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ON A QUATERNIONIC PICARD THEOREM

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ABSTRACT. The classical theorem of Picard states that a non-constant holomorphic function $f : \mathbb{C} \to \mathbb{C}$ can avoid at most one value.

We investigate how many values a non-constant slice regular function of a quaternionic variable $f : \mathbb{H} \to \mathbb{H}$ may avoid.

1. Introduction

A function $f : \mathbb{C} \to \mathbb{C}$ which is given by a globally convergent power series $f(z) = \sum_{k=0}^{\infty} a_k z^k$ $(a_k \in \mathbb{C})$ is called an *entire function.* By the theorem of Picard, a non-constant entire function $f : \mathbb{C} \to \mathbb{C}$ can avoid at most one value [10], [11], [12].

Our goal is a similar statement for *entire slice regular functions*, i.e., for functions $f : \mathbb{H} \to \mathbb{H}$ (where $\mathbb{H}$ denotes the skew field of quaternions) which are given as a globally convergent power series $f(q) = \sum_{k=0}^{\infty} q^k a_k$ $(a_k \in \mathbb{H})$.

For a function $f : \mathbb{H} \to \mathbb{H}$ being "slice regular" is equivalent to the assumption that for every imaginary unit $I \in \mathbb{S}$ its restriction to $\mathbb{C}_I = \{x + yI : x, y \in \mathbb{R}\}$ is holomorphic with respect to the complex structures induced by left multiplication by I; see [4, 5].

Here we show the following:

(i) For every 2-dimensional real affine subspace P of $\mathbb{H} \simeq \mathbb{R}^4$, there exists an entire slice regular function $f : \mathbb{H} \to \mathbb{H}$ such that $f(\mathbb{H}) = \mathbb{H} \setminus P$. In particular, for every triple $q_1, q_2, q_3 \in \mathbb{H}$ there is an entire slice regular function avoiding these three values.

(ii) Let $q_1, \ldots, q_5 \in \mathbb{H}$ be in general position (i.e., these five quaternions are not contained in any 3-dimensional real affine subspace of $\mathbb{H}$). Then every entire slice regular function avoiding all these five values must be constant. In particular, for every non-constant entire slice regular function the image is dense in $\mathbb{H}$.

We do not know whether an entire slice regular function may avoid a generic choice of four quaternionic numbers.

A key tool is the following fundamental correspondence (see Proposition 2.2):

Let f be a slice regular function and let F be its stem function. Let $x, y \in \mathbb{R}$ and $c \in \mathbb{H}$. Then there exists an imaginary unit $I \in \mathbb{S}$ such that $f(x + yI) = c$ if and only if $F(x + yi) - c \otimes 1$ is a zero divisor in the algebra $\mathbb{H} \otimes \mathbb{C}$.

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Maybe this work can be of some inspiration in studying hyperbolic quaternionic slice regular manifolds. Indeed recently many examples of quaternionic slice regular manifolds have been introduced; see for example [2], [1].

2. Preparations

2.1. **Quaternions.** The quaternionic numbers are a real 4-dimensional skew field $\mathbb{H}$, which may be described as the non-commutative $\mathbb{R}$-algebra with 1, generated by I, J, K with $I^2 = J^2 = K^2 = -1$, $K = IJ = -JI$, $I = JK = -KJ$ and $J = KI = -IK$.

The set of all elements $q \in \mathbb{H}$ with $q^2 = -1$ is called the set of *imaginary units* and denoted by $\mathbb{S}$.

One may check easily that

$$\mathbb{S} = \{c_2 I + c_3 J + c_4 K : c_i \in \mathbb{R}, \sum_{i=2}^{4} c_i^2 = 1\}.$$

2.2. **Slice regular functions and stem functions.** We recall the theory of slice regular functions and their stem functions ([5], [6]).

An *entire slice regular function* $f : \mathbb{H} \to \mathbb{H}$ is a function which is given by a globally convergent power series $f(q) = \sum_{k=0}^{\infty} q^k a_k$ (with $a_k \in \mathbb{H}$).

A *stem function* F is a holomorphic map from $\mathbb{C}$ to the $\mathbb{C}$-algebra $\mathbb{H}_{\mathbb{C}} = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ such that $\overline{F(z)} = F(\bar{z})$. The tensor product $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ inherits a complex structure from its second factor, $\mathbb{C}$, hence it makes sense to talk about holomorphicity and complex conjugation.

In explicit terms, the stem function F associated to a slice regular function $f(q) = \sum_{k=0}^{\infty} q^k a_k$ may be defined as $F(z) = \sum_{k=0}^{\infty} a_k \otimes z^k$.

Equivalently, the correspondence may be described as follows:

$$F(x + yi) = F_1(x + yi) \otimes 1 + F_2(x + yi) \otimes i$$

with

$$F_1(x + yi) = \frac{1}{2} \left(f(x + yI) + f(x - yI) \right)$$

and

$$F_2(x + yi) = -\frac{1}{2} I \left(f(x + yI) - f(x - yI) \right).$$

For a slice regular function f the terms on the right hand side can be shown to be independent of the choice of the imaginary unit I.

