Adv. Geom. **13** (2013), 1–27
DOI 10.1515/advgeom-2012-0027

Advances in Geometry

A class of lattices and boolean functions related to the Manickam–Miklös–Singhi conjecture

Cinzia Bisi* and Giampiero Chiaselotti

(Communicated by G. Gentili)

Abstract. The aim of this paper is to build a new family of lattices related to some combinatorial extremal sum problems, in particular to a conjecture of Manickam, Miklös and Singhi. We study the fundamental properties of such lattices and of a particular class of boolean functions defined on them.

Key words. Graded lattices, involution posets, weight functions, boolean maps, extremal sum problems.

2000 Mathematics Subject Classification. 05D05

0 Introduction

Let n, r be two fixed integers such that $0 \leq r \leq n$ and let $I_n = \{1, 2, \ldots, n\}$. In the first part of this paper (Section 1) we define a partial order $\sqsubseteq$ on the power set $\mathcal{P}(I_n)$ having the following property: if X, Y are two subsets of I_n such that $X \sqsubseteq Y$, then $\sum_{i \in X} a_i \leq \sum_{i \in Y} a_i$, for each n-multiset $\{a_1, \ldots, a_n\}$ of real numbers such that $a_1 \geq \cdots \geq a_r \geq 0 > a_{r+1} \geq \cdots \geq a_n$. This order defines a lattice structure on $\mathcal{P}(I_n)$ that we will denote by $(S(n,r), \sqsubseteq)$. We show that this lattice is distributive, graded (Section 2), involutive (Section 3), i.e. $X \sqsubseteq Y$ implies $Y^c \sqsubseteq X^c$, and we also give an algorithmic method to generate uniquely its Hasse diagram (Section 4) and a recursive formula to count the number of its elements having fixed rank (Section 5).

In Section 6 we establish the connection between the lattice $S(n,r)$ and some combinatorial extremal sum problems related to a conjecture of Manickam, Miklös and Singhi. We give an interpretation of these problems in terms of a particular class of boolean maps defined on $S(n,r)$ (Section 7).

*The first author is partially supported by Progetto MIUR di Rilevante Interesse Nazionale *Proprietà geometriche delle varietà reali e complesse* and by GNSAGA - INDAM.

Now we briefly summarize the historical motivations that have led us to build the lattice $S(n,r)$ and the other associated structures. In [19] the authors asked the following question: *let n be an integer strictly greater than 1 and $a_1,\dots,a_n$ be real numbers satisfying the property $\sum_{1=1}^{n} a_i \geq 0$. We may ask: how many subsets of the set $\{a_1,\dots,a_n\}$ will have a non-negative sum?*

Following the notation of [19], the authors denote with $A(n)$ the minimum number of the non-negative partial sums of a sum $\sum_{i=1}^{n} a_i \geq 0$, not counting the empty sum, if we take all the possible choices of the a_i's. They prove (see Theorem 1 in [19]) that $A(n) = 2^{n-1}$ and they explain that Erdös, Ko and Rado investigated a question with an answer similar to this one: *what is the maximum number of pairwise intersecting subsets of an n-elements set?* As in their case, here also the question becomes more difficult if we restrict ourselves to the d-subsets. More details about this remark can be found in the famous theorem of Erdös–Ko–Rado [14] (see also [15] for an easy proof of it).

Formally, with the introduction of the positive integer d, the problem is the following. Let $1 \leq d < n$ be an integer; a function $f : I_n \to \mathbb{R}$ is called an n-weight function if $\sum_{x\in I_n} f(x) \geq 0$. Denote with $W_n(\mathbb{R})$ the set of all the n-weight functions and if $f \in W_n(\mathbb{R})$ we set

$$\begin{aligned} f^+ &= |\{x \in I_n : f(x) \geq 0\}|, \\ \alpha(f) &= \left|\left\{Y \subseteq I_n : \textstyle\sum_{y\in Y} f(y) \geq 0\right\}\right|, \\ \phi(f,d) &= \left|\left\{Y \subseteq I_n : |Y| = d, \textstyle\sum_{y\in Y} f(y) \geq 0\right\}\right|, \end{aligned}$$

and furthermore

$$\psi(n,d) = \min\{\phi(f,d) : f \in W_n(\mathbb{R})\}.$$

If f is such that $f(1) = n-1$, $f(2) = \cdots = f(n) = -1$, it follows that $\psi(n,d) \leq \binom{n-1}{d-1}$.

In [9], Bier and Manickam proved that $\psi(n,d) = \binom{n-1}{d-1}$ if $n \geq d(d-1)^d(d-2)^d + d^4$ and $\psi(n,d) = \binom{n-1}{d-1}$ if $d|n$. Both proofs use the Baranyai theorem on the factorization of complete hypergraphs [4] (see also [24] for a modern exposition of the theorem).

In [19] and [20] it was conjectured that $\psi(n,d) \geq \binom{n-1}{d-1}$ if $n \geq 4d$. In [20] this conjecture has been set in the more general context of association schemes (see [3] for general references on the subject). In the sequel we will refer to this conjecture as the Manickam–Miklös–Singhi (MMS) Conjecture. This conjecture is connected with the first distribution invariant of the Johnson association scheme (see [9], [20], [17], [18]). The distribution invariants were introduced by Bier [7], and later investigated in [8], [16], [17], [20]. In [20] the authors claim that this conjecture is, in some sense, dual to the theorem of Erdös–Ko–Rado [14]. Moreover, as pointed out in [22], this conjecture settles some cases of another conjecture on multiplicative functions by Alladi, Erdös and Vaaler, [2]. Partial results related to the Manickam–Miklös–Singhi conjecture have been obtained also in [6], [5], [11], [12], [21].

Now, if $1 \leq r \leq n$, we set:

$$\gamma(n,r) = \min\{\alpha(f) : f \in W_n(\mathbb{R}), f^+ = r\}, \tag{1}$$

$$\gamma(n,d,r) = \min\{\phi(f,d) : f \in W_n(\mathbb{R}), f^+ = r\}. \tag{2}$$

The numbers $\gamma(n, d, r)$ have been introduced in [11] and they have been studied also in [12], in order to solve the Manickam–Miklös–Singhi conjecture, because it is obvious that:

$$\psi(n, d) = \min\{\gamma(n, d, r) : 1 \leq r \leq n\}. \tag{3}$$

Therefore the complete computation of these numbers gives an answer to the MMS conjecture but this is not the purpose of this paper.

In [19] it has been proved that $\gamma(n, r) \geq 2^{n-1}$ for each r, and that $\gamma(n, 1) = 2^{n-1}$.

Question 0.1. Is it true that $\gamma(n, r) = 2^{n-1}$ for each r?

This is true if, for each r such that $1 \leq r \leq n$, we can find a function $f \in W_n(\mathbb{R})$ with $f^+ = r$ and $\alpha(f) = 2^{n-1}$.

When we have an n-weight function f, the standard ways to produce n-subsets on which f takes non-negative values are the following:

i) if X and Y are two subsets of I_n such that $\sum_{x \in X} f(x) \geq 0$ and $\sum_{x \in X} f(x) \leq \sum_{x \in Y} f(x)$, then also $\sum_{x \in Y} f(x) \geq 0$ (*monotone* property);
ii) if $\sum_{x \in X} f(x) < 0$, then $\sum_{x \in X^c} f(x) \geq 0$ (*complementary* property).

Then we ask:

A) Is it possible to axiomatize the properties (i) and (ii) in some type of abstract structure in such a way that the sum extremal problems described above become particular extremal problems of more general problems?
B) In such an abstract structure, can we find unexpected links with other theories which help us to solve these sum extremal problems?
C) Is it possible to define an algorithmic strategy in such an abstract structure to approach these sum extremal problems in a deterministic way?

In this paper we show that the answer to all the previous questions is affirmative.

We define a partial order $\sqsubseteq$ on the subsets of I_n such that if X and Y are two subsets with $X \sqsubseteq Y$, then $\sum_{x \in X} f(x) \leq \sum_{x \in Y} f(x)$, for each n-weight function f. In the first part of this paper, we study the fundamental properties of this order (see Section 1, 2, 3).

The attempt of computing the numbers in (1) and (2) for each n, d, r gives us the idea to construct two types of lattices, denoted by $S(n, r)$ and $S(n, d, r)$, and to transform the problem of computing the numbers $\gamma(n, r)$ and $\gamma(n, d, r)$ into the problem of computing a minimal cardinality on a family of posets.

This way of consider the problem has many advantages. For example, when we try to prove that $\gamma(n, r)$ is not greater than 2^{n-1} for each r, we need to build a particular n-weight function f with $f^+ = r$ such that $\alpha(f) \leq 2^{n-1}$. In general, one has to examine a certain number of inequalities, and if this number is big the determination of f can be difficult. The case of $\gamma(n, d, r)$ is similar and, obviously, more difficult. In general, if our aim is to prove that $\gamma(n, r) \leq T$ (or $\gamma(n, d, r) \leq T$), for some number T, it is natural to ask: is it possible to determine a minimal number of inequalities which allow us to find an n-weight function f with $f^+ = r$, such that $\alpha(f) \leq T$ (or $\phi(f, d) \leq T$)? If we identify (in some sense) each n-weight function with a particular type of boolean map defined on the lattice of the subsets of I_n, with the order $\sqsubseteq$, the number of these maps will be finite, and even if this number is large, the study of the properties of the lattice could lead to examine a more restricted class of these maps that lends itself to a simpler study.

Example 0.2. Let $n = 8$ and $r = d = 5$. Let f be the following 8-weight function with $f^+ = 5$:

$$\begin{array}{cccccccc} \tilde{5} & \tilde{4} & \tilde{3} & \tilde{2} & \tilde{1} & \overline{1} & \overline{2} & \overline{3} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{array} \tag{4}$$

We have written I_8 in the form $\{\tilde{5}, \tilde{4}, \tilde{3}, \tilde{2}, \tilde{1}, \overline{1}, \overline{2}, \overline{3}\}$, and in the following we write, for example, the 5-tuple $\tilde{5}\tilde{3}\tilde{1}\overline{2}\,\overline{3}$ as $531|23$. It follows easily that $\phi(f, 5) = \binom{5}{5} + 3\binom{5}{4} = 16$. Therefore, by (2) we have $\gamma(8, 5, 5) \leq 16$. To prove that also the inverse inequality holds, we fix an arbitrary 8-weight function f with $f^+ = 5$ and we prove that it has always at least 16 5-tuples on which it takes a non-negative value. Then, if we consider the 5-tuple $4321|3$, it is easy to see that its non-negativity implies the non-negativity of 16 5-tuples (including itself). This means that in the sublattice $S(8, 5, 5)$ of the lattice $S(8, 5)$ the element $4321|3$ spans an up-set having 16 elements. Then we say that the element $4321|3$ has *positive weight* 16. Therefore we can assume that f has a negative sum on $4321|3$. Since f is a weight-function, it takes a non-negative sum on the complementary 3-tuple $5|12$. It is easily seen then that the non-negativity of $5|12$ produces exactly the non-negativity of 15 other 5-tuples. Let us consider now the 5-tuple $4321|1$ (which is not included in the non-negative 5-tuples described above). If f takes a non-negative sum on $4321|1$, we have produced exactly 16 other 5-tuples with non-negative sum for f. If f takes a negative sum on $4321|1$, then it must take a non-negative sum on the complementary 3-tuple $5|23$, and this produces 16 other 5-tuples having non-negative sum and different from the previous 15 5-tuples; therefore we obtain in this case $(15+16 = 31)$ 5-tuples with non-negative sums. This shows that $\gamma(8, 5, 5) = 16$.

In the previous computation of the number $\gamma(8, 5, 5)$, the only properties that we have used are the monotone and the complementary properties. Then, to define an algorithmic procedure which holds for each value of n, d and r, we need an order structure which includes all the subsets of I_n, and not only those with d elements, since this set is not closed with respect to the complementary operation.

