ON QUATERNIONIC TORI AND THEIR MODULI SPACE

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Abstract. Quaternionic tori are defined as quotients of the skew field $\mathbb{H}$ of quaternions by rank-4 lattices. Using slice regular functions, these tori are endowed with natural structures of quaternionic manifolds (in fact quaternionic curves), and a moduli space in a 12-dimensional real space is then constructed to classify them up to biregular diffeomorphisms. The points of the moduli space correspond to suitable *special* bases of rank-4 lattices, which are studied with respect to the action of the group $GL(4, \mathbb{Z})$, and up to biregular diffeomorphisms. All tori with a non trivial group of biregular automorphisms - and all possible groups of their biregular automorphisms - are then identified, and recognized to correspond to five different subsets of boundary points of the moduli space.

1. Introduction

A new notion of regularity for quaternion-valued functions of a quaternionic variable was introduced in 2006, by Gentili and Struppa in [19, 20]. This newly defined class of *slice regular functions* (often called simply *regular functions*) has already proved to be a good candidate in the search for a quaternionic counterpart of complex holomorphic functions. The relative theory, presented in detail in the monograph [18], has been applied to the study of a non-commutative functional calculus, (see for example the monograph [8] and references therein) and to address the problem of the construction and classification of orthogonal complex structures in open subsets of the space $\mathbb{H}$ of quaternions (see [15]). Recent results of geometric theory of regular functions appear in [5],[6], [3],[4], [11], [12], [16], [21], [27].

In this quaternionic setting, the Casorati-Weierstrass Theorem was proved in [32] and it allowed the study of the group $Aut(\mathbb{H})$ of all biregular transformations of the space of quaternions $\mathbb{H}$. This group turned out to coincide with the group of all affine transformations of $\mathbb{H}$ of the form $q \mapsto qa + b$, with $a, b \in \mathbb{H}$ and $a \neq 0$. As we can see, notwithstanding the fact that the group $Aut(\mathbb{C}^2)$ of biholomorphic transformations of $\mathbb{C}^2$ is still unknown, that of biregular transformations of $\mathbb{H} \cong \mathbb{C}^2$ inherits the simplicity of the group $Aut(\mathbb{C})$.

The knowledge of the group $Aut(\mathbb{H})$ permits the direct construction of a class of natural, and interesting, *quaternionic manifolds* (actually *quaternionic curves*): the quaternionic tori. These tori are studied in the present paper; together with the

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quaternionic projective spaces, [28], they are among the few directly constructed quaternionic manifolds, and bear with them the genuine interest that accompanies any analog of elliptic complex Riemann surfaces. In fact it is worthwhile recalling that, as the distinguished Italian mathematician Guido Castelnuovo stated, the theory of elliptic curves is one of the "jewels" of 19-th Century mathematics.

In this paper we construct quaternionic tori, realized as quotients of $\mathbb{H}$ with respect to rank-4 lattices, and endow them with natural structures of quaternionic 1-dimensional manifolds. We then use the basic features of quaternionic regular maps to characterize biregularly diffeomorphic tori by means of properties of their generating lattices; this approach introduces into the scenery the group $GL(4,\mathbb{Z})$, that plays a fundamental role in this context. In fact the use of classical results on the reduction of Gram matrices, based on the Minkowski-Siegel Reduction Algorithm, allows us to express any "normalized" rank-4 lattice of $\mathbb{H}$ in terms of a generating *special basis.* These special bases parameterize the classes of equivalence of biregular diffeomorphism of quaternionic tori, and suggest to define their moduli space (see (7.2)) as the subset of $\mathbb{H}^3 \cong \mathbb{R}^{12}$

$$\mathcal{M} = \{(v_2, v_3, v_4) \in \mathbb{H}^3 : \{1, v_2, v_3, v_4\} \text{ is a special basis}\}.$$

Notice that special bases have properties that urge a comparison with the complex case of elliptic curves:

Proposition 1.1. *If $\{1, v_2, v_3, v_4\}$ is a special basis of a rank-4 lattice, i.e., if $(v_2, v_3, v_4) \in \mathcal{M}$, then*

(1) $1 \leq \langle v_2, v_2\rangle \leq \langle v_3, v_3\rangle \leq \langle v_4, v_4\rangle$*;*
(2) $-\frac{1}{2} \leq Re(v_k) \leq \frac{1}{2}$*, for all* $k = 2, 3, 4$*;*
(3) $-\frac{1}{2}\langle v_l, v_l\rangle \leq \langle v_k, v_l\rangle \leq \frac{1}{2}\langle v_l, v_l\rangle$*, for all* $(k,l) \in \{2,3,4\} \times \{2,3,4\}$ *such that* $l \neq k$*.*

The moduli space $\mathcal{M}$ is not a fundamental domain for the equivalence relation of biregular diffeomorphism of quaternionic tori. In fact there are different moduli, belonging to the boundary $\partial\mathcal{M}$ of the moduli space, that correspond to the same torus; as an example we can take the distinct points (i, j, k) and (j, i, k): the two special bases $\{1, i, j, k\}$ and $\{1, j, i, k\}$ generate the same lattice (the ring of Lipschitz quaternions) and hence the same torus. However, in (7.4) we define the proper subset $\mathcal{T}$ of the moduli space that turns out to be a fundamental domain (and in turn a moduli space) for the subset of equivalence classes of the so called *tame* tori. The complete quotient of the boundary $\partial\mathcal{M}$, with respect to the equivalence relation of biregular diffeomorphism of the corresponding tori, is still unknown. However, as it happens in the complex case of elliptic curves, the classification of all the boundary tori of $\partial\mathcal{M}$ having non trivial groups of biregular automorphisms is of great geometrical interest, and is an important step towards the understanding of the subtle features of the geometry of the moduli space.

In this perspective, by exploiting the classification of the finite subgroups of unitary quaternions, we identify all the groups that can play the role of groups of biregular automorphisms of tori, i.e.,

$$2\mathbb{T}, \quad 2D_4, \quad 2D_6, \quad 2C_1, \quad 2C_2, \quad 2C_3,$$

called respectively *tetrahedral,* 8*-dihedral,* 12*-dihedral, trivial-cyclic, cyclic-dihedral* and *cyclic* group. We then find those points of the boundary of the moduli space $\mathcal{M}$ which correspond to tori whose group of biregular automorphisms either contains, or is isomorphic to, one of the groups listed above. Establishing that the equivalence class of biregular diffeomorphism of a boundary torus inherits the name of the group of its biregular automorphisms, we then prove

Theorem 1.2. *The only quaternionic tori T with a non trivial group of biregular automorphisms $Aut(T) \neq \{\pm 1\}$ correspond to the following moduli of $\mathcal{M}$:*

- $(I, \alpha_3, \alpha_3 I) \in \mathcal{M}$ *with* $|\alpha_3| \geq 1$ *and* $I^2 = -1$ [$2C_2 \subseteq Aut(T)$];
- $(e^{\frac{\pi I}{3}} \alpha_3, \alpha_3 e^{\frac{\pi I}{3}}) \in \mathcal{M}$ *with* $|\alpha_3| \geq 1$ *and* $I^2 = -1$ [$2C_3 \subseteq Aut(T)$].

The most structured groups of biregular automorphisms appear in the tori with the richest symmetries: the 8-dihedral torus (generated by the lattice of *Lipschitz quaternions*), the 12-dihedral torus, and the tetrahedral torus (the latter generated by the lattice of *Hurwitz quaternions*).

We recall that, in the complex case, the tori (or lattices) with non trivial group of holomorphic automorphisms, corresponding to the moduli i and $e^{\frac{\pi i}{3}}$, are called respectively *harmonic* and *equianharmonic* tori.

Appendices present a computational approach to the study of the modulus of a torus: for example, in the last appendix, an algorithm is produced that checks if a given basis of a lattice is tame.

2. Preliminary results

In this section we will briefly present those results on slice regular functions that are essential for what follows.

The 4-dimensional real algebra of quaternions is denoted by $\mathbb{H}$. An element q in $\mathbb{H}$ can be expressed in terms of the standard basis, denoted by $\{1, i, j, k\}$, as $q = x_0 + x_1 i + x_2 j + x_3 k$, where i, j, k are imaginary units, $i^2 = j^2 = k^2 = -1$, related by the multiplication rule $ij = k$. To every non-real quaternion $q \in \mathbb{H} \setminus \mathbb{R}$ we can associate an imaginary unit, with the map

$$q \mapsto I_q = \frac{Im(q)}{|Im(q)|}.$$

If instead $q \in \mathbb{R}$, we can set I_q to be any arbitrary imaginary unit. In this way, for any $q \in \mathbb{H}$ there exist, and are unique, $x, y \in \mathbb{R}$, with $y \geq 0$ ($y = 0$ if $q \in \mathbb{R}$), such that

$$q = x + yI_q.$$

The set of all imaginary units is denoted by $\mathbb{S}$,

$$\mathbb{S} = \{q \in \mathbb{H} \,|\, q^2 = -1\}$$

and, from a topological point of view, it is a 2-dimensional sphere sitting in the 3-dimensional space of purely imaginary quaternions. The symbol $\mathbb{B}$ will denote the open unit ball $\{q \in \mathbb{H} : |q| < 1\}$ of the space $\mathbb{H}$ of quaternions, and the 3-sphere of all the points of its boundary $\partial\mathbb{B}$ will be denoted by $\mathbb{S}^3$.

To each element I of $\mathbb{S}$ there corresponds a copy of the complex plane, namely $L_I = \mathbb{R} + I\mathbb{R} \cong \mathbb{C}$. All these complex planes, also called *slices*, intersect along the real axis, and their union gives back the space of quaternions,

$$\mathbb{H} = \bigcup_{I \in \mathbb{S}} (\mathbb{R} + I\mathbb{R}) = \bigcup_{I \in \mathbb{S}} L_I.$$

The following definition appears in [19, 20].

Definition 2.1. Let Ω be a domain in $\mathbb{H}$ and let $f : \Omega \to \mathbb{H}$ be a function. For all $I \in \mathbb{S}$ let us consider $\Omega_I = \Omega \cap L_I$ and $f_I = f_{|\Omega_I}$. The function f is called *(slice) regular* if, for all $I \in \mathbb{S}$, the restriction f_I has continuous derivatives and the function $\bar{\partial}_I f : \Omega_I \to \mathbb{H}$ defined by

$$\bar{\partial}_I f(x+Iy) = \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x+Iy)$$

vanishes identically.

The same articles introduce the *Cullen derivative* $\partial_c f$ of a slice regular function f as

$$\partial_c f(x+Iy) = \frac{1}{2}\left(\frac{\partial}{\partial x} - I\frac{\partial}{\partial y}\right) f(x+Iy) \tag{2.1}$$

for $I \in \mathbb{S},\ x, y \in \mathbb{R}$.

Remark 2.2. The Cullen derivative of a slice regular function turns out to be still a slice regular function (see, e.g, [18]).

Using the Cullen derivative, it is possible to characterize the slice regular functions defined in the entire space $\mathbb{H}$, or on a ball $B(0,R) = \{q \in \mathbb{H} : |q| < R\}$ centered at $0 \in \mathbb{H}$, as follows (see, e.g., [18]).

Theorem 2.3. *A function f is regular in $B(0,R)$ if and only if f has a power series expansion*

$$f(q) = \sum_{n\geq 0} q^n a_n \quad \textit{with} \quad a_n = \frac{1}{n!}\frac{\partial^n f}{\partial x^n}(0)$$

converging in $B(0,R)$. Moreover its Cullen derivative can be expressed as

$$\partial_c f(q) = \sum_{n\geq 0} q^n b_n \quad \textit{with} \quad b_n = \frac{1}{n!}\frac{\partial^{n+1} f}{\partial x^{n+1}}(0)$$

in $B(0,R)$.

The existence of the power series expansion yields a Liouville Theorem, that we will use in the sequel:

Theorem 2.4 (Liouville)**.** *Let $f : \mathbb{H} \to \mathbb{H}$ be slice regular. If f is bounded then f is constant.*

3. Lattices in the space of quaternions

Let $\omega_1, \dots, \omega_m$ (with $m \leq 4$) be $\mathbb{R}$-linearly independent vectors in $\mathbb{H}$.

Definition 3.1. The additive subgroup of $(\mathbb{H}, +)$ generated by $\omega_1, \dots, \omega_m$ is called a *rank-m lattice, generated* by $\omega_1, \dots, \omega_m$.

We will focus our attention on (topologically) discrete subgroups of $(\mathbb{H}, +)$, for which the following result holds:

Lemma 3.2. *Let M be a discrete (infinite) subgroup of $(\mathbb{H}, +)$. Then M has no accumulation points.*

Proof Since M is discrete, there are no accumulation points of M belonging to M. By contradiction, assume that there exists an accumulation point q of M, belonging to $\mathbb{H} \setminus M$. Then there exists a sequence $\{q_n\}_{n\in\mathbb{N}} \subseteq M$ converging to q. Since $\{q_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, for all $m \in \mathbb{N}$, there exist $r_m,\ s_m \in \mathbb{N}$ such that

$$|q_{r_m} - q_{s_m}| < \frac{1}{m}. \tag{3.1}$$

If we define $\alpha_m = q_{r_m} - q_{s_m} \in M$, inequality (3.1) would imply that $0 \in M$ is not isolated, being the limit of $\{\alpha_m\}_{m\in\mathbb{N}}$ when $m \to +\infty$. □

As a straightforward consequence we obtain:

Corollary 3.3. *Let M be a subgroup of $(\mathbb{H}, +)$, let $R \in \mathbb{R}^+$ and let $\overline{B(0,R)}$ be the closure of $B(0,R)$. Then M is discrete if and only if, for all $R \in \mathbb{R}^+$, the intersection $\overline{B(0,R)} \cap M$ is a finite set.*

The following classical characterization of discrete subgroups of $(\mathbb{H}, +)$ will be used as a basic fact in the sequel (for a proof see, e.g., [31]).