Conversely, one has

$$f(x + yH) = F_1(x + yi) + H F_2(x + yi) \quad \forall x, y \in \mathbb{R}, H \in \mathbb{S}.$$

2.3. **A remarkable quadric in $\mathbb{H}_{\mathbb{C}}$.** The euclidean scalar product on $\mathbb{H} \simeq \mathbb{R}^4$ induces a complex symmetric bilinear form $\langle\ ,\ \rangle$ on $\mathbb{H}_{\mathbb{C}}$. Explicitly: $\langle z, w \rangle = \sum_{i=1}^{4} z_i w_i$.

We observe that $\mathbb{H}_{\mathbb{C}}$ naturally carries the structure of an $\mathbb{R}$-algebra.

Both the field of complex numbers $\mathbb{C}$ and the quaternionic skew field $\mathbb{H}$ embed into $\mathbb{H}_{\mathbb{C}}$ via $z \mapsto 1 \otimes z$, resp., $q \mapsto q \otimes 1$. In this way we may regard $\mathbb{C}$ and $\mathbb{H}$ as subrings of $\mathbb{H}_{\mathbb{C}}$.

Proposition 2.1. *Let* $v = 1 \otimes v_0 + I \otimes v_1 + J \otimes v_2 + K \otimes v_3 = v' \otimes 1 + v'' \otimes i$ *(with* $v_i \in \mathbb{C}$, $v', v'' \in \mathbb{H}$*) be an element of* $\mathbb{H}_\mathbb{C}$*.*

Then the following are equivalent:

(i) v *is a zero divisor, i.e., there exists an element* $w \in \mathbb{H}_\mathbb{C}$, $w \neq 0$ *with* $w \cdot v = 0$.
(ii) $\langle v, v \rangle = 0$, *i.e.,* $\sum_{i=0}^{3} v_i^2 = 0$.
(iii) *There exists an imaginary unit* $H \in \mathbb{S}$ *such that* $Hv' = v''$. *(Geometrically: The vectors* v' *and* v'' *are orthogonal.)*

The above equivalence (i) $\iff$ (ii) is contained in [13] where it is attributed to Hamilton, while (ii) $\iff$ (iii) may be deduced from the work of Mongodi ([7]). In addition, these equivalences may be obtained as a special case of a result of Ghiloni and Perotti ([6, Theorem 17 on page 1679]).

For the convenience of the reader we nevertheless give a proof here.

Proof. (i) $\implies$ (iii): We assume that v is a zero divisor (but $v \neq 0$). Since $\mathbb{H}$ has no zero divisors, it follows that $v', v'' \neq 0$. Now $v' \in \mathbb{H}^*$ and v being a zero divisor, imply that $v \cdot \big((v')^{-1} \otimes 1\big)$ is again a zero divisor. Hence we may assume that $v' = 1$. The same reasoning also shows that we can find an element $w = w' + w'' \otimes i$ with $w' = 1$ and $w \cdot v = 0$. Thus we obtain

$$0 = w \cdot v = (1 + w'' \otimes i) \cdot (1 + v'' \otimes i) = (1 - w''v'') \otimes 1 + (v'' + w'') \otimes i.$$

Hence $v'' = -w''$ and $(v'')^2 = -v''w'' = -1$, i.e., $v'' \in \mathbb{S}$. In particular, $v'' = H \cdot 1 = H \cdot v'$ for some $H \in \mathbb{S}$.

(iii) $\implies$ (i): We have $v = (1 \otimes 1 + H \otimes i) \cdot v'$. Define $w = 1 \otimes 1 - H \otimes i$. Then $w \cdot v = 0$, as easily seen by explicit calculation.

(iii) $\iff$ (ii): Note that

$$\langle v, v \rangle = \langle v' + v'' \otimes i, v' + v'' \otimes i \rangle = \langle v', v' \rangle - \langle v'', v'' \rangle + 2i \langle v', v'' \rangle .$$

Hence $\langle v, v \rangle = 0$ iff v' and v'' have the same norm and are orthogonal to each other. This in turn is equivalent to the existence of an imaginary unit $H \in \mathbb{S}$ with $v'' = Hv'$. □

Thus the set of all zero divisors of $\mathbb{H}_\mathbb{C}$ is a quadric subvariety of $\mathbb{H}_\mathbb{C} \simeq \mathbb{C}^4$. This quadric has also been investigated by Mongodi ([7]), who pointed out the relevance for the zero locus, but not the relation with zero divisors of the algebra $\mathbb{H}_\mathbb{C}$.