In this paper we focus on the numbers $\gamma(n, r)$ and we will approach the study of the $\gamma(n, d, r)$'s in subsequent papers. Here we build a formal context which gives sense to what is said above and also to the question raised in [19]: "What is the structure of the constructions giving this extremal value?". We show also that the problems described above can be considered as problems related to a particular class of boolean functions defined on our order structures. The properties of these boolean functions generalize the essential properties of the weight-functions, i.e. the order preserving and the complementary property.

In Section 7, we state two open problems, which are essentially two statements of representation theorems. If the answer to these problems will be affirmative, the problem of determining the numbers $\gamma(n, r)$ and $\gamma(n, d, r)$ will be equivalent to the problem of determining the minimum number of elements which have value 1 for a particular type of boolean functions. The advantage of this approach consists in the possibility to use the results of combinatorial lattice theory.

To conclude, we believe that the study of the extremal sum problems settled in [19] and in [20] (among which the Manickam–Miklös–Singhi Conjecture), in the setting of the lattices $S(n,r)$, is interesting because it can lead to unexpected links among the combinatorial theory of the lattices, the theory of association schemes (good references for the link between association schemes and the extremal problem on non-negative sums of real numbers are [9], [16], [17], [18], [20]), the transversal theory (see [12], in which the Hall theorem has been used for computing the $\gamma(n,d,r)$'s with $n=2d+2$ and $n-r=3$) and the theory of boolean functions defined on particular classes of posets. For example the lattice structure $S(n,r)$ could be useful in the computation of the higher order distribution invariants of the Johnson association scheme, [20], [9]: this will be the main object of a forthcoming investigation.

In this paper we adopt the classical terminology and notation usually used in the context of partially ordered sets (see [13] and [23] for the general aspects on this subject). In particular, if $(P,\leq)$ is a poset and $Q\subseteq P$, we set $\downarrow Q=\{y\in P \mid (\exists x\in Q)\, y\leq x\}$, $\uparrow Q=\{y\in P \mid (\exists x\in Q)\, y\geq x\}$, and $\downarrow\{x\}=\downarrow x$, $\uparrow\{x\}=\uparrow x$, for each $x\in P$. A subset Q of P is said to be a *down-set* (or *up-set*) of P if $Q=\downarrow Q$ (or $Q=\uparrow Q$).

1 The lattice $S(n,r)$ and its sublattice $S(n,d,r)$

Let n and r be two fixed integers such that $0\leq r\leq n$. We denote with $A(n,r)$ an alphabet composed by the following $(n+1)$ formal symbols: $\tilde{1},\ldots,\tilde{r},0^{\S},\overline{1},\ldots,\overline{n-r}$. We introduce on $A(n,r)$ the following total order:

$$\overline{n-r}\prec\cdots\prec\overline{2}\prec\overline{1}\prec 0^{\S}\prec\tilde{1}\prec\tilde{2}\prec\cdots\prec\tilde{r}, \tag{5}$$

where $\overline{n-r}$ is the minimal element and $\tilde{r}$ is the maximal element in this chain. If $i,j\in A(n,r)$, then we write $i\preceq j$ for $i=j$ or $i\prec j$; $i\curlywedge j$ for the minimum and $i\curlyvee j$ for the maximum between i and j with respect to $\preceq$; $i\vdash j$ if j covers i with respect to $\preceq$ (i.e. if $i\prec j$ and if there does not exist $l\in A(n,r)$ such that $i\prec l\prec j$); $i\nvdash j$ if j does not cover i with respect to $\preceq$; $j\succ i$ for $i\prec j$; $j\succeq i$ for $i\preceq j$.

Let $(\mathcal{C}(n,r),\sqsubseteq)$ be the n-fold cartesian product poset $A(n,r)^n$. An arbitrary element of $\mathcal{C}(n,r)$ can be identified with an n-string $t_1\cdots t_n$ where $t_i\in A(n,r)$ for all $i=1,\ldots,n$. Therefore, if $t_1\cdots t_n$ and $s_1\cdots s_n$ are two strings of $\mathcal{C}(n,r)$, we have

$$t_1\cdots t_n\sqsubseteq s_1\cdots s_n \quad\Longleftrightarrow\quad t_1\preceq s_1,\ldots,t_n\preceq s_n.$$

We introduce now a particular subset $S(n,r)$ of $\mathcal{C}(n,r)$. A string of $S(n,r)$ is constructed as follows: it is a formal expression of the following type

$$i_1\cdots i_r|j_1\cdots j_{n-r}, \tag{6}$$

where $i_1,\ldots,i_r\in\{\tilde{1},\ldots,\tilde{r},0^{\S}\}$, $j_1,\ldots,j_{n-r}\in\{\overline{1},\ldots,\overline{n-r},0^{\S}\}$ and where the choice of the symbols has to respect the following two rules, see (7) and (10):

$$i_1\succeq\cdots\succeq i_r\succeq 0^{\S}\succeq j_1\succeq\cdots\succeq j_{n-r}; \tag{7}$$

furthermore, if we set

$$p = \begin{cases} \max\{l : l \in \{1, \ldots, r\} \text{ with } i_l \succ 0^{\S}\} & \text{if } l \text{ exists} \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

and

$$q = \begin{cases} \min\{l : l \in \{1, \ldots, n-r\} \text{ with } j_l \prec 0^{\S}\} & \text{if } l \text{ exists} \\ n-r+1 & \text{otherwise} \end{cases} \tag{9}$$

then

$$\begin{aligned} &i_1 \succ \cdots \succ i_p \succ 0^{\S}, \qquad && i_{p+1} = \cdots = i_r = 0^{\S} \\ &j_1 = \cdots = j_{q-1} = 0^{\S}, && 0^{\S} \succ j_q \succ \cdots \succ j_{n-r}. \end{aligned} \tag{10}$$

If $p = 0$ we assume that $i_1 = \cdots = i_r = 0^{\S}$ and the condition $i_1 \succ \cdots \succ i_p \succ 0^{\S}$ is empty; if $q = (n-r+1)$, we assume that $j_1 = \cdots = j_{n-r} = 0^{\S}$ and the condition $0^{\S} \succ j_q \succ \cdots \succ j_{n-r}$ is empty. The formal symbols which appear in (6) will be written without $\tilde{}$, $\bar{}$, and §; the vertical bar $|$ in (6) will indicate that the symbols on the left of $|$ are in $\{\tilde{1}, \ldots, \tilde{r}, 0^{\S}\}$ and the symbols on the right of $|$ are in $\{0^{\S}, \overline{1}, \ldots, \overline{n-r}\}$.

Example 1.1. a) If $n = 3$ and $r = 2$, then $A(3,2) = \{\tilde{2} \succ \tilde{1} \succ 0^{\S} \succ \overline{1}\}$ and $S(3,2) = \{21|0, 21|1, 10|0, 20|0, 10|1, 20|1, 00|1, 00|0\}$.

b) If $n = 3$ and $r = 0$, then $A(3,0) = \{0^{\S} \succ \overline{1} \succ \overline{2} \succ \overline{3}\}$ and $S(3,0) = \{|123, |023, |013, |012, |003, |002, |001, |000\}$.

c) If $n = 0$ and $r = 0$, then $S(0,0)$ will be identified with a singleton Γ corresponding to $|$ without symbols.

In the sequel $S(n,r)$ will be considered as a sub-poset of $\mathcal{C}(n,r)$ with the induced order from $\sqsubseteq$ after the restriction to $S(n,r)$. Therefore, if $w = i_1 \cdots i_r | j_1 \cdots j_{n-r}$ and $w' = i_1^{'} \cdots i_r^{'} | j_1^{'} \cdots j_{n-r}^{'}$ are two strings in $S(n,r)$, by definition of induced order we have

$$w \sqsubseteq w' \iff i_1 \preceq i_1^{'}, \ldots, i_r \preceq i_r^{'}, \quad j_1 \preceq j_1^{'}, \ldots, j_{n-r} \preceq j_{n-r}^{'}.$$

As it is well known, $(\mathcal{C}(n,r), \sqsubseteq)$ is a distributive lattice whose binary operations of *inf* and *sup* are given respectively by $(t_1 \cdots t_n) \wedge (s_1 \cdots s_n) = (t_1 \curlywedge s_1) \cdots (t_n \curlywedge s_n)$, and $(t_1 \cdots t_n) \vee (s_1 \cdots s_n) = (t_1 \curlyvee s_1) \cdots (t_n \curlyvee s_n)$.

Example 1.2. If $n = 7$ and $r = 4$, and if $w_1 = 4310|023$ and $w_2 = 2100|012$ are two elements of $S(7,4)$, then $w_1 \wedge w_2 = 2100|023$, and $w_1 \vee w_2 = 4310|012$.

Proposition 1.3. *$(S(n,r), \sqsubseteq)$ is a distributive lattice.*

Proof. $(\mathcal{C}(n,r), \sqsubseteq)$ is a distributive lattice and $S(n,r)$ is closed with respect to $\wedge$ and $\vee$. Hence $S(n,r)$ is a distributive sublattice of $\mathcal{C}(n,r)$. □

Definition 1.4. If $w_1, w_2 \in S(n,r)$, then

i) $w_1 \sqsubset w_2$ if $w_1 \sqsubseteq w_2$ and $w_1 \neq w_2$;

ii) $w_1 \models w_2$ if w_2 covers w_1 with respect to the order $\sqsubseteq$ in $S(n,r)$ (i.e. if $w_1 \sqsubset w_2$ and there does not exist $w \in S(n,r)$ such that $w_1 \sqsubset w \sqsubset w_2$);
iii) $w_1 \not\models w_2$ if w_2 does not cover w_1 with respect to the order $\sqsubseteq$ in $S(n,r)$.

Remark 1.5. The minimal element of $S(n,r)$ is the string $0\cdots 0|12\cdots(n-r)$ and the maximal element is $r(r-1)\cdots 1|0\cdots 0$. Sometimes they are denoted respectively with $\hat{0}$ and $\hat{1}$.

If w is a string in $S(n,r)$ in the form (6) with p and q defined as in (8) and (9) (and (7) and (10) hold), we set:

$$w^* = \{i_1, \ldots, i_p, j_q, \ldots, j_{n-r}\}.$$

For example, if $w = 4310|013 \in S(7,4)$, then $w^* = \{\tilde{1}, \tilde{3}, \tilde{4}, \overline{1}, \overline{3}\}$. In particular, if $w = 0\cdots 0|0\cdots 0$ then $w^* = \emptyset$. We have defined a bijective map

$$* : w \in S(n,r) \mapsto w^* \in \mathcal{P}(A(n,r) \setminus \{0^{\S}\}).$$

Conversely, if $B \in \mathcal{P}(A(n,r) \setminus \{0^{\S}\})$, then $B = B_1 \cup B_2$ (with $B_1 \cap B_2 = \emptyset$) where $B_1 = \{i_1, \ldots, i_p\} \subseteq \{\tilde{1}, \ldots, \tilde{r}\}$ or $B_1 = \emptyset$ and $B_2 = \{j_q, \ldots, j_{n-r}\} \subseteq \{\overline{1}, \ldots, \overline{n-r}\}$ or $B_2 = \emptyset$, for some integer p and q such that $1 \leq p \leq r$, $1 \leq q \leq n-r$, with $i_1 \succ \cdots \succ i_p \succ 0^{\S} \succ j_q \cdots \succ j_{n-r}$. We will set

$$\overline{B} = \begin{cases} i_1 \cdots i_p.\, 0 \cdots 0|0 \cdots 0 j_q \cdots j_{n-r} & B_1 \neq \emptyset, B_2 \neq \emptyset \\ i_1 \cdots i_p.\, 0 \cdots 0|0 \cdots 00 \cdots 0 & B_1 \neq \emptyset, B_2 = \emptyset \\ 0 \cdots 00 \cdots 0|0 \cdots 0 j_q \cdots j_{n-r} & B_1 = \emptyset, B_2 \neq \emptyset \\ 0 \cdots 00 \cdots 0|0 \cdots 00 \cdots 0 & B_1 = \emptyset, B_2 = \emptyset. \end{cases}$$

We have defined a map:

$$^{-} : B \in \mathcal{P}(A(n,r) \setminus \{0^{\S}\}) \mapsto \overline{B} \in S(n,r),$$

which is the inverse of the previous map $*$. For example, if $B = \{\tilde{1}, \overline{1}\} \in \mathcal{P}(A(7,5) \setminus \{0^{\S}\})$, then $\overline{B} = 10000|01$ is the corresponding string in $S(7,5)$.