Theorem 3.4. *A subgroup of $(\mathbb{H}, +)$ is a lattice if and only if it is discrete.*

This last theorem implies that the study of all possible quotient spaces of $(\mathbb{H}, +)$ with respect to a discrete additive subgroup M is reduced to the case in which M is a lattice. With the aim of classifying these quotients, let T^m denote the direct product of m copies of the unit circle S of $\mathbb{R}^2$, and call it the *m-dimensional torus*.

It is well known that, given the rank m of a lattice in $\mathbb{H} \cong \mathbb{R}^4$, there exists only "one" quotient, up to real diffeomorphisms (see, e.g., [31]):

Theorem 3.5. *Let L be a rank-m lattice in $\mathbb{H}$ (with $m \leq 4$). Then the group $\mathbb{H}/L$ is isomorphic to $T^m \times \mathbb{R}^{4-m}$.*

The case of a rank-4 lattice is the one in which the quotient originates "the" real 4-dimensional torus:

Corollary 3.6. *Let L be a rank-4 lattice in $\mathbb{H}$. Then the group $\mathbb{H}/L$ is isomorphic to T^4.*

As we can see, up to real diffeomorphisms the classification is quite simple. If one recalls the beautiful theory of complex elliptic functions - which leads, among other things, to the study of the space of moduli of complex tori (see, e.g., [29], [33]) - he will be inspired to classify the 4-(real)-dimensional, quaternionic tori, up to biregular diffeomorphisms. In the next section we will define slice quaternionic structures on tori, that will be the object of our classification.

4. A regular quaternionic structure on a 4-(real)-dimensional torus

We will give a differential structure on the 4-dimensional torus, whose change of coordinates are regular functions. This structure will be called *a regular quaternionic structure* or simply a *quaternionic* structure on T^4. A torus T^4 endowed with a quaternionic structure will be called a *quaternionic torus*.

To do this we will first of all consider the classical atlas $\mathcal{U}$ of the real torus T^4. Let L be a rank-4 lattice of $\mathbb{H}$, generated by $\omega_1, \omega_2, \omega_3, \omega_4$. Consider the canonical projection $\pi : \mathbb{R}^4 \cong \mathbb{H} \to \mathbb{H}/L = T^4$ and, for any $p \in \mathbb{H}$, an open neighborhood U_p of p small enough to make $\pi_{|U_p}$ an homeomorphism of U_p onto its image $\pi(U_p)$ (see, e.g., [7, 14]). The atlas $\mathcal{U}$ will consist of the local coordinate systems $\left\{\left(\pi(U_p), (\pi_{|U_p})^{-1}\right)\right\}_{p\in\mathbb{H}}$. If we suppose that, for $p, q \in \mathbb{H}$, the intersection $\pi(U_p) \cap \pi(U_q)$ is (open and) connected, then the change of coordinates is such that $\pi^{-1}_{|U_p} \circ \pi_{|U_q}(x) = x + \sum_{l=1}^{4} n_l\omega_l$ for fixed n_1, n_2, n_3, n_4. Hence the change of coordinates is a regular function. Therefore we obtain a quaternionic structure on T^4. Using the classical approach, we can now give the following.

Definition 4.1. Let T_1^4 and T_2^4 be two 4-(real)-dimensional tori. A continuous map $f : T_1^4 \to T_2^4$ is said *regular at* $w \in T_1^4$ if there exist a local coordinate system (C, φ) in a neighborhood of $w \in T_1^4$ and a local coordinate system (D, ψ) in a neighborhood of $f(w) \in T_2^4$ such that $\psi \circ f \circ \varphi^{-1}$ is regular. The function f is *regular* if it is regular for all $w \in T_1^4$. A regular homeomorphism f from T_1^4 to T_2^4, whose inverse is regular, is called a *biregular diffeomorphism of* T_1^4 *onto* T_2^4. Finally, a biregular diffeomorphism g of T_1^4 onto itself is called a *biregular automorphism of* T_1^4.

We are now going to study the quaternonic tori, up to biregular diffeomorphisms. More precisely we will give the following:

Definition 4.2. If there is a biregular diffeomorphism of a 4-(real)-dimensional torus T_1^4 onto a (4-(real)-dimensional) torus T_2^4 we will say that the two tori are *equivalent*.

To proceed, we recall that the group $Aut(\mathbb{H})$ of biregular transformations (or automorphisms) of $\mathbb{H}$ consists of all slice regular affine transformations, that is

$$Aut(\mathbb{H}) = \{f(q) = qa + b : a, b \in \mathbb{H}, a \neq 0\}$$

(see [30], [32]).

The result stated in the next proposition has a complete analog in the complex setting, [14]. Nevertheless we will produce a proof, to acquire familiarity with the quaternionic environment.

Proposition 4.3. *Let L_1 and L_2 be two rank-4 lattices in $\mathbb{H}$, let $\pi_1 : \mathbb{H} \to \mathbb{H}/L_1 = T_1^4$ and $\pi_2 : \mathbb{H} \to \mathbb{H}/L_2 = T_2^4$ be the projections on the quotient tori. For any $F \in Aut(\mathbb{H})$ such that $F(L_1) = L_2$ there exists a biregular diffeomorphism f of T_1^4 onto T_2^4 which allows the equality $f \circ \pi_1 = \pi_2 \circ F$. Conversely, for any biregular diffeomorphism f of T_1^4 onto T_2^4, there exists $F \in Aut(\mathbb{H})$ such that $f \circ \pi_1 = \pi_2 \circ F$ and $F(L_1) = L_2$.*

Proof.

$$\begin{tikzcd} \mathbb{H} \arrow[r, "F"] \arrow[d, "\pi_1"'] & \mathbb{H} \arrow[d, "\pi_2"] \\ T_1^4 \arrow[r, "f"'] & T_2^4 \end{tikzcd} \tag{4.1}$$

Let $F(v) = va + b$. Since $0 \in L_1$ we have $F(0) = b \in L_2$ and hence we can suppose $b = 0$. By definition of regular map between tori, to show that $F(v) = va$ induces a biregular diffeomorphism f of T_1^4 onto T_2^4, it is enough to show that $q \sim p$ implies

$F(q) \sim F(p)$. Indeed, if $q \sim p$ then $q - p \in L_1$ and hence $F(q) - F(p) = qa - pa = (q-p)a = F(q-p) \in L_2$.

To prove the converse statement, we start by recalling that the map $f : T_1^4 \to T_2^4$ lifts to a continuous map $F : \mathbb{H} \to \mathbb{H}$, in such a way that the diagram (4.1) commutes. This latter statement depends on the fact that $(\mathbb{H}, \pi_1)$ is a (topological) universal covering of T_1^4 and hence, if Π_1 denotes the Poincaré fundamental group, we have $(f \circ \pi_1)_*(\Pi_1(\mathbb{H})) = \mathbf{1}$ (for details on this matter see, e.g., [25]). Moreover the map F is regular since f is a regular map of T_1^4 onto T_2^4.

For any $\lambda \in L_1$, consider $G_\lambda(q) = F(q+\lambda) - F(q)$. Since F lifts a map between the quotients, G_λ maps L_1-equivalent points into L_2-equivalent points. Hence its image is contained in the (discrete, see Theorem 3.4) lattice L_2 and, being continuous, is therefore constant. At this point it is clear that, taking the Cullen derivative, we obtain $\partial_c F(q+\lambda) = \partial_c F(q)$, for all $q \in \mathbb{H}$. Thus the map $\partial_c F$ is regular (see Remark 2.2) and L_1-periodic, which makes it bounded. By the Liouville Theorem for regular functions (see Theorem 2.4) the Cullen derivative $\partial_c F$ of F is constant. Since F expands as a power series (see Theorem 2.3)

$$F(q) = \sum_{n \in \mathbb{N}} q^n \frac{1}{n!} \frac{\partial^n F}{\partial x^n}(0)$$

converging in the entire $\mathbb{H}$, we obtain (again by Theorem 2.3)

$$\partial_c F(q) = \sum_{n \in \mathbb{N}} q^n \frac{1}{n!} \frac{\partial^{n+1} F}{\partial x^{n+1}}(0) = \frac{\partial F}{\partial x}(0)$$

and hence

$$F(q) = F(0) + q\frac{\partial F}{\partial x}(0) = b + qa$$

is a first degree regular polynomial. Again, since F lifts a map between quotients, necessarily $L_1 a \subseteq L_2$. If the inclusion $L_1 a \subset L_2$ is proper, then f is not injective: indeed if some $q \in L_2$ satisfies $qa^{-1} \notin L_1$ then $(qa^{-1} + L_1) \neq L_1$ and $f(qa^{-1} + L_1) = \pi_2(q + L_1 a + b) = \pi_2(L_2) = f(L_1)$.

Now we know that $L_1 a = L_2$, that is $L_2 a^{-1} = L_1$. The map $F^{-1} : \mathbb{H} \to \mathbb{H}$ defined by $F^{-1}(w) = (w-b)a^{-1}$ induces the map $f^{-1} : T_2^4 \to T_1^4$. Indeed

$$f^{-1}(f(q + L_1)) = f^{-1}(qa + b + L_2) = (qa + L_2)a^{-1} = q + L_2 a^{-1} = q + L_1.$$

This concludes the proof. □

5. Equivalence of quaternionic tori

To classify the 4-(real)-dimensional, quaternionic tori, up to biregular diffeomorphisms, we start with the following:

Theorem 5.1. *Two rank-4 lattices L_1, L_2 of the space $\mathbb{H}$, generated respectively by the bases $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and $\{\omega_1, \omega_2, \omega_3, \omega_4\}$, determine equivalent tori T_1^4, T_2^4 if and only if there exist $a \in \mathbb{H}^* = \mathbb{H} \setminus \{0\}$ and a linear transformation $A \in GL(4, \mathbb{Z})$ such that*

$$A \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} a$$

Proof. By Proposition 4.3, if f is a biregular diffeomorphism of T_1^4 onto T_2^4, then there exists a biregular transformation F of $\mathbb{H}$ such that the diagram (4.1) commutes. Since F is biregular on $\mathbb{H}$, then $F(q) = qa + b$, with $a \in \mathbb{H}^*$, and $b \in \mathbb{H}$. As we pointed out in the proof of Proposition 4.3, without loss of generality, we can suppose both that $b = 0$ and that the function F maps the set of generators of L_1 to a set of generators of L_2. Taking into account that

$$\begin{cases} F(\alpha_1) &= \alpha_1 a \\ F(\alpha_2) &= \alpha_2 a \\ F(\alpha_3) &= \alpha_3 a \\ F(\alpha_4) &= \alpha_4 a \end{cases}$$

there exists a matrix

$$A = \begin{pmatrix} n_{11} & n_{12} & n_{13} & n_{14} \\ n_{21} & n_{22} & n_{23} & n_{24} \\ n_{31} & n_{32} & n_{33} & n_{34} \\ n_{41} & n_{42} & n_{43} & n_{44} \end{pmatrix}$$

with integer entries, such that

$$\begin{cases} \alpha_1 a = & n_{11}\omega_1 & + & n_{12}\omega_2 & + & n_{13}\omega_3 & + & n_{14}\omega_4 \\ \alpha_2 a = & n_{21}\omega_1 & + & n_{22}\omega_2 & + & n_{23}\omega_3 & + & n_{24}\omega_4 \\ \alpha_3 a = & n_{31}\omega_1 & + & n_{32}\omega_2 & + & n_{33}\omega_3 & + & n_{34}\omega_4 \\ \alpha_4 a = & n_{41}\omega_1 & + & n_{42}\omega_2 & + & n_{43}\omega_3 & + & n_{44}\omega_4 \end{cases} \tag{5.1}$$

or, more concisely,

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} a = A \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{pmatrix}. \tag{5.2}$$

The same argument applied in the opposite direction, implies the existence of a matrix B with integer entries such that

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{pmatrix} a^{-1} = B \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} \tag{5.3}$$

and hence, substituting equation (5.3) into equation (5.2), we get

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} a = AB \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} a$$

which implies $AB = I_4$ and hence that A (and B) is such that $det(A) = \pm 1$, i.e. A (and B) belongs to $GL(4, \mathbb{Z})$.

On the other side, suppose there exists a matrix $A \in GL(4, \mathbb{Z})$, of this form:

$$A = \begin{pmatrix} n_{11} & n_{12} & n_{13} & n_{14} \\ n_{21} & n_{22} & n_{23} & n_{24} \\ n_{31} & n_{32} & n_{33} & n_{34} \\ n_{41} & n_{42} & n_{43} & n_{44} \end{pmatrix}$$

such that

$$A\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\end{pmatrix}=\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{pmatrix}a$$

then we can compute $F(q)$ in four different expressions:

$$\begin{aligned}
F(q) &= q\alpha_1^{-1}(n_{11}\omega_1+n_{12}\omega_2+n_{13}\omega_3+n_{14}\omega_4)\\
F(q) &= q\alpha_2^{-1}(n_{21}\omega_1+n_{22}\omega_2+n_{23}\omega_3+n_{24}\omega_4)\\
F(q) &= q\alpha_3^{-1}(n_{31}\omega_1+n_{32}\omega_2+n_{33}\omega_3+n_{34}\omega_4)\\
F(q) &= q\alpha_4^{-1}(n_{41}\omega_1+n_{42}\omega_2+n_{43}\omega_3+n_{44}\omega_4)
\end{aligned}$$

for all $q\in\mathbb{H}$. Simple computations show that:

$$F(q+\alpha_1)=F(q)+n_{11}\omega_1+n_{12}\omega_2+n_{13}\omega_3+n_{14}\omega_4,$$

$$F(q+\alpha_2)=F(q)+n_{21}\omega_1+n_{22}\omega_2+n_{23}\omega_3+n_{24}\omega_4,$$

$$F(q+\alpha_3)=F(q)+n_{31}\omega_1+n_{32}\omega_2+n_{32}\omega_3+n_{34}\omega_4,$$

$$F(q+\alpha_4)=F(q)+n_{41}\omega_1+n_{42}\omega_2+n_{43}\omega_3+n_{44}\omega_4.$$

Hence F defines a biregular diffeomorphism f between T_1^4 and T_2^4. □

It is natural at this point to give the following

Definition 5.2. Two rank-4 lattices $L_1,\ L_2$ of the space $\mathbb{H}$ are called *equivalent* if the generated quaternionic tori $\mathbb{H}/L_1$ and $\mathbb{H}/L_2$ are equivalent. A basis $\{\omega_1,\omega_2,\omega_3,\omega_4\}$ of a rank-4 lattice L_1 and a basis $\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$ of a rank-4 lattice L_2 are called *equivalent* if L_1 and L_2 are equivalent lattices, i.e. if (according to Theorem 5.1) there exist $a\in\mathbb{H}^*$ and a linear transformation $A\in GL(4,\mathbb{Z})$ such that

$$A\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\end{pmatrix}=\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{pmatrix}a. \tag{5.4}$$

Notice that two (different) equivalent bases $\{\omega_1,\omega_2,\omega_3,\omega_4\}$ and $\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$ of rank-4 lattices may generate exactly the same lattice, and hence exactly the same quaternionic torus. This happens when there exists a linear transformation $A\in GL(4,\mathbb{Z})$ such that

$$A\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\end{pmatrix}=\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{pmatrix}$$

i.e., when $a=1$ in (5.4).

