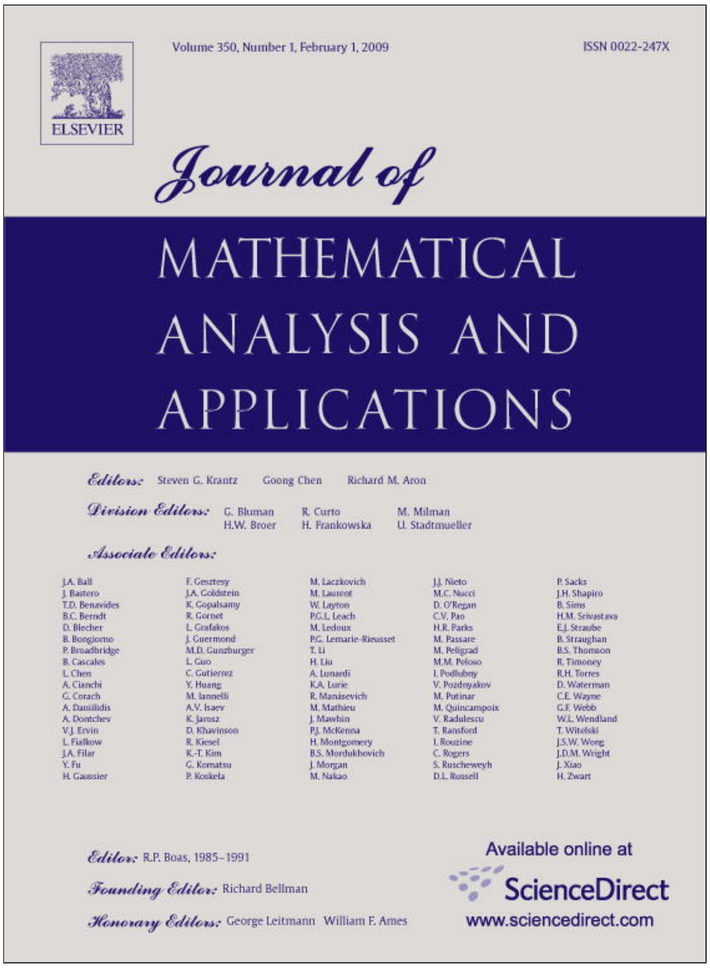
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On closed invariant sets in local dynamics ☆

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ABSTRACT

We investigate the dynamical behaviour of a holomorphic map on an f-invariant subset $\mathcal{C}$ of U, where $f:U\to\mathbb{C}^k$. We study two cases: when U is an open, connected and polynomially convex subset of $\mathbb{C}^k$ and $\mathcal{C}\Subset U$, closed in U, and when ∂U has a p.s.h. barrier at each of its points and $\mathcal{C}$ is not relatively compact in U. In the second part of the paper, we prove a Birkhoff's type theorem for holomorphic maps in several complex variables, i.e. given an injective holomorphic map f, defined in a neighborhood of $\overline{U}$, with U star-shaped and $f(U)$ a Runge domain, we prove the existence of a unique, forward invariant, maximal, compact and connected subset of $\overline{U}$ which touches ∂U.

1. Introduction

Let $f:U\to\mathbb{C}^k$ be a holomorphic map. Here U is an open, connected and bounded (or hyperbolic) subset in $\mathbb{C}^k$. Since the semi-local holomorphic dynamics is not well understood yet, specially when $k>2$ [1,4,8,12], we describe the dynamical behaviour of f on an f-invariant subset $\mathcal{C}$ of U in two different cases:

(a) when $\mathcal{C}\Subset U$, closed in U, and U is polynomially convex;
(b) when $\mathcal{C}$ is not relatively compact in U and every point in ∂U has a p.s.h. barrier.

When there is a recurrent component W in the interior of the polynomially convex hull of $\mathcal{C}$ in case (a) or in the interior of $\overline{\mathcal{C}}$ in case (b), we prove that the dynamical behaviour on W is of three types:

1. W is the basin of attraction of an attractive periodic orbit;
2. W is a Siegel domain;
3. if h is a limit of a subsequence of $\{f^n\}_{n\in\mathbb{N}}$, then $0<\operatorname{rank}(h)<k$.

