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Proper Polynomial Self-maps of the Affine Space: State of the Art and New Results

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Abstract. Two proper polynomial maps $f_1, f_2 \colon \mathbb{C}^n \longrightarrow \mathbb{C}^n$ are said to be *equivalent* if there exist $\Phi_1, \Phi_2 \in \mathrm{Aut}(\mathbb{C}^n)$ such that $f_2 = \Phi_2 \circ f_1 \circ \Phi_1$. In this article, we investigate proper polynomial maps of topological degree $d \geq 2$ up to equivalence. In particular, we describe some of our recent results in the case $n = 2$ and we partially extend them in higher dimension.

0. Introduction

The semi-group of proper polynomial self-maps of the affine space $\mathbb{A}^n$ is a basic object both in complex analysis and algebraic geometry. It is therefore surprising how little is known about its structure. Although there has been some progress in the last few years, many basic questions remain unanswered.

Two proper polynomial maps $f_1, f_2 \colon \mathbb{C}^n \longrightarrow \mathbb{C}^n$ are said to be *equivalent* if there exist $\Phi_1, \Phi_2 \in \mathrm{Aut}(\mathbb{C}^n)$ such that $f_2 = \Phi_2 \circ f_1 \circ \Phi_1$. In this article, we investigate proper polynomial maps of topological degree $d \geq 2$ up to equivalence.

In Section 1 we set up notation and terminology and we state without proof some preliminary results. For further details, we refer the reader to [**BP10**]. In Section 2 we explain our recent work in dimension $n = 2$. In [**Lam05**], Lamy proved that any proper polynomial map $f \colon \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ of topological degree 2 is equivalent to the map $(x, y) \longrightarrow (x, y^2)$; in other words, if $d = 2$ there is just one equivalence class. When $d \geq 3$, we show that the situation is entirely different, since there are always infinitely many equivalence classes (see Theorems A, B and B1). Theorems A and B already appeared in our paper [**BP10**], whereas Theorem B1 is new. Moreover, by using Shephard-Todd's classification of finite

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complex reflection groups ([**ST54**]), we also obtained a complete description of *Galois coverings* $f\colon \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ up to equivalence (Theorem C).

Finally, in Section 3 we give an account on the situation in dimension $n \geq 3$ and we partially extend some of our theorems in this setting. For instance, we prove that for $d \geq 3$ there are still infinitely many equivalence classes (Theorem D). It would be certainly desirable to extend Theorem C in higher dimension, by describing all finite Galois covers $f\colon \mathbb{C}^n \longrightarrow \mathbb{C}^n$ up to equivalence. The main difficulty in carrying out this project is that the linearization theorem proven in [**Ka79**] for $n = 2$ cannot be generalized in dimension $n \geq 3$ (see [**Sch89**], [**Kn91**], [**MaPe91**], [**MaMoPe91**] for some counterexamples), so the classification method of [**BP10**] in this case breaks down. Although this problem is at present far from being solved, we can nevertheless give some partial results (see Theorem 3.1, Theorem 3.2 and Remark 3.3).

1. Proper polynomial maps

DEFINITION 1.1. *Let $f\colon \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a dominant polynomial map. We say that f is* proper *if it is closed and for every point $p \in \mathbb{C}^n$ the set $f^{-1}(p)$ is compact. Equivalently, f is proper if and only if for every compact set $K \subset \mathbb{C}^n$ the set $f^{-1}(K)$ is compact.*

Every proper map is necessarily surjective; the converse is not true, for instance, $(x,\, y) \longrightarrow (x+x^2y,\, y)$ provides an example of surjective self-map of $\mathbb{C}^2$ which is not proper. There is a purely algebraic condition for a polynomial map to be proper, see [**Jel93**, Proposition 3]:

PROPOSITION 1.2. *A dominant polynomial map $f\colon \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is proper if and only if the push-forward map $f_*\colon \mathbb{C}[s_1, \ldots, s_n] \longrightarrow \mathbb{C}[x_1, \ldots, x_n]$ is finite, i.e., $f_*\mathbb{C}[s_1, \ldots, s_n] \subset \mathbb{C}[x_1, \ldots, x_n]$ is an integral extension of rings.*

