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On Runge pairs and topology of axially symmetric domains

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Abstract. We prove a Runge theorem for and describe the homology of axially symmetric open subsets of $\mathbb{H}$.

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1. Introduction

Approximation theory plays a fundamental role in complex analysis, holomorphic dynamics, the theory of minimal surfaces in Euclidean spaces and in many other related fields of mathematics. In this paper, our goal is to study quaternionic analogs of the classical complex Runge theory, in particular analogs of the classical topological characterization of domains in the complex plane on which holomorphic functions may be approximated by entire functions. We recall that the classical theory of holomorphic approximation started in 19th century with the amazing results of Runge and Weierstrass (1885) and continued in the 20th century with the work of Oka and Weil, Mergelyan, Vituskin and others: here we prove the analog of Behnke and Stein theorem in the more modern quaternionic setting, hoping that this paper will bring a new stimulus for future developments in this important area of mathematics.

Throughout this paper the integers, real, complex and quaternionic numbers are denoted by $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ respectively. We recall that $\mathbb{H}$ is a skew field, a four-dimensional associative $\mathbb{R}$-algebra with basis $1, I, J, K$ subject to the rules

$$I^2 = J^2 = K^2 = -1, \quad IJ + JI = IK + KI = KJ + JK = 0, \quad IJK = -1.$$

The set of imaginary units $\mathbb{S} = \{q \in \mathbb{H} : q^2 = -1\}$ is a real two-dimensional sphere, because

$$\mathbb{S} = \{xI + yJ + zK : x^2 + y^2 + z^2 = 1\}.$$

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Our goal is to study (slice) *regular functions* on domains in $\mathbb{H}$ which are the analog of holomorphic functions on $\mathbb{C}$.

Definition 1.1. Let Ω be an open subset of $\mathbb{H}$ with $\Omega \cap \mathbb{R} \neq \{\,\}$. A real differentiable function $f : \Omega \to \mathbb{H}$ is said to be (slice) regular if, $\forall\ I \in \mathbb{S}$ its restriction f_I to the complex line $\mathbb{C}_I = \mathbb{R} + \mathbb{R}I$ passing through the origin and containing 1 and I is holomorphic on $\Omega \cap \mathbb{C}_I$.

This notion was introduced by Gentili and Struppa [16, 17].

For a ball in $\mathbb{H}$ centered at the origin regularity is the same as the condition that the function can be represented by a convergent power series

$$f(q) = \sum_{k=0}^{\infty} q^k a_k.$$

In the last decade the theory of slice regular functions has been investigated in many directions, see, as samples, the papers [1, 2, 4–11].

In this article, we call an open subset $D \subset \mathbb{C}$ *symmetric* if it is invariant under complex conjugation. An open subset $\Omega \subset \mathbb{H}$ is called axially symmetric if it is invariant under all $\mathbb{R}$-algebra automorphisms of $\mathbb{H}$. This is equivalent to the condition that for any $x, y \in \mathbb{R}$, $I, J \in \mathbb{S}$ the condition $x + yI \in \Omega$ holds if and only if $x + yJ \in \Omega$.

There is a one-to-one correspondence between symmetric open subsets $D \subset \mathbb{C}$ and axially symmetric open subsets $\Omega_D \subset \mathbb{H}$ which may described as follows.

Given an axially symmetric open subset $\Omega \subset \mathbb{H}$, we may choose an element $I \in \mathbb{S}$ and define $D \subset \mathbb{C}$ as

$$D = \{x + yi : x + yI \in \Omega,\ x, y \in \mathbb{R}\}.$$

Conversely, given a symmetric open subset $D \subset \mathbb{C}$, we define the corresponding axially symmetric subset $\Omega \subset \mathbb{H}$ (which we often denote as Ω_D) via

$$\Omega = \{x + yI : I \in \mathbb{S},\ x, y \in \mathbb{R},\ x + yi \in D\}.$$

Let D be a symmetric open subset of $\mathbb{C}$. Then a "stem function" on D is a holomorphic function $F : D \to \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ such that $F(\bar{z}) = \overline{F(z)}$ for all $z \in D$. Here "holomorphic" is to be understood with respect to the complex structure on $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ induced by the complex structure on the second factor of the tensor product.

Given a symmetric open subset $D \subset \mathbb{C}$ with $D \cap \mathbb{R} \neq \{\,\}$ and its associated axially symmetric open subset Ω_D we have a one-to-one correspondence between slice regular functions on Ω_D and "stem functions on D".

Given a stem function $F : D \to \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$, we write F as

$$F(z) = F_1(z) \otimes 1 + F_2(z) \otimes \iota$$

with $F_i\colon D \to \mathbb{H}$ and define

$$f(x+yI) = F_1(x+yi) + IF_2(x+yi) \quad (x, y \in \mathbb{R}, I \in \mathbb{S})$$

Conversely, given $f\colon \Omega \to \mathbb{H}$, we fix an element $I \in \mathbb{S}$ and define

$$F_1(x+yi) = \frac{1}{2}\big(f(x+yI) + f(x-yI)\big),$$
$$F_2(x+yi) = -I\frac{1}{2}\big(f(x+yI) - f(x-yI)\big).$$

It can be shown (using the "representation formula") that the F_i are independent of the choice of I, see [18].

For arbitrary axially symmetric domains in $\mathbb{H}$ (for which the intersection with the real axis may be empty) we use the definition below.

Definition 1.2. Let D be a symmetric domain in $\mathbb{C}$ and let Ω_D be its associated axially symmetric domain in $\mathbb{H}$, i.e.,

$$\Omega_D = \{x + yJ : x, y \in \mathbb{R},\ J \in \mathbb{S},\ x + yi \in D\}$$

A function $f\colon \Omega_D \to \mathbb{H}$ is *regular* if it is induced by a holomorphic stem function $F\colon D \to \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$.

Our main result is the following:

Theorem 1.3. *Let $D \subset D_1$ be symmetric open subsets of $\mathbb{C}$ and let $\Omega_D \subset \Omega_{D_1}$ be the corresponding axially symmetric open subsets in $\mathbb{H}$. Then the following are equivalent:*

- (i) *$D \subset D_1$ is a Runge pair, i.e., every holomorphic function on D can be approximated by holomorphic functions on D_1 (uniformly on compact sets),*
- (ii) *Ω_D is Runge in Ω_{D_1} in the sense that every regular function on Ω_D can be approximated (uniformly on compact sets) by regular functions on Ω_{D_1}.*
- (iii) *$i_*\colon H_1(D) \to H_1(D_1)$ is injective, where i_* denotes the homology group homomorphism induced by the inclusion map $i\colon D \to D_1$.*
- (iv) *$i_*\colon H_k(\Omega_D) \to H_k(\Omega_{D_1})$ is injective for $k \in \{1, 3\}$ where i_* is the homomorphism induced by the inclusion map $i\colon \Omega_D \to \Omega_{D_1}$.*
- (v) *Every bounded connected component of $\mathbb{C} \setminus D$ intersects $\mathbb{C} \setminus D_1$.*
- (vi) *Every bounded connected component of $\mathbb{H} \setminus \Omega_D$ intersects $\mathbb{H} \setminus \Omega_{D_1}$.*

The equivalences (i) $\iff$ (iii) $\iff$ (v) are classical (see Proposition 2.1 below). The implication (vi) $\Rightarrow$ (ii) has been proven before by Colombo, Sabadini, and Struppa [12, Theorem 4.13].

The equivalence (i) $\iff$ (ii) is Proposition 2.4. The equivalence (iii) $\iff$ (iv) is Proposition 2.15.

The equivalence (v) $\iff$ (vi) is an easy consequence of the fact that each bounded connected component C of D, resp. D_1, corresponds to a bounded connected component Ω_C of Ω_D, resp. Ω_{D_1}, via

$$\Omega_C = \{x + yI; x, y, \in \mathbb{R}, \ x + yi \in C, \ I \in \mathbb{S}\}.$$

In the context of proving our results on Runge pairs we obtain a precise description of the homology of Ω_D in terms of the topology of D; see Proposition 2.5.