2.4. **Zeros.** Let f be a slice function and let F denote its stem function. Write $F = F_1 \otimes 1 + F_2 \otimes i$, with $F_h : \mathbb{C} \to \mathbb{H}$. Since

$$f(x + yI) = F_1(x + yi) + IF_2(x + yi) \quad \forall x, y \in \mathbb{R}, I \in \mathbb{S},$$

this implies

$$f(x + yI) = 0 \iff F_1(x + yi) = -IF_2(x + yi),$$

The following result is implied by Proposition 2.1, but may also be deduced from [7, Proposition 4.1] in combination with Corollary 3.4 of [7]:

Proposition 2.2. *Let* $f : \mathbb{H} \to \mathbb{H}$ *be a slice regular function and let* $F : \mathbb{C} \to \mathbb{H}_\mathbb{C} = \mathbb{H} \otimes_\mathbb{R} \mathbb{C}$ *be its stem function. Let* $x, y \in \mathbb{R}$*. Then the following conditions are equivalent:*

(i) *There exists an imaginary unit* $H \in \mathbb{S}$ *with* $f(x + yH) = 0$.
(ii) $\langle F(x + yi), F(x + yi) \rangle = 0$.
(iii) $F(x + yi)$ *is a zero divisor in the algebra* $\mathbb{H} \otimes_\mathbb{R} \mathbb{C}$.

This has the following consequence: let $c \in \mathbb{H}$. Then a slice regular function f avoids c as value (i.e., $f(\mathbb{H}) \subset \mathbb{H} \setminus \{c\}$) if and only if $z \mapsto F(z) - c \otimes 1$ has no zero which happens if and only if the entire function

$$Q_c : z \mapsto \langle F(z) - c, F(z) - c \rangle = \langle F(z), F(z) \rangle - 2 \langle F(z), c \rangle + \langle c, c \rangle$$

has no zeros.

3. Avoiding five generic values

The purpose of this section is to show that a non-constant entire slice regular function cannot avoid five values if these are generic in the following sense: there is no real 3-dimensional affine subspace of $\mathbb{H} \simeq \mathbb{R}^4$ containing all of them.

We start with some preparations.

First we recall two results of Noguchi on holomorphic curves in semi-abelian varieties. Here we do not need to deal with arbitrary semi-abelian varieties, it suffices to know that $(\mathbb{C}^*)^g$ is a semi-abelian variety.

Proposition 3.1 (Logarithmic Bloch Ochiai theorem)**.** *Let $f : \mathbb{C} \to G = (\mathbb{C}^*)^g$ be a holomorphic map and let X denote the Zariski closure of its image.*

Then X is an orbit of an algebraic subgroup H of $G = (\mathbb{C}^*)^g$ (acting by left multiplication), i.e., there is an element $\lambda = (\lambda_1, \ldots, \lambda_g) \in G = (\mathbb{C}^*)^g$ such that

$$X = \{\lambda \cdot h : h \in H\}.$$

See Main Theorem (i) in [8].

Proposition 3.2. *Let*

$$f : \Delta^* = \{z \in \mathbb{C} : 0 < |z| < 1\} \to G = (\mathbb{C}^*)^g \subset \bar{G} = (\mathbb{P}_1)^g$$

be a holomorphic map and let X denote the Zariski closure of its image. Define

$$Stab(X) = \{g \in G : g \cdot x \in X \ \forall x \in X\}.$$

If $Stab(X)$ is discrete, then f extends to a holomorphic map from Δ to $\bar{G}$.

Proof. This is a consequence of Theorem 4.5. of [9], applied with taking the Zariski closure of $f(\Delta^*)$ as X. In the notation of [9] non-extendibility of f implies $f(\Delta^*) \subset W$. Since we take X to be the Zariski closure of the image of f, the inclusion $f(\Delta^*) \subset W$ implies $X = W$. In view of Lemma 4.1 in [9] the condition $X = W$ implies that $Stab(X)$ is not discrete. ☐

Proposition 3.3. *Let Z be an algebraic subvariety of $G = (\mathbb{C}^*)^5$. Assume that there exists a non-constant holomorphic map $g : \mathbb{C} \to Z$ with $g(z) = \overline{g(\bar{z})}$ for all $z \in \mathbb{C}$.*

Then there exist $\alpha_1, \ldots, \alpha_5 \in \mathbb{R}^*$ and $(m_1, \ldots, m_5) \in \mathbb{Z}^5 \setminus \{(0, \ldots, 0)\}$ such that $\zeta(\mathbb{C}^*) \subset Z$ for

$$\zeta(z) \stackrel{def}{=} (\alpha_1 z^{m_1}, \ldots, \alpha_5 z^{m_5}).$$

Proof. The Zariski closure of the image $g(\mathbb{C})$ in G is an orbit of an algebraic subgroup H of G acting by multiplication (Proposition 3.1). We choose a connected 1-dimensional algebraic subgroup T of H. Such a subgroup T is isomorphic to $\mathbb{C}^*$ and parametrized by a map $\zeta_0 : \mathbb{C}^* \to G = (\mathbb{C}^*)^5$ given as

$$\zeta_0(z) \stackrel{def}{=} (z^{m_1}, \ldots, z^{m_5}).$$

Define $\alpha = (\alpha_1, \ldots, \alpha_5) \stackrel{def}{=} g(0)$. The condition $g(z) = \overline{g(\bar{z})}$ implies that $\alpha_i \in \mathbb{R}$ for all $i \in \{1, \ldots, 5\}$. By our construction the H-orbit through α must be contained in Z. It follows that $\zeta(\mathbb{C}^*) \subset Z$ for

$$\zeta(z) = \zeta_0(z) \cdot \alpha = (\alpha_1 z^{m_1}, \ldots, \alpha_5 z^{m_5}).$$