We recall that if $f\colon \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is the proper polynomial map

$$f(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)),$$

with $f_1, \ldots, f_n \in \mathbb{C}[x_1, \ldots, x_n]$, then f_* is defined as

$$\begin{aligned} f_*\colon \mathbb{C}[s_1, \ldots, s_n] &\longrightarrow \mathbb{C}[x_1, \ldots, x_n] \\ s_1 &\longrightarrow f_1(x_1, \ldots, x_n) \\ &\vdots \\ s_n &\longrightarrow f_n(x_1, \ldots, x_n). \end{aligned}$$

Moreover, if we denote by J_f the determinant of the Jacobian matrix of f, then the *critical locus* $\mathrm{Crit}(f)$ is defined as the affine hypersurface $V(J_f)$, and the *branch locus* $B(f)$ is the image of $\mathrm{Crit}(f)$ via f. The restriction

$$f\colon \mathbb{C}^n \setminus f^{-1}(B(f)) \longrightarrow \mathbb{C}^n \setminus B(f)$$

is an unramified covering of finite degree d; we will call d the *topological degree* of f.

DEFINITION 1.3. *We say that two proper polynomial maps $f_1,\, f_2\colon \mathbb{C}^n \longrightarrow \mathbb{C}^n$ are* equivalent *if there exist $\Phi_1,\, \Phi_2 \in \mathrm{Aut}(\mathbb{C}^n)$ such that*

$$f_2 = \Phi_2 \circ f_1 \circ \Phi_1. \tag{1}$$

If f_1 and f_2 are equivalent, they have the same topological degree; moreover, the chain rule implies that $\mathrm{Crit}(f_1)$ is biholomorphic to $\mathrm{Crit}(f_2)$ and $B(f_1)$ is biholomorphic to $B(f_2)$. Notice that this equivalence relation in the semi-group of proper polynomial maps is weaker than the conjugacy relation, in which we require $\Phi_2 = \Phi_1^{-1}$. For instance, the two maps $f_1(x, y) = (x, y^2)$ and $f_2(x, y) = (x, y^2 + x)$ are equivalent in our sense but they are not conjugate by any automorphism of $\mathbb{C}^2$, since their sets of fixed points are not biholomorphic. The study of conjugacy classes of proper maps of given topological degree is certainly an interesting problem, but we will not consider it here; some good references are [**FJ07a**] and [**FJ07b**].

2. The case $n = 2$

In [**Lam05**] Lamy proved that any proper polynomial map of topological degree 2 is equivalent to the map $(x, y) \longrightarrow (x, y^2)$; in other words, if $d = 2$ there is just one equivalence class. In [**BP10**], we showed that the situation is entirely different when $d \geq 3$; in fact, we proved the following two results:

Theorem A. *For every $d \geq 3$, consider the polynomial map $f_d \colon \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ given by*

$$f_d(x, y) := (x + y + xy, \, x^{d-1}y).$$

Then f is proper of topological degree d, and it is not *equivalent to any map of the form $(x, y) \longrightarrow (x, Q(x, y))$.*

Theorem B. *For all positive integers d, a, with $d \geq 3$ and $a \geq 2$, consider the polynomial map $f_{d,a} \colon \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ given by*

$$f_{d,a}(x, y) := (x, \, y^d - dx^a y).$$

Then $f_{d,a}$ and $f_{d,b}$ are equivalent if and only if $a = b$. It follows that if $d \geq 3$ there exist infinitely many different equivalence classes of proper polynomial maps $f \colon \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ of fixed topological degree d.

The proof of Theorem B follows from the fact that, when $d \geq 3$ and $a \neq b$, the critical loci of $f_{d,a}$ and $f_{d,b}$ have different Milnor number at their unique singular point $o = (0, 0)$, so they cannot be biholomorphic. Theorem B provides a *discrete* family $\{f_{d,a}\}_{a \geq 2}$ of proper maps of degree d which are pairwise non-equivalent. Now we refine this result, by showing the existence of a *continuous* family of maps with the same property. For all $d \in \mathbb{N}$, $\lambda \in \mathbb{C}$, set

$$\begin{aligned} F_{d,\lambda}(x,y) &:= y^d + \lambda x^{d-1} y + x^d, \\ \Gamma_d &:= \{\lambda \in \mathbb{C} \mid \text{the polynomial } F_{d,\lambda}(x, y) \text{ is square-free, i.e., it is the} \\ &\qquad \text{product of } d \text{ pairwise distinct homogeneous linear factors}\}. \end{aligned}$$

One immediately sees that $\mathbb{C} \setminus \Gamma_d$ is a finite set of points, and that if $\lambda \in \Gamma_d$ then the affine variety $C_{d,\lambda} := V(F_{d,\lambda})$ is the union of d distinct lines through the origin.