1.1. Examples.

Example 1.4. $\mathbb{C}^*$ is a symmetric domain with corresponding axially symmetric domain $\mathbb{H}^*$. $\mathbb{H}^*$ is simply-connected, but not Runge in $\mathbb{H}$, because

$$i_*\colon H_3(\mathbb{H}^*) \simeq \mathbb{Z} \to H_3(\mathbb{H}) = \{0\}$$

is not injective.

Example 1.5. $\mathbb{C} \setminus \mathbb{R}$ is a symmetric domain with corresponding axially symmetric domain $\Omega = \mathbb{H} \setminus \mathbb{R}$. The domain Ω is homotopic to the 2-sphere, thus simply-connected but not contractible. However, Ω is Runge in $\mathbb{H}$: $H_1(\Omega)$ and $H_3(\Omega)$ vanish both, hence $H_k(\Omega) \to H_k(\mathbb{H})$ is injective for $k = 1, 3$. Thus we have a Runge pair although

$$\mathbb{Z} \simeq H_2(\Omega) \to H_2(\mathbb{H}) = \{0\}$$

is not injective.

Example 1.6. Let

$$D = \{z \in \mathbb{C} : |z| > 1\} \quad \text{and} \quad D_1 = D \cup \{z \in \mathbb{C} : -1/2 < \Im m(z) < 1/2\}.$$

Then Ω_D is Runge in Ω_{D_1}.

Evidently Ω_D is the complement of the closed unit ball in $\mathbb{H}$ and therefore homotopic to the 3-sphere. Now $D_1 \neq \mathbb{C}$, hence $\exists\ p \notin \Omega_{D_1}$ and we have inclusion maps

$$\Omega_D \overset{i}{\hookrightarrow} \Omega_{D_1} \overset{j}{\hookrightarrow} \mathbb{H} \setminus \{p\}.$$

Since the composition map $j \circ i$ is a homotopy equivalence, all the homology group homomorphisms i_* induced by i must be injective. Hence our results imply that D is Runge in D_1.

2. Runge

2.1. The complex situation. In the complex case one has the following well known result.

Proposition 2.1. *Let $D \subset D_1$ be open subsets of $\mathbb{C}$. Then the following properties are equivalent:*

(i) *The inclusion map induces an injective group homomorphism $H_1(D) \to H_1(D_1)$.*

(ii) *Every bounded connected component of $\mathbb{C} \setminus D$ intersects $\mathbb{C} \setminus D_1$.*

(iii) *For every holomorphic function f on D, every $\epsilon > 0$ and every compact subset $K \subset D$ there exists a holomorphic function F on D_1 with*

$$\sup_{p \in K} |f(p) - F(p)| < \epsilon.$$

If one (hence all) of these properties are fulfilled, then $D \subset D_1$ is called a Runge pair*, or we say that D is Runge in D_1.*

See [3] and [20, §13.2.1].

2.2. Symmetric complex situation. We recall (see §1) that a subset $D \subset \mathbb{C}$ is "*symmetric*" if it is invariant under complex conjugation.

Lemma 2.2. *Let $D \subset D_1$ be symmetric open subsets of $\mathbb{C}$.*

Then the following are equivalent:

(i) *Every holomorphic function f on D can be approximated (locally uniformly) by holomorphic functions on D_1 (i.e., $D \subset D_1$ is a Runge pair).*

(ii) *Every holomorphic function f on D which is* symmetric*, i.e., for which $f(z) = \overline{f(\overline{z})}$ holds, can be approximated (locally uniformly) by symmetric holomorphic functions on D_1.*

Proof. (i) $\implies$ (ii). Assume that D is Runge in D_1 and that $f: D \to \mathbb{C}$ is holomorphic with $f(z) = \overline{f(\overline{z})}$. If f_n is a sequence of holomorphic functions on D_1 converging to f, then also

$$g_n(z) = \frac{1}{2}\big(f_n(z) + \overline{f_n(\overline{z})}\big)$$

converges to f and in addition fulfills $g_n(z) = \overline{g_n(\overline{z})}$.

(ii) $\implies$ (i). Let $f: D \to \mathbb{C}$ be an arbitrary holomorphic function. We define

$$g(z) = \frac{1}{2}\big(f(z) + \overline{f(\overline{z})}\big)$$
$$h(z) = \frac{1}{2i}\big(f(z) - \overline{f(\overline{z})}\big)$$

Then g and h are both symmetric holomorphic functions and $f(z) = g(z) + ih(z)$. By assumption the functions g and h may be approximated by holomorphic functions on D_1. It follows that $f = g + ih$ can be approximated, too. □

2.3. Passing from D to Ω_D. Let a symmetric open subset $D \subset \mathbb{C}$ be given. The associated axially symmetric subset Ω_D in $\mathbb{H}$ has been defined in §1 as:

$$\Omega_D = \{x + yI : x, y \in \mathbb{R},\ I \in \mathbb{S},\ x + yi \in D\}$$

(with $\mathbb{S} = \{q \in \mathbb{H} : q^2 = -1\}$).

This construction may be reformulated as follows. Define

$$D^+ = D \cap \{z \in \mathbb{C} : \Im m(z) \geq 0\}, \quad D_{\mathbb{R}} = D \cap \mathbb{R}.$$

Let $Z = D^+ \times \mathbb{S}$. Then $\Omega_D \simeq Z/\sim$ where $(p, I) \sim (q, J)$ iff $p = q$ and one of the following conditions is fulfilled:

(i) $I = J$, or

(ii) $p = q \in \mathbb{R}$.

In other words, for each $p \in D_{\mathbb{R}}$, the subset $\{p\} \times \mathbb{S}$ of Z is collapsed to one point.

2.4. Quaternionic situation.

Lemma 2.3. *Let $f : \mathbb{H} \to \mathbb{H}$ be a slice function induced by a stem function F. Then*

$$\frac{1}{\sqrt{2}}\|F(x + yi)\| \leq \max\left\{|f(x + yI)|, |f(x - yI)|\right\} \leq \sqrt{2}\|F(x + yi)\|$$

for every $x, y \in \mathbb{R}$, $I \in \mathbb{S}$.

Proof. From $f(x + yI) = F_1(x + yi) + IF_2(x + yi)$ one deduces

$$\begin{aligned} & |f(x + yI)| \leq \|F_1(x + yi)\| + \|F_2(x + yi)\| \\ \Longrightarrow\ & |f(x + yI)|^2 \leq \big(\|F_1(x + yi)\| + \|F_2(x + yi)\|\big)^2 \\ \Longrightarrow\ & |f(x + yI)|^2 \leq \|F(x + yi)\|^2 + 2\|F_1(x + yi)\| \cdot \|F_2(x + yi)\| \\ & \qquad\qquad \leq 2\|F(x + yi)\|^2 \\ \Longrightarrow\ & |f(x + yI)| \leq \sqrt{2}\|F(x + yi\|. \end{aligned}$$

On the other hand,

$$F_1(x + yi) = \frac{1}{2}\big(f(x + yI) + f(x - yI)\big)$$

implying that

$$\|F_1(x + yi)\| \leq \max\left\{|f(x + yI)|, |f(x - yI)|\right\}.$$

Similarly: $\|F_2(x + yi)\| \leq \max\{|f(x + yI)|, |f(x - yI)|\}$. Combining these bounds we obtain:

$$\|F(x + yi)\|^2 \leq 2 \max\left\{\|f(x + yI)\|^2, \|f(x - yI)\|^2\right\}$$

which implies the first inequality of the lemma. □

Proposition 2.4. *Let $D \subset D_1$ be a symmetric open subsets of $\mathbb{C}$ with corresponding axially symmetric open subsets $\Omega_D \subset \Omega_{D_1}$ in $\mathbb{H}$. Then every regular function on Ω_D may be approximated locally uniformly by regular functions on Ω_{D_1} if and only if D is Runge in D_1.*

Proof. For any symmetric subset $C \subset D$ the corresponding subset

$$\Omega_C = \{x + yI : \exists\, x + yi \in C,\ I \in \mathbb{S}\}$$

of $\mathbb{H}$ is compact if and only if C is compact. We measure the size of a function by using the sup-norm. From the euclidean scalar product on $\mathbb{C} \simeq \mathbb{R}^2$ and $\mathbb{H} \simeq \mathbb{R}^4$ we deduce a scalar product on $\mathbb{H} \otimes \mathbb{C} \simeq \mathbb{R}^8$. The norm induced by this scalar product is denoted by $\| \; \|$. From the preceding lemma we deduce that

$$\frac{1}{\sqrt{2}}\|F\|_C \leq \|f\|_{\Omega_C} \leq \sqrt{2}\|F\|_C$$

for any compact symmetric subset $C \subset D$ (where $\|F\|_C = \sup_{z\in C} \|F(z)\|$.) Therefore the space of slice functions on Ω_D is isomorphic as a topological vector space to the space of stem functions on D (both spaces endowed with topology of locally uniform convergence). This implies the assertion. □

2.5. Homology of axially symmetric domains. In this section we show that (and how) the homology of an axially symmetric domain in $\mathbb{H}$ is determined by that of the corresponding symmetric open set in $\mathbb{C}$. We will study the topology of this procedure aided by the Mayer–Vietoris sequence.