□

Proposition 3.4. *Let $c_1, \ldots, c_4$ be a basis of the real vector space $\mathbb{H}$. Let $M \in Mat(4 \times 4, \mathbb{R})$ be a positive definite symmetric real matrix. Let Z denote the zero set of the function ψ in $G = (\mathbb{C}^*)^5$ where*

$$\psi(v_1, \ldots, v_4; p) = p - w^t M w, \qquad (v = (v_1, \cdots, v_4) \in \mathbb{C}^4, p \in \mathbb{C})$$

with

$$w = v - \begin{pmatrix} p + \langle c_1, c_1 \rangle \\ \vdots \\ p + \langle c_4, c_4 \rangle \end{pmatrix}.$$

Let $\alpha_i \in \mathbb{R}^*$ and $m_i \in \mathbb{Z}$ such that the image of the map $\zeta : \mathbb{C}^* \to G$ given as

$$\zeta(z) \stackrel{def}{=} (\alpha_1 z^{m_1}, \ldots, \alpha_5 z^{m_5})$$

is contained in Z (i.e., $\zeta(\mathbb{C}^*) \subset Z$).

Then $m_i = 0$ for all $i \in \{1, \ldots, 5\}$, i.e., ζ must be constant.

Proof. We discuss the coefficients of the Laurent series $\sum_{k \in \mathbb{Z}} b_k z^k$ of the holomorphic function $z \mapsto (\psi \circ \zeta)(z)$ defined on $\mathbb{C}^*$. Since $\psi \circ \zeta \equiv 0$ due to $\zeta(\mathbb{C}^*) \subset Z$, we know that $b_k = 0$ for all $k \in \mathbb{Z}$. On the other hand, the Laurent coefficients b_k depend on the matrix M and the coefficients α_i, m_i. Using these facts we will see that we arrive at a contradiction if we assume that ζ is not constant.

We start by observing that ψ is a polynomial map of degree 2 whose purely quadratic term is given by

$$\psi_2(v; p) = -(v - pd)^t M (v - pd) \quad \text{with } d = (1, \ldots, 1)^t.$$

We may replace ζ with its composition with the inverse element map $z \mapsto 1/z$ and thereby assume $m_5 \geq 0$. By permuting variables we may also assume that

$$m_1 \leq m_2 \leq m_3 \leq m_4.$$

Let us now assume that ζ is not constant, i.e., let us assume that $(m_1, \ldots, m_5) \neq (0, \ldots, 0)$. Our strategy is to show that the Laurent series of $\psi \circ \zeta$ cannot vanish unless $(m_1, \ldots, m_5) = (0, \ldots, 0)$.

Case 1. We assume $m_1 < 0$.

Fix k such that $m_i = m_1$ for $1 \leq i \leq k$ and $m_i > m_1$ for $k < i \leq 4$. We consider the Laurent coefficient of degree $2m_1$. Note that ζ has no homogeneous component of degree less that m_1. Recall that ψ is a quadratic polynomial. It follows that $\psi \circ \zeta$ has no homogeneous component of degree less that $2m_1$ and that the homogeneous component of degree $2m_1$ equals $(\psi_2 \circ \zeta)_{2m_1}$ where ψ_2 is the purely quadratic part of ψ and $(\psi_2 \circ \zeta)_{2m_1}$ is the homogeneous component of $\psi_2 \circ \zeta$ of degree $2m_1$. Thus $(\psi_2 \circ \zeta)_{2m_1} = b_{2m_1} z^{2m_1}$.

By the definition of ψ and ζ, it follows that $b_{2m_1} = -u^t M u$ with

$$u = (\alpha_1, \ldots, \alpha_k, 0, \ldots, 0).$$

But M is positive definite and the α_i are all real and non-zero. Hence $u^t M u > 0$, contradicting $\psi \circ \zeta \equiv 0$.

Case 2. We assume $m_5 > 0$ and $m_1 \geq 0$.

Fix $k \in \{1, \ldots, 4\}$ such that $m_i = 0$ iff $i \leq k$. Here we investigate the constant term of the Laurent series of $\psi \circ \zeta$, i.e., its degree-0-coefficient.

This is $b_0 = -u^t M u$ with

$$u = (\alpha_1 + \langle c_1, c_1 \rangle, \ldots, \alpha_k + \langle c_k, c_k \rangle, \langle c_{k+1}, c_{k+1} \rangle, \ldots, \langle c_4, c_4 \rangle).$$

We employ again the facts that M is positive definite and u is real. Hence $u^t M u = 0$ requires that u is the zero vector. Because $\langle c_i, c_i \rangle > 0$, it follows that $k = 4$. Thus $m_i = 0$ for all $i < 5$. But now it follows that the degree $2m_5$-term is $-v^t M v$ with

$$v = (\alpha_5, \ldots, \alpha_5)$$

which yields a contradiction.

Case 3. We assume $m_5 = 0$ and $m_1 \geq 0$.