Proposition 2.1. *Assume $d \geq 4$ and λ, $\mu \in \Gamma_d$. Then the two germs of plane curve singularities $(C_{d,\lambda}, o)$ and $(C_{d,\mu}, o)$ are analytically equivalent if and only if $\lambda^d = \mu^d$.*

Proof. See [**K93**, Theorems 1.3 and 2.2]. □

Notice that the Milnor number of $C_{d,\,\lambda}$ at the origin does not depend on λ, since two ordinary d-multiple points are always topologically equivalent. Proposition 2.1 is a particular case of a more general result saying that when $d \geq 4$ there are infinitely many *analytic* types of ordinary d-multiple points. For instance, if $d = 4$ then the analytic type depends precisely on the cross-ratio of the four tangents, see [**GLS07**, Example 3.43.2].

Now, setting

$$Q_{d,\,\lambda}(x,\, y) := \frac{1}{d+1} y^{d+1} + \frac{\lambda}{2} x^{d-1} y^2 + x^d y,$$

we can prove

THEOREM B1. *For all $d \geq 4$ and $\lambda \in \Gamma_d$, consider the proper polynomial map defined by*

$$f_{d,\,\lambda}(x,\, y) := (x,\, Q_{d,\,\lambda}(x,\, y)).$$

If $\lambda^d \neq \mu^d$, then $f_{d,\,\lambda}$ and $f_{d,\,\mu}$ are not *equivalent. In particular, for all $d \geq 4$ there exists a* continuous *family of proper polynomial maps of degree d whose members are pairwise non-equivalent.*

PROOF. The critical locus of $f_{d,\,\lambda}$ is precisely the curve $C_{d,\,\lambda}$. Then the assertion is an immediate consequence of Proposition 2.1. □

The previous results suggest that a satisfactory description of all equivalence classes of proper polynomial maps $f \colon \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ in the case $d \geq 3$ is at the moment out of reach; nevertheless, one could hope at least to classify those proper maps enjoying some additional property. In [**BP10**] we completely solved this problem in the case of *Galois coverings*; some of our computations were carried out by using the Computer Algebra Systems `GAP4` and `Singular`, see [**GAP4**] and [**SING**]. Let $f \colon \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ be a polynomial map which is a Galois covering with finite Galois group G. Then f is proper and its topological degree equals $|G|$; moreover $G \subset \mathrm{Aut}(\mathbb{C}^2)$, and f can be identified with the quotient map $\mathbb{C}^2 \longrightarrow \mathbb{C}^2/G$. Since G is a finite group, we may assume $G \subset \mathrm{GL}(2, \mathbb{C})$ by a polynomial change of coordinates ([**Ka79**, Corollary 4.4]) and, since $\mathbb{C}^2/G \cong \mathbb{C}^2$, it follows that G is a *finite complex reflection group*. Let us denote by $\mathbb{C}[x,\, y]^G$ the subalgebra of G-invariant polynomials; then the following two conditions are equivalent, see [**Coh76**, p. 380]:

- (i) there are two algebraically independent homogeneous polynomials $\phi_1,\, \phi_2 \in \mathbb{C}[x,\, y]^G$ which satisfy $|G| = \deg(\phi_1) \cdot \deg(\phi_2)$;
- (ii) there are two algebraically independent homogeneous polynomials $\phi_1,\, \phi_2 \in \mathbb{C}[x,\, y]^G$ such that $1,\, \phi_1,\, \phi_2$ generate $\mathbb{C}[x,\, y]^G$ as an algebra over $\mathbb{C}$.