In particular when $\mathcal{C}$ is a closed orbit or a countable union of closed orbits, we prove that $\mathcal{C}$ cannot have a non-empty interior with a recurrent point. This has been proved by Fornaess and Stensones in [6] when U has a Lipschitz boundary; here it is proved in a different situation, i.e. when U is polynomially convex or with a p.s.h. barrier at each boundary point, then U has not necessarily Lipschitz boundary.

In the second part of the paper, see Section 4, we give a version of Birkhoff's theorem which was originally stated for surface transformations f having a Lyapunov unstable fixed point p for f or for f^{-1}. Under these hypotheses Birkhoff has shown [3] the existence, in each neighborhood U of p, of a compact set K_+ (or K_-) which is positive (or negative) invariant

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by f and touching the boundary of U. In this general setting there is no forward and backward invariant compact set with this property.

In the same spirit, our Theorem 4.1 asserts that if $f : U \to \mathbb{C}^k$ is a holomorphic injective map of $\mathbb{C}^k$ such that $f(0) = 0$, with U bounded and star-shaped and $f(U)$ a Runge domain, then there exists a unique, maximal, compact, connected set K such that:

1. $0 \in K \subset \overline{U}$;
2. $K \cap \partial U \neq \emptyset$;
3. $f(K) \subset K$.

In general, this compact set K is not totally invariant: we will give an example, see Example 5.1. So the several variables analogue of R. Perez-Marco's *hedgehogs* [15] does not hold: in the one variable case the compact is totally invariant and touches the boundary [15].

2. Preliminaries

We recall some definitions and fix our notations.

Let K be a compact set of $\mathbb{C}^k$, then the polynomially convex hull of K is defined as:

$$\hat{K}_{\mathcal{P}} = \Big\{ z \in \mathbb{C}^k \;\Big|\; \big|p(z)\big| \leqslant \sup_{\zeta \in K} \big|p(\zeta)\big| \ \forall p \text{ polynomial} \Big\}.$$

A compact set K is *polynomially convex* if $K = \hat{K}_{\mathcal{P}}$ [13].

Definition 2.1. An open set U in $\mathbb{C}^k$ is *polynomially convex* if, for every compact K in U, $\hat{K}_{\mathcal{P}} \Subset U$.

For example, the geometrically convex open sets of $\mathbb{C}^k$ are polynomially convex in $\mathbb{C}^k$. The property of being polynomially convex is not invariant by biholomorphisms, as Wermer showed, see Gunning's book [11, p. 46].

If K is polynomially convex, each holomorphic function on a neighborhood of K is the uniform limit on K of polynomials; in the same way if ρ is p.s.h. and continuous on U, polynomially convex open set, then it is the uniform limit on the compact sets of U of p.s.h. functions of $\mathbb{C}^k$.

A consequence, when U is polynomially convex, is that convexity with respect to p.s.h. functions in U is the same as polynomial convexity.

If K is polynomially convex and compact in U, there exists ρ_1 p.s.h. and continuous on $\mathbb{C}^k$, $K = \{\rho_1 \leqslant 0\}$ and $\rho_1 \geqslant 1$ on a neighborhood of $\mathbb{C}^k \setminus U$.

Definition 2.2. A domain U is *Runge* if each holomorphic function on U can be approximated by polynomials, uniformly on compact subsets of U.

In particular any polynomially convex open set is a Runge domain [11].

It is possible to construct Runge domains such that the interior of $\overline{U}$ is not equal to U: for example $U = \{w \in \mathbb{C}^k : |w| < \exp(-\varphi)\}$ with φ subharmonic on the unit disc, $\varphi = 0$ on a dense set of Δ, $\varphi \geqslant 0$ and non-identically zero; in particular U does not have Lipschitz boundary.

