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Arcs in Laguerre planes

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Abstract. k-arcs in Laguerre planes are investigated with particular attention to problems of existence and completeness.

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1 Introduction

Let k be a positive integer. A *k-arc* in a projective plane is a set of k points, no three of which are collinear, and is said to be *complete* if not contained in a $(k+1)$-arc. If the projective plane is finite of order q, the maximum possible value of k is either $q+1$ or $q+2$ according to whether q is odd or even. More precisely, a $(q+1)$-arc is an oval and, when q is even, a $(q+2)$-arc is an hyperoval, that is an oval together with its nucleus. If q is a prime power and the projective plane is Desarguesian, conics are examples of ovals. If q is odd, $q \geq 5$ and the plane is Desarguesian, each oval is a conic, [9], [2]. The study of arcs in finite projective Desarguesian planes has attracted the attention of several authors, and the main problem is that of constructing arcs which are not conics and to see how closely these resemble conics.

In particular, it is proved, see [8], [13], that a *k-arc* in a projective Desarguesian plane of order q which is neither an oval nor an hyperoval contains at most $q - \frac{\sqrt{q}}{4} + \frac{7}{4}$ points for q odd and $q - \sqrt{q} + 1$ points for q even. Furthermore, if q is even, examples of hyperovals which are not conics plus nucleus can exist. The hyperovals of $PG(2,q)$, $q = 2^n$, containing the points $(0,1,0)$,$(1,0,0)$,$(0,0,1)$,$(1,1,1)$, can be described as the set of points $\{(F(t),t,1) \mid t \in GF(q)\} \cup \{(0,1,0),\ (1,0,0)\}$ where F is a permutation polynomial such that:

1) $F(0) = 0$, $F(1) = 1$;

2) For each $s \in GF(q)$, $F_s(x) = \frac{F(x+s)+F(s)}{x}$ is a permutation polynomial.

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This description applies to conics plus nucleus and also hyperovals which do not arise from conics.

A *k-arc* in a projective space of three dimensions is a set of k points no four of which are complanar, and is said to be *complete* if not properly contained in a $(k+1)$-arc. The complete $(q+1)$-arcs of $PG(3,q)$ have been classified. If q is odd, $q \geq 5$, a complete $(q+1)$-arc of $PG(3,q)$ is a twisted cubic. If q is even a complete $(q+1)-$arc of $PG(3,q)$, $q = 2^n$, $n > 1$, is the set of points $C(2^h) = \{(tF(t), F(t), t, 1) \mid t \in GF(2^n)\} \cup \{(1,0,0,0)\}$ with $F(t) = t^{2^h}$, $GCD(h,n) = 1$.

Recently, the notion of a k-arc in finite Benz planes has been introduced, [10], while arcs in Minkowski and in Möbius planes have also been studied, [7], [11], [12]. In this paper we investigate arcs in Laguerre planes.

A Laguerre plane $\mathcal{L} = (\mathcal{P}, \mathcal{B}, \|, \mathcal{I})$ is an incidence structure consisting of a point-set $\mathcal{P}$, a set $\mathcal{B}$ of at least two circles, an incidence relation $\mathcal{I} \subset \mathcal{P} \times \mathcal{B}$ and an equivalence relation $\|$ (parallelism) on $\mathcal{P}$ such that:

(i) three pairwise non-parallel points are incident with a unique circle;

(ii) each circle is incident with at least three points;

(iii) the circles which are "tangent" to a fixed circle C in a point p partition the set of points not parallel to p;

(iv) each parallel class intersects each circle at exactly one point.

We will denote the set of parallel classes under the equivalence relation $\|$ by $\mathcal{G}$, and each class will be called a generator. If $p \in \mathcal{P}$, g_p denotes the unique generator containing p, a set of non-parallel points will be called a set of *independent* points.

