x^5y^{15} + y^{20},\\
\mathsf{g}_{30}(x, y) &= x^{30} + 522x^{25}y^5 - 10005x^{20}y^{10} - 10005x^{10}y^{20} - 522x^5y^{25} + y^{30}.
\end{aligned}$$

Then we have

Theorem 4.8 *Let $f\colon \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ be a polynomial map which is a Galois covering with finite Galois group G. Then, up to equivalence, we are in one of the cases in Table 4 below. Furthermore, these maps are pairwise non-equivalent, with the only exception of* $\mathfrak{f}_{2,1,2}$ *and* $\mathfrak{f}_{4,4,2}$.

Proof By Propositions 4.3 and 4.7, G is conjugate in $\mathrm{U}(2)$ to one of the groups in Table 4. Moreover by Propositions 4.4 and 4.5 these groups are pairwise not isomorphic, with the unique exception of $G(2, 1, 2)$ and $G(4, 4, 2)$. Therefore, by Corollary 4.2 it is sufficient to show that in every case ϕ_1, ϕ_2 form a basic set of invariants for G. This is obvious in the first three cases. For the remaining groups we can do a case-by-case analysis, using the description of G given in Subsects. 4.1 and 4.2. A shorter proof can be obtained by noticing that:

– a_4 is G_4-invariant and, up to a multiplicative constant,

$$\mathsf{b}_6 = \text{Jacobian}(\mathsf{a}_4, \text{Hessian}(\mathsf{a}_4));$$

– b_6 is G_{12}-invariant and, up to multiplicative constants,

$$\mathsf{c}_8 = \text{Hessian}(\mathsf{b}_6) \quad \text{and} \quad \mathsf{d}_{12} = \text{Jacobian}(\mathsf{b}_6, \mathsf{c}_8);$$

– e_{12} is G_{20}-invariant and, up to multiplicative constants,

$$\mathsf{f}_{20} = \text{Hessian}(\mathsf{e}_{12}) \quad \text{and} \quad \mathsf{g}_{30} = \text{Jacobian}(\mathsf{e}_{12}, \mathsf{f}_{20}).$$

Then ϕ_1, ϕ_2 form a basic sets of invariants for $G_4, \dots, G_{22}$ by [21, p. 285–286], [2, 14].

Finally, the computation of the branch locus in each case is a straightforward application of elimination theory and can be carried out with the help of the Computer Algebra System `Singular` [20]. Look at the `Singular` script 3 in Appendix A to see how this applies to an explicit example, namely the map $\tilde{\mathfrak{f}}_4$. □

Table 4 Galois coverings with Galois group G

Map	ϕ_1, ϕ_2	G	Branch locus
$\mathfrak{f}_m$	x, y^m	$\mathbb{Z}_m$	$y = 0$
$\mathfrak{f}_{m,n}$	x^m, y^n	$\mathbb{Z}_m \times \mathbb{Z}_n$	$xy = 0$
$\mathfrak{f}_{m,p,2}$	$x^{m/p}y^{m/p}, x^m + y^m$	$G(m, p, 2)$	$x(y^2 - 4x^p) = 0$ if $p \neq m$
			$y^2 - 4x^p = 0$ if $p = m$
$\tilde{\mathfrak{f}}_4$	$\mathsf{a}_4, \mathsf{b}_6$	$G_4 =$ `[24, 3]`	$x^3 + (-24\xi + 12)y^2 = 0$
$\tilde{\mathfrak{f}}_5$	$\mathsf{b}_6, (\mathsf{a}_4)^3$	$G_5 =$ `[72, 25]`	$y(x^2 + (\frac{1}{18\xi} - \frac{1}{36})y) = 0$
$\tilde{\mathfrak{f}}_6$	$\mathsf{a}_4, (\mathsf{b}_6)^2$	$G_6 =$ `[48, 33]`	$y(x^3 + (-24\xi + 12)y^2) = 0$
$\tilde{\mathfrak{f}}_7$	$(\mathsf{b}_6)^2, (\mathsf{a}_4)^3$	$G_7 =$ `[144, 157]`	$xy(x + (\frac{1}{18\xi} - \frac{1}{36})y) = 0$
$\tilde{\mathfrak{f}}_8$	$\mathsf{c}_8, \mathsf{d}_{12}$	$G_8 =$ `[96, 67]`	$y^2 - x^3 = 0$
$\tilde{\mathfrak{f}}_9$	$\mathsf{c}_8, (\mathsf{d}_{12})^2$	$G_9 =$ `[192, 963]`	$y(y - x^3) = 0$
$\tilde{\mathfrak{f}}_{10}$	$\mathsf{d}_{12}, (\mathsf{c}_8)^3$	$G_{10} =$ `[288, 400]`	$y(y - x^2)$=0
$\tilde{\mathfrak{f}}_{11}$	$(\mathsf{d}_{12})^2, (\mathsf{c}_8)^3$	$G_{11} =$ `[576, 5472]`	$xy(x - y) = 0$
$\tilde{\mathfrak{f}}_{12}$	$\mathsf{b}_6, \mathsf{c}_8$	$G_{12} =$ `[48, 29]`	$y^3 - 108x^4 = 0$
$\tilde{\mathfrak{f}}_{13}$	$\mathsf{c}_8, (\mathsf{b}_6)^2$	$G_{13} =$ `[96, 192]`	$y(x^3 - 108y^2)$=0
$\tilde{\mathfrak{f}}_{14}$	$\mathsf{b}_6, (\mathsf{d}_{12})^2$	$G_{14} =$ `[144, 122]`	$y(y + 108x^4)$=0
$\tilde{\mathfrak{f}}_{15}$	$(\mathsf{b}_6)^2, (\mathsf{d}_{12})^2$	$G_{15} =$ `[288, 903]`	$xy(y + 108x^2) = 0$
$\tilde{\mathfrak{f}}_{16}$	$\mathsf{f}_{20}, \mathsf{g}_{30}$	$G_{16} =$ `[600, 54]`	$y^2 - x^3 = 0$
$\tilde{\mathfrak{f}}_{17}$	$\mathsf{f}_{20}, (\mathsf{g}_{30})^2$	$G_{17} =$ `[1200, 483]`	$y(y - x^3) = 0$
$\tilde{\mathfrak{f}}_{18}$	$\mathsf{g}_{30}, (\mathsf{f}_{20})^3$	$G_{18} =$ `[1800, 328]`	$y(y - x^2) = 0$
$\tilde{\mathfrak{f}}_{19}$	$(\mathsf{g}_{30})^2, (\mathsf{f}_{20})^3$	$G_{19} =$ `[3600, ]`	$xy(x - y) = 0$
$\tilde{\mathfrak{f}}_{20}$	$\mathsf{e}_{12}, \mathsf{g}_{30}$	$G_{20} =$ `[360, 51]`	$y^2 - 1728x^5 = 0$
$\tilde{\mathfrak{f}}_{21}$	$\mathsf{e}_{12}, (\mathsf{g}_{30})^2$	$G_{21} =$ `[720, 420]`	$y(y - 1728x^5) = 0$
$\tilde{\mathfrak{f}}_{22}$	$\mathsf{e}_{12}, \mathsf{f}_{20}$	$G_{22} =$ `[240, 93]`	$y^3 + 1728x^5 = 0$