6. Minkowski-Siegel Reduction Algorithm: reduced and special bases

In this section we will specialize to the quaternionic setting the general Minkowski-Siegel Reduction Algorithm presented in [22, 24], and use it to construct reduced Gram matrices and *reduced* bases associated to lattices. In turn, reduced bases will be used to find *special* bases for lattices, useful in the sequel to identify and parameterize equivalence classes of quaternionic tori.

We explicitly present here some basic facts of this algorithmic construction, both to make the paper as much self-contained as possible, and to have a starting point for the proofs of the results that will follow.

Let $\langle \cdot, \cdot \rangle$ denote the usual scalar product of $\mathbb{R}^4$. Let $p = x_0 + x_1 i + x_2 j + x_3 k$ and $q = y_0 + y_1 i + y_2 j + y_3 k$ be two quaternions. We set, and use in what follows,

$$\langle p, q \rangle = \sum_{\ell=0}^{3} x_\ell y_\ell. \tag{6.1}$$

Let $\{v_1, v_2, v_3, v_4\}$ be a basis of the lattice $L \subset \mathbb{H} \cong \mathbb{R}^4$. For any $u = (n_1, n_2, n_3, n_4) \in \mathbb{Z}^4$ the squared norm of the element $v = n_1 v_1 + n_2 v_2 + n_3 v_3 + n_4 v_4 \in L$ can be expressed by $\langle v, v \rangle = v\ {}^t v = u S_0\ {}^t u$ where the matrix

$$S_0 = \begin{pmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle & \langle v_1, v_3 \rangle & \langle v_1, v_4 \rangle \\ \langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle & \langle v_2, v_3 \rangle & \langle v_2, v_4 \rangle \\ \langle v_3, v_1 \rangle & \langle v_3, v_2 \rangle & \langle v_3, v_3 \rangle & \langle v_3, v_4 \rangle \\ \langle v_4, v_1 \rangle & \langle v_4, v_2 \rangle & \langle v_4, v_3 \rangle & \langle v_4, v_4 \rangle \end{pmatrix} \tag{6.2}$$

is symmetric and positive definite, and is usually called the *Gram matrix* associated to the basis $\{v_1, v_2, v_3, v_4\}$. In this setting and with the notations established, we will use the following procedure (see, e.g., [24], page 122):

Algorithm 6.1 (**Minkowski-Siegel Reduction Algorithm**).

This algorithm acts on a Gram matrix S_0 and produces a matrix $U = U(S_0)$ belonging to $GL(4, \mathbb{Z})$ and a Gram matrix $R = R(S_0) = U S_0\ {}^t U$. Here are the steps of the algorithm:

- The Gram matrix S_0 (of a certain basis $\{v_1, v_2, v_3, v_4\}$) is given.
- Consider the function $Q_1 : \mathbb{Z}^4 \to \mathbb{R}^+$ defined as
$$Q_1(u) = u S_0\ {}^t u.$$
By our assumption, Q_1 attains its strictly positive minimum value at a point $u_1 = (n_{11}, n_{12}, n_{13}, n_{14}) \in \mathbb{Z}^4$.
- To proceed, we need to recall that there exist infinitely many matrices of $GL(4, \mathbb{Z})$ having the first row equal to u_1 (see e.g. [22, 24] for a proof of this assertion, and of the analogous ones, used in this algorithm). With this in mind, we consider the function Q_2 obtained by restricting Q_1 to the elements $u \in \mathbb{Z}^4$ such that there exists a matrix of $GL(4, \mathbb{Z})$ having the first two rows equal to u_1 and u, respectively. Let $u_2 = (n_{21}, n_{22}, n_{23}, n_{24}) \in \mathbb{Z}^4$ be a point in which Q_2 attains its strictly positive minimum value. Up to a change of sign, we can assume that $u_1 S_0\ {}^t u_2 \geq 0$.
- In the next step, we consider the restriction Q_3 of Q_2 to the elements $u \in \mathbb{Z}^4$ such that there exists a matrix of $GL(4, \mathbb{Z})$ having the first three rows equal to u_1, u_2 and u, respectively. Let $u_3 = (n_{31}, n_{32}, n_{33}, n_{34}) \in \mathbb{Z}^4$ be a point

in which Q_3 attains its strictly positive minimum value. Again, up to a change of sign, we can assume that $u_2 S_0\, {}^t u_3 \geq 0$.

- Finally, we take the restriction Q_4 of Q_3 to the elements $u \in \mathbb{Z}^4$ such that there exists a matrix of $GL(4,\mathbb{Z})$ having the four rows equal to u_1, u_2, u_3 and u, respectively, and set $u_4 = (n_{41}, n_{42}, n_{43}, n_{44}) \in \mathbb{Z}^4$ to be a point in which Q_4 has a strictly positive minimum value. As before, we can assume $u_3 S_0\, {}^t u_4 \geq 0$.
- The output of the algorithm consists of the matrix

$$U = U(S_0) = \begin{pmatrix} n_{11} & n_{12} & n_{13} & n_{14} \\ n_{21} & n_{22} & n_{23} & n_{24} \\ n_{31} & n_{32} & n_{33} & n_{34} \\ n_{41} & n_{42} & n_{43} & n_{44} \end{pmatrix},$$

(belonging to $GL(4,\mathbb{Z})$ by construction) and of the Gram matrix $R = US_0\, {}^t U$.

Remark 6.2. Let L be a rank-4 lattice. Consider any basis $\{v_1, v_2, v_3, v_4\}$ of L whose Gram matrix is S_0. The Minkowski-Siegel Reduction Algorithm, applied to the Gram matrix S_0, produces a matrix U of $GL(4,\mathbb{Z})$ which can be used to define the four elements

$$\left\{ \begin{array}{rcccccccc} \omega_1 = & n_{11}v_1 & + & n_{12}v_2 & + & n_{13}v_3 & + & n_{14}v_4 \\ \omega_2 = & n_{21}v_1 & + & n_{22}v_2 & + & n_{23}v_3 & + & n_{24}v_4 \\ \omega_3 = & n_{31}v_1 & + & n_{32}v_2 & + & n_{33}v_3 & + & n_{34}v_4 \\ \omega_4 = & n_{41}v_1 & + & n_{42}v_2 & + & n_{43}v_3 & + & n_{44}v_4. \end{array} \right. \tag{6.3}$$

The elements $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ form a basis of L since the matrix U belongs to $GL(4,\mathbb{Z})$, with its inverse. Notice that

$$U \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{pmatrix}$$

and therefore that the two bases $\{v_1, v_2, v_3, v_4\}$ and $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ are equivalent (in particular they generate the same lattice). We conclude the remark by pointing out that the Gram matrix R associated to the basis $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ is obtained as (recall formula (6.1))

$$US_0\, {}^t U = R = \begin{pmatrix} \langle\omega_1,\omega_1\rangle & \langle\omega_1,\omega_2\rangle & \langle\omega_1,\omega_3\rangle & \langle\omega_1,\omega_4\rangle \\ \langle\omega_2,\omega_1\rangle & \langle\omega_2,\omega_2\rangle & \langle\omega_2,\omega_3\rangle & \langle\omega_2,\omega_4\rangle \\ \langle\omega_3,\omega_1\rangle & \langle\omega_3,\omega_2\rangle & \langle\omega_3,\omega_3\rangle & \langle\omega_3,\omega_4\rangle \\ \langle\omega_4,\omega_1\rangle & \langle\omega_4,\omega_2\rangle & \langle\omega_4,\omega_3\rangle & \langle\omega_4,\omega_4\rangle \end{pmatrix}$$

and is therefore independent of the choice of the particular basis $\{v_1, v_2, v_3, v_4\}$ that generates the Gram matrix S_0. In fact the matrix R depends only on the Gram matrix S_0.

As in [22] and [24], we can give the following definition.

Definition 6.3. If R is a Gram matrix obtained by applying to a given Gram matrix S_0 the Minkowski-Siegel Reduction Algorithm 6.1, then R is called a *reduced Gram matrix (relative to S_0)*. The symbol $\mathcal{R}$ will denote the set of all reduced Gram matrices.

To classify rank-4 lattices and generated tori, we will define the set of bases naturally emphasized by Algorithm 6.1.

Definition 6.4. Let L be a rank-4 lattice in $\mathbb{H}$. A basis $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ of L will be called a *reduced basis* if its Gram matrix is reduced.

A direct application of the Minkowski-Siegel Reduction Algorithm and Remark 6.2 prove a first result in the study of equivalence of lattices.

Theorem 6.5. *Let L be a rank-4 lattice and let $\{v_1, v_2, v_3, v_4\}$ be a basis of L. Then there exists a matrix U of $GL(4,\mathbb{Z})$ such that*

$$U \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{pmatrix}$$

is a reduced basis of the lattice L. As a consequence $L = \mathbb{Z}v_1 + \mathbb{Z}v_2 + \mathbb{Z}v_3 + \mathbb{Z}v_4 = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 + \mathbb{Z}\omega_3 + \mathbb{Z}\omega_4$ and we can suppose that the lattice L is generated by a reduced basis.

We will now present the basic features of reduced Gram matrices (and bases). It turns out that there are two necessary and sufficient conditions that characterize the elements of the set $\mathcal{R}$ of reduced Gram matrices; we recall these conditions here (see [24]):

Proposition 6.6. *A Gram matrix $R = (r_{i,j})_{i,j=1,\cdots 4}$ is a reduced Gram matrix if and only if the two following sets of conditions hold:*

B1) $r_{k,k+1} \geq 0$ *for all* $k = 1, 2, 3$;

B2) *for all fixed $k = 1, 2, 3, 4$, we have $(n_1, n_2, n_3, n_4)\ R\ {}^t(n_1, n_2, n_3, n_4) \geq r_{k,k}$ for any integer vector (n_1, n_2, n_3, n_4) such that $n_k, \cdots n_4$ are without common divisors.*

We point out, and we will use in the sequel, the fact that conditions B2) are equivalent to B2)$'$:

B2)$'$ for all fixed $k = 1, 2, 3, 4$, if a vector $(n_1, n_2, n_3, n_4) \in \mathbb{Z}^4$ has the property that $(n_1, n_2, n_3, n_4)\ R\ {}^t(n_1, n_2, n_3, n_4) < r_{k,k}$, then necessarily $n_k, \cdots, n_4$ have common divisors.

Proposition 6.7. *If $R = (r_{i,j})_{i,j=1,\cdots 4}$ is a reduced Gram matrix, then the two following conditions hold:*

(1) $r_{k,k} \leq r_{l,l}$, *for all* $(k,l) \in \{1,2,3,4\} \times \{1,2,3,4\}$ *such that* $l > k$;
(2) $-\frac{1}{2}r_{l,l} \leq r_{k,l} \leq \frac{1}{2}r_{l,l}$, *for all* $(k,l) \in \{1,2,3,4\} \times \{1,2,3,4\}$ *such that* $l \neq k$.

Proof. To verify (1), we use condition B2) at step k, applied to the vector $e_l = (n_1, n_2, n_3, n_4)$ $(l > k)$ where e_l is the l-th vector of the standard basis of $\mathbb{R}^4$. Inequalities (2) are obtained by applying the same condition B2) at step k, to the vector $(n_1, n_2, n_3, n_4) = e_k \pm e_l$ $(l \neq k)$, where e_l, e_k are, respectively, the l-th and k-th vector of the standard basis of $\mathbb{R}^4$ (see also [24], page 123). $\square$

Remark 6.8. Concerning conditions B2) on the Gram matrix $R = (r_{i,j})_{i,j=1,\cdots 4}$, we observe that: for each $k \in \{1, 2, 3, 4\}$, if e_k is the k-th element of the standard basis of $\mathbb{R}^4$, then we obtain the obvious equality $e_k R\ {}^t e_k = r_{k,k}$ that gives no conditions.

We restate here a deep result that appears in [24], adapting it to our setting and notations.

Theorem 6.9. *The set $\mathcal{R}$ of all reduced Gram matrices is a convex cone with $\mathring{\mathcal{R}} \neq \emptyset$. If $\mathcal{G}$ denotes the set of all Gram matrices, then*

$$\mathcal{G} = \bigcup_{U \in GL(4,\mathbb{Z})} U\mathcal{R}\ {}^t U. \tag{6.4}$$

If $U \in GL(4,\mathbb{Z}), U \neq \pm I_4$, and $\mathcal{R} \cap (U\mathcal{R}\ {}^t U) \neq \emptyset$, then $\mathcal{R} \cap (U\mathcal{R}^t U) \subset \partial\mathcal{R}$. Only for a finite set of matrices $U \in GL(4,\mathbb{Z})$ it is possible that $\mathcal{R} \cap (U\mathcal{R}^t U) \neq \emptyset$.