We introduce some notation which we will keep throughout this section.

Convention. Let D be a symmetric open subset of $\mathbb{C}$ (i.e. a domain such that $z \in D \iff \bar{z} \in D$),

$$D^+ = \{z \in D : \Im m(z) \geq 0\}, \quad D^- = \{z \in D : \Im m(z) \leq 0\},$$
$$D_\mathbb{R} = D \cap \mathbb{R}, \quad D^* = D^+ \setminus \mathbb{R}.$$

For any subset $A \subset \mathbb{C}$ a subset Ω_A of $\mathbb{H}$ is defined as

$$\Omega_A = \{x + yI : x, y \in \mathbb{R},\ x + yi \in A,\ I \in \mathbb{S}\}.$$

Let the boundary of D in $\mathbb{C}$ be denoted by ∂D. Define a real positive function h on $D_\mathbb{R}$ by

$$h(x) = \mathrm{dist}(x, \partial D) = \inf_{z\in\partial D} |z - x|.$$

Using the triangle inequality, it is easy to check that h is continuous. Furthermore, we define

$$W = \{x + yi \in \mathbb{C} : x \in D_\mathbb{R} : 0 \leq y < h(x)\}, \quad W^* = W \setminus D_\mathbb{R}.$$

We observe that

$$\begin{aligned} W &= \{x + rh(x)i : x \in D_{\mathbb{R}},\ r \in [0,1[\,\},\\ W^* &= \{x + rh(x)i : x \in D_{\mathbb{R}},\ r \in]0,1[\,\},\\ D_{\mathbb{R}} &= \{x + rh(x)i : x \in D_{\mathbb{R}},\ r = 0\}. \end{aligned}$$

Since $[0,1[$, $]0,1[$, and $\{0\}$ are all contractible, it is clear that the natural inclusion maps $W^* \to W$ and $D_{\mathbb{R}} \to W$ are homotopy equivalences. The inclusion map $D^* \to D^+$ is likewise a homotopy equivalence.

We recall the definition of $\widetilde{H}_0$: An element α in $H_0(X)$ is a formal finite $\mathbb{Z}$-linear combination of points $\alpha = \sum n_i\{p_i\}$ ($p_i \in X$) and therefore admits a natural degree function by $\deg(\alpha) = \sum n_i$. The "reduced homology group" $\widetilde{H}_0$ is defined as the kernel of the degree map $H_0 \to \mathbb{Z}$.

Proposition 2.5. *Let D be a symmetric open subset of $\mathbb{C}$. We assume that the corresponding axially symmetric set Ω_D is connected. Then $H_2(\Omega_D) = \{0\}$ if $D_{\mathbb{R}} \neq \{\,\}$ and $H_2(\Omega_D) \simeq \mathbb{Z}$ if $D_{\mathbb{R}}$ is empty.*

There are natural exact sequences

$$0 \to H_1(D^+) \to H_3(\Omega_D) \to \widetilde{H}_0(D_{\mathbb{R}}) \to 0 \tag{2.1}$$

and

$$0 \to H_1(D^+) \to H_1(\Omega_D) \to 0. \tag{2.2}$$

Proof. Observe that $\Omega_D = \Omega_{D^*} \cup \Omega_W$ and $\Omega_{D^*} \cap \Omega_W = \Omega_{W^*}$. This yields a Mayer–Vietoris sequence for homology:

$$\cdots \to H_{k+1}(\Omega_D) \to H_k(\Omega_{W^*}) \to H_k(\Omega_{D^*}) \oplus H_k(\Omega_W) \to H_k(\Omega_D) \to \cdots$$

We claim that there are homotopy equivalences

$$\Omega_{W^*} \sim \mathbb{S} \times D_{\mathbb{R}}, \quad \Omega_W \sim D_{\mathbb{R}}, \quad \Omega_{D^*} \sim \mathbb{S} \times D^* \sim \mathbb{S} \times D^+.$$

The first of these homotopy equivalences holds because

$$\Omega_{W^*} = \{x + yI : x \in D_{\mathbb{R}},\ 0 < y < h(x),\ I \in \mathbb{S}\}.$$

We observe that $D_{\mathbb{R}}$ is a deformation retract of Ω_W. Indeed

$$\Omega_W = \{x + yI : x \in D_{\mathbb{R}},\ 0 \le y < h(x),\ I \in \mathbb{S}\}$$

may be retracted to $D_{\mathbb{R}}$ via

$$\Phi_s : (x + yI) \mapsto (x + syI) \quad (0 \le s \le 1).$$

Thus Ω_W is homotopy equivalent to $D_{\mathbb{R}}$.

Finally $\Omega_{D^*} \sim \mathbb{S} \times D^+$ follows from

$$\Omega_{D^*} = \{x + yI,\ x + yi \in D^*,\ I \in \mathbb{S}\} \simeq D^* \times \mathbb{S}$$

and the fact that D^+ and D^* are homotopy equivalent. Thus our Mayer–Vietoris sequence yields this exact sequence:

$$\cdots \to H_{k+1}(\Omega_D) \to H_k(\mathbb{S} \times D_{\mathbb{R}}) \to H_k(\mathbb{S} \times D^+) \oplus H_k(D_{\mathbb{R}}) \to H_k(\Omega_D) \to \cdots$$

Since the homology groups of the sphere $\mathbb{S}$ are torsion-free, the Künneth formula tells us that

$$\begin{aligned} H_*(\mathbb{S} \times X) &\simeq H_*(\mathbb{S}) \otimes_{\mathbb{Z}} H_*(X) \\ &\simeq (H_0(\mathbb{S}) \otimes_{\mathbb{Z}} H_*(X)) \oplus (H_2(\mathbb{S}) \otimes_{\mathbb{Z}} H_*(X)) \\ &\simeq H_*(X) \oplus [\mathbb{S}] \cdot H_*(X), \end{aligned}$$

where $[\mathbb{S}] \in H_2(\mathbb{S})$ is the fundamental class. Hence

$$\begin{aligned} \cdots \to H_{k+1}(\Omega_D) &\to (H_0(\mathbb{S}) \otimes H_k(D_{\mathbb{R}})) \oplus (H_2(\mathbb{S}) \otimes H_{k-2}(D_{\mathbb{R}})) \\ &\to \big(H_0(\mathbb{S}) \otimes H_k(D^+)\big) \oplus \big(H_2(\mathbb{S}) \otimes H_{k-2}(D^+)\big) \oplus H_k(D_{\mathbb{R}}) \\ &\to H_k(\Omega_D) \to \cdots \end{aligned}$$

We know that $H_k(D_{\mathbb{R}}) = \{0\}$ for $k > 0$ and $H_k(D^+) = \{0\}$ for $k > 1$ for dimension reasons. Therefore our long exact Mayer–Vietoris sequences yield the following two exact sequences:

$$\begin{aligned} &0 \to H_2(\mathbb{S}) \otimes H_1(D^+) \to H_3(\Omega_D) \\ &\qquad \to H_2(\mathbb{S}) \otimes H_0(D_{\mathbb{R}}) \to H_2(\mathbb{S}) \otimes H_0(D^+) \to H_2(\Omega_D) \to 0 \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &0 \to H_0(\mathbb{S}) \otimes H_1(D^+) \to H_1(\Omega_D) \\ &\qquad \to H_0(D_{\mathbb{R}}) \to H_0(D^+) \oplus H_0(D_{\mathbb{R}}) \to H_0(\Omega_D) \to 0 \end{aligned} \tag{2.4}$$