Then $m_4 = \max\{m_1, \ldots, m_5\}$ and we discuss the term of degree $2m_4$. Let k be such that $m_i = m_4$ iff $4 \geq i \geq k$. Then the degree $2m_4$-coefficient of the Laurent series equals $-u^t M u$ with

$$u = (0, \ldots, \alpha_k, \ldots, \alpha_4)$$

which cannot be zero by the same arguments as before.

Thus we have checked by contradiction that $(m_1, \ldots, m_5)$ cannot be different from $(0, \ldots, 0)$. □

Corollary 1. *Under the assumptions of Proposition* 3.4*, let X be an algebraic subvariety of Z such that $X \cap (\mathbb{R}^*)^5$ is not empty.*

Then the stabilizer group $Stab(X) = \{g \in G : g \cdot X = X\}$ is discrete.

Proof. If $Stab(X)$ is not discrete, it contains an algebraic subgroup H isomorphic to $\mathbb{C}^*$, i.e., given as

$$H = \{(z^{m_1}, \ldots, z^{m_5}) : z \in \mathbb{C}^*\}$$

with $(m_1, \ldots, m_5) \in \mathbb{Z}^5 \setminus \{(0, \ldots, 0)\}$.

Since $X \cap (\mathbb{R}^*)^5$ is non-empty, there are $\alpha_i \in \mathbb{R}^*$ with $(\alpha_1, \ldots, \alpha_5) \in X$. Then

$$(\alpha_1 z^{m_1}, \ldots, \alpha_5 z^{m_5}) \in X \ \forall z \in \mathbb{C}^*$$

contradicting the preceding proposition. □

Remark. The assumption that X contains a real point is crucial. E.g., for $M = I_4$ consider

$$X = \{(1, 1, z, iz; 2) : z \in \mathbb{C}^*\}.$$

Then $X \cap (\mathbb{R}^*)^5$ is empty and $Stab(X)$ is 1-dimensional.

Theorem 3.5. *Let $c_1, \ldots, c_5 \in \mathbb{H}$ be given such that there is no proper real affine* 3*-subspace of $\mathbb{H}$ containing all c_i.*

Then every slice regular function $f : \mathbb{H} \to \mathbb{H}$ with $f(\mathbb{H}) \subset \mathbb{H} \setminus \{c_1, \ldots, c_5\}$ is constant.

Proof. Without loss of generality we may assume that $c_5 = 0$. By abuse of language we identify $c_i \in \mathbb{H}$ with $c_i \otimes 1 \in \mathbb{H}_{\mathbb{C}}$. Let $\langle\ ,\ \rangle$ denote the complex bilinear form on $\mathbb{H}_{\mathbb{C}}$ induced by the euclidean scalar product on $\mathbb{H} \simeq \mathbb{R}^4$, i.e., $\langle z, w \rangle = \sum_i z_i w_i$.

We define a holomorphic map $\phi : \mathbb{H}_{\mathbb{C}} = \mathbb{C}^4 \to \mathbb{C}^5$ by

$$\phi : \begin{pmatrix} z_1 \\ \vdots \\ z_4 \end{pmatrix} \mapsto \begin{pmatrix} \langle z, z\rangle - 2\langle z, c_1\rangle + \langle c_1, c_1\rangle \\ \vdots \\ \langle z, z\rangle - 2\langle z, c_4\rangle + \langle c_4, c_4\rangle \\ \langle z, z\rangle \end{pmatrix}. \tag{3.1}$$

Observe that $\phi(z) = \overline{\phi(\bar{z})}$.

By assumption the vectors $c_1, \ldots, c_4$ form a real vector space basis for $\mathbb{H}$. It follows that there exists an invertible real 4×4-matrix B such that

$$\begin{pmatrix} \langle z, c_1\rangle \\ \vdots \\ \langle z, c_4\rangle \end{pmatrix} = B^{-1} \cdot z \;\forall z \in \mathbb{R}^4 \simeq \mathbb{H}. \tag{3.2}$$

Let $M = B^t B$. Then M is a positive definite symmetric real matrix M such that for every $z \in \mathbb{C}^4$ we have

$$\langle z, z\rangle = v^t \cdot M \cdot v$$

if

$$v = \begin{pmatrix} \langle z, c_1\rangle \\ \vdots \\ \langle z, c_4\rangle \end{pmatrix}.$$

We observe that

$$\phi_i(z) = \langle z, z\rangle - 2\langle z, c_i\rangle + \langle c_i, c_i\rangle$$

for $z = (z_1, \ldots, z_4)$ and $i \in \{1, 2, 3, 4\}$ implies that

$$\langle z, c_i\rangle = -\frac{1}{2}\left(\phi_i(z) - \langle z, z\rangle - \langle c_i, c_i\rangle\right).$$

Combined with $\phi_5(z) = \langle z, z\rangle$ we obtain that

$$\phi_5(z) = v^t M v$$

for

$$v_i = -\frac{1}{2}\left(\phi_i(z) - \langle z, z\rangle - \langle c_i, c_i\rangle\right).$$

On $\mathbb{C}^5$ we define an algebraic subvariety Z as the zero set of the function

$$\psi(w_1, \ldots, w_4; p) = p - u^t M u, \quad \text{with}$$
$$u = -\frac{1}{2}\left(w_1 - p - \langle c_1, c_1\rangle, \ldots, w_4 - p - \langle c_4, c_4\rangle\right)^t.$$

Due to the definition of ψ it is clear that $\psi(w; p) = 0$ if $(w, p) = \phi(z)$ for some $z \in \mathbb{C}^4$.