The fact that $\mathring{\mathcal{R}} \neq \emptyset$ is not obvious, and a non constructive proof is given in [24]. Now, our second step in the classification of rank-4 lattices and quaternionic tori makes use of a proper subset of the set of reduced bases.

Definition 6.10. A reduced basis $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ of a rank-4 lattice L with the property that $\omega_1 = 1$ will be called a *special basis.*

Theorem 6.11. *Let L_1 be a rank-4 lattice and let $\{v_1, v_2, v_3, v_4\}$ be a basis of L_1. Then $\{v_1, v_2, v_3, v_4\}$ is equivalent to a special basis $\{1, \omega_2, \omega_3, \omega_4\}$ of a rank-4 lattice L_2. As a consequence $L_1 = \mathbb{Z}v_1 + \mathbb{Z}v_2 + \mathbb{Z}v_3 + \mathbb{Z}v_4$ and $L_2 = \mathbb{Z} + \mathbb{Z}\omega_2 + \mathbb{Z}\omega_3 + \mathbb{Z}\omega_4$ are equivalent, and hence they generate equivalent tori.*

Proof. By Theorem 6.5, there there exist a matrix U of $GL(4,\mathbb{Z})$ and a reduced basis $\{u_1, u_2, u_3, u_4\}$ of the lattice L_1 such that

$$U \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}.$$

Therefore

$$U \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 1 \\ u_2 u_1^{-1} \\ u_3 u_1^{-1} \\ u_4 u_1^{-1} \end{pmatrix} u_1 = \begin{pmatrix} 1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{pmatrix} u_1$$

and, by Definition 5.2, the bases $\{v_1, v_2, v_3, v_4\}$ and $\{1, \omega_2, \omega_3, \omega_4\}$ are equivalent. This latter basis is special, since it is obtained multiplying the reduced basis $\{u_1, u_2, u_3, u_4\}$ on the right by $u_1^{-1} \neq 0$ (rigid motion of $\mathbb{H} \cong \mathbb{R}^4$). □

At this point it is possible to associate to each class of equivalence of quaternionic tori at least a special basis of a rank-4 lattice, according to

Corollary 6.12. *Let T be a quaternionic torus. Then, up to biregular diffeomorphisms, we can suppose that $T = \mathbb{H}/L$ where the lattice L is generated by a special basis $\{1, \omega_2, \omega_3, \omega_4\}$.*

We will now pass to identify a natural and useful subset of the possible bases for rank-4 lattices. Let $p = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}$ and let A be a 4×4 matrix with real coefficients. We set the notation $A(p)$ to denote the quaternion whose real components are

$$A\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{6.5}$$

The following is a useful elementary result of linear algebra:

Proposition 6.13. *If two bases $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ and $\{v_1, v_2, v_3, v_4\}$ have the same Gram matrix, then there exists an orthogonal matrix $B \in O(4, \mathbb{R})$ such that $B(\omega_l) = v_l$ for $l = 1, 2, 3, 4$.*

Proof. Since the two bases have the same Gram matrix we have $\langle \omega_l, \omega_p \rangle = \langle v_l, v_p \rangle$ for $l, p = 1, ..., 4$. Let B be the matrix which transforms the first basis into the second one, then $\langle \omega_l, \omega_p \rangle = \langle B(\omega_l), B(\omega_p) \rangle$. Hence B is an isometry with respect to the standard scalar product and the assertion follows. □

A notion that reveals to be useful to deal with lattices in our setting is the following

Definition 6.14. A lattice L is called *normalized* if $1 \in L$ and if every element of L has norm greater or equal than 1.

Remark 6.15. It can be proved that conditions (1)-(2) of Proposition 6.7 together with B1) are sufficient for a Gram matrix to be a reduced Gram matrix if and only if the associated lattice is normalized.

Example 6.16. We provide an example of a non-normalized lattice L having a basis $\mathcal{B}$ whose Gram matrix satisfies conditions (1)-(2) of Proposition 6.7 together with B1), but is not reduced. In fact L is such that an integer combination of three vectors of $\mathcal{B}$ is inside $\mathbb{B}$. To see this, notice that if $I = \frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j$ and $J = \frac{1}{\sqrt{3}}i + \sqrt{\frac{2}{3}}j$, then $\mathcal{B} = \{1, e^{\frac{\pi}{3}I}, e^{\frac{2\pi}{3}J}, k\}$ is a basis for L whose (approximated) Gram matrix

$$G = \begin{pmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & 1 & (0.4891) & 0 \\ -\frac{1}{2} & (0.4891) & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

satisfies conditions (1)-(2) of Proposition 6.7 and B1). Nevertheless it is easy to see that $(1 - v_1 + v_2) \in \mathbb{B}$, and hence G is not reduced.

7. A moduli space for quaternionic tori. Tame tori

The aim of this section is to find a moduli space to "parameterize" the equivalence classes of quaternionic tori, with respect to the action of biregular diffeomorphisms. We will then study the families of *tame* lattices and *tame* tori, whose definition is inspired by Theorem 6.9, and whose moduli correspond to the interior of the moduli space.

We will start by identifying a useful subset of the set $\mathcal{R}$ of reduced Gram matrices.

Remark 7.1. In the sequel we will always consider reduced Gram matrices associated to special bases, i.e. matrices of the form

$$S_0 = \begin{pmatrix} 1 & \langle 1, v_2\rangle & \langle 1, v_3\rangle & \langle 1, v_4\rangle \\ \langle v_2, 1\rangle & \langle v_2, v_2\rangle & \langle v_2, v_3\rangle & \langle v_2, v_4\rangle \\ \langle v_3, 1\rangle & \langle v_3, v_2\rangle & \langle v_3, v_3\rangle & \langle v_3, v_4\rangle \\ \langle v_4, 1\rangle & \langle v_4, v_2\rangle & \langle v_4, v_3\rangle & \langle v_4, v_4\rangle \end{pmatrix} \tag{7.1}$$

where $\langle v_4, v_4\rangle \geq \langle v_3, v_3\rangle \geq \langle v_2, v_2\rangle \geq 1$. This means, in particular, that we restrict to reduced Gram matrices belonging to a hyperplane of $\mathbb{R}^{10}$. We point out that we will consider, in the boundary of the set of reduced Gram matrices, only those elements that represent rank-4 lattices, and hence only definite positive matrices. Instead, as it appears in [24], when considering the entire set of reduced Gram matrices as a subset of the space of symmetric matrices, then its boundary contains also semi-positive definite, reduced matrices.

The promised space of "parameters" for the equivalence classes of biregular diffeomorphism of quaternionic tori is defined as follows.

Definition 7.2. The set $\mathcal{M}$ defined as

$$\mathcal{M} = \{(v_2, v_3, v_4) \in \mathbb{H}^3 : \{1, v_2, v_3, v_4\} \text{ is a special basis}\} \tag{7.2}$$

is called the *moduli space of quaternionic tori*. Let (v_2, v_3, v_4) be a point of $\mathcal{M}$, and let $L = \mathbb{Z} + \mathbb{Z}v_2 + \mathbb{Z}v_3 + \mathbb{Z}v_4$ be the lattice generated by the special basis $\{1, v_2, v_3, v_4\}$. We will say that (v_2, v_3, v_4) is a *modulus* of any quaternionic torus equivalent to $T = \mathbb{H}/L$.

Corollary 6.12 guarantees a fundamental property of the moduli space:

Proposition 7.3. *Every quaternionic torus T has (at least) a modulus in $\mathcal{M}$. In other words: for every quaternionic torus T, there exists $(v_2, v_3, v_4) \in \mathcal{M}$ such that T is equivalent to $\mathbb{H}/L$, where $L = \mathbb{Z} + \mathbb{Z}v_2 + \mathbb{Z}v_3 + \mathbb{Z}v_4$ is the lattice generated by the special basis $\{1, v_2, v_3, v_4\}$.*

With obvious notations, set now

$$\hat{O}(3, \mathbb{R}) = \begin{bmatrix} 1 & \underline{0} \\ {}^t\underline{0} & O(3, \mathbb{R}) \end{bmatrix}$$

and define

$$\mathcal{S} = \bigcup_{B \in \hat{O}(3,\mathbb{R})} B.\mathcal{M} = \hat{O}(3, \mathbb{R}).\mathcal{M} \tag{7.3}$$

where $B.\mathcal{M}$ means the set of all $B.(v_2, v_3, v_4) = (B(v_2), B(v_3), B(v_4))$, for all $(v_2, v_3, v_4) \in \mathcal{M}$. The moduli space $\mathcal{M}$ has a natural axial symmetry with respect to the real axis, namely we have that

Proposition 7.4. *The moduli space $\mathcal{M}$ and the set $\mathcal{S}$ coincide.*

Proof. The proof is a direct computation. □

The geometric symmetry of the moduli space is interesting, and suggests a remark and a few considerations, that help to identify similarities in the moduli of tori.

Remark 7.5. Denote as usual by $\mathcal{E} = \{1, i, j, k\}$ the standard basis for the space $\mathbb{H}$ of quaternions. Recall that, for every two unitary quaternions $I, J \in \mathbb{S}$ with $I \perp J$, the set $\mathcal{A} = \{1, I, J, IJ = K\}$ is also a (positively oriented) basis for the space $\mathbb{H}$, having the same multiplication rules of the basis $\mathcal{E}$. Let us consider two lattices L_1 and L_2 generated, respectively, by the special bases $\mathcal{V} = \{1, v_2, v_3, v_4\}$ and $\mathcal{W} = \{1, w_2, w_3, w_4\}$. It is clear that, if the coefficients of the elements of $\mathcal{V}$ with respect to the basis $\mathcal{E}$ coincide with the coefficients of the elements of $\mathcal{W}$ with respect to the basis $\mathcal{A}$, then the two generated tori T_1 and T_2 - notwithstanding not equivalent according to our definition - have completely similar structures.

Let L be a lattice, and let $\mathcal{A} = \{1, I, J, IJ = K\}$ be a basis of $\mathbb{H}$. What is stated in Remark 7.5 allows us, when necessary or useful, to consider - without loss of generality - a special basis $\mathcal{V} = \{1, v_2, v_3, v_4\}$ of L such that $v_2 \in \mathbb{R} + I\mathbb{R}$.

It is easy to see (and in any case we will see it later on, in this paper) that there are different moduli belonging to $\partial\mathcal{M}$ that correspond to the same equivalence class of quaternionic tori, or equivalently that there are quaternionic tori having more than one modulus. However, this last phenomenon is not present in the case of the family of quaternionic tori that we are going to define.

Definition 7.6. Let L be a rank-4 lattice in $\mathbb{H}$.

(1) The lattice L is called a *tame lattice* if there exists a reduced basis of L whose Gram matrix is an interior point of $\mathcal{R}$. Such a basis will be called a *tame basis.*

(2) A quaternionic torus T is called a *tame torus* if there exists a tame lattice L such that T is equivalent to $\mathbb{H}/L$.

Here is an easy criterion to decide if a given torus is tame or not.

Proposition 7.7. *Let L be a lattice and $\mathcal{B} = \{v_1, v_2, v_3, v_4\}$ be a reduced basis for L. Consider the torus $T = \mathbb{H}/L$. Then T is a tame torus, if, and only if, $\pm\mathcal{B}$ are the unique reduced bases for L.*

Proof. If the torus T^4 is tame, then suppose that there are two different reduced bases, $\mathcal{B} = \{v_1, v_2, v_3, v_4\}$ and $\mathcal{B}_1 = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ for the tame lattice L, with $\mathcal{B}_1 \neq \pm\mathcal{B}$. As a consequence, there exists $U \in GL(4, \mathbb{Z}) \setminus \{\pm I_4\}$ such that

$$U \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{pmatrix}.$$

If R and R_1 denote the reduced Gram matrices associated, respectively, to the bases $\mathcal{B}$ and $\mathcal{B}_1$, then

$$U\ R\ {}^tU = R_1.$$

Therefore, R and R_1 belong to the boundary of $\mathcal{R}$ by Theorem 6.9, and hence the torus is not tame. To prove the converse, suppose that T^4 is not tame, i.e. that the Gram matrix $R = (r_{i,j})$ associated to $\mathcal{B}$ belongs to $\partial\mathcal{R}$. Therefore, equality holds either in B1) or in B2). In the first case, since $r_{k,k+1} = 0$ for some $k = 1, 2, 3$, the two consecutive vectors v_k and v_{k+1} are orthogonal. We can then consider a second special basis $\mathcal{B}'$ obtained by substituting v_{k+1} with $-v_{k+1}$, ... , v_4 with $-v_4$. If instead equality holds in B2), the Minkowski-Siegel Reduction Algorithm directly implies the existence of a second reduced basis.

□

We will now find, inside the moduli space $\mathcal{M}$, a fundamental domain for the classes of equivalence of tame quaternionic tori. In fact if we set

$$\mathcal{T} = \{(v_2, v_3, v_4) \in \mathcal{M} : \{1, v_2, v_3, v_4\} \text{ is a (special) tame basis}\} \tag{7.4}$$

then, with the aid of a preliminary lemma, we can prove a uniqueness result for the modulus of tame torus.