Case (1). Assume now that $D_{\mathbb{R}}$ is not empty. Then inclusion map from $D_{\mathbb{R}}$ into D^+ yields a surjective group homomorphism $H_0(D_{\mathbb{R}}) \to H_0(D^+)$ with $\widetilde{H}_0(D_{\mathbb{R}})$ as kernel. Let α denote the homomorphism $H_2(\mathbb{S}) \otimes H_0(D_{\mathbb{R}}) \to H_2(\mathbb{S}) \otimes H_0(D^+)$ in (2.3). Then the exact sequence (2.3) can be split into two parts

$$0 \to H_2(\mathbb{S}) \otimes H_1(D^+) \to H_3(\Omega_D) \to \ker \alpha \to 0 \tag{2.5}$$

and

$$0 \to (H_2(\mathbb{S}) \otimes H_0(D_{\mathbb{R}})) \,/\, \ker \alpha \xrightarrow{\alpha} H_2(\mathbb{S}) \otimes H_0(D^+) \to H_2(\Omega_D) \to 0. \tag{2.6}$$

Since $\ker \alpha \simeq \widetilde{H}_0(D_{\mathbb{R}})$, (2.5) now implies (2.1). Furthermore (2.6) implies that $H_2(\Omega_D)$ is zero, because α is surjective.

Case (2). Now let us discuss the case where $D_{\mathbb{R}}$ is empty. Then $H_0(D_{\mathbb{R}}) = \{0\}$ and consequently from (2.3) we obtain two sequences

$$0 \to H_2(\mathbb{S}) \otimes H_1(D_{\mathbb{R}}) \to H_3(\Omega_D) \to 0 = H_2(\mathbb{S}) \otimes H_0(D_{\mathbb{R}})$$

and

$$0 = H_2(\mathbb{S}) \otimes H_0(D_{\mathbb{R}}) \to \mathbb{Z} \simeq H_2(\mathbb{S}) \otimes H_0(D^+) \to H_2(\Omega_D) \to 0.$$

Using $H_2(\mathbb{S}) \simeq \mathbb{Z} \simeq H_0(\mathbb{S})$ we get (2.1) and $H_2(\Omega_D) = \{\mathbb{Z}\}$.

It remains to show (2.2). For this purpose we return to (2.4). The map

$$H_0(D_{\mathbb{R}}) \to H_0(D^+) \oplus H_0(D_{\mathbb{R}})$$

in (2.4) is obviously injective, therefore (due to exactness of the sequence) the preceding map is zero and $H_1(\Omega_D)$ is isomorphic to $H_0(\mathbb{S}) \otimes H_1(D^+)$. However, $H_0(\mathbb{S}) \simeq \mathbb{Z}$ and therefore

$$H_0(\mathbb{S}) \otimes H_1(D^*) \simeq H_1(D^*).$$

Hence

$$H_1(\Omega_D) \simeq H_1(D^*).$$ □

Corollary 2.6. *Assume in addition that D is a bounded domain with smooth boundary. Then all the homology groups are finitely generated and Proposition 2.5 implies the following description of the Betti numbers $b_k = \dim_{\mathbb{R}} H_k(\ ,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$: Let $r = b_0(D_{\mathbb{R}}) - 1$ if $D_{\mathbb{R}}$ is not empty and set $r = 0$ if $D_{\mathbb{R}}$ is empty. Then*

$$b_1(\Omega_D) = \frac{1}{2}\big(b_1(D) - r\big)$$
$$b_3(\Omega_D) = \frac{1}{2}\big(b_1(D) + r\big)$$

and

$$b_2(\Omega_D) = \begin{cases} 1 & \text{if } D_{\mathbb{R}} \text{ is empty,} \\ 0 & \text{if } D_{\mathbb{R}} \text{ is not empty.} \end{cases}$$

Corollary 2.7. *Let D be a symmetric open subset and let Ω_D denote the corresponding axially symmetric set (not necessarily connected). Then $H_2(\Omega_D) \simeq \mathbb{Z}^k$ where k denote the number of connected components of D^+ which do not intersect $\mathbb{R}$.*

Let $\hat{H}_0(D_{\mathbb{R}})$ denote the kernel of the homomorphism $i_*: H_0(D_{\mathbb{R}}) \to H_0(D^+)$. There are natural exact sequences

$$0 \to H_1(D^+) \to H_3(\Omega_D) \to \hat{H}_0(D_{\mathbb{R}}) \to 0 \tag{2.7}$$

and

$$0 \to H_1(D^+) \to H_1(\Omega_D) \to 0. \tag{2.8}$$

Proof. This is an easy consequence of Proposition 2.5, since the homology of a disconnected space is isomorphic to the direct sum of the homology of its connected components. □

Corollary 2.8. *For an axially symmetric open subset $\Omega \subset \mathbb{H}$ all homology groups are torsion-free.*

Proof. First observe that there is no loss in generality in assuming that Ω_D is connected, because the homology groups of Ω_D are isomorphic to the direct sum of the homology groups of its connected components.

For connected Ω_D the assertion follows from the preceding proposition, because the homology groups of open sets in $\mathbb{R}$ and $\mathbb{R}^2$ are known to be always torsion-free and $D_{\mathbb{R}}$, resp. D^*, is an open subset in $\mathbb{R}$ resp. $\mathbb{R}^2$. □

We now explain the geometric meaning of the short exact sequence (2.1). Given an element $\alpha \in H_1(D^+)$ we may represent α as a finite formal $\mathbb{Z}$-linear combination of closed curves $\gamma_j : S^1 \to D^+$. Each such curve γ_j defines a map η from $S^1 \times \mathbb{S}$ to Ω_D via

$$\eta(t, I) = \Re e(\gamma_j(t)) + I\, \Im m(\gamma_j(t)).$$

The fundamental class of the real three-dimensional manifold $S^1 \times \mathbb{S}$ then defines the corresponding element in $H_3(\Omega_D)$.

An element $\beta \in H_0(D_{\mathbb{R}})$ may be represented as a formal $\mathbb{Z}$-linear combination of points $\sum n_i \{p_i\}$. Assume that β is in the kernel of the natural map to $\mathbb{Z}$ which is given by

$$\sum n_i \{p_i\} \mapsto \sum n_i.$$

Then β is the sum of elements of the form $+1\{p_i\} - 1\{q_i\}$. Given such an element, we choose a curve $\gamma : [0,1] \to D^+$ with $\gamma(0) = p_i$, $\gamma(1) = q_i$, $\gamma(t) \in D^+ \setminus \mathbb{R}$ for $0 < t < 1$. Then $\Omega_{\gamma([0,1])}$ is a 3-sphere defining an element in $H_3(\Omega_D)$. Note that this construction depends on the choice of the curve γ. Therefore the sequence (2.1) has no natural splitting.

Lemma 2.9. *Let $D \subset \mathbb{C}$ be a symmetric open subset. With D^+, $D_{\mathbb{R}}$ and $\widehat{H}_0(D_{\mathbb{R}})$ defined as in Corollary 2.7 there is natural exact sequence*

$$0 \to H_1(D^+) \oplus H_1(D^-) \to H_1(D) \to \widehat{H}_0(D_{\mathbb{R}}) \to 0. \tag{2.9}$$

Proof. Let W be as above in the proof of Proposition 2.5 and define

$$V = \{z \in \mathbb{C} : z \in W \text{ or } \overline{z} \in W\},$$
$$U^+ = D^+ \cup V, \quad U^- = D^- \cup V.$$

Observe that we have homotopy equivalences

$$U^+ \sim D^+, \quad U^- \sim D^-, \quad (U^+ \cap U^-) = V \sim D_{\mathbb{R}}.$$

We use the Mayer–Vietoris sequence associated to $D = U^+ \cup U^-$:

$$\cdots \to H_{k+1}(D) \to H_k(D_{\mathbb{R}}) \to H_k(D^+) \oplus H_k(D^-) \to H_k(D) \to \cdots$$

The details (which we omit) are very much similar to the proof of Proposition 2.5. □

Corollary 2.10. *Let $D \subset D_1$ be symmetric open subsets in $\mathbb{C}$. Assume that $H_1(D) \to H_1(D_1)$ is injective. Then $H_1(D^+) \to H_1({D_1}^+)$ is injective, too.*