Therefore $\phi(\mathbb{C}^4) \subset Z$.

We claim that $\phi : \mathbb{C}^4 \to Z$ is biholomorphic. Indeed, consider

$$\mu : \begin{pmatrix} v_1 \\ \vdots \\ v_4 \\ v_5 \end{pmatrix} \mapsto B \cdot \begin{pmatrix} -\frac{1}{2}\left(v_1 - \langle c_1, c_1\rangle - v_5)\right) \\ \vdots \\ -\frac{1}{2}\left(v_4 - \langle c_4, c_4\rangle - v_5)\right) \end{pmatrix}$$

with B defined as in (3.2). Due to the definitions of ϕ and B ((3.1), resp., (3.2)) this map $\mu : Z \to \mathbb{C}^4$ is an inverse for $\phi : \mathbb{C}^4 \to Z$. Thus $\mathbb{C}^4$ and Z are biholomorphic and even isomorphic as algebraic varieties.

Now let f be a non-constant slice regular function avoiding the values $c_1,\ldots, c_4$, $c_5 = 0$ and let $F : \mathbb{C} \to \mathbb{H}_{\mathbb{C}} \simeq \mathbb{C}^4$ be its stem function. Since $\phi(\mathbb{C}^4) \subset Z = \{\psi = 0\}$, we obtain a holomorphic map $g = \phi \circ F : \mathbb{C} \to Z$. By construction $g(z) = \overline{g(\bar{z})}$ for all $z \in \mathbb{C}$. Furthermore g is non-constant, because F is non-constant and ϕ is injective.

Because $f : \mathbb{H} \to \mathbb{H}$ is assumed to avoid c_i for every i, we know (thanks to Proposition 2.2) that $\phi_i(F(z)) \neq 0$ for all $z \in \mathbb{C}$ and all i, i.e., $\phi(F(\mathbb{C})) \subset Z \cap (\mathbb{C}^*)^5$.

Thus we may apply Proposition 3.3 and conclude that there exist $\alpha_1 \ldots, \alpha_5 \in \mathbb{R}^*$ and $(m_1, \ldots, m_5) \in \mathbb{Z}^5 \setminus \{(0, \ldots, 0)\}$ such that $\zeta(\mathbb{C}^*) \subset Z$ for

$$\zeta(z) \stackrel{def}{=} (\alpha_1 z^{m_1}, \ldots, \alpha_5 z^{m_5}) .$$

But such a holomorphic map cannot exist due to Proposition 3.4. Contradiction! Thus there is no non-constant slice regular function $f : \mathbb{H} \to \mathbb{H}$ avoiding all the c_i. □

Remark. If $f : \mathbb{H} \to \mathbb{H}$ is non-constant and slice preserving (i.e., it preserves each slice), then it can avoid only real points and at most one.

If f is non-constant and one-slice preserving (i.e., it preserves a unique slice), then it can avoid only one point on the slice which is preserved.

4. Big Picard

In complex analysis, the "Big Picard theorem" states the following: If f is a holomorphic function on $\Delta^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$ with an essential singularity at 0, then f assumes every value in $\mathbb{P}_1$ infinitely often with at most two exceptions.

Proposition 4.1. *Let Z be defined as in Proposition 3.4. Let η be a holomorphic map from Δ^* to $Z \subset (\mathbb{C}^*)^5 \subset (\mathbb{P}_1)^5$ with $\eta(\bar{z}) = \overline{\eta(z)}$ for all z.*

Then η extends through 0 to a holomorphic map to $(\mathbb{P}_1)^5$, i.e., the isolated singularity of η at 0 is not essential.

Proof. Let X denote Zariski closure of $\eta(\Delta^*)$ in Z. Note that $\eta(z) \in (\mathbb{R}^*)^5$ for $z \in \mathbb{R} \cap \Delta^*$. Thus X has non-trivial intersection with $(\mathbb{R}^*)^5$. It follows that $Stab(X)$ is discrete (see Corollary 1 of Section 3). This implies that η extends to a holomorphic map defined on Δ (Proposition 3.2). □

Theorem 4.2 (Quaternionic Big Picard)**.** *Let $\mathbb{B}$ denote the open unit ball in $\mathbb{H}$ and let $f : \mathbb{B} \setminus \{0\} \to \mathbb{H}$ be a slice regular function with stem function $F : \Delta^* \to \mathbb{H}_{\mathbb{C}}$. Assume that F has an essential singularity at 0 (i.e., at least one of the components of F has an essential singularity).*

Let S denote the set of all $v \in \mathbb{H}$ for which the level set $f^{-1}(v) = \{q \in \mathbb{H} : f(q) = v\}$ is finite.