If $\mathcal{P}$ is finite, any two circles are incident with the same number $q+1$ of points and q is said to be the *order* of $\mathcal{L}$. In a finite Laguerre plane of order q there are q^2+q points, $q+1$ generators and q^3 circles, and each generator contains q points. If $\mathcal{L}$ is of order q and p is a fixed point, the derived incidence structure of $\mathcal{L}$ at p is denoted by $\mathcal{L}_p$. The points of $\mathcal{L}_p$ are the points of $\mathcal{L}$ which are not contained in g_p, the blocks are the elements of $\mathcal{G}-\{g_p\}$ together with the circles containing p, with p deleted. $\mathcal{L}_p$ is an affine plane of order q under the incidence relation naturally induced by $\mathcal{I}$. We will denote the projective closure of $\mathcal{L}_p$ by $\mathcal{L}_p^*$, the improper line of $\mathcal{L}_p$ by l_∞, and the common point of the lines of $\mathcal{G}-\{g_p\}$ by x_∞. When q is a prime power, examples of Laguerre planes are given taking a quadratic cone in the 3-dimensional space $PG(3,q)$: the points of the plane are the points of the cone except the vertex; the generators are the generators of the cone without the vertex; the circles are the non-trivial plane sections. Such a plane is said to be Miquelian and the derived plane at any of its points is Desarguesian. If q is even, a Laguerre plane is constructed in the same way by taking a cone over an oval of $PG(2,q)$. In this case the Laguerre plane is said to be ovoidal. If we consider the oval plus its nucleus, we construct $q+2$ ovoidal Laguerre planes, some of which, however, are isomorphic. Let q be an odd prime power, $q \geq 5$. If $\mathcal{L}$ is a Laguerre plane of order q and there exists a point p such that $\mathcal{L}_p$ is Desarguesian, then $\mathcal{L}$ is Miquelian, [4]. This result fails when q is even (see the non-Miquelian translation Laguerre plane of order

16 found in [3]).

2 Arcs in Laguerre planes

Let $\mathcal{L}$ be a Laguerre plane of order q, and $\mathcal{P}$, $\mathcal{B}$, $\mathcal{G}$ its set of points, circles and generators respectively. According to [10], we have the following definition:

Definition 1 Let $k \geq 3$ be a positive integer and let $K \subset \mathcal{P}$ be a set of k independent points. K is called a *k-arc* of $\mathcal{L}$ if every circle contains at most three points of it. K is said to be *complete* if not properly contained in a *(k+1)-arc*.

Remark 1. If p is a point of K, then $\overline{K}= K - \{p\} \cup \{x_\infty\}$ is a *k-arc* of $\mathcal{L}_p^*$. Furthermore, $|\mathcal{G}| = q + 1$ implies $k \leq q + 1$.

Investigating the maximal cardinality of arcs in finite Laguerre planes, we obtain the following general result:

Proposition 1 *An arc in a Laguerre plane $\mathcal{L}$ of even order q, with $q \equiv 1$ $(mod\ 3)$, contains at most q points.*

Proof. Suppose K to be a *(q+1)-arc* , let a and b be two independent points such that b is a point of K and a is not. $\overline{K}= K - \{b\} \cup \{x_\infty\}$ is an oval of $\mathcal{L}_b^*$ with nucleus on the improper line. Therefore a unique line contains a and is tangent to $\overline{K}$ at a point different from x_∞. This means that exactly one circle containing a, b and with two points in common with K does exist. Let now $K^* = K - (K \cap g_a)$, we construct an incidence structure $\mathcal{S}$ as follows: K^* is the set of points, the blocks are the 3-tuples of points of K^* obtained by intersecting K^* with the circles of $\mathcal{L}$ containing a and 3-secant K^*. The incidence relation is obviously induced by that of $\mathcal{L}$. Each point of $\mathcal{S}$ is contained in exactly $r = \dfrac{q-2}{2}$ blocks of $\mathcal{S}$; in fact, if $b \in K^*$, as observed above, there is exactly one circle of $\mathcal{L}$ containing a, b and 2-secant K in b and $b' \neq b$. Let $y \in K^* - \{b, b'\}$, the block containing a, b, y is 3-secant K^*; as $|K^* - \{b, b'\}| = q - 2$, there are exactly $\dfrac{q-2}{2}$ block of $\mathcal{S}$ containing b. We conclude that $\mathcal{S}$ is an incidence structure with q points, each block contains three points, each point is on $\dfrac{q-2}{2}$ blocks. Therefore the total number of blocks of $\mathcal{S}$ is $q \cdot \dfrac{q-2}{2} \cdot \dfrac{1}{3} = \dfrac{q(q-2)}{6}$. This contradicts the hypothesis $q \equiv 1$ $(mod\ 3)$, and hence K does not exist. □

