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Micro and macro models of granular computing induced by the indiscernibility relation

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ABSTRACT

In rough set theory (RST), and more generally in granular computing on information tables (GRC-IT), a central tool is the Pawlak's indiscernibility relation between objects of a universe set with respect to a fixed attribute subset. Let us observe that Pawlak's relation induces in a natural way an equivalence relation $\approx$ on the attribute power set that identifies two attribute subsets yielding the same indiscernibility partition. We call *indistinguishability relation* of a given information table $\mathcal{I}$ the equivalence relation $\approx$, that can be considered as a kind of global indiscernibility. In this paper we investigate the mathematical foundations of indistinguishability relation through the introduction of two new structures that are, respectively, a complete lattice and an abstract simplicial complex. We show that these structures can be studied at both a micro granular and a macro granular level and that are naturally related to the core and the reducts of $\mathcal{I}$. We first discuss the role of these structures in GrC-IT by providing some interpretations, then we prove several mathematical results concerning the fundamental properties of such structures.

1. Introduction

We denote by $\mathcal{I} := \langle U, Att, F, Val- \rangle$ a knowledge representation system (information table in the finite case [41–44]), having universe set U, attribute set Att, information map F: $U \times Att \to Val$ and value set Val. In this paper we introduce and study some micro-macro granular mathematical structures uniquely associated to any knowledge representation system $\mathcal{I}$.

1.1. General premise

Tabular representation of data appears in several fields of research, related to many problems of taxonomy in biology, economics, social sciences and so forth. At present, many researchers are dedicating themselves to the analysis of Pawlak's information tables only through heuristic interpretations and by using an informatic vocabulary. We mainly aim to introduce and investigate Pawlak's indiscernibility relation between attributes instead of the usual relation between objects. In order

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Table 1
Information Table $\mathcal{C}$.

$\mathcal{C}$	*Speed*	*Color*	*RoadH*
m_1	Medium	Green	Good
m_2	Medium	Blue	Discrete
m_3	Medium	Green	Good
m_4	High	Blue	Discrete

to distinguish the two relations, the term *indistinguisuishability* will be used in case of attributes. That is, the indistinguishability is simply the equivalence relation $\approx$ on the power set $\mathcal{P}(Att)$ that identifies two attribute subsets inducing on U the same indiscernibility relation. The surprising fact is that the indistinguishability relation induces a very rich interrelation of several mathematical structures, that we will call *indiscernibility hypergraphic structures*, and that we will study to different granularity levels. In this paper, we are mainly interested in developing the mathematical foundations relying on a series of old and new hypergraphic structures induced by the Pawlak's indiscernibility relation on information tables. We study the basic mathematical properties of such structures that arise in a natural way without assuming any extra hypothesis. One of our basic motivations is to understand which kind of formal theory can be developed in terms of order structures, hypergraphic structures and their potential links arising when we investigate an information table with GrC methodologies (abbreviated GrC-IT). From this standpoint, we think that a granular perspective in the study of an information table can be fruitfully applied also when one tries to develop notions derived from the Pawlak indiscernibility relation that are not directly connected with rough set theory (RST). In fact, RST main features are the lower and upper approximations and the reducts, all of which are defined starting from the notion of indiscernibility. Implicitly, the indiscernibility notion is underlying also to the relation database theory arisen by the classical Codd's model (see [17]). However, in relational database context, theory lacks the explicit presence of the object set U, because the rows of a table are simply considered as tuples of an n-ary relation without a specific identity. In this case the indiscernibility must be expressed by means of the notion of projection on tuples therefore, in a relational context, the indiscernibility relation has not the same epistemological consistence that has in the Pawlak's context. In this paper we intend to contribute to the development of a possible new research line based on the generality and simplicity (at the same time) of the Pawlak's indiscernibility notion on information tables. To this regard, we can observe that usually in mathematics the nodes of the order structures are studied as points of some lattices or posets and, in these cases, they have not in itself special types of inner structures interacting with the order relation that connects them. In our case, the nodes of our construction have an inner structure, that we can call *local structure* (the *micro-granular* level) that is strictly connected with the global order structure (the *macro-granular* level). In this work we introduce two new basic constructions based on the maximal and minimal members of the indiscernibility relation considered to a global level. The first of these structures, denoted by $\mathbb{M}(\mathcal{I})$ is a complete lattice (Theorem 3.6 of [16]); the second, denoted by $\mathfrak{m}(\mathcal{I})$ and called *minimal partitioner hypergraph* is an abstract simplicial complex (see Theorem 5.6). In this paper we first discuss broadly in the initial example of the next subsection the possible interpretative developments of these new structures. Next, we devote our efforts mainly to the discovery of new formal relations between the two previous structures. Finally, it is convenient here to highlight that also well-known attribute subset families (such as for example the reduct family) find a more general collocation within the above discussed structures.

1.2. A concrete example

Suppose that an individual should buy a car and has four possible choices: m_1, m_2, m_3, m_4. We assume that he is interested in the following properties (attributes): speed, color and roadholding. Therefore, we can consider the information table $\mathcal{C} = \langle U, Att, F, Val- \rangle$ given in Table 1, where $U := \{m_1, m_2, m_3, m_4\}$ is the car set and $Att := \{Speed, Color, RoadH\}$.