We define now the following operations on $S(n,r)$: if $w_1, w_2 \in S(n,r)$, we will set

i) $w_1 \sqcup w_2 = \overline{w_1^* \cup w_2^*}$;
ii) $w_1 \sqcap w_2 = \overline{w_1^* \cap w_2^*}$;
iii) $w_1^c = \overline{(w_1^*)^{\pi}}$;

where $(w_1^*)^{\pi}$ means the complement of w_1^* in $A(n,r) \setminus \{0^{\S}\}$.

For example, if $w_1 = 4310|001$ and $w_2 = 2000|012$ are two strings of $S(7,4)$, then $w_1 \sqcup w_2 = \overline{w_1^* \cup w_2^*} = \overline{\{\tilde{1}, \tilde{3}, \tilde{4}, \overline{1}\} \cup \{\tilde{2}, \overline{1}, \overline{2}\}} = \overline{\{\tilde{1}, \tilde{2}, \tilde{3}, \tilde{4}, \overline{1}, \overline{2}\}} = 4321|012$; $w_1 \sqcap w_2 = \overline{w_1^* \cap w_2^*} = \overline{\{\tilde{1}, \tilde{3}, \tilde{4}, \overline{1}\} \cap \{\tilde{2}, \overline{1}, \overline{2}\}} = \overline{\{\overline{1}\}} = 0000|001$; $w_1^c = \overline{(w_1^*)^{\pi}} = \overline{\{\tilde{1}, \tilde{3}, \tilde{4}, \overline{1}\}^{\pi}} = \overline{\{\tilde{2}, \overline{2}, \overline{3}\}} = 2000|023$, and $w_2^c = \overline{(w_2^*)^{\pi}} = \overline{\{\tilde{2}, \overline{1}, \overline{2}\}^{\pi}} = \overline{\{\tilde{1}, \tilde{3}, \tilde{4}, \overline{3}\}} = 4310|003$.

Remark 1.6. By the previous definitions, it is immediate to verify that $(w^c)^c = w$ for all $w \in S(n,r)$.

Now suppose that $1 \leq r \leq n$ and that d is a fixed integer such that $1 \leq d \leq n$. We denote with $S(n,d,r)$ the set of all the strings of $S(n,r)$ such that their form (6) contains exactly d symbols of the alphabet $A(n,r)$ different from $0^{\S}$.

Proposition 1.7. *$S(n,d,r)$ is a distributive sublattice of $S(n,r)$.*

Proof. It is sufficient to prove that, given $w_1, w_2 \in S(n,d,r)$, it holds that $w_1 \wedge w_2 \in S(n,d,r)$ and $w_1 \vee w_2 \in S(n,d,r)$. Let

$$\begin{aligned} w_1 &= i_1 \cdots i_k 0 \cdots 0 | 0 \cdots 0 i_q \cdots i_{n-r} \\ w_2 &= i_1^{'} \cdots i_p^{'} 0 \cdots 0 | 0 \cdots 0 i_s^{'} \cdots i_{n-r}^{'} \end{aligned}$$

with $i_1, \ldots, i_k$, k symbols different from $0^{\S}$ and $i_q, \ldots, i_{n-r}$, $(d-k)$ symbols different from $0^{\S}$; $i'_1 \cdots i'_p$, p symbols different from $0^{\S}$ and $i'_s, \ldots, i'_{n-r}$, $(d-p)$ symbols different from $0^{\S}$. If $k = p$, then $(d-k) = (d-p)$ and hence $w_1 \wedge w_2$ and $w_1 \vee w_2$ have exactly d symbols different from $0^{\S}$.

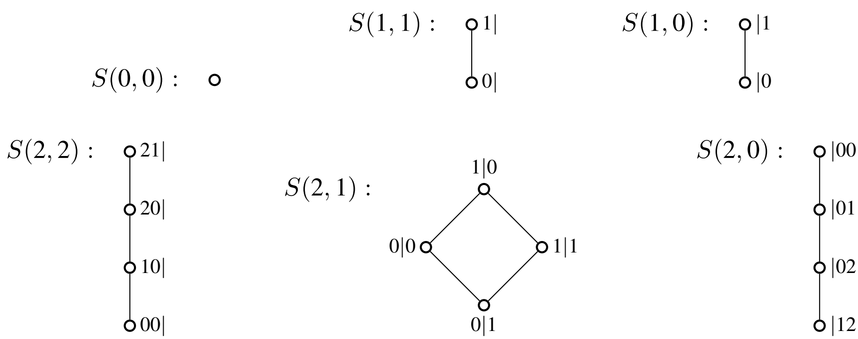
If $k > p$, then $w_1 \vee w_2$ has k symbols different from $0^{\S}$ on the left of $|$ and since $(d-k) < (d-p)$, $w_1 \vee w_2$ has $(d-k)$ symbols different from $0^{\S}$ on the right of $|$; hence $w_1 \vee w_2$ has exactly d symbols different from $0^{\S}$. On the other hand, $w_1 \wedge w_2$ has p symbols different from $0^{\S}$ on the left of $|$, and since $(d-k) < (d-p)$, $w_1 \wedge w_2$ has $(d-p)$ symbols different from $0^{\S}$ on the right of $|$; hence $w_1 \wedge w_2$ has exactly d symbols different from $0^{\S}$.

Analogously if $k < p$. □

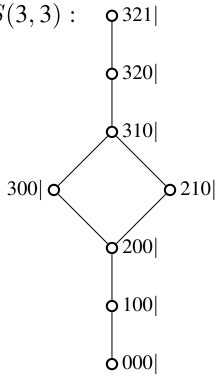
Remark 1.8. The map $*$ induces a bijection between the set of subsets with d elements of $A(n,r) \setminus \{0^{\S}\}$, denoted with $\mathcal{P}_d(A(n,r) \setminus \{0^{\S}\})$, and the distributive lattice $S(n,d,r)$.

2 Fundamental properties of the lattice $S(n,r)$

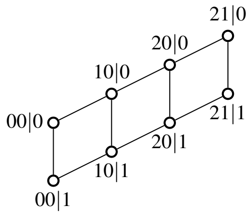
The Hasse diagrams of the lattices $S(n,r)$ for the first values of n and r are the following:



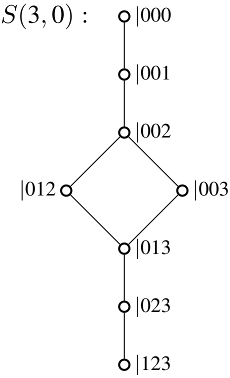
$S(3,3)$:



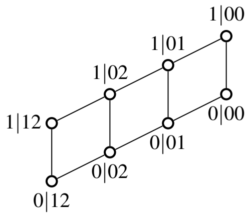
$S(3,2)$:



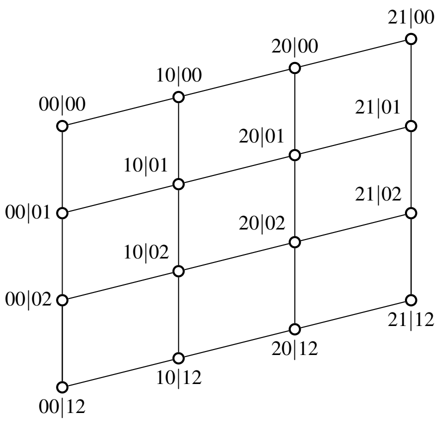
$S(3,0)$:



$S(3,1)$:



$S(4,2)$:



Proposition 2.1. *If $0 \leq r \leq n$, then $S(n,r) \cong S(r,r) \times S(n-r,0)$.*

Proof. Let $(w_P, w_N) \in S(r,r) \times S(n-r,0)$, with $w_P = i_1 \cdots i_r|$ and $w_N = |j_1 \cdots j_{n-r}$, where $i_1, \dots, i_r \in \{\tilde{1}, \tilde{2}, \cdots \tilde{r}, 0^{\S}\}$ and $j_1, \dots, j_{n-r} \in \{0^{\S}, \overline{1}, \overline{2}, \dots, \overline{n-r}\}$.

We set $\varphi(w_P, w_N) = i_1 \cdots i_r | j_1 \cdots j_{n-r}$. It is easy to verity that φ is an isomorphism between $S(r,r) \times S(n-r,0)$ and $S(n,r)$. $\square$

If we do not want to specify which elements of a string w are in $\{\tilde{1}, \dots, \tilde{r}, 0^{\S}\}$ and which are in $\{0^{\S}, \overline{1}, \dots, \overline{n-r}\}$, we simply write $w = l_1 \cdots l_n$, without specifying which l_i's are in $\{\tilde{1}, \dots, \tilde{r}, 0^{\S}\}$ and which are in $\{0^{\S}, \overline{1}, \dots, \overline{n-r}\}$. In any case, the order will be $l_1 \succeq l_2 \succeq \cdots \succeq l_n$.

If $l, q \in A(n,r)$, we will set

$$\delta(l,q) = \begin{cases} \emptyset & \text{if } l = q \\ (l,q) & \text{if } l \neq q. \end{cases}$$

If $w = l_1 \cdots l_n$ and $w' = l_1^{'} \cdots l_n^{'}$ are two strings in $S(n,r)$, we will set

$$\Delta(w,w') = (\delta(l_1, l_1^{'}), \dots, \delta(l_n, l_n^{'})).$$

Proposition 2.2. *Let $w = l_1 \cdots l_n$ and $w' = l_1^{'} \cdots l_n^{'}$ be two strings in $S(n,r)$. Then:*

$$w \models w' \quad \Longleftrightarrow$$
$$\Delta(w,w') = (\emptyset, \dots, \emptyset, (l_k, l_k'), \emptyset, \dots, \emptyset) \textit{ for some } k \in \{1, \dots, n\} \textit{ where } l_k \vdash l_k'.$$

Proof. $\Longrightarrow$ By contradiction, we distinguish three cases:

1) there exists a pair (l_k, l_k') different from $\emptyset$ in $\Delta(w,w')$ such that $l_k \nvdash l_k'$. Since by hypothesis $w \models w'$, we have $w \sqsubset w'$; therefore $l_k \prec l_k'$ and, for some $l \in A(n,r)$, we have $l_k \prec l \prec l_k'$.

Hence the string $w_l = l_1 \cdots l_{k-1} l l_{k+1} \cdots l_n$ is such that $w \sqsubset w_l \sqsubset w'$, against the hypothesis.

2) there exist at least two pairs (l_k, l_k'), (l_s, l_s') with $s > k$ different from $\emptyset$ in $\Delta(w,w')$, such that $l_k \vdash l_k'$ and $l_s \vdash l_s'$. Then, if we consider the string:

$$u = l_1 \cdots l_{k-1} l_k l_{k+1} \cdots l_{s-1} l_s^{'} l_{s+1} \cdots l_n,$$

it follows that $w \sqsubset u \sqsubset w'$, against the hypothesis.

3) all the components of $\Delta(w,w')$ are equal to $\emptyset$. In this case, by definition of $\Delta(w,w')$ we will have that $w = w'$, against the hypothesis.