Proof. The inclusion map from D to D_1 combined with (2.9) yields the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & H_1(D^+) \oplus H_1(D^-) & \to & H_1(D) & \to & \hat{H}_0(D_{\mathbb{R}}) & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & H_1({D_1}^+) \oplus H_1({D_1}^-) & \to & H_1(D_1) & \to & \hat{H}_0(D_{1,\mathbb{R}}) & \to & 0 \end{array}$$

Now the assertion follows from the snake lemma (see e.g. [19, III.§9]). □

Proposition 2.11. *Let D be a symmetric open subset of $\mathbb{C}$. Then there is a natural exact sequence*

$$0 \longrightarrow H_1(D^+) \xrightarrow{\alpha} H_1(D) \xrightarrow{\beta} H_3(\Omega_D) \longrightarrow 0. \qquad (2.10)$$

Here α, β are as follows: Let $\tau: \mathbb{C} \to \mathbb{C}$ denote complex conjugation on $\mathbb{C}$ and let $\zeta: D \times \mathbb{S} \to \Omega_D$ be the map given by

$$\zeta(x + yi, J) = x + yJ.$$

Then $\alpha(\gamma) = \gamma - \tau_*\gamma$ and $\beta(\gamma) = \zeta_*(\gamma \times [\mathbb{S}])$, where $[\mathbb{S}] \in H_2(\mathbb{S})$ denotes the fundamental class.

Proof. There is no loss in generality in assuming that D^+ is connected (and therefore Ω_D, too). We cover D^+ by the two open subsets D^* and W as in the proof of Proposition 2.5. This induces corresponding coverings of D, $D \times \mathbb{S}$ and Ω_D:

$$\begin{aligned} D &= (D \setminus D_{\mathbb{R}}) \cup V \quad \text{with } V = \{z \in \mathbb{C} : z \in W \text{ or } \overline{z} \in W\}, \\ D \times \mathbb{S} &= ((D \setminus D_{\mathbb{R}}) \times \mathbb{S}) \ \cup \ (V \times \mathbb{S}), \\ \Omega_D &= \Omega_{D^*} \ \cup \ \Omega_W. \end{aligned}$$

For each of these coverings we obtain a Mayer–Vietoris sequence for homology.

We utilize the map $\zeta: D \times \mathbb{S} \to \Omega_D$ given by

$$(x + yi; J) \mapsto x + yJ.$$

This yields a morphism between the respective Mayer–Vietoris sequences:

$$\begin{array}{ccccccccc}
\cdots \to & H_k((V\setminus D_{\mathbb{R}})\times\mathbb{S}) & \to & H_k((D\setminus D_{\mathbb{R}})\times\mathbb{S})\oplus H_k(V\times\mathbb{S}) & \to & H_k(D\times\mathbb{S}) & \to \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & H_k(\Omega_{W^*}) & \to & H_k(\Omega_{D^*})\oplus H_k(\Omega_W) & \to & H_k(\Omega_D) & \to \cdots
\end{array}$$

In particular, we get

$$\begin{array}{ccccccccc}
H_3((V\setminus D_{\mathbb{R}})\times\mathbb{S}) & \to & H_3((D\setminus D_{\mathbb{R}})\times\mathbb{S})\oplus H_3(V\times\mathbb{S}) & \to & H_3(D\times\mathbb{S}) & \to & C & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_3(\Omega_{W^*}) & \to & H_3(\Omega_{D^*})\oplus H_3(\Omega_W) & \to & H_3(\Omega_D) & \to & C' & \to & 0
\end{array}$$

with

$$C = \ker\big[H_2((V\setminus D_{\mathbb{R}})\times\mathbb{S}) \to H_2((D\setminus D_{\mathbb{R}})\times\mathbb{S})\oplus H_2(V\times\mathbb{S})\big]$$

and

$$C' = \ker\big[H_2(\Omega_{W^*}) \to H_2(\Omega_{D^*})\oplus H_2(\Omega_W)\big].$$

Recall that $H_3(M\times\mathbb{S})\simeq H_1(M)$ and $H_2(M\times\mathbb{S})\simeq H_0(M)$ for any $M\subset\mathbb{C}$ due to Künneth formula and dimension reasons. Observe also that $V\setminus D_{\mathbb{R}}$ is the disjoint union of two open subsets (namely $D^+\cap(V\setminus D_{\mathbb{R}})$ and $D^-\cap(V\setminus D_{\mathbb{R}})$) both of which are homotopic to $D_{\mathbb{R}}$. Recall moreover that V and $D_{\mathbb{R}}$ are homotopy equivalent. Hence

$$C \simeq \ker\big[H_0(V\setminus D_{\mathbb{R}}) \to H_0(D\setminus D_{\mathbb{R}})\oplus H_0(V)\big]$$

and consequently

$$H_0(D_{\mathbb{R}}) \sim \ker\big[H_0(V\setminus D_{\mathbb{R}}) \to H_0(V)\big],$$

where the isomorphism may be describe as

$$\begin{aligned}
&H_0(D_{\mathbb{R}}) \ni \xi = \sum_J n_j\{p_j\} \\
&\mapsto \sum_J n_j\big(\{p_j-\epsilon\}-\{p_j+\epsilon\}\big) \in \ker\big[H_0(V\setminus D_{\mathbb{R}}) \to H_0(V)\big] \quad (p_j\in D_{\mathbb{R}})
\end{aligned}$$

for a sufficiently small ϵ.

Let

$$\eta = \sum_J n_j\big(\{p_j-\epsilon\}-\{p_j+\epsilon\}\big) \in \ker\big[H_0(V\setminus D_{\mathbb{R}}) \to H_0(V)\big].$$

Then the homomorphism to $H_0(D \setminus D_{\mathbb{R}})$ may be described as

$$\eta \mapsto \Big(\sum_J n_j, -\sum_J n_j \Big) \in \mathbb{Z}^2 \simeq H_0(D \setminus D_{\mathbb{R}}).$$

It follows that

$$C \simeq \widetilde{H}_0(D_{\mathbb{R}}).$$

Now

$$\begin{aligned} C' &= \ker\big[H_2(\Omega_{W^*}) \to H_2(\Omega_{D^*}) \oplus H_2(\Omega_W)\big] \\ &\simeq \ker\big[H_2(D_{\mathbb{R}} \times \mathbb{S}) \to H_2(D^+ \times \mathbb{S}) \oplus H_2(D_{\mathbb{R}})\big] \end{aligned}$$

due to the homotopy equivalences (which were verified in the proof of Proposition 2.5)

$$\Omega_{W^*} \simeq D_{\mathbb{R}} \times \mathbb{S}, \quad \Omega_{D^*} \simeq D^+ \times \mathbb{S}, \quad \Omega_W \simeq D_{\mathbb{R}}.$$

It follows that

$$C' \simeq \ker\big[H_0(D_{\mathbb{R}}) \to H_0(D^+) \oplus \{0\}\big] \simeq \ker\big[H_0(D_{\mathbb{R}}) \to H_0(D^+)\big] \simeq \widetilde{H}_0(D_{\mathbb{R}}).$$

The aforementioned homotopy equivalences also imply $H_3(\Omega_{D^*}) \simeq H_1(D^+)$ and $H_3(\Omega_W) = \{0\}$. Combining all these facts, the above commutative diagram turns into the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_1(D^+) \oplus H_1(D^-) & \xrightarrow{\eta_1} & H_1(D) & \xrightarrow{\eta_2} & \widetilde{H}_0(D_{\mathbb{R}}) & \longrightarrow & 0 \\ \big\downarrow & & \rho_1 \big\downarrow & & \rho_2 \big\downarrow & & \big\downarrow \rho_3 = \mathrm{id} & & \big\downarrow \\ 0 & \longrightarrow & H_1(D^+) & \xrightarrow{\mu_1} & H_3(\Omega_D) & \xrightarrow{\mu_2} & \widetilde{H}_0(D_{\mathbb{R}}) & \longrightarrow & 0 \end{array}$$