3. Invariant sets

3.1. f-Invariant relatively compact subsets

Let $f : U \to \mathbb{C}^k$ be a holomorphic map with $U \Subset \mathbb{C}^k$ or U Kobayashi hyperbolic. We assume that U is an open, connected and polynomially convex set. We say that a closed set $\mathcal{C}$ is f-invariant if $f(\mathcal{C}) \subset \mathcal{C}$.

Proposition 3.1. *Let $\mathcal{C} \subset\subset U$ be a closed f-invariant set, then $\hat{\mathcal{C}}_{\mathcal{P}}$ is f-invariant.*

Proof. By hypothesis, $\mathcal{C} \Subset U$. Choose $z_0 \in \hat{\mathcal{C}}_{\mathcal{P}}$ and suppose $f(z_0) \notin \hat{\mathcal{C}}_{\mathcal{P}}$. Then there is a p.s.h. smooth function ρ_0 in $\mathbb{C}^k$, such that $\rho_0 \leqslant 0$ on $\hat{\mathcal{C}}_{\mathcal{P}}$ and $\rho_0(f(z_0)) > 1$.

The function $\rho_0 \circ f$ is p.s.h. on U, $\rho_0 \circ f \leqslant 0$ on $\mathcal{C}$ and $\rho_0 \circ f$ is also p.s.h. on the holomorphic hull of $\mathcal{C}$ with respect to U, which is the same as $\hat{\mathcal{C}}_{\mathcal{P}}$. It follows, by Maximum Principle, that $\rho_0(f(z_0)) \leqslant 0$, which is a contradiction. □

Definition 3.2. A connected component $\Omega \subset U$, of the set of points where $\{f^n\}_{n \in \mathbb{N}}$ is equicontinuous, is *recurrent* if there exists $p_0 \in \Omega$ such that $f^{n_i}(p_0)$ is relatively compact in Ω for some subsequence n_i, i.e. if Ω contains a recurrent point p_0.

Proposition 3.3. *If $V = Int(\hat{\mathcal{C}}_{\mathcal{P}}) \neq \emptyset$ then the sequence $\{f^n\}_{n\in\mathbb{N}}$ defined on V is a normal family and if V has a recurrent component W then there are three possibilities*:

(i) *f has an attractive periodic orbit,*
(ii) *there is a Siegel domain, i.e. there is W, a component of V and a subsequence n_i, s.t. $f^{n_i}_{|W} \to Id$,*
(iii) *if h is a limit of a subsequence of $\{f^n\}_{n\in\mathbb{N}}$, then $0 < \operatorname{rank}(h) < k$.*

Proof. We assume that for some p_0, $f^{n_i}(p_0) \to p \in W$, and f^{n_i} converges uniformly on compact sets. We now write $f^{n_{i+1}-n_i} \circ f^{n_i} = f^{n_{i+1}}$. Extracting a subsequence we get a limit h of $f^{n_{i+1}-n_i}$ such that $h(p) = p$ [7]. If h is of rank 0, we show that p is an attractive fixed point [7]. If h is of maximal rank, then we get a Siegel domain [7]. The theorem of Carathéodory–Cartan–Kaup–Wu, see [18, p. 438] and [14, p. 66], describes the permitted eigenvalues. Otherwise for all possible h, $0 < \operatorname{rank}(h) < k$.

In [7], Fornaess and Sibony prove a more precise result when f is an endomorphism of $\mathbb{P}^2$. Their stronger result is valid only in dimension two. □

3.2. f-Invariant non-relatively compact subsets

Theorem 3.4. *Let $f: U \to \mathbb{C}^k$ be a holomorphic open map defined on U, a bounded (or hyperbolic) open and connected subset of $\mathbb{C}^k$. Assume that every point in ∂U has a p.s.h. barrier, i.e. if $q \in \partial U$, there exists a p.s.h. function ρ_q, $\rho_q < 0$ on U, continuous such that $\lim_{p\to q} \rho_q(p) = 0$. Suppose $\mathcal{C}$ is an f-invariant set in U. Let V be the non-empty interior of $\bar{\mathcal{C}}$, where the adherence is with respect to U. We also assume that a connected component of V, W, contains a recurrent point p_0. Then there are three possibilities for W*:

(1) *it is the basin of attraction of an attracting periodic orbit*;
(2) *it is a Siegel domain*;
(3) *if h is a limit of a subsequence of $\{f^n\}_{n\in\mathbb{N}}$, on W, then $0 < \operatorname{rank}(h) < k$.*

Proof. We start proving that the sequence $\{f^n\}_{n\in\mathbb{N}}$ is well defined on V. Since $V \subset U$ is invariant, by continuity $f(V) \subset \overline{U}$: indeed if $p \in V$ there exists a sequence of points $p_n \in \mathcal{C}$ such that $p_n \to p$ and hence $f(p_n) \to f(p) = q \in \overline{U}$. We show now that $f(V) \subset U$. Suppose $q \in \partial U$. Consider the barrier ρ_q at q. The function $\rho_q \circ f$ is p.s.h. and continuous on V, and $\rho_q \circ f \leqslant 0$ on V. But $(\rho_q \circ f)(p) = \lim_{n\to+\infty}(\rho_q \circ f)(p_n) = \lim_{n\to+\infty} \rho_q(f(p_n)) = 0$. Hence, by Maximum Principle, $\rho_q \circ f \equiv 0$, i.e. $f(V) \subset (\rho_q = 0) \subset \partial U$. This is impossible because f is open. Hence $f(V) \subset U$ and $f^n(V) \subset U$, therefore the sequence $\{f^n\}_{n\in\mathbb{N}}$ is normal, since U is bounded.

Now suppose that there exists a recurrent point p_0 in W, a connected component of V. This means that there exists a sequence of $n_i \to +\infty$ s.t. $f^{n_i}(p_0) \to p_0 \in W$. We can always suppose that $n_{i+1} - n_i \to +\infty$. Taking a subsequence $\{i = i(j)\}$ we can suppose that the sequence $\{f^{n_{i+1}-n_i}\}_i$ converges uniformly on compact sets of W to a holomorphic map $h: W \to \overline{U}$ s.t. $h(p_0) = p_0$. Indeed let $p_i = f^{n_i}(p_0)$. Then $f^{n_{i+1}-n_i}(p_i) = f^{n_{i+1}}(p_0) = p_{i+1}$. Hence $f^{n_{i+1}-n_i}(p_0) = p_{i+1} + O(|p_i - p_0|)$ so converges to p_0 and therefore, necessarily, $h(p_0) = p_0$ [7].

Consider all maps h obtained in this way. If some h is of rank 0, then some iterate of f has p_0 as an attractive fixed point and f has p_0 as an attractive periodic point.

If some h is of maximal rank k, then W is a Siegel domain, otherwise all the limit maps have lower rank r, $0 < r < k$. In [7] the authors analyze the case of holomorphic endomorphisms of $\mathbb{P}^2$ and thanks to the restriction to the dimension 2 and to the endomorphism case, the result there is much more precise: for example in case (iii), $h(W)$ is always independent of h and attracts all orbits. □

Remark 3.5. If f is not open it is enough to assume that $(\rho_q = 0)$ does not contain the image of f.

Corollary 3.6. *Under the hypotheses of Theorem* 3.4, *if $\bar{\mathcal{C}}$ is an invariant closed set with a dense orbit in it or a countable union of closed invariant sets each one with a dense orbit, then the interior V of $\bar{\mathcal{C}}$ does not contain recurrent points.*

Proof. Indeed in the possible dynamical behaviours described in Theorem 3.4, when $\bar{\mathcal{C}}$ is closed with a dense orbit cannot have interior; when we consider a countable union of closed sets with empty interior then, by Baire's theorem, the union of them is still with empty interior. □