$\Longleftarrow$ By hypothesis, $w \sqsubseteq w'$ because $\Delta(w,w') = (\emptyset, \dots, \emptyset, (l_k, l_k'), \emptyset, \dots, \emptyset)$ with $l_k \vdash l_k'$. Suppose that the assertion is false; then there exists a $w'' \in S(n,r)$ such that $w \sqsubset w'' \sqsubset w'$. Let $w'' = l_1^{''} \cdots l_n^{''}$. By hypothesis it follows that $l_i = l_i^{''} = l_i^{'}$ if $i \neq k$, and $l_k \prec l_k^{''} \prec l_k^{'}$, and hence $l_k \nvdash l_k^{'}$, against the hypothesis. $\square$

We define now the function $\rho : S(n,r) \to \mathbb{N}_0$ as follows: if $w = i_1 \cdots i_r | j_1 \cdots j_{n-r} \in S(n,r)$ and we consider the symbols $i_1, \ldots, i_r, j_1, \ldots, j_{n-r}$ as non-negative integers (without $\tilde{}$ and $\bar{}$), then we set:

$$\begin{aligned}\rho(w) &= i_1 + \cdots + i_r + |j_1 - 1| + \cdots + |j_{n-r} - (n-r)| \\ &= i_1 + \cdots + i_r + (1 - j_1) + \cdots + ((n-r) - j_{n-r}).\end{aligned}$$

Proposition 2.3. *The function ρ satisfies the following two properties:*

i) $\rho(\hat{0}) = 0$*;*
ii) *if $w, w' \in S(n,r)$ and $w \models w'$, then $\rho(w') = \rho(w) + 1$.*

Proof. i) Since $\hat{0} = 0 \cdots 0|12 \cdots (n-r)$, the assertion follows by the definition of ρ.
ii) If $(w = l_1 \cdots l_n) \models (w' = l_1^{'} \cdots l_n^{'})$, by Proposition 2.2 we have that $\Delta(w, w') = (\emptyset, \ldots, \emptyset, (l_t, l_t'), \emptyset, \cdots \emptyset)$, for some $t \in \{1, \ldots, n\}$, with $l_t \vdash l_t'$. We distinguish different cases:

1) Suppose that $1 \le t \le r$ and $l_t \succ 0^{\S}$. Then $w = l_1 \cdots l_{t-1} l_t l_{t+1} \cdots l_r | l_{r+1} \cdots l_n$ and $w' = l_1 \cdots l_{t-1}(l_t + 1) l_{t+1} \cdots l_r | l_{r+1} \cdots l_n$. Hence $\rho(w') = l_1 + \cdots + l_{t-1} + (l_t + 1) + l_{t+1} + \cdots + l_r + \sum_{k=1}^{n-r}(k - l_{r+k}) = \sum_{k=1}^{r} l_k + \sum_{k=1}^{n-r}(k - l_{r+k}) + 1 = \rho(w) + 1$.

2) Suppose that $1 \le t \le r$ and $l_t = 0^{\S}$. In this case we have that $l_t' = 1$, hence $w = l_1 \cdots l_{t-1} 00 \cdots 0 | l_{r+1} \cdots l_n$ and $w' = l_1 \cdots l_{t-1} 10 \cdots 0 | l_{r+1} \cdots l_n$, from which it follows that $\rho(w') = \rho(w) + 1$.

3) Suppose that $(r+1) \le t \le n$ and that $l_t = 0^{\S}$. In this case we have a contradiction because there does not exist an element l_t' in $\{0^{\S}, \overline{1}, \ldots, \overline{n-r}\}$ which covers $0^{\S}$.

4) Suppose that $(r+1) \le t \le n$ and that $l_t \prec 0^{\S}$; since we consider l_t as an integer, it means that $1 \le l_t \le (n-r)$. In this case we have $w = l_1 \cdots l_r | l_{r+1} \cdots l_{t-1} l_t l_{t+1} \cdots l_n$ and $w' = l_1 \cdots l_r | l_{r+1} \cdots l_{t-1}(l_t - 1) l_{t+1} \cdots l_n$, and therefore $\rho(w') = \sum_{i=1}^{r} l_i + \sum_{k=1, k \ne t-r}^{n-r}(k - l_{r+k}) + [(t-r) - (l_t - 1)] = \sum_{i=1}^{r} l_i + \sum_{k=1}^{n-r}(k - l_{r+k}) + 1 = \rho(w) + 1$. $\square$

Proposition 2.4. *$S(n,r)$ is a graded lattice having rank $R(n,r) = \binom{r+1}{2} + \binom{n-r+1}{2}$ and its rank function coincides with ρ.*

Proof. A finite distributive lattice is also graded (see [23]), therefore $S(n,r)$ is graded by Proposition 1.3. In order to calculate the rank of $S(n,r)$, we need to determine a maximal chain and its length. We consider the following chain C in $S(n,r)$:

$\hat{1} = r(r-1) \cdots 21|00 \cdots 0$ 1 string with r elements different from 0 on the left of |

$$\vdots$$

$$\begin{cases} r(r-1)(r-2)0 \cdots 0|00 \cdots 0 & (r-2) \text{ strings with 3 elem. diff. from 0 on the left of } | \\ \vdots & \\ r(r-1)10 \cdots 0|00 \cdots 0 & \end{cases}$$

$$\begin{cases} r(r-1)0\cdots 0|00\cdots 0 & (r-1) \text{ strings with 2 elem. diff. from 0 on the left of } | \\ \vdots & \\ r10\cdots 0|00\cdots 0 & \end{cases}$$

$$\begin{cases} r0\cdots 0|00\cdots 0 & r \text{ strings with 1 elem. diff. from 0 on the left of } | \\ \vdots & \\ 10\cdots 0|00\cdots 0 & \end{cases}$$

$$00\cdots 0|00\cdots 0$$

$$\begin{cases} 00\cdots 0|00\cdots 1 & (n-r) \text{ strings with 1 elem. diff. from 0 on the right of } | \\ \vdots & \\ 00\cdots 0|00\cdots (n-r) & \end{cases}$$

$$\vdots$$

$$\begin{cases} 00\cdots 0|001(4\cdots (n-r)) & 3 \text{ strings with } (n-r-2) \text{ elements different} \\ 00\cdots 0|002(4\cdots (n-r)) & \text{from 0 on the right of } | \\ 00\cdots 0|003(4\cdots (n-r)) & \end{cases}$$

$$\begin{cases} 00\cdots 0|01(34\cdots (n-r)) & 2 \text{ strings with } (n-r-1) \text{ elements different} \\ 00\cdots 0|02(34\cdots (n-r)) & \text{from 0 on the right of } | \end{cases}$$

$$\hat{0} = 00\cdots 00|(12\cdots (n-r)) \quad 1 \text{ string with } (n-r) \text{ elem. diff. from 0 on the right of } |$$

Therefore C has length $(1+2+\cdots+(r-1)+r)+1+(1+2+\cdots+(n-r)) = \frac{r(r+1)}{2}+\frac{(n-r+1)(n-r)}{2}+1 = \binom{r+1}{2}+\binom{n-r+1}{2}+1$. By Proposition 2.2, each element of the chain covers the previous one with respect to the order $\sqsubseteq$ in $S(n,r)$. Furthermore, C has minimal element $\hat{0}$ (the minimum of $S(n,r)$) and maximal element $\hat{1}$ (the maximum of $S(n,r)$), hence C is a maximal chain in $S(n,r)$. Since C has $R(n,r)+1$ elements and $S(n,r)$ is graded, it follows that $S(n,r)$ has rank $R(n,r)$. Finally, since $S(n,r)$ is a graded lattice of rank $R(n,r)$ and it has $\hat{0}$ as minimal element, its rank function has to be the unique function defined on $S(n,r)$ and with values in $\{0,1,\ldots,R(n,r)\}$ which satisfies i) and ii) of Proposition 2.3 (see [23]). Hence such a function coincides with ρ, by the uniqueness property. □

The following proposition shows that w and w^c are symmetric in the Hasse diagram of $S(n,r)$.

Proposition 2.5. *If $w \in S(n,r)$, then $\rho(w)+\rho(w^c) = R(n,r)$.*

Proof. Let $w = i_1\cdots i_r|j_1\cdots j_{n-r}$ and $w^c = i_1^{'}\cdots i_r^{'}|j_1^{'}\cdots j_{n-r}^{'}$. Then $\rho(w)+\rho(w^c) = \sum_{k=1}^{r} i_k + \sum_{k=1}^{n-r}(k-j_k) + \sum_{k=1}^{r} i_k^{'} + \sum_{k=1}^{n-r}(k-j_k^{'})$. By definition of w^c in $S(n,r)$, it follows that $\sum_{k=1}^{r} i_k + \sum_{k=1}^{r} i_k^{'} = \sum_{k=1}^{r} k$ and $\sum_{k=1}^{n-r} j_k + \sum_{k=1}^{n-r} j_k^{'} = \sum_{k=1}^{n-r} k$.

Hence $\rho(w)+\rho(w^c)=\sum_{k=1}^{r}k+2\sum_{k=1}^{n-r}k-\sum_{k=1}^{n-r}k=\sum_{k=1}^{r}k+\sum_{k=1}^{n-r}k=\binom{r+1}{2}+\binom{n-r+1}{2}=R(n,r)$. $\square$

3 The order reversing property of $\sqsubseteq$

In general, $(S(n,r),\sqsubseteq,\hat{0},\hat{1})$ is not a boolean lattice. If $w=54210|012\in S(8,5)$, for example, it is easy to verify that there does not exist an element $w'\in S(8,5)$ such that $w\wedge w'=\hat{0}$ and $w\vee w'=\hat{1}$. In this section we will prove that the function $w\in S(n,r)\mapsto w^c\in S(n,r)$ is *order reversing* with respect to the order $\sqsubseteq$, in the sense that if $w_1\sqsubseteq w_2$, then $w_2^c\sqsubseteq w_1^c$. In the sequel, we will see that this property is fundamental to prove many results of this paper.

Proposition 3.1. *Let $w,w'\in S(n,r)$ be such that $w'\models w$. Then $w^c\models(w')^c$.*

Proof. Case 1) Let w and w' be distinct in the following way:

$$\begin{aligned}w&=i_1\cdots i_{r-s-1}10\cdots0|\cdots\\ w'&=i_1\cdots i_{r-s-1}00\cdots0|\cdots,\end{aligned}$$

where $i_1\succ\cdots\succ i_{r-s-1}\succ\tilde{1}$. Consider now $(w^*)^\pi$ and $((w')^*)^\pi$: they are two elements of $\mathcal{P}(A(n,r)\setminus\{0^\S\})$. In $(w^*)^\pi$ there are (s) elements of $A(n,r)\succ 0^\S$ and in $((w')^*)^\pi$ there are $(s+1)$ elements of $A(n,r)\succ 0^\S$. Furthermore, $\tilde{1}\in((w')^*)^\pi$ and $\tilde{1}\notin(w^*)^\pi$, hence the symmetric difference between $(w^*)^\pi$ and $((w')^*)^\pi$ is equal to $\{\tilde{1}\}$. From this, it follows that:

$$\begin{aligned}(w')^c&=\overline{((w')^*)^\pi}=t_1\cdots t_s10\cdots0|\cdots\\ w^c&=\overline{(w^*)^\pi}=t_1\cdots t_s00\cdots0|\cdots,\end{aligned}$$

where $\{t_1,\ldots,t_s,\tilde{1}\}=\{i_1,\ldots,i_{r-s-1},\overline{1},\ldots,\overline{n-r}\}^c$ in $A(n,r)\setminus\{0^\S\}$ and $t_1\succ\cdots\succ t_s\succ\tilde{1}$. By Proposition 2.2, it follows that $(w')^c$ covers w^c.

Case 2) Adaptation of Case 1) to the elements on the right of $|$.

Case 3) Let k be the index in which w and w' are distinct, $1\le k\le r$, then:

$$\begin{aligned}w&=i_1\cdots i_{k-1}(i_k+1)i_{k+1}\cdots i_p.\,0\cdots0|\cdots\\ w'&=i_1\cdots i_{k-1}i_ki_{k+1}\cdots i_p.\,0\cdots0|\cdots,\end{aligned}$$

with $i_1\succ\cdots\succ i_{k-1}\succ i_k+1\succ i_k\succ i_{k+1}\succ\cdots\succ i_p\succ 0^\S$.