We say that $\phi_1,\, \phi_2$ are a *basic set of invariants* for G. Furthermore, putting $d_1 := \deg(\phi_1)$, $d_2 := \deg(\phi_2)$, the set $\{d_1,\, d_2\}$ is independent of the particular choice of $\phi_1,\, \phi_2$. We call $d_1,\, d_2$ the *degrees* of G. Complex reflection groups were classified in all dimensions by Shephard and Todd, see [**ST54**] and [**Coh76**]. Let us explain their classification in the case $n = 2$. If G is reducible, i.e., if there exists a 1-dimensional linear subspace $V \subset \mathbb{C}^2$ which is invariant under G, then we are in one of the following cases:

(1) $G = \mathbb{Z}_m$, generated by $g = \begin{pmatrix} 1 & 0 \\ 0 & \exp(2\pi i/m) \end{pmatrix}$;

(2) $G = \mathbb{Z}_m \times \mathbb{Z}_n$, generated by $g_1 = \begin{pmatrix} \exp(2\pi i/m) & 0 \\ 0 & 1 \end{pmatrix}$ and $g_2 = \begin{pmatrix} 1 & 0 \\ 0 & \exp(2\pi i/n) \end{pmatrix}$.

If G is irreducible, there exists an infinite family $G(m, p, 2)$, depending on two positive integer parameters m, p, with $p|m$, and 19 exceptional cases, that in [**ST54**] are numbered from 4 to 22. We start by describing the groups belonging to the infinite family. One has

$$G(m, p, 2) = \mathbb{Z}_2 \ltimes A(m, p, 2),$$

where $A(m, p, 2)$ is the abelian group of order m^2/p whose elements are the matrices $\left(\begin{smallmatrix} \theta^{\alpha_1} & 0 \\ 0 & \theta^{\alpha_2} \end{smallmatrix}\right)$, with $\theta = \exp(2\pi i/m)$ and $\alpha_1 + \alpha_2 \equiv 0 \pmod p$, whereas $\mathbb{Z}_2$ is generated by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. In particular, $G(m, m, 2)$ is the dihedral group of order $2m$.

Now let us consider the exceptional groups in the Shephard-Todd's list. We closely follow the treatment given in [**BP10**], which was in turn inspired by [**ST54**]. For $p = 3, 4, 5$, the abstract group

$$\langle s, t \,|\, s^2 = t^3 = (st)^p = 1 \rangle$$

is isomorphic to $\mathcal{A}_4$, $\mathcal{S}_4$ and $\mathcal{A}_5$, respectively. These are the well-known groups of symmetries of regular polyhedra: $\mathcal{A}_4$ is the symmetry group of the tetrahedron, $\mathcal{S}_4$ is the symmetry group of the cube (and of the octahedron) and $\mathcal{A}_5$ is the symmetry group of the dodecahedron (and the icosahedron). We take Klein's representation of these groups by complex matrices ([**Kl84**]), and we call S_1, T_1 the matrices corresponding to the generators s and t, respectively. Therefore the exceptional finite complex reflection groups are generated by matrices

$$S = \lambda S_1, \quad T = \mu T_1, \quad Z = \exp(2\pi i/k) I,$$

where λ, μ are suitably chosen roots of unity and k is a suitable integer. The corresponding abstract presentations are of the form

$$(2) \quad \langle S, T, Z \,|\, S^2 = Z^{k_1}, T^3 = Z^{k_2}, (ST)^p = Z^{k_3}, [S, Z] = I, [T, Z] = I, Z^k = I \rangle$$

where $p = 1, 2, 3$ and k_1, k_2, k_3, k are suitably chosen integers. We shall arrange the possible values of λ, μ, k_1, k_2, k_3, k in tabular form, according to Shephard-Todd's list ([**ST54**, p. 280-286]).

Exceptional groups derived from $\mathcal{A}_4$. Set $\omega = \exp(2\pi i/3)$, $\varepsilon = \exp(2\pi i/8)$. We have

$$S_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad T_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} \varepsilon & \varepsilon^3 \\ \varepsilon & \varepsilon^7 \end{pmatrix}.$$

The four corresponding groups are shown in Table 1 below. Here `IdSmallGroup`(G) denotes the label of G in the `GAP4` database of small groups, which includes all groups of order less than 2000, with the exception of 1024 ([**GAP4**]). For instance, one has `[24,3]`$= \mathrm{SL}_2(\mathbb{F}_3)$ and this means that $\mathrm{SL}_2(\mathbb{F}_3)$ is the third in the list of groups of order 24.