The homomorphism ρ_1 is induced by the embedding

$$D \setminus D_{\mathbb{R}} = D^+ \cup D^- \longrightarrow \Omega_{D^*}$$

and

$$H_3(\Omega_{D^*}) \simeq H_3(D^+ \times \mathbb{S}) \simeq H_1(D^+).$$

Hence $\rho_1(c_1, c_2) = c_1 + \tau_* c_2$ if c_1 is a 1-cycle in D^+ and c_2 a 1-cycle in D^-. In particular, ρ_1 is surjective with kernel

$$\ker \rho_1 = \{(c, -\tau_* c) : c \in H_1(D^+)\}$$

ρ_2 is defined by

$$H_1(D) \simeq H_3(D \times \mathbb{S}) \xrightarrow{\zeta_*} H_3(\Omega_D).$$

We set $\beta = \rho_2$ and define α via $\alpha(c) = \eta_1(c, -\tau_* c)$. Injectivity of α is implied by injectivity of η_1. To check surjectivity of β, let $s \in H_3(\Omega_D)$. Since ρ_3 is an isomorphism, we find an element $c \in H_1(D)$ with $\eta_2(c) = \mu_2(s)$. Then

$$s - \rho_2(c) \in \ker \mu_2 = \text{image}(\mu_1).$$

Now ρ_1 is surjective. Therefore there exists $a \in H_1(D^+) \oplus H_1(D^-)$ with

$$s - \rho_2(c) = \mu_1(\rho_1(a)) = \rho_2(\eta_1(a)) \Rightarrow s = \rho_2(c + \eta_1(a)).$$

Let us check that $\beta \circ \alpha = 0$:

$$\beta(\alpha(c)) = \rho_2(\alpha(c)) = \rho_2(\eta_1(c, -\tau_* c)) = \mu_1(\rho_1(c, -\tau_* c)) = \mu_1(0) = 0.$$

Finally, assume $b \in \ker \beta$. We have to show that b is in the image of α. Now $\beta(b) = \rho_2(b) = 0$ implies

$$\mu_2(\rho_2(b)) = \rho_3(\eta_2(b)) = \eta_2(b) = 0.$$

Thus

$$b \in \ker(\eta_2) = \text{image}(\eta_1),$$

i.e., there is an element $(c', c'') \in H_1(D^+) \oplus H_1(D^-)$ with $\eta_1(c', c'') = b$. Since μ_1 is injective, and $\rho_2(b) = 0$, we know that

$$0 = \rho_1(c', c'') = c' + \tau_* c''.$$

Hence $c'' = -\tau_* c'$. It follows that $b = \alpha(c')$. □

Corollary 2.12. *Let $D \subset D_1$ be symmetric open subsets in $\mathbb{C}$ such that $H_1(\Omega_D) \to H_1(\Omega_{D_1})$ and $H_3(\Omega_D) \to H_3(\Omega_{D_1})$ are both injective. Then $H_1(D) \to H_1(D_1)$ is injective, too.*

Proof. First recall that $H_1(\Omega_D) \simeq H_1(D^+)$ (and $H_1(\Omega_{D_1}) \simeq H_1({D_1}^+)$) due to (2.2).

Second, we consider the following commutative diagram induced from (2.10) via the map $D \hookrightarrow D_1$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_1(D^+) & \longrightarrow & H_1(D) & \longrightarrow & H_3(\Omega_D) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H_1({D_1}^+) & \longrightarrow & H_1(D_1) & \longrightarrow & H_3(\Omega_{D_1}) & \longrightarrow & 0. \end{array}$$

Now the snake lemma (see e.g. [19, III.§9]) yields the statement. □

Lemma 2.13. *Let P be a symmetric compact connected subset of $\mathbb{C}$ such that $P \cap \mathbb{R}$ is non-empty and connected. Let P' be a non-empty symmetric closed subset of P and define*

$$D = \mathbb{C} \setminus P,$$
$$D_1 = \mathbb{C} \setminus P'.$$

Then $H_3(\Omega_D) \to H_3(\Omega_{D_1})$ is injective.

Proof. By construction we have

$$H_1(D) \simeq \mathbb{Z}, \quad \widetilde{H}_0(D_{\mathbb{R}}) \simeq \mathbb{Z}.$$

Using (2.9) it follows that $H_1(D^+) = \{0\}$. Then we may apply (2.1) to conclude that $H_3(\Omega_D) \simeq \mathbb{Z}$.

Let $R > \max\{|z| : z \in P\}$. Regard the 3-sphere S with center 0 and radius R in $\mathbb{H}$. Because P is contained in the interior of the sphere, S defines a non-trivial homology class in $H_3(\Omega_D)$. Since P' is also non-empty and in the interior of the sphere, the homology class of S in $H_3(\Omega_{D_1})$ is likewise non-zero. Thus the homomorphism

$$i_* \colon H_3(\Omega_D) \to H_3(\Omega_{D_1})$$

maps a non-trivial element of $H_3(\Omega_D)$ to a non-trivial element of $H_3(\Omega_{D_1})$. This implies the statement because $H_3(\Omega_D) \simeq \mathbb{Z}$. □

Proposition 2.14. *Let $D \subset D_1$ be symmetric open subsets of $\mathbb{C}$ such that the natural homomorphism $H_1(D) \to H_1(D_1)$ is injective. Then $H_3(\Omega_D) \to H_3(\Omega_{D_1})$ is injective, too.*

Proof. Assume the contrary. Let

$$\alpha \in \ker\big(H_3(\Omega_D) \to H_3(\Omega_{D_1})\big), \quad \alpha \neq 0.$$

The injectivity of $H_1(D) \to H_1(D_1)$ implies that $H_1(D^+) \to H_1({D_1}^+)$ is injective too (Corollary 2.10). The inclusion map $D \to D_1$ applied to (2.1) yields the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_1(D^+) & \longrightarrow & H_3(\Omega_D) & \longrightarrow & \widehat{H}_0(D_{\mathbb{R}}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H_1({D_1}^+) & \longrightarrow & H_3(\Omega_{D_1}) & \longrightarrow & \widehat{H}_0(D_{1,\mathbb{R}}) & \longrightarrow & 0.
\end{array}$$

Let α_0 denote the image of α in $\widehat{H}_0(D_{\mathbb{R}})$. First, we claim that α_0 can not vanish. Indeed, if $\alpha_0 = 0$, then α is induced by an element $\beta \in H_1(D^+)$. Evidently $\alpha \neq 0$ implies $\beta \neq 0$. But now we obtain a contradiction, since $H_1(D^+) \to H_1({D_1}^+)$

and $H_1({D_1}^+) \to H_3(\Omega_{D_1})$ are both injective, but α is mapped to zero in $H_3(\Omega_{D_1})$. Hence $\alpha_0 \neq 0$.

Second, by assumption the image of α in $H_3(\Omega_{D_1})$ vanishes, implying that the image in $\widehat{H}_0(D_{1,\mathbb{R}})$ also vanishes. Thus α_0 is in the kernel of $\widehat{H}_0(D_{\mathbb{R}}) \to \widehat{H}_0(D_{1,\mathbb{R}})$. Let α_0 be represented by the formal $\mathbb{Z}$-linear combination $\sum_{x \in I} n_x\{x\}$ where I is a finite subset of $D_{\mathbb{R}}$. Since $\alpha_0 \neq 0$, but $\sum n_k = 0$ (because α is in the kernel of the morphism from $H_0(D_{\mathbb{R}})$ to $H_0(D)$), we can find a point $q \in \mathbb{R} \setminus D$ such that

$$\sum_{p \in I; p > q} n_p \neq 0.$$

Fix such a point q. Let B denote the connected component of $D^c = \mathbb{C} \setminus D$ containing q. Fix $p_1, p_2 \in I$ with $p_1 < q < p_2$ and such that $I \cap]p_1, p_2[= \{ \}$. Note that α_0 is mapped onto zero in $\widehat{H}_0(D_{1,\mathbb{R}})$ which implies that $[p_1, p_2]$ is contained in $D_{1,\mathbb{R}}$.

Because α is mapped to zero in $H_0(D^+)$, we know that p_1 and p_2 are contained in the same connected component of D^+. Therefore p_1 and p_2 can be connected by a path γ in D^+. This path, combined with its image under conjugation, yields a closed curve inside D which surrounds q. Therefore B must be bounded, and $B \cap \mathbb{R} \subset]p_1, p_2[$.