Then, in $(w^*)^\pi$ there are exactly $q=(r-p)$ elements $\succ 0^\S$ and in $((w')^*)^\pi$ there are also $q=(r-p)$ elements $\succ 0^\S$. Moreover, it follows that $i_k\in(w^*)^\pi\setminus((w')^*)^\pi$ and $i_{k+1}\in((w')^*)^\pi\setminus(w^*)^\pi$, hence the symmetric difference between $(w^*)^\pi$ and $((w')^*)^\pi$ is equal to $\{i_k,i_{k+1}\}$. From this, it follows that:

$$\begin{aligned}(w')^c&=\overline{((w')^*)^\pi}:t_1\cdots t_{l-1}(i_k+1)t_{l+1}\cdots t_q0\cdots0|\cdots\\ w^c&=\overline{(w^*)^\pi}:t_1\cdots t_{m-1}i_kt_{m+1}\cdots t_q0\cdots0|\cdots,\end{aligned}$$

where $i_k + 1$ appears in the l-th place ($1 \leq l \leq r$) in $(w')^c$ and i_k appears in the m-th place ($1 \leq m \leq r$) in w^c, with

$$\{t_1, \ldots, t_{l-1}, t_{l+1}, \ldots, t_q\} = \{t_1, \ldots, t_{m-1}, t_{m+1}, \ldots, t_q\}, \tag{11}$$

where $\{t_1, \ldots, t_q\} = \{i_1, \ldots, i_k, i_k + 1, i_{k+1}, \ldots, i_p, \overline{1}, \ldots, \overline{n-r}\}^\pi$ in $A(n,r) \setminus \{0^\S\}$. We prove now that the l-th place coincides with the m-th place. Let $t \in \{t_{l+1}, \ldots, t_q\}$ and suppose by contradiction that $t \notin \{t_{m+1}, \ldots, t_q\}$. By (11) it follows that $t \in \{t_1, \ldots, t_{m-1}\}$, hence we will have $i_k + 1 \succ t$ and $t \succ i_k$, and hence $i_k + 1 \succ t \succ i_k$ in $A(n,r)$ and this contradicts $i_k \vdash (i_k + 1)$.

Let now $t \in \{t_1, \ldots, t_{m-1}\}$. Suppose by contradiction that $t \notin \{t_1, \ldots, t_{l-1}\}$. By (11) it follows that $t \in \{t_{l+1}, \ldots, t_q\}$, hence we will have $t \succ i_k$ and $i_k + 1 \succ t$, by which $i_k + 1 \succ t \succ i_k$, and this contradicts $i_k \vdash i_k + 1$. By (11) hence follows that $m = l$, and this proves that $w^c \models (w')^c$.

Case 4) Analogously to Case 3) with k such that $r + 1 \leq k \leq n$. □

Proposition 3.2. *If $w, w' \in S(n,r)$ are such that $w' \sqsubseteq w$, then $w^c \sqsubseteq (w')^c$.*

Proof. It is enough to consider a sequence of elements $w_0, w_1, \ldots, w_n$ such that $w' = w_0 \sqsubseteq w_1 \sqsubseteq \cdots \sqsubseteq w_{n-1} \sqsubseteq w_n = w$ where w_i covers w_{i-1} for $i = 1, \ldots, n$ and apply Proposition 3.1 to $w_{i-1} \models w_i$. □

A poset $\mathbb{P} = (P, \leq)$ is called an *involution poset* if there exists a map $\,' : P \to P$ such that (i) $(x')' = x$ and (ii) if $x \leq y$, then $y' \leq x'$ for all $x, y \in P$. Recent studies related to this particular class of posets can be found in [1] and in [10]. Hence by Proposition 3.2 and Remark 1.6, $(S(n,r), \sqsubseteq, ^c, \hat{0}, \hat{1})$ is an involution poset and a distributive bounded lattice. If $w = i_1 \cdots i_r | j_1 \cdots j_{n-r}$ is an element of $S(n,r)$, with $0 \leq r \leq n$, we can also consider the symbols $i_1, \cdots i_r, j_1, \ldots, j_{n-r}$ as elements in the alphabet $A(n, n-r)$, where $j_1, \ldots, j_{n-r} \in \{\widetilde{n-r} \succ \cdots \succ \tilde{1} \succ 0^\S\}$ and $i_1, \ldots, i_r \in \{0^\S \succ \overline{1} \succ \cdots \succ \overline{r}\}$; in such a case we will set $w^t = j_{n-r} \cdots j_1 | i_r \cdots i_1$. Then the map $w \in S(n,r) \mapsto w^t \in S(n, n-r)$ is bijective and it is such that

$$w \sqsubseteq w' \text{ in } S(n,r) \quad \Longleftrightarrow \quad (w')^t \sqsubseteq w^t \text{ in } S(n, n-r), \tag{12}$$

since $(w^t)^t = w$, for each $w \in S(n,r)$. Also the map $w \in S(n,r) \mapsto w^c \in S(n,r)$ is bijective, and since $(w^c)^c = w$, by Proposition 3.2, it follows that

$$w \sqsubseteq w' \text{ in } S(n,r) \quad \Longleftrightarrow \quad (w')^c \sqsubseteq w^c \text{ in } S(n,r). \tag{13}$$

Therefore we have the following isomorphism of lattices:

Proposition 3.3. *If $0 \leq r \leq n$, then $S(n,r) \cong S(n, n-r)$.*

Proof. It is enough to consider the map $\varphi : S(n,r) \to S(n, n-r)$ defined by $\varphi(w) = (w^t)^c$. Since the map φ is the composition of the map $w \in S(n,r) \mapsto w^t \in S(n, n-r)$

with the map $u \in S(n, n-r) \mapsto u^c \in S(n, n-r)$, it follows that φ is bijective. Furthermore, by (12) and (13), it holds that

$$w \sqsubseteq w' \text{ in } S(n,r) \iff \varphi(w) \sqsubseteq \varphi(w') \text{ in } S(n, n-r).$$

Hence φ is an isomorphism of lattices. □

Example 3.4. $S(3,1) \cong S(3,2)$ and for example $\varphi(0|01) = ((0|01)^t)^c = (10|0)^c = (20|1)$, or $\varphi(1|02) = ((1|02)^t)^c = (20|1)^c = (10|0)$, see the Hasse diagrams of Section 2.

4 An algorithmic method for generating $S(n, r)$

In this section we describe a generating algorithm for $S(n,r)$, which will permit us to fix an order, from the left to the right on each subset of the lattice composed by elements with fixed rank. In the Hasse diagram we will provide an algorithm for giving, on each line, a total order from the left to the right.

Set $w = l_1 \cdots l_n \in S(n,r)$ and let $k \in \{1, \ldots, n\}$ be fixed. If there exists an element $l_k' \in A(n,r)$ which covers l_k with respect to the order $\succ$ and such that $(l_1, \ldots, l_{k-1}, l_k', l_{k+1}, \ldots, l_n) \in S(n,r)$, we will say that k is a *generating index* for the string w. If k is a generating index for w and if l_k' is an element of $A(n,r)$ which covers l_k, the string $(l_1, \ldots, l_{k-1}, l_k', l_{k+1}, \ldots, l_n)$ will be called *string of index k generated by* w and it will be denoted with the symbol $w[k]$. If k is a generating index of w contained in $\{1, \ldots, r\}$, we will say that k is a *positive generating index* of w; if k is a generating index of w contained in $\{r+1, \ldots, n\}$, we will say that k is a *negative generating index* of w. Let now $s_1, \ldots, s_p$ be the positive generating indexes of w (if $p = 0$ there are no positive generating indexes of w) and $t_1, \ldots, t_q$ the negative generating indexes of w (if $q = 0$ there are no negative generating indexes of w), with $s_1 < \cdots < s_p < t_1 < \cdots < t_q$.

On the set $\{w[s_1], \ldots, w[s_p], w[t_1], \ldots, w[t_q]\}$ we introduce the following formal order $\lessdot$:

$$w[s_1] \lessdot w[s_2] \lessdot \cdots \lessdot w[s_p] \lessdot w[t_q] \lessdot \cdots \lessdot w[t_1] \tag{14}$$

In the Hasse diagram of $S(n,r)$ we will write the string generated by w following the order given in (14): $w[s_1]$ on the left of $w[s_2], \ldots, w[s_p]$ on the left of $w[t_q], \cdots, w[t_2]$ on the left of $w[t_1]$.

Example 4.1. If $n = 9$, $r = 5$ and $w = 52000|0024$, the positive generating indexes of w are 2 and 3, while the negative generating indexes are 8 and 9, therefore, by (14), we write $w[2] = 53000|0024 \lessdot w[3] = 52100|0024 \lessdot w[9] = 52000|0023 \lessdot w[8] = 52000|0014$.

Let now $k \in \{0, 1, \ldots, R(n,r)\}$ be fixed. Denote by $S_k(n,r)$ the set of elements of $S(n,r)$ with constant rank k. We want to define a total order $\leftharpoondown$ on $S_k(n,r)$. If $k = 0$ there is nothing to say because there is a unique element of rank 0. If $k = 1$, $S_1(n,r)$ coincides with the set of strings generated by $0 \cdots 0|12 \cdots (n-r)$ and, in this case, $\leftharpoondown$ will coincide with the order $\lessdot$ given in (14).

Let now k be an integer such that $1 \leq k < R(n,r)$ and suppose we have ordered with the total order $\leftharpoondown$ all the strings of $S_k(n,r)$. Suppose that $S_k(n,r) = \{w_1, \ldots, w_m\}$ and that $w_1 \leftharpoondown w_2 \leftharpoondown \cdots \leftharpoondown w_m$ (in the Hasse diagram of $S(n,r)$ this implies that $w_1, \ldots, w_m$ are written from the left to the right).

Let $w_i^1, \ldots, w_i^{k_i}$ be the strings of $S(n,r)$ generated by w_i, for $i = 1, \ldots, m$. By (14), we can suppose that

$$w_1^1 \lessdot \cdots \lessdot w_1^{k_1}, \ldots, w_m^1 \lessdot \cdots \lessdot w_m^{k_m}.$$

We construct now $\leftharpoondown$ as follows: at first we set

$$w_1^1 \leftharpoondown \cdots \leftharpoondown w_1^{k_1} \leftharpoondown w_2^1 \leftharpoondown \cdots \leftharpoondown w_2^{k_2} \leftharpoondown \cdots \leftharpoondown w_m^1 \leftharpoondown \cdots \leftharpoondown w_m^{k_m}. \tag{15}$$

Then we have to eliminate in (15) all the repeated strings.

We examine the sequence (15) starting from w_1^1 and continuing to the right until $w_m^{k_m}$. For $i = 1, \ldots, m$ if the string $w_i^{k_i}$ already appears among the strings on its left, then it will be deleted from the list (15), otherwise it stays. At the end of the process there remain the strings of $S_{k+1}(n,r)$, each one appearing only one time in the list (15).

We have chosen to order the strings generated by w as in (14) because this choice gives great emphasis to the partition of $S(n,r)$ into two sublattices that we will describe in the next section; however we can also choose a different order with respect to (14), in fact, in some cases it is more useful to consider the following order on the subset of strings generated by w:

$$w[s_1] \leqslant w[s_2] \leqslant \cdots \leqslant w[s_p] \leqslant w[t_1] \leqslant \cdots \leqslant w[t_q]. \tag{16}$$

In any case, no matter what is the chosen order, (14) or (16), for the subset of the strings generated by w, the previous algorithm stays unchanged in all the other aspects. We will say that the previous generating algorithm for $S(n,r)$ is of type $\leftrightarrows$ if it is based on the order (14), and of type $\rightrightarrows$ if it is based on the order (16). In this paper we use the generative algorithm $\leftrightarrows$.