No.	IdSmall Group(G)	λ	μ	k_1	k_2	k_3	k	Degrees
4	[24,3]	-1	$-\omega$	1	2	2	2	4, 6
5	[72,25]	$-\omega$	$-\omega$	1	6	6	6	6, 12
6	[48,33]	i	$-\omega$	4	4	1	4	4, 12
7	[144,157]	$i\omega$	$-\omega$	8	12	3	12	12, 12

Table 1

Exceptional groups derived from $\mathcal{S}_4$. We have

$$S_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ -1 & -i \end{pmatrix}, \quad T_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} \varepsilon & \varepsilon \\ \varepsilon^3 & \varepsilon^7 \end{pmatrix}.$$

The eight corresponding groups are shown in Table 2 below.

No.	IdSmall Group(G)	λ	μ	k_1	k_2	k_3	k	Degrees
8	[96,67]	ε^3	1	1	2	4	4	8, 12
9	[192,963]	i	ε	8	7	8	8	8, 24
10	[288,400]	$\varepsilon^7\omega^2$	$-\omega$	7	12	12	12	12, 24
11	[576,5472]	i	$\varepsilon\omega$	24	21	8	24	24, 24
12	[48,29]	i	1	2	1	1	2	6, 8
13	[96,192]	i	i	4	1	2	4	8, 12
14	[144,122]	i	$-\omega$	6	6	5	6	6, 24
15	[288,903]	i	$i\omega$	12	3	10	12	12, 24

Table 2

Exceptional groups derived from $\mathcal{A}_5$. Set $\eta = \exp(2\pi i/5)$. We have

$$S_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} \eta^4 - \eta & \eta^2 - \eta^3 \\ \eta^2 - \eta^3 & \eta - \eta^4 \end{pmatrix}, \quad T_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} \eta^2 - \eta^4 & \eta^4 - 1 \\ 1 - \eta & \eta^3 - \eta \end{pmatrix}.$$

The seven corresponding groups are shown in Table 3 below.

No.	IdSmall Group(G)	λ	μ	k_1	k_2	k_3	k	Degrees
16	[600,54]	$-\eta^3$	1	7	10	10	10	20, 30
17	[1200,483]	i	$i\eta^3$	20	11	20	20	20, 60
18	[1800,328]	$-\omega\eta^3$	ω^2	11	30	30	30	30, 60
19	[3600,]	$i\omega$	$i\eta^3$	40	33	40	60	60, 60
20	[360,51]	1	ω^2	3	6	5	6	12, 30
21	[720,420]	i	ω^2	12	12	1	12	12, 60
22	[240, 93]	i	1	4	4	3	4	12, 20

Table 3

This allows us to obtain the classification, up to equivalence, of finite Galois coverings $f\colon \mathbb{C}^2 \longrightarrow \mathbb{C}^2$. Set

$$\begin{aligned}
\mathsf{a}_4(x,\, y) &= x^4 + (4\xi - 2)x^2y^2 + y^4, \quad \xi = \exp(2\pi i/6),\\
\mathsf{b}_6(x,\, y) &= x^5y - xy^5,\\
\mathsf{c}_8(x,\, y) &= x^8 + 14x^4y^4 + y^8,\\
\mathsf{d}_{12}(x,\, y) &= x^{12} - 33x^8y^4 - 33x^4y^8 + y^{12},\\
\mathsf{e}_{12}(x,\, y) &= x^{11}y + 11x^6y^6 - xy^{11},\\
\mathsf{f}_{20}(x,\, y) &= x^{20} - 228x^{15}y^5 + 494x^{10}y^{10} + 228x^5y^{15} + y^{20},\\
\mathsf{g}_{30}(x, y) &= x^{30} + 522x^{25}y^5 - 10005x^{20}y^{10} - 10005x^{10}y^{20} - 522x^5y^{25} + y^{30}.
\end{aligned}$$

Then we have

Theorem C. *Let $f\colon \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ be a polynomial map which is a Galois covering with finite Galois group G. Then f is equivalent to one of the normal forms described in Table 4 below. Furthermore, these maps are pairwise non-equivalent, with the only exception of $\mathfrak{f}_{2,\,1,\,2}$ and $\mathfrak{f}_{4,\,4,\,2}$.*