4. Forward invariant compact sets

Theorem 4.1. *Let U be a bounded star-shaped domain with respect to 0 in $\mathbb{C}^k$ and let U' be an open neighborhood of $\overline{U}$. Let $f: U' \to \mathbb{C}^k$, be a holomorphic map, $f(0) = 0$, f injective on U (i.e. $f: U \to f(U)$ is a biholomorphic map) and $f(U)$ is a Runge domain. Assume $f(z) = Az + O(z^2)$, with A a linear invertible map and with all the eigenvalues λ_j, for $1 \leqslant j \leqslant k$, of modulus 1. Then there exists a unique maximal connected compact set K, with $0 \in K \subset \overline{U}$ s.t. $(K \cap \partial U) \neq \emptyset$ and $f(K) \subset K$. Furthermore f is linearizable iff $0 \in Int(K)$.*

Proof. Define $f_{\mu_n}(z) = f(\mu_n \cdot z)$ with $\mu_n \in \mathbb{R}$, $0 < \mu_n < 1$ and $\mu_n \to 1$ for $n \to +\infty$. Then $f_{\mu_n} \to f$ uniformly on $\overline{U}$ and $|Jac(f_{\mu_n})(0)| = |\mu_n| \cdot |Jac(A)| < 1$ because $|\mu_n| < 1$ and $|Jac(A)| = 1$; indeed $f_{\mu_n}(z) = \mu_n \cdot A \cdot z + O((\mu_n \cdot z)^2)$.

For simplicity, we call $\mu := \mu_n$.

Let $f_\mu : \frac{1}{\mu} \cdot U \to f(U)$ indeed $f_\mu(\frac{1}{\mu} \cdot U) \equiv f(U)$. Hence f_μ is a biholomorphism from a star-shaped domain $\frac{1}{\mu} \cdot U$ to a Runge domain $f(U) = f_\mu(\frac{1}{\mu} \cdot U)$. Now applying a result of Andersen and Lempert ([2, Theorem 2.1], [9,10]) to the biholomorphism $f_\mu : \frac{1}{\mu} \cdot U \to f(U)$, we find a sequence of automorphisms g_m of $\mathbb{C}^k$, such that $g_m \to f_\mu$ for $m \to +\infty$ uniformly on compact subsets of $\overline{U}$, i.e. the g_m's converge to f_μ, uniformly on compact sets and $g_m(0) = 0$ for all m.

Since $|Jac(f_\mu)(0)| < 1$, then $|Jac(g_m)(0)| < 1$.

Hence $g_m \in Aut(\mathbb{C}^k)$ and $g_m : U \to g_m(U)$ with $0 \in U \cap g_m(U)$.

Let B be a domain which is a homothetic of U, i.e. $B = \epsilon U$, sufficiently small s.t. $g_m^{-1}(B) \subset U$ i.e. $0 \in B \subset (U \cap g_m(U))$. Since the basin of attraction of 0 for g_m (i.e. $\bigcup_{n\in\mathbb{N}} g_m^{-n}(B)$) is biholomorphic to $\mathbb{C}^k$ [16] and in particular is unbounded, there exists $n_0 \in \mathbb{N}$ s.t. $g_m^{-n_0}(B) \subset U$ but $g_m^{-(n_0+1)}(B) \not\subset U$ $(n_0 \geqslant 1)$.

We consider the one-parameter family $\{B_t\}_{t\geqslant 1}$ where $B_t = t \cdot B$ [15]. Then we consider the t's for which:

$$g_m^{-n_0}(B_t) \subset U.$$

The set is not empty because for $t = 1$ the inclusion is true. By continuity, there exists $\bar{t}$ s.t.

$$g_m^{-n_0}(B_{\bar{t}}) \subset U$$

and

$$g_m^{-n_0}(\overline{B_{\bar{t}}}) \cap (\partial U) \neq \emptyset.$$

We call $F_m := \overline{g_m^{-n_0}(B_{\bar{t}})}$.