Example 4.2. Let $n = 6$, $r = 3$ and $k = 3$. Then $S_0(6,3) = \{000|123\}$. The generating indexes of $000|123$ are 1 and 4, therefore, by (14),

$$000|123[1] = 100|123 \lessdot 000|123[4] = 000|023;$$

hence

$$S_1(6,3) = \{100|123 \leftharpoondown 000|023\}.$$

The generating indexes of $100|123$ are 1 and 4, therefore, as above,

$$100|123[1] = 200|123 \lessdot 100|123[4] = 100|023,$$

the generating indexes of $000|023$ are 1 and 5:

$$000|023[1] = 100|023 \lessdot 000|123[5] = 000|013;$$

hence (after having deleted the repeated strings)

$$S_2(6,3) = \{200|123 \leftharpoondown 100|023 \leftharpoondown 000|013\}.$$

The generating indexes of 200|123 are 1, 2 and 4:

$$200|123[1] = 300|123 \lessdot 200|123[2] = 210|023 \lessdot 200|123[4] = 200|023,$$

the generating indexes of 100|023 are 1 and 5:

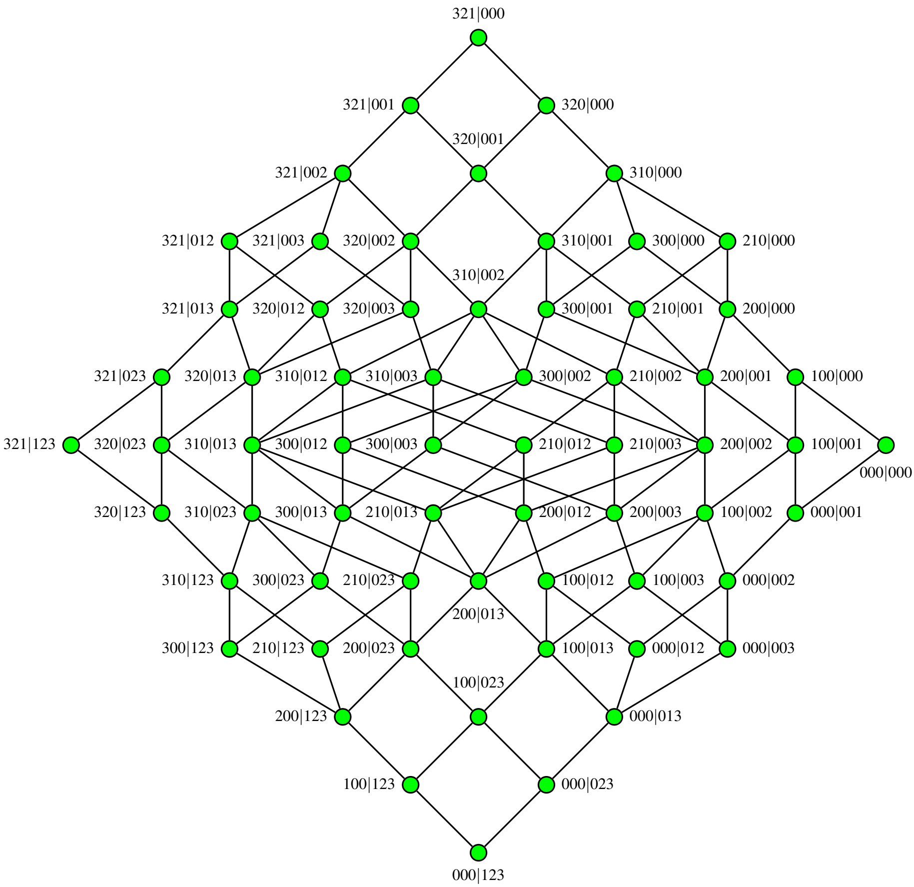
$$100|023[1] = 200|023 \lessdot 100|023[5] = 100|013,$$

the generating indexes of 000|013 are 1, 5 and 6:

$$000|013[1] = 100|013 \lessdot 000|013[6] = 000|012 \lessdot 000|013[5] = 000|003;$$

hence (after having deleted the repeated strings)

$$S_3(6,3) = \{300|123 \leftharpoondown 210|123 \leftharpoondown 200|023 \leftharpoondown 100|013 \leftharpoondown 000|012 \leftharpoondown 000|003\}.$$



In the figure above we have drawn the complete Hasse diagram of the lattice $S(6,3)$ in which each horizontal line represents the sub-poset $S_k(6,3)$ of the elements of rank k, with $0 \le k \le 12$, written in a totally ordered way from the left to the right following the total order $\leftharpoondown$ previously described.

5 A recursive formula for the number of elements in $S(n,r)$ of rank k

In this section we give a recursive formula which counts the number of elements in $S(n,r)$ having fixed rank. At first we show that $S(n,r)$ can be seen as a translate union of two copies of $S(n-1,r)$ if $0 \le r < n$ and of $S(n-1,n-1)$ if $r = n$.

Proposition 5.1. *Let $n \ge 1$ and $r \in \mathbb{N}$ such that $0 \le r \le n$. Then there exist two disjoint sublattices $S_1(n,r)$, $S_2(n,r)$ of $S(n,r)$ such that $S(n,r) = S_1(n,r) \cup S_2(n,r)$, where:*
- i) *$S_i(n,r) \cong S(n-1,r)$ for $i = 1,2$, if $0 \le r < n$;*
- ii) *$S_i(n,n) \cong S(n-1,n-1)$ for $i = 1,2$, if $r = n$.*

Proof. We distinguish two cases:

i) $0 \le r < n$; we denote by $S_1(n,r)$ the subset of $S(n,r)$ of all the strings of the form $w = i_1 \cdots i_r | j_1 \cdots j_{n-1-r}(n-r)$, with $j_1 \cdots j_{n-1-r} \in \{0^{\S}, \overline{1}, \dots, \overline{n-r-1}\}$; moreover, we denote by $S_2(n,r)$ the subset of $S(n,r)$ of all the strings of the form $w = i_1 \cdots i_r | 0 j_2 \cdots j_{n-r}$, with $j_2 \cdots j_{n-r} \in \{0^{\S}, \overline{1}, \dots, \overline{n-r-1}\}$.
It is clear that $S(n,r)$ is a disjoint union of $S_1(n,r)$ and $S_2(n,r)$. We prove now that $S_i(n,r) \cong S(n-1,r)$ for $i = 1,2$. Let $i = 1$ (the case $i = 2$ is analogous). It is obvious that there exists a bijective correspondence between $S_1(n,r)$ and $S(n-1,r)$. Furthermore, if $w, w' \in S_1(n,r)$ are such that $w = i_1 \cdots i_r | j_1, \cdots j_{n-r-1}(n-r)$, $w^{'} = i_1^{'} \cdots i_r^{'} | j_1^{'} \cdots j_{n-r-1}^{'}(n-r)$, it follows that $w \sqsubseteq w'$ (with respect to the order on $S(n,r)$) if and only if $i_1 \cdots i_r | j_1 \cdots j_{n-r-1} \sqsubseteq i_1^{'} \cdots i_r^{'} | j_1^{'} \cdots j_{n-r-1}^{'}$ (with respect to the order in $S(n-1,r)$). Hence $S_1(n,r)$ is isomorphic to $S((n-1),r)$.
Finally, since the order on $S(n,r)$ is component by component, it follows that each $S_i(n,r)$ (for $i = 1,2$) is a sublattice of $S(n,r)$.

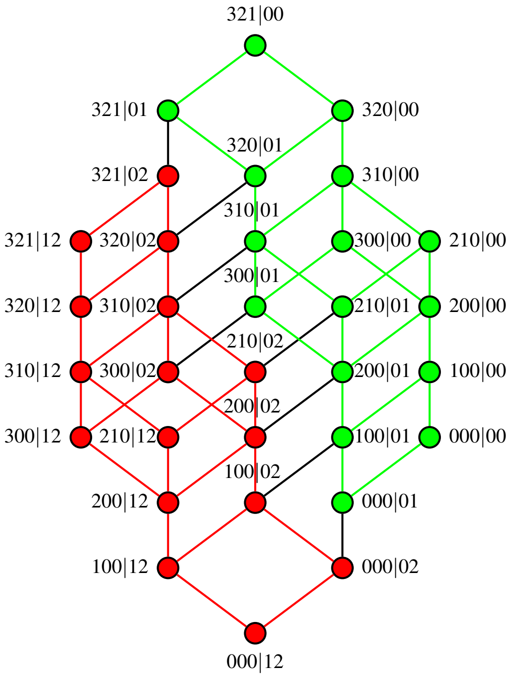
ii) $r = n$; by i), there exist two disjoint sublattices $S_1(n,0)$, $S_2(n,0)$, of $S(n,0)$ such that $S(n,0) = S_1(n,0) \cup S_2(n,0)$, with $S_i(n,0) \cong S_i(n-1,0)$, for $i = 1,2$. By Proposition 3.3, it follows that $S(n,n) \cong S(n,0)$, therefore there also exist two disjoint sublattices $S_1(n,n)$, $S_2(n,n)$, of $S(n,n)$ such that $S(n,n) = S_1(n,n) \cup S_2(n,n)$, where $S_i(n,n) \cong S_i(n,0) \cong S(n-1,0) \cong S(n-1,n-1)$, for $i = 1,2$, again by Proposition 3.3. □

If $n \ge 1$, the element of minimal rank of the sublattice $S_2(n,r)$ is obviously $\hat{w} = 0 \cdots 0 | 012 \cdots (n-r-1)$. This element has rank 0 as an element of $(S_2(n,r), \sqsubseteq)$, but in $S(n,r)$ it has rank given by

$$\rho(\hat{w}) = (1-0) + (2-1) + (3-2) + \cdots + ((n-r) - (n-r-1)) = n-r.$$

Therefore we can visualize $S_2(n,r)$ (in the Hasse diagram of $S(n,r)$) as an upper-translation of the sublattice $S_1(n,r)$, of height $(n-r)$.

Example 5.2. For example, this is the Hasse diagram of $S(5,3)$ as a translate union of $S_1(5,3) \cong S(4,3)$ (red lattice) and of $S_2(5,3) \cong S(4,3)$ (green lattice).



Given the lattice $S(n,r)$, for each k with $0 \leq k \leq R(n,r)$, we denote with $s(n,r,k)$ the number of elements of $S(n,r)$ with rank k. The following recursive formula holds for $s(n,r,k)$:

Proposition 5.3. *Let $n \geq 1$. If $r \in \mathbb{N}$ is such that $0 \leq r < n$, then*

$$s(n,r,k) = \begin{cases} s(n-1,r,k) & \text{if } 0 \leq k < (n-r) \\ s(n-1,r,k) + s(n-1,r,k-(n-r)) & \text{if } (n-r) \leq k \leq R(n-1,r) \\ s(n-1,r,k-(n-r)) & \text{if } R(n-1,r) < k \leq R(n,r) \end{cases}$$

If $r = n$, then $s(n,n,k) = s(n,0,k)$.

Proof. Case 1) Let k be such that $0 \leq k < (n-r)$. By what we have asserted before, the element $\hat{w}$ (i.e. the minimum of $S_2(n,r)$) has rank $(n-r)$ in $S(n,r)$, hence, by Proposition 5.1, it follows that $s(n,r,k)$ coincides with the number of elements of rank k in $S_1(n,r)$ and since $S_1(n,r) \cong S(n-1,r)$, it follows that $s(n,r,k) = s(n-1,r,k)$.

Case 2) Let k be such that $(n-r) \leq k \leq R(n-1,r)$. In this case, the number of elements of rank k in $S(n,r)$ coincides with the sum of the number of elements of rank k in $S_1(n,r)$ and of the number of elements of rank $k-(n-r)$ in $S_2(n,r)$. Since $S_1(n,r) \cong S_2(n,r) \cong S(n-1,r)$, it follows that $s(n,r,k) = s(n-1,r,k) + s(n-1,r,k-(n-r))$.

Case 3) Let k be such that $R(n-1,r) < k \leq R(n,r)$. In this case $s(n,r,k)$ coincides with the number of elements of rank $k-(n-r)$ in $S_2(n,r)$, and since $S_2(n,r) \cong S(n-1,r)$ it follows that $s(n,r,k) = s(n-1,r,k-(n-r))$.