In the Laguerre plane $\mathcal{L}$ is obtained from a cone in $PG(3, q)$ and if K is a k-arc of $\mathcal{L}$, then K is a *k-arc* of $PG(3, q)$. However, the completeness of K in $\mathcal{L}$ does not guarantee the completeness in $PG(3, q)$. The arc K together with the vertex of the cone is itself an arc in $PG(3, q)$. As already observed in [10], each twisted cubic lying on a quadratic cone of $PG(3, q)$ contains the vertex of the cone, [5 pag.238], and, when $q \geq 5$, gives an example of complete *q-arc* in the associated Miquelian Laguerre plane.

Nevertheless, when q is an odd prime power, each *(q+1)-arc* in $PG(3, q)$ is a twisted cubic, [5 pag.243]. This, together with the result mentioned in [5 pag. 238], allows us to state that an arc in a Miquelian Laguerre plane of odd order q, $q \geq 5$ contains at most q points, see [10]. In the following proposition 2 we redemonstrate this result with a different technique. Furthermore, we find the number of complete *q-arcs*. Hence this number coincides with the number of twisted cubics lying on a quadratic cone, as we state in a corollary to Proposition 2.

Recall that in a Desarguesian projective plane of order q (q prime power, $q \geq 5$), there is exactly one conic containing any five fixed points no three of which collinear. So, given four points no three of which are collinear, and a line containing only one of them, there is exactly one irreducible conic among the $q-2$ through the four given points which is tangent to the given line. Then:

PROPOSITION 2 *Let $\mathcal{L}$ be a Miquelian Laguerre plane of odd order q, $q \geq 5$. An arc in $\mathcal{L}$ contains at most q points. $\mathcal{L}$ contains exactly $q^3(q+1)(q-1)$ complete q-arcs.*

Proof. Suppose K to be a *(q+1)-arc* in $\mathcal{L}$ and let p be a point of it; then $\overline{K} = K - \{p\} \cup \{x_\infty\}$ is a *(q+1)-arc* in the projective Desarguesian plane $\mathcal{L}_p^*$. As q is odd, with $q \geq 5$, $\overline{K}$ is a conic, [9], [2]. Let p_1,p_2,p_3 be three distinct points of $\overline{K} - \{x_\infty\}$ and let C be the unique circle of $\mathcal{L}$ containing them. As K is an arc, then p is not a point of C; let $\{p'\} = C \cap g_p$, therefore $\overline{C} = C - \{p'\} \cup \{x_\infty\}$ is a $(q+1)$-*arc*, that is a conic of $\mathcal{L}_p^*$. Both $\overline{C}$ and $\overline{K}$ contain p_1,p_2,p_3 and are tangent to the line l_∞ in x_∞. Therefore it is $\overline{C} = \overline{K}$; the relation $(q-1) \geq 4$ implies $|K \cap C| \geq 4$: a contradiction.