Map	$\phi_1,\, \phi_2$	G	Branch locus
$\mathfrak{f}_m$	$x,\, y^m$	$\mathbb{Z}_m$	$y = 0$
$\mathfrak{f}_{m,\,n}$	$x^m,\, y^n$	$\mathbb{Z}_m \times \mathbb{Z}_n$	$xy = 0$
$\mathfrak{f}_{m,\,p,\,2}$	$x^{m/p}y^{m/p},\, x^m + y^m$	$G(m,\, p,\, 2)$	$x(y^2 - 4x^p) = 0$ if $p \neq m$ $y^2 - 4x^p = 0$ if $p = m$
$\tilde{\mathfrak{f}}_4$	$\mathsf{a}_4,\, \mathsf{b}_6$	$G_4 =$ `[24, 3]`	$x^3 + (-24\xi + 12)y^2 = 0$
$\tilde{\mathfrak{f}}_5$	$\mathsf{b}_6,\, (\mathsf{a}_4)^3$	$G_5 =$ `[72, 25]`	$y(x^2 + (\frac{1}{18\xi} - \frac{1}{36})y) = 0$
$\tilde{\mathfrak{f}}_6$	$\mathsf{a}_4,\, (\mathsf{b}_6)^2$	$G_6 =$ `[48, 33]`	$y(x^3 + (-24\xi + 12)y^2) = 0$
$\tilde{\mathfrak{f}}_7$	$(\mathsf{b}_6)^2,\, (\mathsf{a}_4)^3$	$G_7 =$ `[144, 157]`	$xy(x + (\frac{1}{18\xi} - \frac{1}{36})y) = 0$
$\tilde{\mathfrak{f}}_8$	$\mathsf{c}_8,\, \mathsf{d}_{12}$	$G_8 =$ `[96, 67]`	$y^2 - x^3 = 0$
$\tilde{\mathfrak{f}}_9$	$\mathsf{c}_8,\, (\mathsf{d}_{12})^2$	$G_9 =$ `[192, 963]`	$y(y - x^3) = 0$
$\tilde{\mathfrak{f}}_{10}$	$\mathsf{d}_{12},\, (\mathsf{c}_8)^3$	$G_{10} =$ `[288, 400]`	$y(y - x^2)=0$
$\tilde{\mathfrak{f}}_{11}$	$(\mathsf{d}_{12})^2,\, (\mathsf{c}_8)^3$	$G_{11} =$ `[576, 5472]`	$xy(x - y) = 0$
$\tilde{\mathfrak{f}}_{12}$	$\mathsf{b}_6,\, \mathsf{c}_8$	$G_{12} =$ `[48, 29]`	$y^3 - 108x^4 = 0$
$\tilde{\mathfrak{f}}_{13}$	$\mathsf{c}_8,\, (\mathsf{b}_6)^2$	$G_{13} =$ `[96, 192]`	$y(x^3 - 108y^2)=0$
$\tilde{\mathfrak{f}}_{14}$	$\mathsf{b}_6,\, (\mathsf{d}_{12})^2$	$G_{14} =$ `[144, 122]`	$y(y + 108x^4)=0$
$\tilde{\mathfrak{f}}_{15}$	$(\mathsf{b}_6)^2,\, (\mathsf{d}_{12})^2$	$G_{15} =$ `[288, 903]`	$xy(y + 108x^2) = 0$
$\tilde{\mathfrak{f}}_{16}$	$\mathsf{f}_{20},\, \mathsf{g}_{30}$	$G_{16} =$ `[600, 54]`	$y^2 - x^3 = 0$
$\tilde{\mathfrak{f}}_{17}$	$\mathsf{f}_{20},\, (\mathsf{g}_{30})^2$	$G_{17} =$ `[1200, 483]`	$y(y - x^3) = 0$
$\tilde{\mathfrak{f}}_{18}$	$\mathsf{g}_{30},\, (\mathsf{f}_{20})^3$	$G_{18} =$ `[1800, 328]`	$y(y - x^2) = 0$
$\tilde{\mathfrak{f}}_{19}$	$(\mathsf{g}_{30})^2,\, (\mathsf{f}_{20})^3$	$G_{19} =$ `[3600, ]`	$xy(x - y) = 0$
$\tilde{\mathfrak{f}}_{20}$	$\mathsf{e}_{12},\, \mathsf{g}_{30}$	$G_{20} =$ `[360, 51]`	$y^2 - 1728x^5 = 0$
$\tilde{\mathfrak{f}}_{21}$	$\mathsf{e}_{12},\, (\mathsf{g}_{30})^2$	$G_{21} =$ `[720, 420]`	$y(y - 1728x^5) = 0$
$\tilde{\mathfrak{f}}_{22}$	$\mathsf{e}_{12},\, \mathsf{f}_{20}$	$G_{22} =$ `[240, 93]`	$y^3 + 1728x^5 = 0$

Table 4

The following corollary is a generalization of Lamy's result to the case of Galois coverings of arbitrary degree.