Lemma 7.8. *Let two lattices L_1 and L_2 of $\mathbb{H}$ be generated respectively, by the special bases $\{1, \alpha_2, \alpha_3, \alpha_4\}$ and $\{1, \omega_2, \omega_3, \omega_4\}$. If $F(q) = qa$, with $a \in \mathbb{H}^*$, is an automorphism of $\mathbb{H}$ such that $F(L_1) = L_2$, then $|a| = 1$.*

Proof. By Theorem 5.1, and more precisely by the first equation of the system (5.1) with $\alpha_1 = 1$, it follows that $a \in L_2$, and hence $|a| \geq 1$. Moreover, since $\{a, \alpha_2 a, \alpha_3 a, \alpha_4 a\}$ are linearly independent vectors which generate the lattice L_2, then there exist $n_1, n_2, n_3, n_4 \in \mathbb{Z}$ such that $n_1(a) + n_2(\alpha_2 a) + n_3(\alpha_3 a) + n_4(\alpha_4 a) = 1$. This equality implies that $(n_1 + n_2\alpha_2 + n_3\alpha_3 + n_4\alpha_4)a = 1$. But $|n_1 + n_2\alpha_2 + n_3\alpha_3 + n_4\alpha_4| \geq 1$ because $n_1 + n_2\alpha_2 + n_3\alpha_3 + n_4\alpha_4$ is an element of L_1. Hence $|a| \leq 1$. □

Theorem 7.9. *The set $\mathcal{T} \subset \mathcal{M}$ is a fundamental domain for the equivalence classes of tame tori. In other words, every tame torus has exactly one modulus.*

Proof. Suppose that the two moduli $V = (v_2, v_3, v_4) \in \mathcal{T}$ and $W = (\omega_2, \omega_3, \omega_4) \in \mathcal{T}$ correspond to equivalent tame tori. If this is the case, then (see Definition 5.2) there exist $U \in GL(4, \mathbb{Z})$ and a quaternion $a \neq 0$ such that

$$U \begin{pmatrix} 1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{pmatrix} = \begin{pmatrix} 1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} a.$$

Lemma 7.8 implies that $|a| = 1$, and hence that (a, Va) is a reduced basis. Now, since (a, Va) and $(1, W)$ are both reduced bases, then by Proposition 7.7 we reach the conclusion that $a = \pm 1$ and hence that $V = W$. □

8. On the groups of automorphisms of "boundary" tori

According to Theorem 6.9, a reduced Gram matrix R belongs to $\partial\mathcal{R}$ if, and only if, there exists a reduced Gram matrix S such that

$$S = UR\,{}^tU$$

for some $U \in GL(4, \mathbb{Z})$, $U \neq \pm I_4$. Notice that to each of these reduced Gram matrices R, S there correspond infinitely many $GL(4, \mathbb{Z})$-nonequivalent bases (see (7.3)). Therefore the study of the equivalence classes of non tame tori consists in the identification and classification of reduced Gram matrices belonging to the boundary of $\mathcal{R}$, and corresponding to non equivalent special bases. We plan to address this fascinating problem in a forthcoming paper.

However, as it happens in the complex case, the very interesting and fundamental step in this direction is the search and classification of boundary tori with non trivial groups of (biregular) automorphisms. These tori, which are the quaternionic counterpart of tori with complex multiplication (classically indicated as *harmonic* and *equianharmonic*), will be found and classified in the rest of this section.

Remark 8.1. We recall the geometric interpretation of the quaternionic multiplication by a on the right, with $a \in \mathbb{H}$, $|a| = 1$. Every vector of $\mathbb{H} \cong \mathbb{R}^4$ lies in an invariant real plane of the rotation $q \mapsto qa$, where $a = \cos\alpha + I_a \sin\alpha$. This fact can be verified directly as follows: for any quaternion q, we define $q' = qI_a$, and notice that, since 1 and I_a are perpendicular vectors in $\mathbb{H} \cong \mathbb{R}^4$, so are q and q'. Moreover $I_a^2 = -1$, $q'I_a = -q$; thus

$$a = \cos\alpha + I_a \sin\alpha$$

$$qa = q\cos\alpha + q' \sin\alpha, \quad q'a = q'\cos\alpha - q\sin\alpha.$$

Therefore, the plane containing the vectors q and q' is invariant, and the rotation in this plane is of an angle α, [13].

Lemma 8.2. *Let L be a lattice of $\mathbb{H}$ generated by the special basis $\{1, \alpha_2, \alpha_3, \alpha_4\}$ and let $F(q) = qa$, with $a \in \mathbb{S}^3$, be an automorphism of $\mathbb{H}$ such that $F(L) = L$. Then a has finite order, i.e. there exists $n \in \mathbb{N}$ such that $a^n = 1$, and the order of a divides either 4 or 6.*

Proof. Since 1 is an element of the lattice L and since $F(q) = qa$ maps L onto L, it follows that $1a = a \in L$; similarly for all $m \in \mathbb{N}$, it holds that $a^m \in L$. By compactness, the sequence $\{a^m\}_{m\in\mathbb{N}}$ of unit vectors in L, has a convergent subsequence. Unless $\{a^m\}_{m\in\mathbb{N}}$ is a finite set, this is in contradiction with Lemma 3.2 and Theorem 3.4 which assert respectively that L is a (closed) discrete subgroup of $\mathbb{H}$. In order to prove the second assertion, we use what is stated in Remark 8.1: since 1 and a are elements of L, it follows that the complex plane L_{I_a}, which contains 1 and a, is invariant by right multiplication for a, i.e. the integer combinations of 1 and a form a rank-2 sublattice of L, contained in the complex plane L_{I_a}, with $F(q) = qa$ as an automorphism of the sublattice L_{I_a}; therefore a is a root of unity of order n, where n divides either 4 or 6, because in the complex setting these are the only possibilities (see, e.g., [1, 26]). □

Proposition 4.3 directly suggests how to define the automorphisms of a quaternionic torus.

Definition 8.3. Let $T = \mathbb{H}/L$ be the quaternionic torus associated to the rank-4 lattice L. The group of *biregular automorphisms of the torus* T is defined as

$$Aut(T) = \{F \in Aut(\mathbb{H}) \mid F(q) = qa \ \text{ with } \ a \in \mathbb{S}^3 \ \text{ and } \ F(L) = L\}.$$

We point out that the group $Aut(T)$ of biregular automorphisms of the torus $T = \mathbb{H}/L$ can also be interpreted as the group of biregular automorphisms $Aut_0(L)$ of a rank-4 lattice L fixing the point $0 \in L \subset \mathbb{H}$.

Proposition 8.4. *Let $T = \mathbb{H}/L$ be a quaternionic torus, and let $A_T = \{a \in \mathbb{S}^3 :$ $\exists\ F \in Aut(T)$ defined as $F(q) = qa\}$. Then:*

(1) *the set $A_T \subset \mathbb{S}^3 \cap L$ is a subgroup (with respect to quaternionic multiplication) of the group $\mathbb{S}^3$ of unitary quaternions;*
(2) *the group A_T is isomorphic to $Aut(T)$;*
(3) *for any fixed $R \geq 0$, each $F \in Aut(T)$ acts as a permutation on the finite set of all vectors of $L \cap \partial B(0, R)$.*

Proof. The group structure of the set A_T with respect to the quaternionic multiplication is inherited by the one of $Aut(T)$ with respect to composition. The fact that each $F \in Aut(T)$ acts as a permutation on vectors of fixed norm is straightforward by Lemma 7.8 and 8.2 and by the fact that an automorphism F of T maps T onto itself. □

We now pass to recall and list all the finite subgroups of the unitary quaternions, following the classical books [9], [10], [13] and [23]. We point out that in the literature there is a diffused confusion in the use of notations that concern the groups we are dealing with; here we will mainly refer to, and use the notations of, the book [9] by Conway and Smith.

We begin by listing all (up to conjugation) finite subgroups of the group of rotations $SO(3,\mathbb{R})$:

(1) the *icosahedral* group, $\mathbb{I} \cong A_5$, 60 elements;
(2) the *octahedral* group, $\mathbb{O} \cong S_4$, 24 elements;
(3) the *tetrahedral* group, $\mathbb{T} \cong A_4$, 12 elements;
(4) the *dihedral* group, $D_{2n} \cong D(n)$, $2n$ elements;
(5) the *cyclic* group, $C_n \cong C(n)$, n elements.

Every unitary quaternion q is associated to a precise rotation of $SO(3,\mathbb{R})$ by means of the $2-$to-1 correspondence that maps q to the rotation $[q] : x \to \bar{q}xq$ (see, e.g., [9]). As a consequence, every finite group Q of unitary quaternions is mapped to a group $[Q] = \{[q] : q \in Q\}$(isomorphic to) $C_n, D_{2n}, \mathbb{T}, \mathbb{O}, \mathbb{I}$. The number of elements of Q is 2 or 1 times the number of elements of $[Q]$, according to whether -1 is or is not in Q.

If G denotes one of the finite subgroups of the group of rotations, then we set

$$2G = \{q \in \mathbb{S}^3 : [q] \in G\}.$$

The only possible cases in which $-1 \in Q$ are those where $Q = 2C_n, 2D_{2n}, 2\mathbb{T}, 2\mathbb{O}, 2\mathbb{I}$. On the other hand, let us suppose that $-1 \notin Q$. In this case, G can contain no order 2 rotation g: if $[q] = g$ then $q^2 = -1$ must be in Q. The only group G without order 2 elements is C_n with n odd; this gives rise to a group $Q = 1C_n$ in $\mathbb{S}^3$ isomorphic to C_n. In fact, the following result holds (see, e.g., [9]).

Theorem 8.5. *The finite subgroups of unitary quaternions are*

$$2\mathbb{I}, \quad 2\mathbb{O}, \quad 2\mathbb{T}, \quad 2D_{2n}, \quad 2C_n, \quad 1C_n (n \text{ odd}).$$

With the usual notations for quaternions, let $I, J \in \mathbb{S}$, with $I \perp J$, and let $\{1, I, J, IJ = K\}$ be a basis for $\mathbb{H}$ having the usual multiplication rules. We then set

$$\begin{aligned}
&I_{\mathbb{I}} = \frac{I + \sigma J + \tau K}{2}, \quad \sigma = \frac{\sqrt{5}-1}{2}, \quad \tau = \frac{\sqrt{5}+1}{2};\\
&I_{\mathbb{O}} = \frac{J+K}{\sqrt{2}};\\
&\omega = \frac{-1+I+J+K}{2};\\
&I_{\mathbb{T}} = I;\\
&e_n = e^{\frac{\pi I}{n}}.
\end{aligned}$$

Theorem 8.6. *The finite subgroups of unitary quaternions are generated as follows:*

$$\begin{aligned}
&2\mathbb{I} = \langle\langle I_{\mathbb{I}}, \omega\rangle\rangle, \quad 2\mathbb{O} = \langle\langle I_{\mathbb{O}}, \omega\rangle\rangle, \quad 2\mathbb{T} = \langle\langle I_{\mathbb{T}}, \omega\rangle\rangle,\\
&2D_{2n} = \langle\langle e_n, j\rangle\rangle, \quad 2C_n = \langle\langle e_n\rangle\rangle, \quad 1C_n = \langle\langle e_{\frac{n}{2}}\rangle\rangle \ (n \text{ odd}).
\end{aligned}$$

Theorem 8.7. *There is no quaternionic torus whose group of automorphisms is isomorphic to* $2\mathbb{I},\ 2\mathbb{O},\ 2D_{2n}(n \geq 4),\ 2C_n(n \geq 4),\ 1C_n(n \text{ odd})$.

Proof. Since the subgroup $\mathbb{I}$ contains an element of order 5, then $2\mathbb{I}$ has an element of order 10. Hence, by Lemma 8.2, it cannot be the group of automorphisms of a quaternionic torus. The same argument holds to exclude $2\mathbb{O}$ that has an element of order 8. Analogously the groups $2D_{2n}(n \geq 4)$, $2C_n(n \geq 4)$ are excluded since they both contain the element e_n whose order is $2n$. Finally, the group $1C_n(n \text{ odd})$ cannot be isomorphic to the group of automorphisms of a quaternonic torus since it does not contain -1. □

This last result reduces the possible groups of automorphisms of a quaternionic torus to the list

$$2\mathbb{T}, \quad 2D_2, \quad 2D_4, \quad 2D_6, \quad 2C_1, \quad 2C_2, \quad 2C_3.$$

Since, as it is well known and easy to check, the groups $2C_2$ and $2D_2$ are isomorphic, the final list of these groups becomes

$$2\mathbb{T}, \quad 2D_4, \quad 2D_6, \quad 2C_1, \quad 2C_2, \quad 2C_3. \tag{8.1}$$

Remark 8.8. We observe that the following group inclusions hold:

$$2C_1 \subset 2C_2 \subset 2D_4 \subset 2\mathbb{T}$$

and

$$2C_1 \subset 2C_3 \subset 2D_6.$$

For each group $2G$ in the list (8.1), we will exhibit all tori whose group of automorphisms contains, or coincides with, $2G$. We will begin with the group $2C_1 = \{1, -1\}$, which appears for each quaternionic torus. As the reader may imagine, for reasons of neat presentation we will from now on suppose, without loss of generality, that the lattices which generate the tori involved have 1 as a vector of minimum modulus.

Proposition 8.9. *The group* $2C_1$ *is (isomorphic to) a subgroup of the group of biregular automorphisms of any quaternionic torus* T.

Proof. Let L be a rank-4 lattice of $\mathbb{H}$ generated by the special basis $\{1, \alpha_2, \alpha_3, \alpha_4\}$ and such that $T = \mathbb{H}/L$. Then $2C_1 = \{1, -1\}$ consists of automorphisms of T since $\{-1, -\alpha_2, -\alpha_3, -\alpha_4\}$ generates the lattice L. □

Proposition 8.10. *Let* T *be a quaternionic torus. The group of biregular automorphisms of the torus* T *contains a subgroup (isomorphic to)* $2C_2 \cong 2D_2$*, if and only if there exists* $I \in \mathbb{S}$ *and a quaternion* α_3 *with* $|\alpha_3| \geq 1$ *such that* $(I, \alpha_3, \alpha_3 I)$ *is a modulus of* T.