Then $(F_m)_{m\in\mathbb{N}}$ is a sequence of compact sets in $\overline{U}$ s.t. $g_m(F_m) \subset F_m$ because $g_m^{-n_0+1}(B_{\bar{t}}) \subset g_m^{-n_0}(B_{\bar{t}})$: this follows from the description of the basin of attraction of 0.

Each F_m is a connected set because it is the closure of the pre-image by a biholomorphism of a connected set.

By compactness of the space $\mathcal{K}_c(\overline{U}) = \{$*connected compact subsets* of $\overline{U}\}$, there exists a subsequence $(m_k)_{k\in\mathbb{N}}$ t.c. $F_{m_k} \to K_\mu \in \mathcal{K}_c(\overline{U})$. Finally we prove that $f_\mu(K_\mu) \subset K_\mu$.

We use that:

(i) $g_m \to f_\mu$ uniformly on compact subsets of $\overline{U}$;
(ii) $\lim_{k\to+\infty} F_{m_k} = K_\mu$.

Let $x \in K_\mu$, then we want to prove that $f_\mu(x) \in K_\mu$.

Since $x \in K_\mu$, there exists a sequence $x_k \to x$ with $x_k \in F_{m_k}$ by (ii).

Then $g_{m_k}(x_k) \in F_{m_k}$ and we can assume $g_{m_k}(x_k) \to y \in K_\mu$, by (ii).

But $g_{m_k} \to f_\mu$ for $k \to +\infty$ by (i), so $f_\mu(x) = \lim_{k\to\infty} g_{m_k}(x_k) = y \in K_\mu$.

Hence K_μ is f_μ-invariant.

Therefore for each μ we have found a forward invariant connected compact set for f_μ and K_μ intersects ∂U. Now, with an argument similar to the one already used for $\{g_m\}_{m\in\mathbb{N}}$ and $\{F_{m_k}\}_{k\in\mathbb{N}}$, we prove that, up to considering a subsequence, $K_{\mu_n} \to K$ in the Hausdorff metric. Since $f_{\mu_n} \to f$ uniformly on compact sets, we have that $f(K) \subset K$ and K touches ∂U. In order to have the unique, maximal, connected, invariant compact set, it is enough to take the closure of the union of all such compact sets K. Obviously, the closure of a union of f-invariant sets is still f-invariant and it is also connected because each compact set contains 0. Since K_{μ_n} intersects ∂U for all μ_n, also its limit K in the Hausdorff topology does. Suppose $0 \in Int(K)$, we show that f is linearizable. The family $(f^n)_{n\in\mathbb{N}}$ is locally equicontinuous on $Int(K)$ and $f(0) = 0$. Following a standard trick, we define

$$h(z) := \lim_{n_j\to+\infty} \frac{1}{n_j} \sum_{j=0}^{n_j-1} A^{-j} f^j(z).$$

The limit exists in a neighborhood of zero. Indeed there is a $c > 1$ such that $f^n(\mathbb{B}(0,r)) \subset \mathbb{B}(0,cr) \subset K$ for all n. Then we can consider a limit map h for an appropriate subsequence n_j. We have $h(0) = 0$, $Jac(h)(0) = Id$ and we easily check that $h(f) = Ah$. □

Remark 4.2. If we take a sequence $\mu_n > 1$, $\mu_n \to 1$, we can prove that there exists a maximal connected compact set invariant for f^{-1}. In general the forward and backward invariant compact subsets are different, as the case of Hénon maps shows, see Example 5.1 below.

Remark 4.3. We want to point out that K is not necessarily a proper subset of $\overline{U}$, indeed if f is an automorphism of the ball $\mathbb{B}^k \subset \mathbb{C}^k$ fixing 0, then $K = \overline{\mathbb{B}^k}$.

Remark 4.4. Suppose that f, g are two commuting maps satisfying all the hypotheses of Theorem 4.1, then they share the same maximal, compact, connected, invariant set $K \ni 0$.