Now let p_1,p_2,p_3 be three fixed independent points of $\mathcal{L}$ and let C be the unique circle containing them. Let p be a point not contained in C and such that $\{p_1,p_2,p_3,p\}$ is still a set of independent points. Denoting with p' the point of C contained in g_p, the set $C - \{p'\} \cup \{x_\infty\}$ is an oval in $\mathcal{L}_p^*$, that is a conic tangent to l_∞ in x_∞. Let D be one of the $q-3$ conics of $\mathcal{L}_p^*$ containing p_1,p_2,p_3,x_∞ and not tangent to l_∞, then $D \cap l_\infty = \{x_\infty, y_\infty\}$. $\overline{D} = D - \{x_\infty, y_\infty\} \cup \{p\}$ is a *complete q-arc* of $\mathcal{L}$. In fact, $\overline{D}$ contains q points; these points are independent (the generators of $\mathcal{L}$ are lines of $\mathcal{L}_p^*$ containing x_∞). Furthermore, each circle of $\mathcal{L}$ has at most three points on $\overline{D}$. Supposing B to be a circle of $\mathcal{L}$ such that $|B \cap \overline{D}| \geq 4$, then there are two possibilities:

(1) p is a point of B and $B - \{p\}$ is a line of $\mathcal{L}_p$ which intersects the conic D at three points: a contradiction;

(2) p is not a point of B, then $\overline{B} = B - (B \cap g_p) \cup \{x_\infty\}$ is an oval, that is a conic of $\mathcal{L}_p^*$, which is tangent to l_∞ in x_∞ and such that $\overline{B} - \{x_\infty\}$ has at least four points in common with $\overline{D}$. Therefore $\overline{B}$ and $\overline{D}$ have at least five points in common (one being x_∞) and this is a contradiction as they do not coincide (the conic $\overline{B}$ is tangent to l_∞ while D is not).

Finally, as $\overline{D}$ contains q points, it is complete. In this way, at least $q-3$ complete *q-arcs* containing the fixed independent points p_1, p_2, p_3, p are found.

These are the only *q-arcs* containing these four points. In fact, suppose K to be a *q-arc* containing p_1, p_2, p_3, p and let A be the unique block containing p_1, p_2, p_3 with $A \cap g_p = \{p'\}$, $p' \neq p$. Therefore $\overline{A} = A - \{p'\} \cup \{x_\infty\}$ is a conic of $\mathcal{L}_p^*$ and $\overline{K} = K - \{p\} \cup \{x_\infty\}$ is a q-arc of $\mathcal{L}_p^*$. A well known result, [6], assures that $\overline{K}$ is contained in a unique conic, say C. This conic C contains p_1, p_2, p_3, x_∞, but, as $q \geq 5$, it does not coincide with $\overline{A}$. Therefore, the only point to adjoin $\overline{K}$ to obtain C is on the improper line l_∞, and C is one of the $q-3$ conics containing p_1, p_2, p_3, x_∞ and not tangent to l_∞. There are exactly $q^4(q+1)(q-1)^2(q-2)$ ordered 4–tuples of independent points of $\mathcal{L}$ not contained in a circle. The number of ordered 4–tuples of points on a *q-arc* is $q(q-1)(q-2)(q-3)$. Thus the total number of *complete q-arcs* is $q^3(q+1)(q-1)$. □

A strightforward consequence is the following:

COROLLARY. *The number of twisted cubics lying on a quadratic cone of $PG(3,q)$, q odd, $q \geq 5$ is $q^3(q+1)(q-1)$.*

REMARK 2. Propositon 2 holds when $q \geq 5$. A Laguerre plane of order $q = 3$ contains 4-*arcs*. It is sufficient to fix three independent points: p_1, p_2, p_3 and, denoting with g the only generator which does not contain any of them, the set $\{p_1, p_2, p_3, p\}$ (with p a point of g not contained in the circle through p_1, p_2, p_3) is a 4-*arc*. There are exactly two 4-*arcs* containing three fixed independent points.