Then S is contained in an affine real hyperplane in $\mathbb{H}$.

Proof. Assume the contrary. Then there are five values $c_0, \ldots, c_4$ for which the level set is finite such that these five values generate $\mathbb{H}$ as an affine real space. Since

$\bigcup_{m=0}^{4} f^{-1}(c_m)$ is finite, we may define

$$r = \min\left\{ |q| : q \in \bigcup_{m=0}^{4} f^{-1}(c_m), q \neq 0 \right\}, \qquad \mathbb{B}_r = \{q \in \mathbb{H} : |q| < r\}.$$

Now $f|_{\mathbb{B}_r \setminus \{0\}}$ avoids $c_0, \ldots, c_4$. Hence $\phi(F(z)) \in (\mathbb{C}^*)^5 \cap Z$ for all $z \in \mathbb{C}$, $|z| < r$ (with ϕ and Z defined as in Theorem 3.5). Due to Proposition 4.1 the holomorphic map $\phi \circ F : \{z \in \mathbb{C} : 0 < |z| < r\} \to Z$ extends to a holomorphic map with values in $(\mathbb{P}_1)^5$. But $\phi : \mathbb{H}_\mathbb{C} \to Z$ is a biholomorphic map, whose inverse map $\phi^{-1} = \mu$ is polynomial (see the proof of Theorem 3.5). It follows immediately that $\phi^{-1} \circ (\phi \circ F) = F$ extends to a holomorphic map from Δ to $(\mathbb{P}_1)^4$. This yields a contradiction to our assumptions. □

Since over the complex field, Picard's theorems are the global version of the local Landau's Theorem, we point out that a quaternionic Landau's Theorem for slice regular functions already exists in the literature; see [3].

Proposition 4.3. *For every non-constant slice regular function $f : \mathbb{H} \to \mathbb{H}$ the image is dense in $\mathbb{H}$.*

Proof. If the image is not dense, its complement contains a non-empty open set. But it is trivially possible to choose five points in general position inside any given non-empty open set, leading to a contradiction with Theorem 3.5. □

In particular, a bounded slice regular function $f : \mathbb{H} \to \mathbb{H}$ must be constant, a fact which was first proved in [5, Theorem 3.7].

5. The example of a function avoiding $\mathbb{C}_I$

Here we provide an example of a slice regular function avoiding infinitely many values.

Proposition 5.1. *Let $f : \mathbb{H} \to \mathbb{H}$ be the slice regular function induced by the stem function*

$$F(z) = J \otimes \sin(z) + K \otimes \cos(z).$$

Then

$$f(\mathbb{H}) = \mathbb{H} \setminus \mathbb{C}_I = \{c_1 + c_2 I + c_3 J + c_4 K; c_i \in \mathbb{R}, (c_3, c_4) \neq (0,0)\}.$$

Proof. We start with some preparations concerning complex trigonometric functions.

We recall that $\sin(iy) = i\sinh(y)$ and $\cos(iy) = \cosh(y)$ for all $y \in \mathbb{R}$.

For $z = x + iy$ $(x, y \in \mathbb{R})$ we obtain

$$\begin{aligned}\sin(z) = \sin(x+iy) &= \sin(x)\cos(iy) + \cos(x)\sin(iy) \\ &= \sin(x)\cosh(y) + i\cos(x)\sinh(y)\end{aligned}$$

and

$$\cos(z) = \cos(x+iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y).$$

Given $c = c_1 + c_2 I + c_3 J + c_4 K \in \mathbb{H}$, there exists a quaternionic number q with $f(q) = c$ iff there exists a complex number $z = x + iy$ with

$$\langle F(z) - c \otimes 1, F(z) - c \otimes 1 \rangle = 0.$$

Now

$$\begin{aligned}&\langle F(z)-c\otimes 1, F(z)-c\otimes 1\rangle\\ =&\langle F(z),F(z)\rangle - 2\langle c\otimes 1, F(z)\rangle + ||c||^2\\ =&1-2\left(c_3\sin(z)+c_4\cos(z)\right)+||c||^2\end{aligned}$$

implying

$$\Im\left(\langle F(z)-c\otimes 1, F(z)-c\otimes 1\rangle\right) = -2\sinh(y)\left(c_3\cos(x)-c_4\sin(x)\right) \tag{5.1}$$

and

$$\begin{aligned}\Re\left(\langle F(z)-c\otimes 1, F(z)-c\otimes 1\rangle\right)&\\ = 1-2\cosh(y)\left(c_3\sin(x)+c_4\cos(x)\right)+||c||^2.&\end{aligned} \tag{5.2}$$

It follows that

$$\Re\left(\langle F(z)-c\otimes 1, F(z)-c\otimes 1\rangle\right) = 1+||c||^2 \geq 1 > 0$$

if $c_3=c_4=0$. This proves that f does not assume any value in $\mathbb{C}_I$.

It remains to prove that all other values are assumed.

We claim: For every $c\in\mathbb{H}\simeq\mathbb{R}^4$ with $(c_3,c_4)\neq(0,0)$ there exist $x,y\in\mathbb{R}$ such that $\langle F(x+yi)-c\otimes 1, F(x+yi)-c\otimes 1\rangle = 0$.