Combining the latter fact with $[p_1, p_2] \subset D_{1,\mathbb{R}}$ implies that

$$\mathbb{R} \cap (B \setminus D_1) = \{ \}.$$

Since we assumed that $H_1(D) \to H_1(D_1)$ is injective, boundedness of B implies

$$B \cap {D_1}^c \neq \{ \}.$$

We choose a path $\zeta\colon [0, 1] \to B$ such that $\zeta(0) = q$, $\zeta(1) \notin D_1$ and $\zeta(t) \notin \mathbb{R}$ for $t > 0$. Define

$$P = \{z \in \mathbb{C} : \exists\, t \in [0, 1],\ z = \zeta(t) \text{ or } \overline{\zeta(t)}\}.$$

Observe that $P \cap \mathbb{R} = \{q\}$.

Now we consider the following diagram of inclusion maps

$$\begin{array}{ccc} D & \longrightarrow & D_1 \\ \downarrow & & \downarrow \\ \mathbb{C} \setminus P & \longrightarrow & \mathbb{C} \setminus (P \cap {D_1}^c). \end{array}$$

From Lemma 2.13 we obtain injectivity of

$$H_3(\Omega_{\mathbb{C} \setminus P}) \to H_3(\Omega_{\mathbb{C} \setminus (P \cap {D_1}^c)})$$

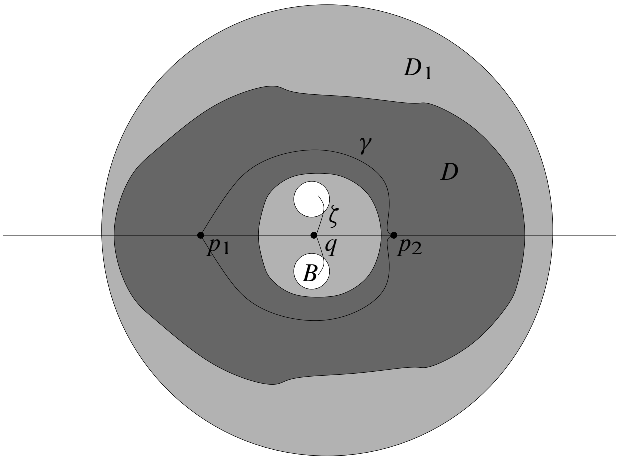

Figure 1.

which leads to a contradiction: First, by construction α_0 is mapped to a non-zero element of $\widetilde{H}_0(\mathbb{R} \setminus P)$. Due to (2.1) it follows that α is mapped to a non-zero element of $H_3(\Omega_{\mathbb{C}\setminus P})$. Second, its image in $H_3(\Omega_{D_1})$ is zero, which forces its image in $H_3(\Omega_{\mathbb{C}\setminus(P\cap {D_1}^c)})$ to be zero, because $D_1 \subset \mathbb{C} \setminus (P \cap {D_1}^c)$. □

Proposition 2.15. *Let $D \subset D_1$ be symmetric open subsets of $\mathbb{C}$ with corresponding axially symmetric subsets $\Omega_D \subset \Omega_{D_1}$ in $\mathbb{H}$. Then $H_1(D) \to H_1(D_1)$ is injective if and only if both $H_1(\Omega_D) \to H_1(\Omega_{D_1})$ and $H_3(\Omega_D) \to H_3(\Omega_{D_1})$ are injective.*

Proof. First we recall that the homology of a disjoint union $X = A \cup B$ is simply the direct sum of the homology of A and B. For this reason there is no loss in generality in assuming that Ω_D is connected. If both $H_1(\Omega_D) \to H_1(\Omega_{D_1})$ and $H_3(\Omega_D) \to H_3(\Omega_{D_1})$ are injective, injectivity of $H_1(D) \to H_1(D_1)$ follows from Corollary 2.12.

Now assume $H_1(D) \to H_1(D_1)$ is injective. Then $H_3(\Omega_D) \to H_3(\Omega_{D_1})$ is injective due to Proposition 2.14. Furthermore injectivity of $H_1(\Omega_D) \to H_1(\Omega_{D_1})$ follows from Corollary 2.10 combined with (2.2). □

3. Appendix

3.1. Some planar topology. Here we show that for a pair of domains $G \subset H$ in $\mathbb{C}$ the group homomorphism $i_*\colon H_1(G) \to H_1(H)$ induced by the inclusion map i is injective if and only if every bounded connected component of $G^c = \mathbb{C} \setminus G$ hits a bounded connected component of H^c. This is well known, but we provide a new proof based on an identification of $H_1(G)$ with a certain function space, namely $\mathcal{C}_c(G^c, \mathbb{Z})$.

Proposition 3.1. *Let G be an open subset of $\mathbb{C}$ and denote its complement by G^c. Then there is a natural isomorphism ξ between $H_1(G, \mathbb{Z})$ and $\mathcal{C}_c(G^c, \mathbb{Z})$ (i.e. the space of $\mathbb{Z}$-valued continuous (locally constant) functions with compact support on G^c).*

Proof. A cycle $\gamma \in H_1(G, \mathbb{Z})$ defines a function n_γ on $\mathbb{C} \setminus \operatorname{supp}(\gamma)$ by the *winding number*

$$n_\gamma(z) = \int_\gamma \frac{dw}{w - z}.$$

The winding number n_γ is locally constant on $\mathbb{C} \setminus |\gamma|$, therefore n_γ is continuous on G^c. It is compactly supported, because $n_\gamma(z) = 0$ for all z with $|z| > \max\{|w| : w \in |\gamma|\}$.

Now assume that γ is in the kernel of this map $\xi: \gamma \mapsto n_\gamma$. For each $k \in \mathbb{Z}$ let Z_k denote the cycle defined by the open set $\{z \in G : n_\gamma(z) = k\}$. Then the homology class of γ in $H_1(G, \mathbb{Z})$ vanishes, because $\gamma = \partial\big(\sum_k kZ_k\big)$ (here ∂ denotes the boundary operator in homology). This proves injectivity of the group homomorphism $\xi: H_1(G, \mathbb{Z}) \to \mathcal{C}_c(G^c, \mathbb{Z})$.

Conversely let $f \in \mathcal{C}_c(G^c, \mathbb{Z})$. Since f has compact support and takes values in $\mathbb{Z}$, f is a finite sum of functions $\pm f_i$ with $f_i \in \mathcal{C}_c(G^c, \mathbb{Z})$ and $f_i(z) \in \{0, 1\}$ for all z, i. We may therefore without loss of generality assume that $f(G^c) = \{0, 1\}$. Let $R > \sup\{|z| : f(z) \neq 0\}$. Now we define a function g on $G^c \cup \{z : |z| \geq R\}$ as follows

$$g(z) = \begin{cases} f(z) & \text{if } z \in G^c, \\ 0 & \text{if } |z| \geq R. \end{cases}$$

We extend g to a (real-valued) smooth function F defined on all of $\mathbb{C}$. Sards theorem implies that $\{z : F(z) = c\}$ is a smooth submanifold of $\mathbb{C}$ for almost all $c \in]0, 1[$. Each level set $\{z : F(z) = c\}$ $(0 < c < 1)$ is compact, because $F(z) = 0$ if $|z| \geq R$. Therefore almost every $c \in]0, 1[$ defines a finite union of disjoint closed smooth real curves in $\mathbb{C}$ which circumscribe $F = 1$. The homology class of this curve defines the element of $H_1(G, \mathbb{Z})$ corresponding to the function f. □

Lemma 3.2. *Let A be a closed subset of $\mathbb{C}$ and let B be a bounded connected component of A. Assume that $B \neq A$ and let $q \in A \setminus B$. Then there exists a function $f \in \mathcal{C}_c(A, \mathbb{Z})$ which is identically 1 on B such that $f(q) = 0$.*

Proof. Connected components are closed. Hence B is compact. Let $R > \max\{|z| : z \in B\}$. Define $C = \{z \in A : |z| = R\}$ and for each $w \in C$ choose disjoint open subsets U_w, V_w of A with $A = U_w \cup V_w$, $B \subset U_w$ and $w \in V_w$. Define f_w as the indicator function of U_w, i.e.,

$$f_w(z) = \begin{cases} 1 & \text{if } z \in U_w, \\ 0 & \text{if } z \in A \setminus U_w = V_w. \end{cases}$$

Now C is a compact set covered by the open sets V_w $(w \in C)$. Hence there is a finite set $S \subset C$ with

$$C \subset \cup_{w \in S} V_w.$$

We define

$$g(z) = \Pi_{w \in S} f_w(z)$$

observing that $g \equiv 1$ on B and $g \equiv 0$ on C.