If $r = n$, the last equality follows from the isomorphism $S(n,n) \cong S(n,0)$. □

It is clear that we would prefer a closed formula for the numbers $s(n,r,k)$, however at present the previous recursive formula is the best result that we have. By Proposition 5.3, the first values of $s(n,r,k)$ are given by:

$$\begin{aligned}
&s(0,0,0) = 1,\\
&s(1,0,0) = s(0,0,0) = 1, \quad s(1,0,1) = s(0,0,0) = 1,\\
&s(1,1,0) = s(1,0,0) = 1, \quad s(1,1,1) = s(1,0,1) = 1,\\
&s(2,0,0) = 1, \quad s(2,0,1) = s(1,0,1) = 1, \quad s(2,0,2) = s(1,0,0) = 1,\\
&s(2,0,3) = s(1,0,1) = 1,\\
&s(2,1,0) = s(1,1,0) = 1, \quad s(2,1,1) = s(1,1,1) + s(1,1,0) = 2,\\
&s(2,1,2) = s(1,1,1) = 1,\\
&s(2,2,0) = s(2,0,0) = 1, \quad s(2,2,1) = s(2,0,1) = 1, \quad s(2,2,2) = s(2,0,2) = 1,\\
&s(2,2,3) = s(2,0,3) = 1.
\end{aligned}$$

If P is a graded poset of rank m and has p_i elements of rank i, for $0 \leq i \leq m$, then the polynomial $F(P,t) = \sum_{i=1}^{m} p_i t^i$ is called the *rank-generating function* of P. If P and Q are two graded posets with rank-generating functions $F(P,t)$ and $F(Q,t)$, respectively, then $P \times Q$ is also graded and $F(P \times Q, t) = F(P,t) \cdot F(Q,t)$ (see [23]). This leads to the following Cauchy-type formula for $s(n,r,k)$.

Proposition 5.4. *If* $0 \leq r \leq n$ *and* $0 \leq k \leq R(n,r)$ *then* $s(n,r,k) = \sum_{i=0}^{k} s(r,r,i) \cdot s(n-r,n-r,k-i)$.

Proof. The rank-generating function of $S(n,r)$ is $F(S(n,r),t) = \sum_{k=0}^{R(n,r)} s(n,r,k)t^k$. By Propositions 2.1 and 3.3 it follows that

$$S(n,r) \cong S(r,r) \times S(n-r,0) \cong S(r,r) \times S(n-r,n-r).$$

Hence

$$F(S(n,r),t) = F(S(r,r) \times S(n-r,n-r),t) = F(S(r,r),t) \cdot F(S(n-r,n-r),t).$$

Then

$$\begin{aligned}
F(S(n,r),t) &= \left(\sum_{l=0}^{R(r,r)} s(r,r,l)t^l\right) \cdot \left(\sum_{j=0}^{R(n-r,n-r)} s(n-r,n-r,j)t^j\right)\\
&= \sum_{k=0}^{R(r,r)+R(n-r,n-r)} \sum_{i=0}^{k} s(r,r,i)s(r,r,k-i)t^k
\end{aligned}$$

$$= \sum_{k=0}^{R(n,r)} \sum_{i=0}^{k} s(r,r,i)s(n-r,n-r,k-i)t^k,$$

hence the assertion follows. □

The last result of this section shows a symmetric property of $S(n,r)$.

Proposition 5.5. *If* $0 \leq r \leq n$ *and* $k = R(n,r)$*, then* $s(n,r,i) = s(n,r,k-i)$ *for* $0 \leq i \leq k$.

Proof. We recall that $S_l(n,r)$ is the set of elements of $S(n,r)$ with rank l, for each $0 \leq l \leq k$. It is enough to consider the map $f : S_i(n,r) \to S_{k-i}(n,r)$ defined by $f(w) = w^c$.

At first we observe that f is well-defined, because if $w \in S_i(n,r)$ then $\rho(w) = i$ and, by Proposition 2.5, $\rho(w^c) = k - i$, therefore $w^c \in S_{k-i}(n,r)$. The map f is injective, because by $(w^c)^c = w$ it follows that $w_1^c = w_2^c \Rightarrow w_1 = w_2$. To show that f is also onto, we take $v \in S_{k-i}(n,r)$ and $w = v^c$. Since $\rho(v) = k - i$, by Proposition 2.5 we have that $k = \rho(v^c) + \rho(v) = \rho(w) + \rho(v) = \rho(w) + (k-i)$, hence $\rho(v^c) = i$, i.e. $w \in S_i(n,r)$ and $f(w) = w^c = (v^c)^c = v$, so f is onto and hence f is bijective. □

6 Relation between weight functions, the lattices $S(n,r)$ and $S(n,d,r)$ and the numbers $\gamma(n,r)$ and $\gamma(n,d,r)$

Definition 6.1. An (n,r)-function is an application $f : A(n,r) \to \mathbb{R}$ which is increasing and such that $f(0^\S) = 0$, i.e.:

$$f(\tilde{r}) \geq \cdots \geq f(\tilde{1}) \geq f(0^\S) = 0 > f(\overline{1}) \geq \cdots \geq f(\overline{n-r}). \tag{17}$$

We denote by $F(n,r)$ the set of the (n,r)-functions.

Definition 6.2. The function f is an (n,r)-weight function if (17) holds and if:

$$f(\tilde{1}) + \cdots + f(\tilde{r}) + f(\overline{1}) + \cdots + f(\overline{n-r}) \geq 0. \tag{18}$$

We denote by $WF(n,r)$ the set of the (n,r)-weight functions.

Definition 6.3. If f is an (n,r)-function, we define the *sum function* induced by f on $S(n,r)$

$$\Sigma_f : S(n,r) \to \mathbb{R}$$

to be the function that associates to $w \in S(n,r)$, $w = i_1 \cdots i_r | j_1 \cdots j_{n-r}$, the real number $\Sigma_f(w) = f(i_1) + \cdots + f(i_r) + f(j_1) + \cdots + f(j_{n-r})$.

Proposition 6.4. *If* f *is an* (n,r)*-function and if* $w, w' \in S(n,r)$ *are such that* $w \sqsubseteq w'$*, then* $\Sigma_f(w) \leq \Sigma_f(w')$.

Proof. If $w = i_1 \cdots i_r | j_1 \cdots j_{n-r} \sqsubseteq w' = i_1^{'} \cdots i_r^{'} | j_1^{'} \cdots j_{n-r}^{'}$, then $i_1 \preceq i_1^{'}, \ldots, i_r \preceq i_r^{'}$, $j_1 \preceq j_1^{'}, \ldots, j_{n-r} \preceq j_{n-r}^{'}$; hence, since f is increasing on $A(n,r)$, the assertion follows immediately by definition of the sum function Σ_f. □

Proposition 6.5. *If f is an (n,r)-weight function and if $w \in S(n,r)$ is such that $\Sigma_f(w) < 0$, then $\Sigma_f(w^c) > 0$.*

Proof. By definition of the two binary operations $\sqcap$, $\sqcup$ and of the complement operation ^{c} on $S(n,r)$, we have that

$$w \sqcup w^c = r \cdots 1 | 1 \cdots (n-r) \quad \text{and} \quad w \sqcap w^c = 00 \cdots 0 | 0 \cdots 0.$$

Hence, by definition of Σ_f and since $f(0^{\S}) = 0$, we have that

$$\Sigma_f(w) + \Sigma_f(w^c) = \Sigma_f(w \sqcup w^c) = f(\tilde{1}) + \cdots + f(\tilde{r}) + f(\overline{1}) + \cdots + f(\overline{n-r}) \geq 0$$

by (18). Hence, if $\Sigma_f(w) < 0$, we will have that $\Sigma_f(w^c) > 0$. □

If f is an (n,r)-function, we set:

$$S_f^+(n,r) = \{w \in S(n,r) : \Sigma_f(w) \geq 0\};$$

furthermore, if d and r are integers such that $1 \leq d, r \leq n$, we set:

$$S_f^+(n,d,r) = \{w \in S(n,d,r) : \Sigma_f(w) \geq 0\}.$$

Observe that, in general, neither $S_f^+(n,r)$ nor $S_f^+(n,d,r)$ are sublattices of $S(n,r)$, because they are not closed with respect to the operation of inf ($\wedge$). They are simply subposets of $S(n,r)$ with the induced order.

We set

(β) $\gamma(n,r) = \min\{|S_f^+(n,r)|$: f is an (n,r)-weight function $\}$;
(δ) $\gamma(n,d,r) = \min\{|S_f^+(n,d,r)|$: f is an (n,r)-weight function $\}$.

It is easy to observe that the numbers defined in (β) are exactly those in (1) of the introduction, while the numbers defined in (δ) are the same as those in (2) of the introduction. We use therefore both the notations.

Theorem 1 of [19] applied to our context, gives the following

Proposition 6.6. *For each $r \in \mathbb{N}$ with $1 \leq r \leq n$, we have that:*

i) $\gamma(n,r) \geq 2^{n-1} + 1$,
ii) $\gamma(n,1) \leq 2^{n-1} + 1$.

The difference between our situation and Theorem 1 of [19] is that we admit the string $0 \cdots 0 | 0 \cdots 0$ in the set $S_f^+(n,r)$, i.e. we admit the empty set. For this reason in i) and ii) the number $2^{n-1} + 1$ appears instead of 2^{n-1} of [19].

In the next section, we will link the numbers $\gamma(n,r)$ and $\gamma(n,d,r)$ to a minimum problem on a family of boolean functions defined on the lattices $S(n,r)$ and $S(n,d,r)$.

7 Weight functions and boolean functions on $S(n, r)$

In this section we show how to associate to any (n, r)-function and to any (n, r)-weight function a Boolean function on $S(n, r)$. Our aim is to connect the study of the (n, r)-weight functions and of the related extremal problems (in particular the computation of $\gamma(n, r)$ and $\gamma(n, d, r)$) to some boolean functions on $S(n, r)$.

If f is an (n, r)-function (or an (n, r)-weight function), we can define the map

$$A_f : S(n, r) \to \mathbf{2}$$

setting

$$A_f(w) = \begin{cases} P & \text{if } \Sigma_f(w) \geq 0 \\ N & \text{if } \Sigma_f(w) < 0. \end{cases}$$

In order to underline the essential properties of the map A_f, we introduce the concept of (n, r)-boolean map.

Definition 7.1. An (n, r)-*boolean map* (briefly (n, r)-BM) is a map $A : S(n, r) \to \mathbf{2}$ with the following properties:

$a_1)$ if $w_1, w_2 \in S(n, r)$ and $w_1 \sqsubseteq w_2$, with $A(w_1) = P$, then $A(w_2) = P$;
$a_2)$ if $w_1, w_2 \in S(n, r)$ and $w_1 \sqsubseteq w_2$, with $A(w_2) = N$, then $A(w_1) = N$;
$a_3)$ $A(0 \cdots 0|0 \cdots 0) = P$ and $A(0 \cdots 0|0 \cdots 01) = N$.

Definition 7.2. An (n, r)-*weighted boolean map* (briefly (n, r)-WBM) is a (n, r)-BM $A : S(n, r) \to \mathbf{2}$ which satisfies the following two properties:

$a_4)$ if $w \in S(n, r)$ is such that $A(w) = N$, then $A(w^c) = P$;
$a_5)$ $A(r(r-1) \cdots 21|12 \cdots (n-r)) = P$.

We denote by $B(n, r)$ the family of all the (n, r)-BM's and by $WB(n, r)$ the family of all the (n, r)-WBM's.

Proposition 7.3. i) *If $A : S(n, r) \to \mathbf{2}$ then:*

$$A \textit{ satisfies } a_1) \iff A \textit{ satisfies } a_2) \iff A \textit{ is order-preserving}.$$

ii) *if f is an (n, r)-function, then A_f is an (n, r)-BM.*
iii) *if f is an (n, r)-weight function, then A_f is an (n, r)-WBM.*

Proof. i) The assertion is straightforward, thanks to an argument by contradiction.
ii) Let f be an (n, r)-function. Suppose that $w_1, w_2 \in S(n, r)$ and that $w_1 \sqsubseteq w_2$. By Proposition 6.4, it follows that $\Sigma_f(w_1) \leq \Sigma_f(w_2)$. Suppose that $A_f(w_1) > A_f(w_2)$. This would imply that $A_f(w_1) = P$ and $A_f(w_2) = N$, i.e. (by definition of A_f) $\Sigma_f(w_1) \geq 0$ and $\Sigma_f(w_2) < 0$ and this is a contradiction. Hence A_f is order-preserving. The property $a_3)$ holds by definition of Σ_f and A_f.
iii) Let f be an (n, r)-weight function. Let $w \in S(n, r)$ be such that $A_f(w) = N$. By definition of A_f, we have that $\Sigma_f(w) < 0$, and hence, by Proposition 6.5, $\Sigma_f(w^c) > 0$, i.e. $A_f(w^c) = P$. Hence A_f satisfies $a_4)$. The property $a_5)$ is obviously satisfied, by definition of Σ_f and A_f since f is an (n, r)-weight function. □

Definition 7.4. A map $A \in B(n,r)$ is said to be *numerically represented* if there exists an (n,r)-function $f \in F(n,r)$ such that $A = A_f$.