PROPOSITION 3 *A Miquelian Laguerre plane of odd order q, $q > 121$, does not contain complete (q-1)-arcs.*

Proof. Let $\mathcal{C}$ be the cone of $PG(3,q)$ corresponding to the Miquelian plane and let V be the vertex of $\mathcal{C}$. Suppose the Miquelian plane to contain a $(q-1)$-*arc* K and let $\overline{K} = K \cup \{V\}$. The point-set $\overline{K}$ is an arc in $PG(3,q)$. Every projection of $\overline{K}$ from one of its points onto a plane is a plane $(q-1)$-*arc*. When $q > 121$, the relation $q - 1 > q - \frac{\sqrt{q}}{4} + \frac{7}{4}$ holds, thus the plane $(q-1)$-*arc* is contained in a conic. This condition forces $\overline{K}$ to be contained in a unique twisted cubic (see [5 pag.243]), say $\overline{K} \cup \{P\}$. The twisted cubic $\overline{K} \cup \{P\}$ has more than six points on $\mathcal{C}$, this forces P to be a point of $\mathcal{C}$ and $K \cup \{P\}$ is a *q-arc* of the Miquelian Laguerre plane. Therefore the $(q-1)$-*arc* K is not complete in the Miquelian Laguerre plane. □

Suppose now $\mathcal{L}$ to be a Laguerre plane of even order $q = 2^n$. If n is even, $n = 2\lambda$, we have: $q - 2 = 2^{2\lambda} - 2 = (2^2 - 1)(1 + \cdots + 2^{2(\lambda-1)}) - 1 = 3r - 1$ and $q \equiv 1 \ (mod\ 3)$. The previous Proposition 1 assures that an arc in $\mathcal{L}$ contains at most q points.

If $\mathcal{L}$ is ovoidal, that is, $\mathcal{L}$ arises in $PG(3,q)$ from a cone over an oval, the following result holds:

PROPOSITION 4 *An arc in an ovoidal Laguerre plane of even order $q = 2^n$, $q > 2$, contains at most q points.*

Proposition 4 *An arc in an ovoidal Laguerre plane of even order $q = 2^n$, $q > 2$, contains at most q points.*

Proof. Suppose K to be a $(q+1) - arc$ in $\mathcal{L}$. Then K is an arc in $PG(3,q)$ and the generators of $\mathcal{L}$ are special unisecants to K. Recalling the result of [5 pag.247], the special unisecants to K are the generators of a hyperbolic quadric. On the other hand, these lines are concurrent at the vertex of the cone, hence a contradiction. □

Let $C(2^h)$ be a *(q+1)-arc* in $PG(3,q)$, $q = 2^n$, $q > 2$, $GCD(n,h) = 1$. The projection of $C(2^h)$ from $U_0 = (1,0,0,0)$ onto the plane $x_0 = 0$ gives the translation hyperoval $\theta = \{(0, t^{2^h}, t, 1) \mid t \in GF(q)\} \cup \{U_1, U_2\}$ with $\{U_1 = (0,0,1,0)$ and $U_2 = (0,1,0,0)\}$, [5 pag.249]. Let $\mathcal{L}$ be the ovoidal Laguerre plane constructed taking the cone of vertex U_0 on either the oval $\theta - \{U_1\}$ or $\theta - \{U_2\}$, then $K - \{U_0\}$ is a *complete q-arc* in $\mathcal{L}$. This proves the following:

Proposition 5 *Let $\mathcal{L}$ be an ovoidal Laguerre plane of order $q = 2^n$ obtained by projecting the points of a $(q+1) - arc$ of $PG(3,q)$ from one of its points onto a plane. The plane $\mathcal{L}$ contains complete q-arcs.*

A Miquelian Laguerre plane can be obtained by projecting the points of a twisted cubic, as previously described. Then, the property that a Miquelian Laguerre plane of order $q = 2^n$, $q \geq 2$, contains *complete q-arcs,* is a corollary of Proposition 5. Follow the previous notation and let $P_{\bar{t}} = (0, \bar{t}^{2^h}, \bar{t}, 1)$, $\bar{t} \in GF(q)$. Let $\overline{\mathcal{L}}$ be the ovoidal Laguerre plane which arises from the cone of vertex U_0 on the oval $\theta - \{P_{\bar{t}}\}$. Let $P'_{\bar{t}} = (\bar{t}^{2^h+1}, \bar{t}^{2^h}, \bar{t}, 1)$, then:

Proposition 6 *Let $q = 2^n \geq 8$. The set $C(2^h) - \{U_0, P'_{\bar{t}}\}$ is a complete (q-1)-arc in $\overline{\mathcal{L}}$.*