Proof. If $(I, \alpha_3, \alpha_3 I)$ is a modulus of T then $\mathcal{B} = \{1, I, \alpha_3, \alpha_3 I\}$ is a special basis of a lattice L such that T is equivalent to $\mathbb{H}/L$. Then the four bases

$$\begin{aligned} \mathcal{B}I^0 &= \mathcal{B} = \{1, I, \alpha_3, \alpha_3 I\} \\ \mathcal{B}I^1 &= \mathcal{B}I = \{I, -1, \alpha_3 I, -\alpha_3\} \\ \mathcal{B}I^2 &= -\mathcal{B} = \{-1, -I, -\alpha_3, -\alpha_3 I\} \\ \mathcal{B}I^3 &= -\mathcal{B}I = \{-I, 1, -\alpha_3 I, \alpha_3\} \end{aligned}$$

all generate the lattice L, and hence the subgroup $2C_2 \cong \{\pm 1, \pm I\}$ is a subgroup of the group of automorphisms $Aut(T)$. On the other hand, if $2C_2 \subseteq Aut(T)$, then thanks to Theorem 8.6, Lemma 8.2 and Proposition 8.4, the vectors $\{\pm 1, \pm I\}$ must belong to, and generate, the rank-2 sublattice $L \cap L_I$ of L. Therefore (by the classification of rank-2 lattices of $\mathbb{C}$), while using the Minkowski-Siegel Reduction Algorithm, we can choose $\alpha_2 = I$ in the special basis $\mathcal{B} = \{1, \alpha_2, \alpha_3, \alpha_4\}$ that generates L. We can also suppose that the third vector α_3 is chosen (again according to Algorithm 6.1) among those vectors that can complete the special basis $\mathcal{B}$. As a consequence, $\alpha_3 I, -\alpha_3, -\alpha_3 I$, must belong to L together with α_3. Since these three elements of L have the same norm of α_3, and since $Aut(T)$ contains a subgroup isomorphic to $2C_2$, then the Minkowski-Siegel Reduction Algorithm 6.1 can produce a special basis that generates L by using suitable α_3 and taking, automatically, $\alpha_4 = \alpha_3 I$; this concludes the proof. $\square$

Let $\{1, I, J, K\}$ be a standard basis for the skew field of quaternions. We recall that the ring of *Lipschitz quaternions* (or *Lipschitz integers*) consists of the set $\mathcal{L} = \{m + nI + pJ + qK : m, n, p, q \in \mathbb{Z}\} \subset \mathbb{H}$. The ring $\mathcal{L}$ is, in turn, a subring of the ring of *Hurwitz quaternions* (or *Hurwitz integers*) $\mathcal{H} = \{a + bI + cJ + dK : a, b, c, d \in \mathbb{Z} \text{ or } a, b, c, d \in \mathbb{Z} + \frac{1}{2}\}$. The surprising properties of these rings are described, for instance, in [9].

Remark 8.11. Concerning the proof of Proposition 8.10, notice that only in the case in which the lattice L consists of the ring of Hurwitz integers $\mathcal{H}$, it can happen that there exists α_3 such that the special basis $\mathcal{B} = \{1, I, \alpha_3, \alpha_3 I\}$ simply generates a proper sublattice of L and not the whole L : for example if $\alpha_3 = J$ is an imaginary unit quaternion orthogonal to I, then the set $\mathcal{B} = \{1, I, J, JI\}$ generates the sublattice of the Lipschitz quaternions instead of the whole lattice of Hurwitz quaternions. According to Remark 8.8 and to the classification (8.1) of the finite subgroups of unitary quaternions which can be contained in $L \cap \mathbb{S}^3$, this is the only case in which a set of linearly independent vectors of type $\mathcal{B} = \{1, I, \alpha_3, \alpha_3 I\} \subset L$ (with $|\alpha_3| = 1$) can generate a proper sublattice instead of the whole lattice. In this particular case, it is enough to change α_3 with another vector of $L \cap \mathbb{S}^3$ which can be reached by means of the Minkowski-Siegel Reduction Algorithm and such that $1, I, \alpha_3$ are not in the same multiplicative subgroup of $L \cap \mathbb{S}^3$: we know that at least an α_3 of this kind exists (this last fact depends on the well known structure of the subgroups of $2\mathbb{T}$).

If the group of automorphisms $Aut(T)$ of the torus T contains a subgroup isomorphic to $2C_2$, and if the torus T has a special basis of type $\{1, I, \alpha_3, \alpha_3 I\}$ with $|\alpha_3| > 1$, then $Aut(T) \cong 2C_2$: this is a consequence of the classification of the rank-2 lattices of $\mathbb{C}$, and of the fact that, in these hypotheses, there are only four points in $L \cap \mathbb{S}^3$, all belonging to $L_I \cap \mathbb{S}^3$ (see Proposition 8.4).

Definition 8.12. A quaternionic torus whose group of biregular automorphisms is isomorphic to $2C_2 \cong 2D_2$ is called a *cyclic-dihedral torus*.

Proposition 8.13. *Let T be a quaternionic torus. The group of biregular automorphisms of the torus T contains a subgroup (isomorphic to) $2C_3$ if and only if there exists $I \in \mathbb{S}$ and a quaternion α_3 with $|\alpha_3| \geq 1$ such that $(e^{\frac{\pi I}{3}}, \alpha_3, \alpha_3 e^{\frac{\pi I}{3}})$ is a modulus of T.*

Proof. If $(e^{\frac{\pi I}{3}}, \alpha_3, \alpha_3 e^{\frac{\pi I}{3}})$ is a modulus of T then $\mathcal{B} = \{1, e^{\frac{\pi I}{3}}, \alpha_3, \alpha_3 e^{\frac{\pi I}{3}}\}$ is a special basis of a lattice L such that T is equivalent to $\mathbb{H}/L$. Then the six bases

$$\begin{aligned}
\mathcal{B}(e^{\frac{\pi I}{3}})^0 &= \mathcal{B} = \{1, e^{\frac{\pi I}{3}}, \alpha_3, \alpha_3 e^{\frac{\pi I}{3}}\}\\
\mathcal{B}(e^{\frac{\pi I}{3}})^1 &= \mathcal{B}e^{\frac{\pi I}{3}} = \{e^{\frac{\pi I}{3}}, e^{\frac{2\pi I}{3}}, \alpha_3 e^{\frac{\pi I}{3}}, \alpha_3 e^{\frac{2\pi I}{3}}\}\\
\mathcal{B}(e^{\frac{\pi I}{3}})^2 &= \mathcal{B}e^{\frac{2\pi I}{3}} = \{e^{\frac{2\pi I}{3}}, -1, \alpha_3 e^{\frac{2\pi I}{3}}, -\alpha_3\}\\
\mathcal{B}(e^{\frac{\pi I}{3}})^3 &= -\mathcal{B} = \{-1, -e^{\frac{\pi I}{3}}, -\alpha_3, -\alpha_3 e^{\frac{\pi I}{3}}\}\\
\mathcal{B}(e^{\frac{\pi I}{3}})^4 &= -\mathcal{B}e^{\frac{\pi I}{3}} = \{-e^{\frac{\pi I}{3}}, -e^{\frac{2\pi I}{3}}, -\alpha_3 e^{\frac{\pi I}{3}}, -\alpha_3 e^{\frac{2\pi I}{3}}\}\\
\mathcal{B}(e^{\frac{\pi I}{3}})^5 &= -\mathcal{B}e^{\frac{2\pi I}{3}} = \{-e^{\frac{2\pi I}{3}}, 1, -\alpha_3 e^{\frac{2\pi I}{3}}, \alpha_3\}
\end{aligned}$$

all generate the lattice L, and hence the subgroup $2C_3 \cong \{\pm 1, \pm e^{\frac{\pi I}{3}}, \pm e^{\frac{2\pi I}{3}}\}$ is a subgroup of the group of automorphisms $Aut(T)$. On the other hand, if $2C_3 \subseteq Aut(T)$, then thanks to Theorem 8.6, Lemma 8.2 and Proposition 8.4, the vectors $\{\pm 1, \pm e^{\frac{\pi I}{3}}, \pm e^{\frac{2\pi I}{3}}\}$ must belong to, and generate, the rank-2 sublattice $L \cap L_I$ of L. Therefore (using the classification of the rank-2 lattices of $\mathbb{C}$), while using the Minkowski-Siegel Reduction Algorithm to construct the special basis $\mathcal{B} = \{1, \alpha_2, \alpha_3, \alpha_4\}$ that generates L, we can choose $\alpha_2 = e^{\frac{\pi I}{3}}$. We can also suppose that the third vector α_3 is chosen (according to Algorithm 6.1) among those vectors that can complete the special basis $\mathcal{B}$. As a consequence, the points $\alpha_3 e^{\frac{\pi I}{3}}, \alpha_3 e^{\frac{2\pi I}{3}}, -\alpha_3, -\alpha_3 e^{\frac{\pi I}{3}}, -\alpha_3 e^{\frac{2\pi I}{3}}$ must belong to L together with α_3. Since these five elements have the same norm of α_3, and since $Aut(T)$ contains a subgroup isomorphic to $2C_3$, then the Minkowski-Siegel Reduction Algorithm 6.1 can produce a special basis that generates L by using suitable α_3 and, automatically, $\alpha_4 = \alpha_3 e^{\frac{\pi I}{3}}$; this completes the proof. □

If the group of automorphisms $Aut(T)$ of the torus T contains a subgroup isomorphic to $2C_3$, and if the torus T has a special basis of type $\{1, e^{\frac{\pi I}{3}}, \alpha_3, \alpha_3 e^{\frac{\pi I}{3}}\}$ with $|\alpha_3| > 1$, then $Aut(T) \cong 2C_3$: this is a consequence of the classification of the rank-2 lattices of $\mathbb{C}$, and of the fact that, in these hypotheses, there are only six points in $L \cap \mathbb{S}^3$, all belonging to $L_I \cap \mathbb{S}^3$ (see Proposition 8.4).

Definition 8.14. A quaternionic torus whose group of biregular automorphisms is isomorphic to $2C_3$ is called a *cyclic torus*.

Proposition 8.15. *Let T be a quaternionic torus. The group of biregular automorphisms of the torus T is isomorphic to the group $2D_4$, if and only if, for $I, J \in \mathbb{S}$ with $J \perp I$, the point (I, J, JI) is a modulus of T.*

Proof. Let $\mathcal{B} = \{1, I, J, JI\}$ be the special basis associated to the modulus (I, J, JI), and let L be the generated lattice such that T is equivalent to $\mathbb{H}/L$. Thanks to Proposition 8.10, we know that the multiplication by I on the right generates a $2C_2$ subgroup of $Aut(T)$. Using Theorem 8.6, we are left to prove that the multiplication by J on the right generates a second subgroup of type $2C_2$ of $Aut(T)$. To this aim notice that the four bases

$$\begin{aligned}
\mathcal{B}J^0 &= \mathcal{B} = \{1, I, J, JI\} = \{1, I, J, -K\} \\
\mathcal{B}J^1 &= \mathcal{B}J = \{J, K, -1, I\} \\
\mathcal{B}J^2 &= -\mathcal{B} = \{-1, -I, -J, K\} \\
\mathcal{B}J^3 &= -\mathcal{B}J = \{-J, -K, 1, -I\}
\end{aligned}$$

all generate the lattice L. It is then easy to conclude that the subgroup $2D_4 \cong \{\pm 1, \pm I, \pm J, \pm K\}$ is a subgroup of the group of automorphisms $Aut(T)$. On the other hand, if $2D_4 \subseteq Aut(T)$, then $2C_2 \subset 2D_4$ is a subgroup of $Aut(T)$, and hence thanks to Theorem 8.10, in the special basis $\mathcal{B} = \{1, \alpha_2, \alpha_3, \alpha_4\}$ that generates L, we can choose $\alpha_2 = I$ and $\alpha_4 = \alpha_3 I$. Theorem 8.6, Lemma 8.2 and Proposition 8.4 imply that the vectors $\{\pm 1, \pm J\}$ must belong to, and generate, the rank-2 sublattice $L \cap L_J$ of L. Since J and $-J$ have the same norm of I, then by the Minkowski-Siegel Reduction Algorithm used to construct a special basis for L, we can suppose that $\alpha_3 = J$ and find the desired basis. To prove that $2D_4 \cong Aut(T)$, begin by noticing that $\mathcal{B}$ is an orthonormal basis of $\mathbb{H}$; it is then easy to see that $L \cap \mathbb{S}^3 = \{\pm 1, \pm I, \pm J, \pm JI\}$ and that this set has exactly the same cardinality of the group $2D_4$. Proposition 8.4 leads now to the conclusion. □

Definition 8.16. A quaternionic torus whose group of biregular automorphisms is isomorphic to $2D_4$ is called a 8-*dihedral torus* (or a *dihedral torus of order* 8).

Remark 8.17. As we already mentioned, the notations concerning finite subgroups of unit quaternions vary very much. We observe that the group $2D_4$, that we (following [9]) called dihedral group of order 8, coincides with the so called multiplicative group of unit quaternions (and not with D_8, sometimes called dihedral group of order 8 , $D_8 = \langle\langle a, b\rangle\rangle$ with the relations $a^4 = b^2 = 1, bab^{-1} = a^{-1}$, see [2]).

Notice that the lattice of the dihedral torus of order 8 is generated by the group $2D_4$ and coincides with the ring of Lipschitz quaternions, defined after Proposition 8.10.