We choose a continuous function $h\colon A \to \{0, 1\}$ such that h equals 1 on B and $h(q) = 0$ (which is possible, since q lies in a connected component of A different from B). Now we can define the function f we are looking for as

$$f(z) = \begin{cases} g(z)h(z) & \text{if } z \in A \text{ and } |z| \leq R, \\ 0 & \text{if } z \in A \text{ and } |z| > R. \end{cases}$$

The function f is continuous on A, because $g(z) = 0$ for all $z \in A$ with $|z| = R$, which implies that $g(z)h(z) = 0$ for $|z| = R$. By construction its support is contained in the closed disc of radius R (and therefore compact) and we have $f \equiv 1$ on B and $f(q) = 0$. □

Proposition 3.3. *Let $G \subset H \subset \mathbb{C}$ be open subsets. Then the following properties are equivalent:*

(i) *$H^c = \mathbb{C} \setminus H$ intersects each bounded connected component of G^c.*

(ii) *The restriction map from $\mathcal{C}_c(G^c, \mathbb{Z})$ to $\mathcal{C}_c(H^c, \mathbb{Z})$ is injective.*

(iii) *$H_1(G, \mathbb{Z}) \to H_1(H, \mathbb{Z})$ is injective.*

Proof. The equivalence of properties (ii) and (iii) has been shown above.

We prove the equivalence of (i) and (ii). Let B be a bounded connected component of G^c with $B \subset H$. Let $f \in \mathcal{C}_c(G^c, \mathbb{Z})$ be a function which equals 1 on B and assumes only 0 and 1 as values. (Such a function exists due to Lemma 3.2). Let

$$K = \operatorname{supp}(f) = \overline{\{z : f(z) \neq 0\}}$$

be its support and define $C = K \setminus H$. For every $x \in C$ we choose a function $g_x \in \mathcal{C}_c(G^c, \mathbb{Z})$ with $g_x(x) = 0$ and $g_x \equiv 1$ on B. (This is possible by Lemma 3.2, since B is compact). Due to compactness of C we may choose a finite subset S of C such that

$$C \subset \cup_{x \in S} \{z \in G^c : g_x(z) = 0\}.$$

Define

$$g(z) = f(z) \cdot \Pi_{x \in S} g_x(z).$$

Then g equals one on B and vanishes identically on C. Since $\operatorname{supp}(g) \subset \operatorname{supp}(f) \subset K$, $C = K \setminus H$ and $g|_C \equiv 0$, it is clear that g vanishes identically on H^c. Thus we have found a non-zero function $g \in \mathcal{C}_c(G^c, \{0,1\})$ whose restriction to H^c is zero. Therefore the existence of a bounded connected component B of G^c with $B \subset H$ implies that the restriction homomorphism $\mathcal{C}_c(G^c, \mathbb{Z}) \to \mathcal{C}_c(H^c, \mathbb{Z})$ is not injective.

To prove the opposite direction, let us assume that $B \cap H^c \neq \{\ \}$ for every bounded connected component B of G^c. Let $f \in \mathcal{C}_c(G^c, \mathbb{Z})$. Since f is locally constant and has compact support, it must vanish identically on every unbounded connected component of G^c. Thus, if $f \not\equiv 0$, there must be a bounded connected component B of G^c on which f is not zero. Since by assumption $B \cap H^c$ is not empty, it follows that the restriction of f to H^c is not everywhere zero. This proves injectivity. □

References

[1] A. Altavilla and C. Bisi, Log-biharmonicity and a Jensen formula in the space of quaternions, *Ann. Acad. Sci. Fenn. Math.*, **44** (2019), no. 2, 805–839. Zbl 1422.30072 MR 3973543

[2] D. Angella and C. Bisi, Slice-quaternionic Hopf surfaces, *J. Geom. Anal.*, **29** (2019), no. 3, 1837–1858. Zbl 1435.30137 MR 3969415

[3] H. Behnke and K. Stein, Entwicklung analytischer Funktionen auf Riemannschen Flächen (German), *Math. Ann.*, **120** (1949), 430–461. Zbl 0038.23502 MR 29997

[4] C. Bisi and G. Gentili, Möbius transformations and the Poincaré distance in the quaternionic setting, *Indiana Univ. Math. J.*, **58** (2009), no. 6, 2729–2764. Zbl 1193.30067 MR 2603766

[5] C. Bisi and G. Gentili, On the geometry of the quaternionic unit disc, in *Hypercomplex analysis and applications*, 1–11, Trends Math., Birkhäuser/Springer Basel AG, Basel, 2011. Zbl 1220.30067 MR 3026129

[6] C. Bisi and G. Gentili, On quaternionic tori and their moduli space, *J. Noncommut. Geom.*, **12** (2018), no. 2, 473–510. Zbl 1405.30051 MR 3825194

[7] C. Bisi and C. Stoppato, The Schwarz–Pick lemma for slice regular functions, *Indiana Univ. Math. J.*, **61** (2012), no. 1, 297–317. Zbl 1271.30024 MR 3029399

[8] C. Bisi and C. Stoppato, Regular vs. classical Möbius transformations of the quaternionic unit ball, in *Advances in hypercomplex analysis*, 1–13, Springer INdAM Ser., 1, Springer, Milan, 2013. Zbl 1270.30018 MR 3014606

[9] C. Bisi and C. Stoppato, Landau's theorem for slice regular functions on the quaternionic unit ball, *Internat. J. Math.*, **28** (2017), no. 3, 1750017, 21pp. Zbl 1368.30023 MR 3629144

[10] C. Bisi and J. Winkelmann, The harmonicity of slice regular functions, to appear in *J. Geom. Anal.* arXiv:1902.08165

[11] C. Bisi and J. Winkelmann, On a quaternionic Picard theorem, *Proc. Amer. Math. Soc. Ser. B*, **7** (2020), 106–117. MR 4137036

[12] F. Colombo, I. Sabadini, and D. Struppa, The Runge theorem for slice hyperholomorphic functions, *Proc. Amer. Math. Soc.*, **139** (2011), no. 5, 1787–1803. Zbl 1220.30068 MR 2763766

[13] J. E. Fornaess, F. Forstneric, and E. Fornaess Wold, Holomorphic approximation: the legacy of Weierstrass, Runge, Oka-Weil, and Mergelyan, in *Advancements in complex analysis. From theory to practice)*, 133–192, Springer, Cham, 2020. Zbl 07216541

[14] S. G. Gal and I. Sabadini, Arakelian's approximation theorem of Runge type in the hypercomplex setting, *Indag. Math. (N.S.)*, **26** (2015), no. 2, 337–345. Zbl 1311.30032 MR 3317238

[15] S. G. Gal and I. Sabadini, Approximation by polynomials on quaternionic compact sets, *Math. Methods Appl. Sci.*, **38** (2015), no. 14, 3063–3074. Zbl 1348.30028 MR 3382692

[16] G. Gentili and D. Struppa, A new approach to Cullen-regular functions of a quaternionic variable, *C. R. Math. Acad. Sci. Paris*, **342** (2006), no. 10, 741–744. Zbl 1105.30037 MR 2227751

[17] G. Gentili and D. Struppa, A new theory of regular functions of a quaternionic variable, *Adv. Math.*, **216** (2007), no. 1, 279–301. Zbl 1124.30015 MR 2353257

[18] R. Ghiloni and A. Perotti, Slice regular functions on real alternative algebras, *Adv. Math.*, **226** (2011), no. 2, 1662–1691. Zbl 1217.30044 MR 2737796

[19] S. Lang, *Algebra*. Revised third edition, Graduate Texts in Mathematics, 211, Springer-Verlag, New York, 2002. Zbl 0984.00001 MR 1878556

[20] R. Remmert, *Classical topics in complex function theory*. Translated from the German by Leslie Kay, Graduate Texts in Mathematics, 172, Springer-Verlag, New York, 1998. Zbl 0895.30001 MR 1483074

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