Proof. Let $\overline{K} = C(2^h) - \{U_0, P'_{\bar{t}}\}$. The set $\overline{K}$ contains $q-1$ independent points of $\overline{\mathcal{L}}$. Suppose $\overline{K}$ to be non complete and let T be a point of $\overline{\mathcal{L}}$ such that $\overline{K} \cup \{T\}$ is still an arc in $\overline{\mathcal{L}}$. Observe that T is either a point of the line U_0U_1 or of the line U_0U_2. Furthermore $\overline{K} \cup \{T\}$ is an arc in $PG(3,q)$ as well as $\overline{K} \cup \{T, U_0\}$. Therefore $\overline{K} \cup \{U_0\}$ is a *q-arc* of $PG(3,q)$ which is contained in two different *(q+1)-arcs*: $\overline{K} \cup \{U_0, T\}$ and $C(2^h)$. This contradicts the result of [1]. □

Remark 3. Recall that there exist two Laguerre planes of order 8. They are both ovoidal and can be obtained from the same cone of $PG(3,8)$ over the unique $10 - arc$ of $PG(2,8)$, which is a conic plus its nucleus. If we delete the line of the cone containing the nucleus, we obtain the Miquelian plane; by deleting any other line we obtain the non-Miquelian plane. Following the construction of Proposition 5, these planes are obtained by projecting the points of a twisted cubic of $PG(3,8)$ from one of its points onto a plane. Therefore, the previous Propositions 5 and 6 assure that the Miquelian plane contains a *complete 8-arc* and the non-Miquelian plane contains *complete 8-arcs* and *complete 7-arcs*.

oval of $PG(2,16)$, which is a conic minus one point plus its nucleus. So, both these planes contain *complete 16-arcs* and the non-Miquelian one also contains *complete 15-arcs.*

Furthermore, we can prove the following:

PROPOSITION 7 *Let $\mathcal{L}$ be an ovoidal Laguerre plane of order $q = 2^n$. Suppose $\mathcal{L}$ to be obtained from a cone over a non-translation oval θ. Then $\mathcal{L}$ does not contain q-arcs.*

Proof. Suppose K to be a $q-arc$ in $\mathcal{L}$ and let V be the vertex of the cone; then $K \cup \{V\}$ is a $(q+1)-arc$ in $PG(3,q)$. We can assume $V = (1,0,0,0)$ and $K \cup \{V\} = C(2^h)$, $GCD(h,n) = 1$. Projecting K from V onto the plane $x_0 = 0$, we obtain a translation oval $\overline{\theta}$. This is a contradiction, as $\overline{\theta}$ coincides with θ. □

The first example of a non translation oval can be found when $q = 16$. In the next section we will study this particular case.

PROPOSITION 8 *Let $\mathcal{L}$ be a Miquelian Laguerre plane of order $q = 2^n$, $n \geq 3$. The number of complete q-arcs of $\mathcal{L}$ is $q^3(q-1)(q+1)$.*

Proof. Suppose K to be a *q-arc* in $\mathcal{L}$. Let V be the vertex of the cone corresponding to $\mathcal{L}$. Then $K \cup \{V\}$ is a $(q+1)-arc$ of $PG(3,q)$. Projecting $K \cup \{V\}$ from V onto a plane, we obtain a plane $q-arc$ with q points in common with a conic. Therefore, this $q-arc$ is contained in a translation oval with associated polynomial $F(x) = x^{2^h}$ and either $h = 1$ or $h = n-1$. In both cases, $K \cup \{V\}$ is a twisted cubic (see [5 pag.252]). The number N of twisted cubics lying on a cone, and such that each generator contains at most one point of the cubic, can be computed. First observe that $N \leq q^3(q-1)(q+1)$. In fact, in $PG(3,q)$ there is exactly one twisted cubic containing five fixed points no four of which coplanar. Furthermore, the number of ordered $5-tuples$ of independent points of $\mathcal{L}$ no four of which on a circle is $q^4(q+1)(q-1)^2(q-2)(q-3)(q-4)$. The number of ordered $5-tuples$ on a $q-arc$ is $q(q-1)(q-2)(q-3)(q-4)$. This yields $N \leq q^3(q-1)(q+1)$. Now consider the following representation of $\mathcal{L}$:

$\mathcal{P} = \{(a) \mid a \in GF(2^n)\} \cup \{(x,y) | x,y \in GF(2^n)\}$ is the point-set;
$\mathcal{G} = \{g_\infty\} \cup \{g_a \mid a \in GF(2^n)\}$ is the generator-set and:
$g_\infty = \{(a) \mid a \in GF(2^n)\}$; $\quad g_a = \{(a,x) \mid x \in GF(2^n)\}$;
$\mathcal{B} = \{B_{abc} \mid a,b,c \in GF(2^n)\}$ is the circle-set and :
$B_{abc} = \{(x,y) \mid y = ax^2 + bx + c\} \cup \{(a)\}$.

Set $\overline{B}_{abc} = B_{abc} - \{(a)\} \cup \{(0,1,0)\}$ and observe that $\{\overline{B}_{abc} \mid a,b,c \in GF(2^n), a \neq 0\}$ is the set of all the conics of $\mathcal{L}^*_{(0)}$ which are tangent to l_∞ in $x_\infty = (0,1,0)$.

Let p_1,p_2,p_3 be three fixed independent points of $\mathcal{L}$ and let C be the only circle containing them. Let p be a point not contained in C and such that $\{p_1,p_2,p_3,p\}$ is still a set of independent points. Let $\{p'\} = C \cap g_p$. We may always suppose $p = (0)$. Let D be one of the $(q-3)$ irriducible conics of $\mathcal{L}^*_{(0)}$ containing p_1,p_2,p_3, x_∞ and $2-secant$ l_∞, so that $D \cap l_\infty = \{x_\infty, y_\infty\}$. Observe that if B is a circle not containing p, then $B - (B \cap g_{(0)}) \cup \{x_\infty\}$ is a conic of

$\mathcal{L}^*_{(0)}$, now follow Proposition 2 and prove that $D - \{x_\infty, y_\infty\} \cup \{(0)\}$ is a $q-arc$ of $\mathcal{L}$. By counting as in Proposition 2 we find at least $q^3(q-1)(q+1)$ arcs in $\mathcal{L}$. This ends the proof. □

Again we have the following :

COROLLARY *The number of twisted cubics lying on a quadratic cone of* $PG(3,q)$, *q even,* $q \geq 8$, *is* $q^3(q+1)(q-1)$.

3 Examples of arcs in "small" Laguerre planes

The Miquelian Laguerre plane of order q over the conic $y = x^2$ can be described as follows:

$\mathcal{P} = \{(a) \mid a \in GF(q)\} \cup \{(x,y) \mid x,y \in GF(q)\}$ is the point-set;

$\mathcal{G} = \{g_\infty\} \cup \{g_a \mid a \in GF(q)\}$ is the generator-set, $\mathcal{B} = \{C_{abc} \mid a,b,c \in GF(q)\}$ is the circle-set, with: $g_\infty = \{(a) \mid a \in GF(q)\}$; $g_a = \{(a,x) \mid x \in GF(q)\}$;

$C_{abc} = \{(x,y) \mid y = ax^2 + bx + c\} \cup \{(a)\}$.

When q is even and $\theta = \{(x, F(x), 1) | x \in GF(q)\} \cup \{(0,1,0)\}$ is an oval of $PG(2,q)$, the ovoidal Laguerre plane over θ can be described as above, taking:

$C_{abc} = \{(x,y) \mid y = aF(x) + bx + c\} \cup \{(a)\}$.

Using this description, we have developed a search with computer-aided experiments. Our aim was to find examples of arcs in Laguerre planes of order q containing less than q points.

First we examine the Miquelian Laguerre planes of order $q \in \{8, 9, 11, 13, 16\}$.