Proposition 8.18. *Let T be a quaternionic torus. The group of biregular automorphisms of the torus T is isomorphic to the group $2D_6$, if and only if, for $I, J \in \mathbb{S}$ with $J \perp I$ the point $(e^{\frac{\pi I}{3}}, J, Je^{\frac{\pi I}{3}})$ is a modulus of T.*

Proof. Let $\mathcal{B} = \{1, e^{\frac{\pi I}{3}}, J, Je^{\frac{\pi I}{3}}\}$ be the special basis associated to the modulus $(e^{\frac{\pi I}{3}}, J, Je^{\frac{\pi I}{3}})$ and let L be the generated lattice such that T is equivalent to $\mathbb{H}/L$. Thanks to Proposition 8.13, we know that the multiplication by $e^{\frac{\pi I}{3}}$ on the right generates a $2C_3$ subgroup of $Aut(T)$. We are then left to prove that the multiplication by J on the right generates a subgroup of type $2C_2$ of $Aut(T)$. To this aim notice that the four bases

$$\begin{aligned}
\mathcal{B}J^0 &= \mathcal{B} = \{1, e^{\frac{\pi I}{3}}, J, Je^{\frac{\pi I}{3}}\} \\
\mathcal{B}J^1 &= \mathcal{B}J = \{J, Je^{\frac{-\pi I}{3}}, -1, -e^{\frac{-\pi I}{3}}\} \\
\mathcal{B}J^2 &= -\mathcal{B} = \{-1, -e^{\frac{\pi I}{3}}, -J, -Je^{\frac{\pi I}{3}}\} \\
\mathcal{B}J^3 &= -\mathcal{B}J = \{-J, -Je^{\frac{-\pi I}{3}}, 1, e^{\frac{-\pi I}{3}}\}
\end{aligned}$$

all generate the lattice L. We then conclude that the dihedral subgroup of order 12, $2D_6 \cong \{\pm 1, \pm e^{\frac{\pi I}{3}}, \pm e^{\frac{2\pi I}{3}}, \pm J, \pm Je^{\frac{\pi I}{3}}, \pm Je^{\frac{2\pi I}{3}}\}$, is a subgroup of the group of automorphisms $Aut(T)$. On the other hand, if $2D_6 \subseteq Aut(T)$, then $2C_3 \subset 2D_6$ is a subgroup of $Aut(T)$, and hence, thanks to Proposition 8.13, in the special basis $\mathcal{B} = \{1, \alpha_2, \alpha_3, \alpha_4\}$ that generates L, we can choose $\alpha_2 = e^{\frac{\pi I}{3}}$ and $\alpha_4 = \alpha_3 e^{\frac{\pi I}{3}}$. Theorem 8.6, Lemma 8.2 and Proposition 8.4 imply that the vectors $\{\pm 1, \pm J\}$ must belong to, and generate, the rank-2 sublattice $L \cap L_J$ of L. Since J and $-J$ have the same norm of $e^{\frac{\pi I}{3}}$, then by the Minkowski-Siegel Reduction Algorithm used to construct a special basis for L, we can suppose that $\alpha_3 = J$ and find the desired basis. To prove that $2D_6 \cong Aut(T)$, begin by noticing that L_I is orthogonal to JL_I; it is then easy to see that $L \cap \mathbb{S}^3 = \{\pm 1, \pm e^{\frac{\pi I}{3}}, \pm e^{\frac{2\pi I}{3}}, \pm J, \pm Je^{\frac{\pi I}{3}}, \pm Je^{\frac{2\pi I}{3}}\}$ and that this set has exactly the same cardinality of the group $2D_6$. Proposition 8.4 leads now to the conclusion. □

Definition 8.19. A quaternionic torus whose group of biregular automorphisms is isomorphic to $2D_6$ is called a 12-*dihedral torus* (or a *dihedral torus of order* 12).

Proposition 8.20. *Let T be a quaternionic torus. The group of biregular automorphisms of the torus T is isomorphic to the group $2\mathbb{T}$, if and only if, for some $I, J \in \mathbb{S}$ with $J \perp I$, setting $K = IJ$ and $\omega = \frac{-1+I+J+K}{2}$, the point $(-\omega, -I, I\omega)$ is a modulus of T.*

Proof. Notice, first of all, that if $M = \frac{\sqrt{3}}{3}(I + J + K) \in \mathbb{S}$, then $\omega = e^{\frac{2\pi M}{3}}$. Consider the special basis $\mathcal{B} = \{1, -\omega, -I, I\omega\}$ associated to the modulus $(-\omega, -I, I\omega)$ and let L be the generated lattice such that T is equivalent to $\mathbb{H}/L$. Thanks to Theorem 8.6, we know that $-\omega$ and $-I$ generate a $2\mathbb{T}$ subgroup of $Aut(T)$. Set $\tilde{\omega} = \frac{-1+I-J-K}{2}$, and notice that $\tilde{\omega} = -\omega - 1 + I$ belongs to L. It is now necessary (and it is only a direct computation) to verify that the iterated multiplication by powers of $e^{\frac{2\pi M}{3}}$ and by powers of I (on the right) maps the basis $\mathcal{B}$ onto bases that generate the lattice L. For example, as for the multiplication by I on the right, we get that

$$\begin{aligned}
\mathcal{B}(-I)^0 &= \mathcal{B} = \{1, -\omega, -I, I\omega\} \\
\mathcal{B}(-I)^1 &= -\mathcal{B}I = \{-I, I\tilde{\omega}, -1, \tilde{\omega}\} \\
\mathcal{B}(-I)^2 &= -\mathcal{B} = \{-1, \omega, I, -I\omega\} \\
\mathcal{B}(-I)^3 &= \mathcal{B}I = \{I, -I\tilde{\omega}, 1, -\tilde{\omega}\}
\end{aligned}$$

are all generating bases for L. We then conclude that the subgroup $2\mathbb{T} = \langle\langle -\omega, -I \rangle\rangle$ is a subgroup of the group of automorphisms $Aut(T)$. On the other hand, if $2\mathbb{T} \subseteq Aut(T)$, then $2C_3 \subset 2\mathbb{T}$ is a subgroup of $Aut(T)$, and hence, thanks to Proposition 8.13, in the special basis $\mathcal{B} = \{1, \alpha_2, \alpha_3, \alpha_4\}$ that generates L, we can choose $\alpha_2 = -\omega$ and $\alpha_4 = \alpha_3(-\omega)$. Theorem 8.6, Lemma 8.2 and Proposition 8.4 imply that the vectors $\{\pm 1, \pm I\}$ must belong to, and generate, the rank-2 sublattice $L \cap L_I$ of L. Since I and $-I$ have the same norm of $-\omega$, then by the Minkowski-Siegel Reduction Algorithm used to construct a special basis for L, we can suppose that $\alpha_3 = -I$ and find the desired basis. To prove that $2\mathbb{T} \cong Aut(T)$, begin by noticing that the 16 elements $\{\frac{\pm 1 \pm I \pm J \pm K}{2}\}$ belong to $L \cap \mathbb{S}^3$ as well as the 8 elements $\{\pm 1, \pm I, \pm J, \pm K\}$. Therefore the set $L \cap \mathbb{S}^3$ has to have the same cardinality of the group $2\mathbb{T}$. As in the previous cases, at this point Proposition 8.4 leads to the conclusion. □

Definition 8.21. A quaternionic torus whose group of biregular automorphisms is isomorphic to $2\mathbb{T}$ is called a *tetrahedral torus*.

Notice that the lattice of the tetrahedral torus is the ring $\mathcal{H}$ of the Hurwitz quaternions, defined after Proposition 8.10. This lattice is generated by the group $2\mathbb{T}$.

The next result can be obtained directly as a consequence of the investigation performed up to now, on the possible groups of biregular automorphisms of "boundary" tori. Recall that, for a quaternionic torus T, in Proposition 8.4 we have introduced the group $A_T = \{a \in \mathbb{S}^3 : \exists\ F \in Aut(T) \text{ defined as } F(q) = qa\}$.

Proposition 8.22. *Let T be a quaternionic torus. When the group $A_T \cong Aut(T)$ is not reduced to $\{\pm 1\}$, then it coincides with the biggest subgroup of $L \cap \mathbb{S}^3$.*

Proof. Suppose $Aut(T) \neq \{\pm 1\} = 2C_1$. Thanks to the classification (8.1), we get that $Aut(T)$ has to coincide with $2C_2, 2C_3, 2D_4, 2D_6$ or $2\mathbb{T}$. If $Aut(T) \cong 2C_2$ and $2C_2$ is, by contradiction, strictly contained in a larger subgroup of $\mathbb{S}^3$, then using Remark 8.8 we obtain $Aut(T) \cong 2D_4$ or $Aut(T) \cong 2\mathbb{T}$. In both cases $2D_4$ and $2\mathbb{T}$ generate a lattice associated to a different torus (see Propositions 8.15 and 8.20). If $Aut(T) \cong 2D_4$ and $2D_4$ is, by contradiction, strictly contained in a larger subgroup of $\mathbb{S}^3$, then using Remark 8.8 we obtain $Aut(T) \cong 2\mathbb{T}$. In this last case $2\mathbb{T}$ generates a lattice associated to a different torus (see Proposition 8.20). In the remaining case in which $Aut(T) \cong 2C_3$ the proof is totally analogous. □

Example 8.23. We give an example which shows, in connection to Proposition 8.22, that when $Aut(T) \cong 2C_1 = \{\pm 1\}$, then it can be strictly contained in a larger subgroup of $\mathbb{S}^3$: consider the lattice L generated by the special basis $\{1, i, 3j + \frac{1}{10}, 4k + \frac{1}{100}\}$. If $T = \mathbb{H}/L$, then the unitary vectors of L are $\{\pm 1, \pm i\}$ and they form a group isomorphic to $2C_2$.

We conclude this section by stating a summarizing result.

Theorem 8.24. *The cyclic, cyclic dihedral, 8-dihedral, 12-dihedral, tetrahedral tori defined in this section are the unique (up to biregular diffeomorphisms) tori with $Aut(T) \neq \{\pm 1\}$.*

Proof. Follows directly from Propositions 8.10, 8.13, 8.15, 8.18, 8.20. □

Remark 8.25. The lattices generating tori T with $Aut(T) \cong 2C_2$ or $Aut(T) \cong 2C_3$ are called *regular tessellations of* $\mathbb{R}^2$; the lattices generating tori T with $Aut(T) \cong 2D_4$, $Aut(T) \cong 2D_6$ or $Aut(T) \cong 2\mathbb{T}$ are called *regular tessellations of* $\mathbb{R}^4$.

9. Appendix A: an algorithm to check if a basis is reduced

Let $R = (r_{i,j})_{i,j=1,\cdots,4}$ be the Gram matrix associated to a given basis $\mathcal{B} = \{v_1, v_2, v_3, v_4\}$ of the rank-4 lattice L. Reordering the four vectors $\{v_1, v_2, v_3, v_4\}$, without loss of generality, we can always suppose that $r_{1,1} \leq r_{2,2} \leq r_{3,3} \leq r_{4,4}$. To check that $\mathcal{B}$ is a reduced basis, we will check the fact that R is a reduced Gram matrix.

The first step of our algorithm is the easiest:

Step 0: We check if $r_{k,k+1} = \langle v_k, v_{k+1} \rangle$ (for $k = 1, 2, 3$) are all non negative quantities. If this is true, we proceed in the algorithm; otherwise we conclude that the Gram matrix R, and the basis $\mathcal{B}$, are not reduced, and stop. Since R is a (4×4) symmetric, real and positive definite matrix, there exists a positive definite diagonal matrix

$$D = \begin{bmatrix} \lambda_1^2 & 0 & 0 & 0 \\ 0 & \lambda_2^2 & 0 & 0 \\ 0 & 0 & \lambda_3^2 & 0 \\ 0 & 0 & 0 & \lambda_4^2 \end{bmatrix}$$

and an orthogonal matrix Q such that ${}^tQRQ = D$. Moreover, we can suppose that $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4$. In order to verify if R is a reduced Gram matrix we proceed as follows:

Step 1: Since $R = QD\,{}^tQ$, the quadratic form $(n_1, n_2, n_3, n_4)R\,{}^t(n_1, n_2, n_3, n_4)$ can also be written as

$$(n_1, n_2, n_3, n_4)QD\,{}^tQ\,{}^t(n_1, n_2, n_3, n_4) = \lambda_1^2 x_1^2 + \cdots + \lambda_4^2 x_4^2$$

where $\lambda_1^2, \cdots, \lambda_4^2$ are the ordered positive eigenvalues of D and $(x_1 \cdots, x_4) = (n_1, \cdots, n_4)Q$. The geometric locus of vectors $(x_1, \cdots, x_4) \in \mathbb{R}^4$ for which the diagonalized quadratic form is equal to $r_{1,1} = \langle v_1, v_1 \rangle$ is an ellipsoid having the length of the maximal axis of symmetry equal to $\frac{2|v_1|}{\lambda_1}$. Therefore, the quadruplets $(n_1, n_2, n_3, n_4) \in \mathbb{Z}^4$ such that

$$(n_1, n_2, n_3, n_4)R\,{}^t(n_1, n_2, n_3, n_4) < r_{1,1} \tag{9.1}$$

belong necessarily to the finite set $E_1 = \mathbb{Z}^4 \cap I_1^4$ where

$$I_1 = \left[-\frac{|v_1|}{\lambda_1}, \frac{|v_1|}{\lambda_1} \right].$$

At this point we recall the first step of the construction of the Minkowski-Siegel Reduction Algorithm: we check if there exists a point (n_1, n_2, n_3, n_4) in the finite set $E_1 \setminus \{\underline{0}\}$ such that inequality (9.1) is fulfilled. If the answer is yes, then we conclude that the Gram matrix R, and hence the basis $\mathcal{B}$, are not reduced, and stop. Otherwise we proceed in the algorithm.

Step 2: In this step we consider the quadratic equation

$$(n_1, n_2, n_3, n_4)R\,{}^t(n_1, n_2, n_3, n_4) = r_{2,2}.$$

Let

$$I_2 = \left[-\frac{|v_2|}{\lambda_1}, \frac{|v_2|}{\lambda_1} \right]$$

and set E_2 to be the finite set $\mathbb{Z}^4 \cap I_2^4$. By Definition 6.3 and by condition B2)$'$, a reduced Gram matrix is such that: if there exists $(n_1, n_2, n_3, n_4) \in E_2 \setminus \{\underline{0}\}$ with

$$(n_1, n_2, n_3, n_4)R\,{}^t(n_1, n_2, n_3, n_4) < r_{2,2} \tag{9.2}$$

then n_2, n_3, n_4 have common divisors. Therefore we check if B2)$'$ holds true. If the answer is no, then we conclude that the Gram matrix R, and hence the basis $\mathcal{B}$, are not reduced, and stop. Otherwise we proceed in the algorithm.