COROLLARY 2.2. *For all $d \geq 2$, there exist only finitely many equivalence classes of Galois coverings $f\colon \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ of topological degree d.*

3. The case $n \geq 3$

We have only a few general results about proper polynomial self-maps of $\mathbb{C}^n$ for $n \geq 3$. First of all, we can prove the following analogue of Theorem B:

THEOREM D. *Let $\mathbf{a} := (a_1, \ldots, a_{n-1}) \in \mathbb{N}^{n-1}$ be such that $a_i \geq 2$ for all i. For all $d \geq 3$, consider the proper polynomial map $f_{d,\mathbf{a}}\colon \mathbb{C}^n \longrightarrow \mathbb{C}^n$ defined by*

$$f_{d,\mathbf{a}}(x_1, \ldots, x_n) := (x_1,\, x_2, \ldots, x_{n-1},\, x_n^d - d(x_1^{a_1} + x_2^{a_2} + \cdots + x_{n-1}^{a_{n-1}})x_n).$$

If $\prod_{i=1}^{n-1}(a_i - 1) \neq \prod_{i=1}^{n-1}(b_i - 1)$, then $f_{d,\mathbf{a}}$ and $f_{d,\mathbf{b}}$ are not *equivalent. It follows that for all $d \geq 3$ there exist infinitely many different equivalence classes of proper polynomial maps $f\colon \mathbb{C}^n \longrightarrow \mathbb{C}^n$ of topological degree d.*

PROOF. The critical locus of $f_{d,\mathbf{a}}$ is the affine hypersurface $C_{d,\mathbf{a}}$ of equation $x_n^{d-1} - (x_1^{a_1} + x_2^{a_2} + \cdots + x_{n-1}^{a_{n-1}}) = 0$, whose unique singular point is $o := (0, \ldots, 0)$. The Milnor number of $C_{d,\mathbf{a}}$ in o is

$$\mu(C_{d,\mathbf{a}},\, o) = (d-2)\prod_{i=1}^{n-1}(a_i - 1).$$

It follows that if $d \geq 3$ and $\prod_{i=1}^{n-1}(a_i - 1) \neq \prod_{i=1}^{n-1}(b_i - 1)$, then $C_{d,\mathbf{a}}$ and $C_{d,\mathbf{b}}$ are not biholomorphic, hence $f_{d,\mathbf{a}}$ and $f_{d,\mathbf{b}}$ are not equivalent. $\square$

It would be also desirable to extend Theorem C in higher dimension, in other words to classify all the finite Galois covers $f\colon \mathbb{C}^n \longrightarrow \mathbb{C}^n$ up to equivalence. The main difficulty in carrying out this project is that the linearization theorem stated in [**Ka79**] for $n = 2$ cannot be generalized in dimension $n \geq 3$. So the classification method of [**BP10**] in this case breaks down. For the reader's convenience, let us give a short account of these topics; for further details we refer to the survey paper [**Kr95**].

In [**Ka79**] it was conjectured that if G is a linearly reductive algebraic group acting regularly on $\mathbb{C}^n$, then G has a fixed point, say p, and the action of G is linear with respect to a suitable coordinate system of $\mathbb{C}^n$ having p as its origin (the so-called Algebraic Linearization Conjecture). The first results in this direction were very promising, indeed any such action on $\mathbb{C}^2$ is linearizable as a consequence of the Jung's Theorem on the structure $\mathrm{Aut}(\mathbb{C}^2)$. Any torus action with an orbit of codimension one is linearizable by Bialynicki-Birula, see [**BiBi66**], [**BiBi67**], and Kraft, Popov and Panyushev showed that every semisimple group action is linearizable on $\mathbb{C}^3$ and $\mathbb{C}^4$, see [**KrP85**] and [**Pa84**].