The following corollary generalizes Theorem 1.1 to the case of Galois coverings of arbitrary degree.

Corollary 4.9 *For all* $d \geq 2$, *there exist only finitely many equivalence classes of Galois coverings* $f: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ *of topological degree* d.

Proof For all $d \geq 2$, there are only finitely many integers m, n, p such that any of the equalities

$$|\mathbb{Z}_m| = d, \qquad |\mathbb{Z}_m \times \mathbb{Z}_n| = d, \qquad |G(m, p, 2)| = d$$

holds. □

Remark 4.10 The computation of the invariant polynomials $\mathsf{a}_4, \ldots, \mathsf{g}_{30}$ goes back to Klein, see *[14]*. Nowadays, it can be easily carried out by using the `Singular` script 2 in Appendix A.

Remark 4.11 Some of the coverings in Table 4 already appeared in the literature. For instance, those with groups $G(m, m, 2)$, G_4, G_8, G_{16}, G_{20} were studied (by different methods) in [18], whereas those with groups $G(m, 1, 2)$, G_5, G_6, G_9, G_{10}, G_{14}, G_{17}, G_{18}, G_{21} were studied in [17].

Remark 4.12 The critical locus of every map in Table 4 is a finite union of lines through the origin, since the two components ϕ_1, ϕ_2 are always homogeneous polynomials. Therefore in each case the origin is a total ramification point and this in turn implies that all these examples are "polynomial-like" self-maps of $\mathbb{C}^2$, see [6, Example 2.1.1]. In particular, their dynamical behaviour has been largely investigated, see [5].

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Appendix A

In this appendix we include for the reader's convenience some of the `GAP4` and `Singular` scripts that we have used in our computations; all the others are similar and can be easily obtained modifying the ones below.