Indeed let K_f and K_g be the maximal, compact, connected invariant sets containing 0, for f and g respectively, which exist by Theorem 4.1. Then consider $f \circ g(K_f) = g \circ f(K_f) \subset g(K_f)$, hence $g(K_f) \subset K_f$ which implies that $K_f \subset K_g$. Analogously, considering $g \circ f(K_g) = f \circ g(K_g)$, we can prove that $K_g \subset K_f$.

5. Examples

In this section we are going to prove that our Theorem 4.1 is optimal, we mean that there exists a map $f : \mathbb{B} \to \mathbb{C}^k$ which satisfies all the hypotheses of Theorem 4.1 such that it has a forward invariant compact and connected set containing 0 which touches the boundary of $\mathbb{B}$ but it does not admit a totally invariant compact and connected set containing 0 which touches the boundary of $\mathbb{B}$.

Example 5.1. Let f be the following Hénon map:

$$f(z, w) = \left(z^2 + w, z\right).$$

Then $f(0, 0) = (0, 0)$ and

$$Jac(f) = \begin{pmatrix} 2z & 1 \\ 1 & 0 \end{pmatrix}.$$

So, at 0, $\lambda_1 = 1$, $\lambda_2 = -1$, i.e. $|\lambda_j| = 1$ for $j = 1, 2$. Clearly $f \in Aut(\mathbb{C}^2)$. From the well-known study of the dynamics of f, there exist the following closed invariant subsets of $\mathbb{C}^2$:

$$K_f^+ = \left\{z \in \mathbb{C}^2 \;\middle|\; f^n(z) \text{ is bounded}\right\},$$
$$K_f^- = \left\{z \in \mathbb{C}^2 \;\middle|\; f^{-n}(z) \text{ is bounded}\right\}$$

and the following compact set of $\mathbb{C}^2$ containing 0:

$$K = K_f^+ \cap K_f^-.$$

Consider a ball $\mathbb{B}(0, R) \subset \mathbb{C}^2$ with $R \gg 1$ such that $\mathbb{B}(0, R) \Supset K$. If we consider the restriction $f : \mathbb{B}(0, R) \to \mathbb{C}^2$, by Theorem 4.1 there exists a connected compact subset X of $\mathbb{B}(0, R)$ which touches the $\partial\mathbb{B}(0, R)$, which is f-invariant and which contains 0. For any such X, we have $X \subset K_f^+$ [17], because if $z \in X$, $f^n(z)$ is bounded since X is f-invariant and compact. Hence $X \subset (K_f^+ \cap \overline{\mathbb{B}(0, R)})$. It is well known from the study of the dynamics of Hénon maps that:

$$dist\left(f^n(X), K\right) \to 0$$

uniformly on compact sets. Hence there exists $n_0 \in \mathbb{N}$ such that $dist(f^{n_0}(X), K) < \frac{1}{2} \cdot dist(K, \partial\mathbb{B}(0, R))$. So X cannot be at the same time forward and backward invariant i.e. $f(X) \subset X$, but $f(X) \neq X$.

If $f^{n_0}(X)$ is distant from K less than $dist(\partial\mathbb{B}(0, R), K)$, then it means that $f^{n_0}(X) \subset X$ and they are different.

Hence, if we consider $g := f^{n_0}$, then $g(X) \Subset X$.

Example 5.2. In some cases it is possible that the forward and the backward invariant compact sets coincide. For example, if in the previous example we consider a ball $\overline{\mathbb{B}}(0, r)$ which contains $K = K_f^+ \cap K_f^-$ and such that $K \cap \partial\mathbb{B}(0, r) \neq \emptyset$, then the restriction of the Hénon map f to $\mathbb{B}(0, r)$ admits a forward and backward invariant compact set K which touches the boundary of $\mathbb{B}(0, r)$.

Remark 5.3. Let K be one of the f-invariant, connected and compact set of Theorem 4.1, and let $X = \bigcap_{n \in \mathbb{N}} f^n(K)$ [5]. The set X is connected because it is a decreasing intersection of connected sets, $X \ni 0$, X is compact and $f(X) = X$. For example if f is an Hénon map, $X = K_f^+ \cap K_f^-$.

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