Step 3: Similar procedure applies to the quadratic equation

$$(n_1, n_2, n_3, n_4)R\ {}^t(n_1, n_2, n_3, n_4) = r_{3,3}.$$

Let

$$I_3 = \left[-\frac{|v_3|}{\lambda_1}, \frac{|v_3|}{\lambda_1}\right]$$

and set E_3 to be the finite set $\mathbb{Z}^4 \cap I_3^4$. By Definition 6.3 and by condition B2)$'$, a reduced Gram matrix is such that: if there exists $(n_1, n_2, n_3, n_4) \in E_3 \setminus \{\underline{0}\}$ with

$$(n_1, n_2, n_3, n_4)R\ {}^t(n_1, n_2, n_3, n_4) < r_{3,3} \tag{9.3}$$

then n_3, n_4 have common divisors. Therefore we check if B2)$'$ holds true. If the answer is no, then we conclude that the Gram matrix R, and hence the basis $\mathcal{B}$, are not reduced, and stop. Otherwise we proceed in the algorithm.

Step 4: In the last step our procedure is applied to the quadratic equation

$$(n_1, n_2, n_3, n_4)R\ {}^t(n_1, n_2, n_3, n_4) = r_{4,4}.$$

Let

$$I_4 = \left[-\frac{|v_4|}{\lambda_1}, \frac{|v_4|}{\lambda_1}\right]$$

and set E_4 to be the finite set $\mathbb{Z}^4 \cap I_4^4$. By Definition 6.3 and by condition B2)$'$, a reduced Gram matrix is such that: if there exists $(n_1, n_2, n_3, n_4) \in E_4 \setminus \{\underline{0}\}$ with

$$(n_1, n_2, n_3, n_4)R\ {}^t(n_1, n_2, n_3, n_4) < r_{4,4} \tag{9.4}$$

then n_4 must be different from ± 1. Therefore we check if B2)$'$ holds true. If the answer is no, then we conclude that the Gram matrix R, and hence the basis $\mathcal{B}$, are not reduced. Otherwise we finally conclude that the Gram matrix R, and hence the basis $\mathcal{B}$, are reduced.

We also provide here a Matlab Script which "implements" part of the algorithm that we have just described. Let us consider in particular Step 3, and let $p_3 = \lceil \frac{|v_3|}{\lambda_1} \rceil$ (the smallest integer greater or equal than $\frac{|v_3|}{\lambda_1}$). Set:

$i = 1;$
for $n_1 = -p_3 : 1 : p_3,$
for $n_2 = -p_3 : 1 : p_3,$
for $n_3 = -p_3 : 1 : p_3,$
for $n_4 = -p_3 : 1 : p_3,$
$m = (n_1, n_2, n_3, n_4)R\ {}^t(n_1, n_2, n_3, n_4);$
if $(m < r_{3,3})$ $A(i, :) = [n_1, n_2, n_3, n_4];$
$i = i + 1;$
end
end
end
end
end
for j=1:1:i-1
if $(gcd(A(j, 3), A(j, 4)) == 1)$ disp('the matrix is not reduced')

break
end
end

A few significant examples, useful to illustrate the meaning of the results obtained, are the following.

Example 9.1. An example of a cyclic-dihedral torus is the one associated to the modulus $(i, e^{\frac{2}{5}\pi j}, e^{\frac{2}{5}\pi j} i)$. Indeed, running the Algorithm presented in this section, we find out that Step 0 is satisfied and that the only vector of the lattice L generated by the special basis $\mathcal{B} = \{1, i, e^{\frac{2}{5}\pi j}, e^{\frac{2}{5}\pi j} i\}$ inside the unit ball is the null vector; besides, on $\mathbb{S}^3 \cap L$ we only find the set of vectors $\mathcal{B} \cup -\mathcal{B}$.

Example 9.2. A second example of a cyclic-dihedral torus is the one associated to the modulus $(i, 4j+3k, -4k+3j)$. Indeed, running the Algorithm presented in this section, we find out that Step 0 is satisfied and all conditions in B2) are verified. The only vector of the lattice L generated by the special basis $\mathcal{B} = \{1, i, 4j+3k, -4k+3j\}$ inside the unit ball is the null vector; besides, on $\mathbb{S}^3 \cap L$ we only find the set of vectors $\{\pm 1, \pm i\}$. On the sphere of radius 5 there are 16 elements of L and the product by $a \in \{\pm 1, \pm i\}$ permutes them, as explained in Proposition 8.4.

Example 9.3. To present an example of a cyclic torus we use the one associated to the modulus $(e^{\frac{\pi}{3} i}, e^{\frac{2}{5}\pi j}, e^{\frac{2}{5}\pi j} e^{\frac{\pi}{3} i})$, and hence to the lattice L generated by the special basis $\mathcal{B} = \{1, e^{\frac{\pi}{3} i}, e^{\frac{2}{5}\pi j}, e^{\frac{2}{5}\pi j} e^{\frac{\pi}{3} i}\}$. Indeed, running the Algorithm presented in this section, we find out that Step 0 is satisfied and the only vector of the lattice L inside the unit ball is the null vector; besides, on $L \cap \mathbb{S}^3$ we have 12 vectors: 8 of them are in $\mathcal{B} \cup -\mathcal{B}$ and 4 other vectors correspond to $\{\pm e^{\frac{2}{3}\pi i}, \pm e^{\frac{2}{5}\pi j} e^{\frac{2}{3}\pi i}\}$.

10. Appendix B: an algorithm to check if a basis is tame

We want now to provide an algorithm to establish when a reduced basis is a tame basis.

Let $R = (r_{i,j})_{i,j=1,\cdots,4}$ be the reduced Gram matrix associated to a reduced basis $\mathcal{B} = \{v_1, v_2, v_3, v_4\}$ of a rank-4 lattice L. We will use precisely the same notations as in the algorithm of the previous section. The first step of our new algorithm is the following:

Step 0: We check if $r_{k,k+1} = \langle v_k, v_{k+1} \rangle$ (for $k = 1, 2, 3$) are all strictly positive quantities. If this is true, we proceed in the algorithm; otherwise we conclude that the basis $\mathcal{B}$ is not tame, and stop.

Step 1: Since $R = QD^tQ$, the quadratic form $(n_1, n_2, n_3, n_4) R\, {}^t(n_1, n_2, n_3, n_4)$ can also be written as

$$(n_1, n_2, n_3, n_4) QD\, {}^tQ\, {}^t(n_1, n_2, n_3, n_4) = \lambda_1^2 x_1^2 + \cdots + \lambda_4^2 x_4^2$$

where $\lambda_1^2, \cdots, \lambda_4^2$ are the ordered positive eigenvalues of D and $(x_1 \cdots, x_4) = (n_1, \cdots, n_4)Q$. The geometric locus of vectors $(x_1, \cdots, x_4) \in \mathbb{R}^4$ for which the diagonalized quadratic form is equal to $r_{1,1} = \langle v_1, v_1 \rangle$ is an ellipsoid having the length of the maximal axis of symmetry equal to $\frac{2|v_1|}{\lambda_1}$. Therefore, the quadruplets $(n_1, n_2, n_3, n_4) \in \mathbb{Z}^4$ such that

$$(n_1, n_2, n_3, n_4) R\, {}^t(n_1, n_2, n_3, n_4) = r_{1,1} \tag{10.1}$$

belong necessarily to the finite set $E_1 = \mathbb{Z}^4 \cap I_1^4$, where

$$I_1 = \left[-\frac{|v_1|}{\lambda_1}, \frac{|v_1|}{\lambda_1} \right].$$

At this point we recall the first step of the construction of the Minkowski-Siegel Reduction Algorithm: we check if there exists a point (n_1, n_2, n_3, n_4) in the finite set $E_1 \setminus \{(\pm 1, 0, 0, 0)\}$ such that equality (10.1) is fulfilled. If the answer is yes, then the integers (n_1, n_2, n_3, n_4) have common divisors: indeed, if no non-trivial common divisor exists, we can find a vector in the lattice L whose squared norm is equal to $r_{1,1}$ and which can substitute v_1 in $\mathcal{B}$. If this is the case the basis $\mathcal{B}$ is not unique and it is not tame. Otherwise we proceed in the algorithm.

Step 2: In this step we consider the quadratic equation

$$(n_1, n_2, n_3, n_4) R\, {}^t(n_1, n_2, n_3, n_4) = r_{2,2}.$$

Let

$$I_2 = \left[-\frac{|v_2|}{\lambda_1}, \frac{|v_2|}{\lambda_1} \right]$$

and set E_2 to be the finite set $\mathbb{Z}^4 \cap I_2^4$. Suppose there exists no quadruplet $(n_1, n_2, n_3, n_4) \in E_2 \setminus \{(0, \pm 1, 0, 0)\}$ with

$$(n_1, n_2, n_3, n_4) R\, {}^t(n_1, n_2, n_3, n_4) = r_{2,2}. \tag{10.2}$$

Then we go to the next step of the algorithm. Suppose that instead we find a (finite) set B of quadruplets $(n_1, n_2, n_3, n_4) \in E_2 \setminus \{(0, \pm 1, 0, 0)\}$ with

$$(n_1, n_2, n_3, n_4) R\, {}^t(n_1, n_2, n_3, n_4) = r_{2,2}. \tag{10.3}$$

We use now Definition 7.6, Proposition 7.7 and condition B2)$'$: if, for all elements $(n_1, n_2, n_3, n_4) \in B$, (n_2, n_3, n_4) have common divisors, we go to the next step of the algorithm. Otherwise we conclude that the basis $\mathcal{B}$ is not tame, and stop.

Step 3: Similar procedure applies to the quadratic equation

$$(n_1, n_2, n_3, n_4) R\, {}^t(n_1, n_2, n_3, n_4) = r_{3,3}.$$

Let

$$I_3 = \left[-\frac{|v_3|}{\lambda_1}, \frac{|v_3|}{\lambda_1} \right]$$

and set E_3 to be the finite set $\mathbb{Z}^4 \cap I_3^4$. Suppose there exists no quadruplet $(n_1, n_2, n_3, n_4) \in E_3 \setminus \{(0, 0, \pm 1, 0)\}$ with

$$(n_1, n_2, n_3, n_4) R\, {}^t(n_1, n_2, n_3, n_4) = r_{3,3}. \tag{10.4}$$

Then we go to the next step of the algorithm. Suppose that instead we find a (finite) set C of quadruplets $(n_1, n_2, n_3, n_4) \in E_3 \setminus \{(0, 0, \pm 1, 0)\}$ with

$$(n_1, n_2, n_3, n_4) R\, {}^t(n_1, n_2, n_3, n_4) = r_{3,3}. \tag{10.5}$$

We use again Definition 7.6, Proposition 7.7 and condition B2)$'$: if, for all elements $(n_1, n_2, n_3, n_4) \in C$, (n_3, n_4) have common divisors, we go to the next step of the algorithm. Otherwise we conclude that the basis $\mathcal{B}$ is not tame, and stop.

Step 4: In the last step our procedure is applied to the quadratic equation

$$(n_1, n_2, n_3, n_4)R\ {}^t(n_1, n_2, n_3, n_4) = r_{4,4}.$$

Let

$$I_4 = \left[-\frac{|v_4|}{\lambda_1}, \frac{|v_4|}{\lambda_1}\right]$$

and set E_4 to be the finite set $\mathbb{Z}^4 \cap I_4^4$. Suppose there exists no quadruplet $(n_1, n_2, n_3, n_4) \in E_4 \setminus \{(0,0,0,\pm 1)\}$ with

$$(n_1, n_2, n_3, n_4)R\ {}^t(n_1, n_2, n_3, n_4) = r_{4,4}. \tag{10.6}$$

Then we conclude that $\mathcal{B}$ is tame. Suppose that instead we find a (finite) set D of quadruplets $(n_1, n_2, n_3, n_4) \in E_4 \setminus \{(0,0,0,\pm 1)\}$ with

$$(n_1, n_2, n_3, n_4)R\ {}^t(n_1, n_2, n_3, n_4) = r_{4,4}. \tag{10.7}$$

We use again Definition 7.6, Proposition 7.7 and condition B2)$'$: if, for all elements $(n_1, n_2, n_3, n_4) \in D$, $n_4 = \pm 1$, then the basis $\mathcal{B}$ is not tame. Otherwise we conclude that the basis $\mathcal{B}$ is tame.

We provide a Matlab Script which implements part of the third step of the Algorithm we just presented, under the hypothesis that the basis is reduced.
Let $p_3 = \lceil \frac{|v_3|}{\lambda_1} \rceil$ and set:

$i = 1;$
for $n_1 = -p_3 : 1 : p_3,$
for $n_2 = -p_3 : 1 : p_3,$
for $n_3 = -p_3 : 1 : p_3,$
for $n_4 = -p_3 : 1 : p_3,$
$m = (n_1, n_2, n_3, n_4)R\ {}^t(n_1, n_2, n_3, n_4);$
if $(m == r_{3,3})$ $A(i,:) = [n_1, n_2, n_3, n_4];$
$i = i + 1;$
end
end
end
end
end
for j=1:1:i-1
if $(gcd(A(j,3), A(j,4)) == 1)\&((A(j,:) \sim= [0,0,1,0])|(A(j,:) \sim= [0,0,-1,0])$
disp('the basis is not tame')
break
end
end

To conclude, we present an explicit example of special tame lattice (i.e. a lattice whose reduced Gram matrix belongs to $\mathring{\mathcal{R}}$).

Example 10.1. The lattice L generated by the special basis

$$\mathcal{B} = \{1, 2i + \frac{1}{10}, 3j + \frac{1}{100}, 4k + \frac{1}{1000}\}$$

is a tame lattice. The proof follows by a direct application of the algorithm and of the Matlab Script presented in this section.

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