The Miquelian Laguerre plane of order 8 *contains complete 6-arcs.*

Let i be a root of the polynomial $x^3+x+1 \in Z_2[x]$ and let $GF(8) = \{a+bi+ci^2 | a,b,c \in Z_2\}$. The set $K = \{(0); (0,0); (1,0); (1+i,1); (1+i^2, 1+i^2); (i, 1+i^2)\}$ gives an example of a *complete* $6-arc$.

The Miquelian Laguerre plane of order 9 *contains complete 7-arcs.*

Let i be a root of the polynomial $x^2+x+2 \in Z_3[x]$ and let $GF(9) = \{a+bi | a,b \in Z_3\}$.

The set $K = \{(0); (0,0); (1,0); (1+i,1); (2+i,1); (2i,2i); (2,2+2i)\}$ gives an example of a *complete* $7-arc$.

The Miquelian Laguerre plane of order 11 *contains complete 6-arcs and complete 7-arcs.*

The set $K_1 = \{(0); (0,0); (1,0); (2,1); (3,1); (4,4)\}$ is an example of a *complete* $6-arc$.

The set $K_2 = \{(0); (0,0); (1,0); (2,1); (3,1); (4,9); (10,3)\}$ is an example of a *complete* $7-arc$.

The Miquelian Laguerre plane of order 13 *contains complete 7-arcs and complete 8-arcs.*

The set $K_1 = \{(0); (0,0); (1,0); (2,1); (3,1); (4,4); (5,8)\}$ is a *complete* $7-arc$.

The set $K_1 = \{(0); (0,0); (1,0); (2,1); (3,1); (4,4); (5,8)\}$ is a *complete* $7-$ *arc*.

The set $K_2 = \{(0); (0,0); (1,0); (2,1); (3,1); (4,11); (6,6); (11,6)\}$ is a *complete* $8-$ *arc*.

The Miquelian Laguerre plane of order 16 *contains complete 8-arcs and complete 10-arcs.*

Let i denote a root of the polynomial $x^4 + x + 1 \in Z_2[x]$ and let $GF(16) = \{a + bi + ci^2 + di^3 | a, b, c, d \in Z_2\}$.

The set $K_1 = \{(0); (0,0); (1,0); (i,1); (1+i,i); (i^2,1); (1+i^2,1+i); (1+i+i^2,i^2)\}$ is a *complete* $8-$ *arc*.

The set $K_2 = \{(0); (0,0); (1,0); (i,1); (1+i,1+i); (i^2,i^3); (1+i^2,i^2); (i+i^2,i+i^3); (1+i+i^2,i^2); (i+i^3,1+i+i^2+i^3)\}$ is a *complete* $10-$ *arc*.

A Laguerre plane of order $q = 2^n$, which is obtained from a cone over a non-translation oval, does not contain $q-$ *arcs* (see Proposition 7). The first example of non-translation oval can be found when the order is $q = 16$.

Let i be a root of the polynomial $x^4+x+1 \in Z_2[x]$. The hyperoval obtained with the permutation polynomial $F(x) = (i^2x^7 + i^{12}x^6 + i^6x^5 + i^9x^4 + i^5x^3 + i^5x^2 + i^6x)^2$ is an irregular hyperoval (see [6 pag.177]).

We have found examples of *complete* $8-$ *arcs* and $9-$ *arcs* in the Laguerre plane which is ovoidal over the non-translation oval $\{(x, F(x), 1) | x \in GF(16)\} \cup \{(0,1,0)\}$.

The set $K_1 = \{(0); (0,0); (1,0); (1+i,1); (i^2,1); (1+i^2,1+i^3); (i+i^2,1+i); (1+i+i^2,i+i^2+i^3)\}$ is a *complete* $8-$ *arc*.

The set $K_2 = \{(0); (0,0); (1,0); (1+i,1); (i^2,1); (1+i^2,1+i+i^2+i^3); (i+i^2,1+i^2); (1+i+i^2,i); (i^3,i+i^2+i^3)\}$ is a *complete* $9-$ *arc*.

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