On the other hand, in 1989 Schwarz discovered the first examples of non-linearizable actions of the orthogonal group $O(2)$ on $\mathbb{C}^4$ and of SL_2 on $\mathbb{C}^7$, [**Sch89**]. Using these results, Knop showed that every connected reductive group which is not a torus admits a faithful non-linearizable action on some affine space $\mathbb{C}^n$, [**Kn91**]. Using a different approach, Masuda, Moser-Jauslin and Petrie produced more examples and discovered the first non-linearizable actions of finite groups, namely, dihedral groups of order ≥ 10 on $\mathbb{C}^4$, see [**MaMoPe91**]. So far, all these examples of non-linearizable actions have been obtained from non-trivial G-vector bundles on representation spaces V of G using an idea of Bass and Haboush: for example in [**MaMoPe91**] it is proven that if G is a dihedral group of order ≥ 10, then there exists a positive-dimensional continuous family of isomorphism classes of G-vector bundles to which corresponds a positive-dimensional continuous family of inequivalent actions on $\mathbb{C}^4$. This method does not work in the holomorphic setting, however in [**DerKut98**] it is shown how to construct non-linearizable holomorphic actions on $\mathbb{C}^n$ for all reductive groups.

These results are not conclusive, and in particular the problem of describing all finite, non-linearizable automorphism subgroups of $\mathbb{A}^n$ for $n \geq 3$ is at present far from being solved. For instance, it is not even known whether there exist non-linearizable *involutions* on $\mathbb{A}^3$.

It is not our purpose to investigate these deep questions here, so we just present the following two results:

THEOREM 3.1. *Let $n \geq 2$ and $f : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a polynomial map which is a Galois covering with finite Galois group $G \cong \mathbb{Z}_m = \langle \sigma \rangle$, where σ is a triangular automorphism of $\mathbb{C}^n$ of the form*

$$\sigma(x_1, \cdots, x_n) = (s_1 x_1 + a_1, s_2 x_2 + a_2(x_1), \cdots, s_n x_n + a_n(x_1, \cdots, x_{n-1})), \quad s_i \in \mathbb{C}^*$$

such that $\sigma^m = I$. Then f is equivalent to $\mathfrak{f}_m(x_1, \cdots, x_n) = (x_1, x_2, \cdots, x_{n-1}, x_n^m)$.

PROOF. By [**Ivan98**] the group generator σ is linearizable, so the group action is also linearizable. By using Shephard-Todd's classification of finite complex reflection groups, we see that G is conjugated in $U(n)$ to the group generated by $\tilde{\sigma}(x_1, \cdots, x_n) = (x_1, x_2, \cdots, x_{n-1}, \theta_m x_n)$, where θ_m is a primitive m-th root of unity. $\square$

THEOREM 3.2. *Let $f : \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ be a proper polynomial map which is a Galois covering with finite Galois group G, and assume that the action of G is linearizable and reducible. Then G is one of the groups in Table 4 and f is equivalent to the map $(x_1^m, \mathfrak{f}(x_2, x_3))$, where $\mathfrak{f}$ is the normal form on $\mathbb{C}^2$ corresponding to G and $m \geq 1$.*

PROOF. Since the action is reducible, there exists either a $1-$dimensional or a $2-$dimensional linear subspace $V \subset \mathbb{C}^3$ which is invariant under G; then its orthogonal complement $V^{\perp}$ is also invariant, see [**Se71**], and up to a linear change of coordinates we may assume $V = \langle e_1 \rangle$, $V^{\perp} = \langle e_2, e_3 \rangle$ where $\{e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{C}^3$. Then the assertion follows by using the classification given in Theorem C. $\square$

REMARK 3.3. By using the same methods of [**BP10**], it is possible to completely classify the Galois coverings $f \colon \mathbb{C}^n \longrightarrow \mathbb{C}^n$ such that the G-action on $\mathbb{C}^n$ is *linearizable*. Indeed, this is equivalent to compute a minimal base of generators of the invariant algebra $\mathbb{C}[x_1, \ldots, x_n]^G$ for each of the 34 exceptional groups in the Shephard-Todd's list. This is a standard calculation that can be carried out by using either invariant theory (as in [**ST54**]) or some Computer Algebra Systems (e.g., `GAP4` and `Singular`). However, some of these groups have very large order (for instance, in the last case of the list we have $G = W(E_8)$, whose order is 696729600), so the problem is computationally hard and we think that the outcome is not worthy of the effort.

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