Definition 7.5. A map $A \in WB(n,r)$ is said to be *numerically represented* if there exists an (n,r)-weight function $f \in WF(n,r)$ such that $A = A_f$.

Set

$$RB(n,r) = \{A \in B(n,r) : A \text{ is numerically represented}\},$$
$$RWB(n,r) = \{A \in WB(n,r) : A \text{ is numerically represented}\}.$$

Proposition 7.6. i) *$RB(n,r)$ is identified with a quotient of $F(n,r)$.*
ii) *$RWB(n,r)$ is identified with a quotient of $WF(n,r)$.*

Proof. i) We define on $F(n,r)$ the following binary relation: if $f, g \in F(n,r)$, we set $f \sim g$ if for each $w \in S(n,r)$, we have that $\Sigma_f(w) \geq 0 \Leftrightarrow \Sigma_g(w) \geq 0$.

Then $\sim$ is an equivalence relation on $F(n,r)$. By Proposition 7.3ii), if $f \in F(n,r)$ it follows that $A_f \in B(n,r)$. Therefore there is a map $\varphi : F(n,r) \to B(n,r)$ such that $\varphi(f) = A_f$. Then, if $f, g \in F(n,r)$, it follows that:

$$f \sim g \iff \text{for each } w \in S(n,r), \Sigma_f(w) \text{ and } \Sigma_g(w) \text{ have the same sign}$$
$$\iff A_f(w) = A_g(w) \text{ for each } w \in S(n,r) \iff \varphi(f) = \varphi(g).$$

By the Universal Property of the quotient, there exists a unique injective map

$$\tilde{\varphi} : F(n,r) \to B(n,r)$$

such that the following diagram commutes:

$$\begin{tikzcd} F(n,r) \arrow[rr, "\varphi"] \arrow[dr, "\nu"'] & & B(n,r) \\ & F(n,r)/\sim \arrow[ur, "\tilde{\varphi}"'] & \end{tikzcd}$$

where ν is the projection on the quotient. Since the image of φ is exactly $RB(n,r)$ and since it coincides with the image of $\tilde{\varphi}$, it follows that $\tilde{\varphi}$ is a bijective map between $F(n,r)/\sim$ and $RB(n,r)$.

Analogously we prove ii), using Proposition 7.3iii). $\square$

If $A \in WB(n,r)$, we will set

$$S_A^+(n,r) = \{w \in S(n,r) : A(w) = P\},$$

and if $d \geq 1$ is such that $d \leq n$, we will set:

$$S_A^+(n,d,r) = \{w \in S(n,d,r) : A(w) = P\}.$$

Furthermore, we set:

$$\begin{aligned}\tilde{\gamma}(n,r) &= \min\{|S_A^+(n,r)| : A \in WB(n,r)\},\\ \tilde{\gamma}(n,d,r) &= \min\{|S_A^+(n,d,r)| : A \in WB(n,r)\},\\ \overline{\gamma}(n,r) &= \min\{|S_A^+(n,r)| : A \in RWB(n,r)\},\\ \overline{\gamma}(n,d,r) &= \min\{|S_A^+(n,d,r)| : A \in RWB(n,r)\}.\end{aligned}$$

Proposition 7.7. i) $\gamma(n,r) = \overline{\gamma}(n,r) \geq \tilde{\gamma}(n,r)$.
ii) $\gamma(n,d,r) = \overline{\gamma}(n,d,r) \geq \tilde{\gamma}(n,d,r)$.

Proof. i) The inequality $\overline{\gamma}(n,r) \geq \tilde{\gamma}(n,r)$ is obvious because $RWB(n,r)$ is a subset of $WB(n,r)$. We prove that $\gamma(n,r) = \overline{\gamma}(n,r)$. Let f be an (n,r)-weight function for which it holds $\gamma(n,r) = |S_f^+(n,r)|$. Then $A_f \in RWB(n,r)$ and (by definition of A_f) we have that $S_{A_f}^+(n,r) = \{w \in S(n,r) : A_f(w) = P\} = \{w \in S(n,r) : \Sigma_f(w) \geq 0\} = S_f^+(n,r)$. Hence $\gamma(n,r) = |S_f^+(n,r)| = |S_{A_f}^+| \geq \overline{\gamma}(n,r)$, because $A_f \in RWB(n,r)$.

Let $A \in RWB(n,r)$ such that $\overline{\gamma}(n,r) = |S_A^+(n,r)|$. Since A is numerically represented, there will exist an (n,r)-weight function $f \in WF(n,r)$ such that $A = A_f$. Then $S_A^+(n,r) = S_{A_f}^+(n,r) = S_f^+(n,r)$, and hence $\overline{\gamma}(n,r) = |S_A^+(n,r)| = |S_f^+(n,r)| \geq \gamma(n,r)$. This proves that $\gamma(n,r) = \overline{\gamma}(n,r)$.

The proof of ii) is similar to i). □

It is natural now to pose the following two problems:

First open problem: $B(n,r) = RB(n,r)$?

Second open problem: $WB(n,r) = RWB(n,r)$?

If $RWB(n,r)$ coincides with $WB(n,r)$ (i.e. if any (n,r)-weighted boolean map is numerically represented) then $\gamma(n,r) = \tilde{\gamma}(n,r)$ and $\gamma(n,d,r) = \tilde{\gamma}(n,d,r)$, by Proposition 7.7. If the answer to the second open problem is affirmative, this would imply that each time we give the boolean formal values N or P to each string of $S(n,r)$ in such a way that the rules a_1)–a_5) are respected, then there exists a numerical attribution to the singletons which permits the reconstruction of the configuration of N's and P's in a unique way. In other words, if the assertion of the second open problem holds we have an effective representation theorem.

References

[1] K. J. Al-Agha, R. J. Greechie, The involutory dimension of involution posets. *Order* **18** (2001), 323–337 (2002). MR1884425 (2003b:06002) Zbl 0993.03082

[2] K. Alladi, P. Erdős, J. D. Vaaler, Multiplicative functions and small divisors. In: *Analytic number theory and Diophantine problems* (*Stillwater, OK,* 1984), 1–13, Birkhäuser 1987. MR1018367 (90h:11089) Zbl 0626.10004

[3] R. A. Bailey, *Association schemes*. Cambridge Univ. Press 2004. MR2047311 (2005d:05001) Zbl 1051.05001

[4] Z. Baranyai, On the factorization of the complete uniform hypergraph. In: *Infinite and finite sets* (*Colloq., Keszthely,* 1973*), Vol. I*, 91–108. Colloq. Math. Soc. Jánōs Bolyai, Vol. 10, North-Holland 1975. MR0416986 (54 #5047) Zbl 0306.05137

[5] A. Bhattacharya, Some problems in combinatorics, Ph. D. Thesis, Indian Institute of Technology, Bombay, India, 2004.

[6] A. Bhattacharya, On a conjecture of Manickam and Singhi. *Discrete Math.* **272** (2003), 259–261. MR2009545 (2004h:05002) Zbl 1030.05116

[7] T. Bier, A distribution invariant for association schemes and strongly regular graphs. *Linear Algebra Appl.* **57** (1984), 105–113. MR729265 (85b:05049) Zbl 0529.05011

[8] T. Bier, P. Delsarte, Some bounds for the distribution numbers of an association scheme. *European J. Combin.* **9** (1988), 1–5. MR938815 (89e:05054) Zbl 0646.05010

[9] T. Bier, N. Manickam, The first distribution invariant of the Johnson-scheme. *Southeast Asian Bull. Math.* **11** (1987), 61–68. MR914366 (89b:05061) Zbl 0723.05120

[10] K. Brenneman, R. Haas, A. G. Helminck, Implementing an algorithm for the twisted involution poset for Weyl groups. In: *Proceedings of the Thirty-Seventh Southeastern International Conference on Combinatorics, Graph Theory and Computing*, volume 182, 137–144, 2006. MR2311295 (2008c:05189) Zbl 1112.20033

[11] G. Chiaselotti, On a problem concerning the weight functions. *European J. Combin.* **23** (2002), 15–22. MR1878770 (2002k:05236) Zbl 0998.05064

[12] G. Chiaselotti, G. Infante, G. Marino, New results related to a conjecture of Manickam and Singhi. *European J. Combin.* **29** (2008), 361–368. MR2388371 (2008m:05293) Zbl 1131.05093

[13] B. A. Davey, H. A. Priestley, *Introduction to lattices and order*. Cambridge Univ. Press 2002. MR1902334 (2003e:06001) Zbl 1002.06001

[14] P. Erdős, C. Ko, R. Rado, Intersection theorems for systems of finite sets. *Quart. J. Math. Oxford Ser.* (2) **12** (1961), 313–320. MR0140419 (25 #3839) Zbl 0100.01902

[15] G. O. H. Katona, A simple proof of the Erdős-Chao Ko-Rado theorem. *J. Combin. Theory Ser. A* **13** (1972), 183–184. MR0304181 (46 #3316) Zbl 0262.05002

[16] N. Manickam, First distribution invariants of association shemes. Ph. D. Thesis, The Ohio State University, 1986.

[17] N. Manickam, Distribution invariants of association schemes. *Congr. Numer.* **61** (1988), 121–131. MR961647 (89i:05071) Zbl 0676.05026

[18] N. Manickam, First distributed sets in the association scheme of bilinear forms. In: *Sets, graphs and numbers* (*Budapest,* 1991), volume 60 of *Colloq. Math. Soc. János Bolyai*, 465–468, North-Holland 1992. MR1218212 (94b:05221) Zbl 0785.05088

[19] N. Manickam, D. Miklós, On the number of nonnegative partial sums of a nonnegative sum. In: *Combinatorics* (*Eger,* 1987), volume 52 of *Colloq. Math. Soc. János Bolyai*, 385–392, North-Holland 1988. MR1221577 (94d:05140) Zbl 0726.11014

[20] N. Manickam, N. M. Singhi, First distribution invariants and EKR theorems. *J. Combin. Theory Ser. A* **48** (1988), 91–103. MR938860 (89d:05044) Zbl 0645.05023

[21] G. Marino, G. Chiaselotti, A method to count the positive 3-subsets in a set of real numbers with non-negative sum. *European J. Combin.* **23** (2002), 619–629. MR1931945 (2003i:05012) Zbl 1021.05003

[22] S. Srinivasan, On an arithmetical inequality. II. In: *Number theory* (*Tiruchirapalli,* 1996), volume 210 of *Contemp. Math.*, 299–301, Amer. Math. Soc. 1998. MR1478499 (98k:11133) Zbl 0893.11003

[23] R. P. Stanley, *Enumerative combinatorics. Vol.* 1. Cambridge Univ. Press 1997. MR1442260 (98a:05001) Zbl 0889.05001

[24] J. H. van Lint, R. M. Wilson, *A course in combinatorics*. Cambridge Univ. Press 2001. MR1871828 (2002i:05001) Zbl 0980.05001

Received 9 July, 2010

C. Bisi, Dipartimento di Matematica ed Informatica, Universitá di Ferrara, Via Machiavelli 35, 44100, Ferrara, Italy
Email: bsicnz@unife.it, bisi@math.unifi.it

G. Chiaselotti, Dipartimento di Matematica, Universitá della Calabria, Via Pietro Bucci, Cubo 30B, 87036 Arcavacata di Rende (CS), Italy
Email: chiaselotti@unical.it