The `GAP4` script 1 finds the label `[24, 3]` of the group G_4 in Table 4 and shows that it is isomorphic to $\mathrm{SL}_2(\mathbb{F}_3)$. The `Singular` script 2 computes the basic set of invariants a_4, b_6 for G_4, whereas the `Singular` script 3 shows that the branch locus of the map $\tilde{\mathfrak{f}}_4(x, y) = (\mathsf{a}_4(x, y), \mathsf{b}_6(x, y))$ is the curve $x^3 + (-24\exp(2\pi i/6) + 12)y^2 = 0$.

```
gap> ####### GAP4 SCRIPT 1:
               Identifying the groups #######
gap> F:=FreeGroup("s", "t", "z");; gap> s:=F.1;;
     t:=F.2;; z:=F.3;;
```

```
gap> #insert the presentation of G
gap> G:=F/[s^2*z^-1, t^3*z^-2, (s*t)^3*z^-2,
> z*s*z^-1*s^-1, z*t*z^-1*t^-1, z^2];;
gap> # compute the label of G
gap> IdSmallGroup(G);
[24, 3]
gap> # check that G is isomorphic to SL(2,3)
gap> G1:=SL(2,3);; IdSmallGroup(G1);
[24, 3]
```

```
> ;  // ------- SINGULAR SCRIPT 2:
                Finding the invariants -------
> LIB("finvar.lib");
> ring R =(0,a), (x, y), dp;
> ;  // minimal polynomial of a=exp(2 pi i/24)
> minpoly = a^8-a^4+1;
> number e=a^3;
> number w=a^8;
> number i=e^2;
> number r2=e-e^3; // r2=sqrt(2)
> ;  // define the matrices S1 and T1
> matrix S1[2][2]= i, 0, 0, -i;
> matrix T1[2][2]= e*r2^-1, e^3*r2^-1,
                   e*r2^-1, e^7*r2^-1;
> ;  // define the matrices S and T
> matrix S = -S1;
> matrix T = -w * T1;
> ;  // compute a basic set of invariants
> ;  // for the group generated by S and T
> invariant_ring(S, T);
_[1,1]=x4+(4a4-2)*x2y2+y4
_[1,2]=x5y-xy5 _[1,1]=1 _[1,1]=0
```

```
> ;  // ------- SINGULAR SCRIPT 3:
                Computing the branch locus -------
> ring R = (0, a), (s, t, x, y), dp;
> ; // minimal polynomial of a=exp(2 pi i/6)
> minpoly = a^2-a+1;
> ;  // define the map f(s,t)=(X(s,t), Y(s,t))
> poly X = s4+(4a-2)*s2t2+t4;
> poly Y = s5t-st5;
> ;  // compute the Jacobian of f
> poly j = diff(X,s)*diff(Y,t)-diff(X,t)*diff(Y,s);
> ideal I = j, x-X, y-Y;
```

```
> ;  // compute the equation of the branch curve B(f)
> ;  // by eliminating the variables s, t
> ideal J = eliminate(I,  st);
> J;
J[1]=x3+(-24a+12)*y2
```

References

1. Bass, H., Connel, E.H., Wright, D.: The Jacobian conjecture: reduction of degree and formal expansion of the inverse. Bull. Am. Math. Soc. (N.S.) **7**(2), 287–330 (1982)
2. Chevalley, C.: Invariants of finite groups generated by reflections. Am. J. Math. **77**(4), 778–782 (1955)
3. Cohen, A.M.: Finite complex reflection groups. Ann. Sci. Ec. Norm. Sup. **4**(9), 379–436 (1976)
4. Dimca, A.: Singularities and Topology of Hypersurfaces. Springer Universitext. Springer, Berlin (1992)
5. Dinh, T.C., Sibony, N.: Dynamique des applications d'allure polynomiale [Dynamics of polynomial-like mappings]. J. Math. Pures Appl. (4) **82**(9), 367–423 (2003)
6. Dinh, T.C., Sibony, N.: Dynamics in several complex variables: endomorphisms of projective spaces and polynomial-like mappings. arXiv:0810.0811 (2008)
7. de Jong, T., Pfister, G.: Local Analytic Geometry. Advanced Lectures in Mathematics. Vieweg & Sohn, Braunschweig (2000)
8. Favre, C., Jonsson, M.: Eigenvaluations. Ann. Sci. Ec. Norm. Super. (2) **40**(4), 309–349 (2007)
9. Favre, C., Jonsson, M.: Dynamical compactifications of $\mathbb{C}^2$. arXiv:0711.2770 (2007)
10. Friedland, S., Milnor, J.: Dynamical properties of plane polynomial automorphisms. Ergod. Theory Dyn. Syst. **9**(1), 67–99 (1989)
11. The GAP Group: GAP—Groups, Algorithms and Programming, Version 4.4. http://www.gap-system.org
12. Jelonek, Z.: The set of points at which a polynomial map is not proper. Ann. Pol. Math. **LVIII**(3) (1993)
13. Kambayashi, T.: Automorphism group of a polynomial ring and automorphism group action on an affine space. J. Algebra **60**, 439–451 (1979)
14. Klein, F.: Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade. Teubner, Leipzig (1884)
15. Lamy, S.: Sur la structure du groupe d'automorphismes de certaines surfaces affines. Publ. Math. **49**, 3–20 (2005)
16. Looijenga, E.J.N.: Isolated Singular Points on Complete Intersections. London Mathematical Society Lecture Note Series, vol. 77. Cambridge University Press, Cambridge (1984)
17. Nakano, T., Nishikubo, H.: On some maximal Galois coverings over affine and projective planes II. Tokyo J. Math. **23**(2), 295–310 (2000)
18. Nakano, T., Tamai, K.: On some maximal Galois coverings over affine and projective planes. Osaka J. Math. **33**, 347–364 (1996)
19. Serre, J.P.: Représentations linéaires des groupes finis. Hermann, Paris (1971)
20. Singular: a Computer Algebra System for polynomial computations. http://www.singular.uni-kl.de/
21. Shephard, G.C., Todd, J.A.: Finite unitary reflection groups. Can. J. Math. **6**, 274–304 (1954)