First we choose $x\in\mathbb{R}$ such that

$$c_3\cos(x)-c_4\sin(x)=0.$$

Due to (5.1) this guarantees that

$$\Im(\langle F(x+yi)-c\otimes 1, F(x+yi)-c\otimes 1\rangle)=0.$$

If $c_3\sin(x)+c_4\cos(x)<0$, we replace x by $x+\pi$. This ensures that

$$c_3\sin(x)+c_4\cos(x)>0.$$

Define

$$t=\frac{1+||c||^2}{2\left(c_3\sin x+c_4\cos x\right)}.$$

We have to show that there exists a number $y\in\mathbb{R}$ with $\cosh(y)=t$, because then it follows from (5.1) and (5.2) that $\langle F(x+iy),F(x+iy)\rangle=0$.

An application of the Cauchy Schwarz Inequality to the vectors (c_3,c_4) and $(\sin(x),\cos(x))$ yields the inequality

$$|c_3\sin(x)+c_4\cos(x)|\leq\sqrt{c_3^2+c_4^2}.$$

Using $c_3\sin(x)+c_4\cos(x)>0$ it follows that

$$t=\frac{1+||c||^2}{2\left(c_3\sin x+c_4\cos x\right)}\geq\frac{1+\left(c_3\sin x+c_4\cos x\right)^2}{2\left(c_3\sin x+c_4\cos x\right)}\geq 1.$$

Now $t\geq 1$ implies that there exists a real number y with $\cosh(y)=t$. This completes the proof. □

6. Avoiding three points

Proposition 6.1. *Let c_1, c_2, c_3 be three arbitrary quaternionic numbers.*

Then there exists a non-constant slice regular function $f(q) = \sum q^k a_k$ such that $f(\mathbb{H}) \subset \mathbb{H} \setminus \{c_1, c_2, c_3\}$.

Proof. We have seen that there exists a slice regular function $f(q) = \sum_k q^k a_k$ with $f(\mathbb{H}) \subset \mathbb{H} \setminus \mathbb{C}_I$ (Proposition 5.1).

We modify this function in the following way: Let $\lambda \in \mathbb{H}^*$, $p \in \mathbb{H}$ and let ϕ be a ring automorphism of $\mathbb{H}$.

Then we define a slice regular function g by

$$g(q) \stackrel{def}{=} \left(\sum_k q^k \phi(a_k) \right) \lambda + p.$$

For any $c \in \mathbb{H}$ we have

$$\begin{aligned} & c = g(\phi(q)) \\ \iff & c = \phi(f(q))\lambda + p \\ \iff & \phi^{-1}(c) = f(q)\phi^{-1}(\lambda) + \phi^{-1}(p) \\ \iff & f(q) = \left(\phi^{-1}(c) - \phi^{-1}(p)\right) \phi^{-1}(1/\lambda). \end{aligned}$$

Let $c_1, c_2, c_3 \in \mathbb{H}$ be three given distinct quaternionic numbers. (Evidently it suffices to consider only the case of three *distinct* numbers.)

We choose p, λ, ϕ such that:

(i) $p = c_1$,
(ii) $\lambda = c_2 - c_1$,
(iii) $\phi^{-1}\left((c_3 - c_1)(c_2 - c_1)^{-1}\right) \in \mathbb{C}_I$.

In order to verify that this is possible, let $H \in \mathbb{H}$ be an imaginary unit (i.e., $H^2 = -1$) such that

$$(c_3 - c_1)(c_2 - c_1)^{-1} \in \mathbb{C}_H = \mathbb{R} \oplus H\mathbb{R}.$$

Let ϕ be an orientation preserving linear orthogonal transformation of $\mathbb{H}$ fixing $\mathbb{R}$ pointwise and such that $\phi(I) = H$. Then ϕ is a ring automorphism of $\mathbb{H}$ satisfying (iii).

It is easily verified that

$$\left(\phi^{-1}(c_i) - \phi^{-1}(p)\right) \phi^{-1}(1/\lambda) \in \mathbb{C}_I$$

for all three indices $i \in \{1, 2, 3\}$. Since f avoids values in $\mathbb{C}_I$, it follows that g avoids the three values c_1, c_2, c_3. □

Remark. Since any 2-dimensional real affine subspace P of $H \simeq \mathbb{R}^4$ is spanned by three points, it follows from the above that there exists an entire slice regular function $f : \mathbb{H} \to \mathbb{H}$ such that $f(\mathbb{H}) = \mathbb{H} \setminus P$.

Open Problem. Is or isn't there a non-constant slice regular entire function of $\mathbb{H}$ avoiding four general points?

7. Octonions

In view of the results of [6], in particular theorem 17, one may easily modify our arguments in order to obtain a Picard theorem for the algebra of octonions, namely we have the following.

Theorem 7.1. *For every non-constant* slice regular function $f : \mathbb{O} \to \mathbb{O}$ *the set* $\mathbb{O} \setminus f(\mathbb{O})$ *is contained in a real affine hyperplane of* $\mathbb{O}$.

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