Then, the indiscernibility partitions (classifications) of the car set with respect to all attribute subsets are the following:

$$\pi_{\mathcal{C}}(\emptyset) = m_1m_2m_3m_4,$$

$$\pi_{\mathcal{C}}(Speed) = m_1m_2m_3|m_4,$$

$$\pi_{\mathcal{C}}(Color) = \pi_{\mathcal{C}}(RoadH) = \pi_{\mathcal{C}}(Color, RoadH) = m_1m_3|m_2m_4,$$

$$\pi_{\mathcal{C}}(Speed, Color) = \pi_{\mathcal{C}}(Speed, RoadH) = \pi_{\mathcal{C}}(Speed, Color, RoadH) = m_1m_3|m_2|m_4.$$

It is clear that the above indiscernibility partitions induce respectively the information sub-tables $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_4$ of $\mathcal{C}$ described in Table 2 where:

$U_{\mathcal{C}_1} := \{m_1m_2m_3m_4\}$, $Att_{\mathcal{C}_1} := \{\emptyset\}$; $U_{\mathcal{C}_2} := \{m_1m_2m_3, m_4\}$, $Att_{\mathcal{C}_2} := \{Speed\}$; $U_{\mathcal{C}_3} := \{m_1m_3, m_2m_4\}$, $Att_{\mathcal{C}_3} := \{Color, Road\}$; $U_{\mathcal{C}_4} := \{m_1m_3, m_2, m_4\}$, $Att_{\mathcal{C}_4} := \{Speed, Color, RoadH\}$.

We call the information tables $\mathcal{C}_1, \ldots, \mathcal{C}_4$ *indiscernibility sub-tables* of $\mathcal{C}$.

Table 2
Car information sub tables.

$\mathcal{C}_1$	$\emptyset$	$\mathcal{C}_2$	*Speed*
$m_1m_2m_3m_4$	$\emptyset$	$m_1m_2m_3$	Medium
		m_4	High

$\mathcal{C}_3$	*Color*	*RoadH*	$\mathcal{C}_4$	*Speed*	*Color*	*RoadH*
m_1m_3	Green	Good	m_1m_3	Medium	Green	Good
m_2m_4	Blue	Discrete	m_2	Medium	Blue	Discrete
			m_4	High	Blue	Discrete

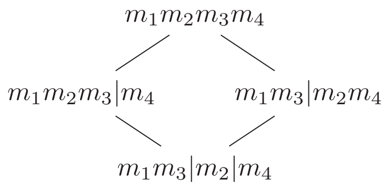

Fig. 1. Diagram of the lattice $\mathbb{P}_{ind}(\mathcal{C})$.

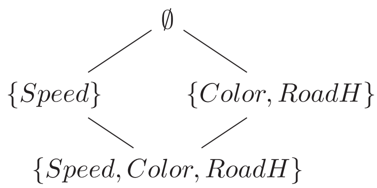

Fig. 2. Diagram of the lattice $\mathbb{M}(\mathcal{C})$.

It is well known [76] that for any information table $\mathcal{I}$ we can build a lattice structure $\mathbb{P}_{ind}(\mathcal{I})$ on the set $\Pi_{ind}(\mathcal{I})$ of all the indiscernibility partitions of $\mathcal{I}$ with the usual refinement order $\preceq$ (the diagram of $\mathcal{C}$ is drawn in Fig. 1). This lattice provides to the user the possibility to compare all the distinct indiscernibility partitions of the information table by means of the refinement order between set partitions on the universe set U. For example, for the information table $\mathcal{C}$, we can see that the only two incomparable indiscernibility partitions are $m_1m_2m_3|m_4$ and $m_1m_3|m_2m_4$. In general, by using the diagram of the lattice $\mathbb{P}_{ind}(\mathcal{I})$ the user looses all the information concerning the attribute subsets that induce the different indiscernibility partitions. Again with reference to the diagram in Fig. 1, we can observe that the user cannot see that the attribute subsets $\{Color\}$, $\{RoadH\}$ and $\{Color, RoadH\}$ induce the same indiscernibility partition $m_1m_3|m_2m_4$.

It has been proved that the set $\Pi_{ind}(\mathcal{I})$ of all the indiscernibility partitions of $\mathcal{I}$ with the usual refinement order $\preceq$ is a complete lattice $\mathbb{P}_{ind}(\mathcal{I})$ that is order isomorphic to another lattice $\mathbb{M}(\mathcal{I})$ (see [16]. The elements of this new lattice $\mathbb{M}(\mathcal{I})$ are called *maximum partitioners* of $\mathcal{I}$ and their set is denoted by $MAXP(\mathcal{I})$. The maximum partitioners of $\mathcal{I}$ are the greatest subsets of *Att* that induce all the distinct indiscernibility partitions of $\mathcal{I}$ (in our car information table, the maximum partitioners of $\mathcal{C}$ are $\emptyset$, $\{Speed\}$, $\{Color, RoadH\}$ and $\{Speed, Color, RoadH\}$). The partial order on $\mathbb{M}(\mathcal{I})$ is the dual relation $\subseteq^*$ of the usual set inclusion relation $\subseteq$. The lattices $\mathbb{P}_{ind}(\mathcal{I})$ and $\mathbb{M}(\mathcal{I})$ have been called respectively the *indiscernibility partition lattice* and the *maximum partitioner lattice* of the information table $\mathcal{I}$ (see [16]; a representation of the diagram of $\mathbb{M}(\mathcal{C})$ is given in Fig. 2).

Another lattice structure $\mathbb{G}(\mathcal{I})$ introduced in [16] is defined on the set of all the ordered pairs $(\pi, Max(\pi))$, where $\pi \in \Pi_{ind}(\mathcal{I})$ and $Max(\pi)$ is the maximum partitioner of $\mathcal{I}$ that induces the indiscernibility partition π. The partial order on $\mathbb{G}(\mathcal{I})$ is the direct product order $\subseteq^* \times \preceq$. The lattice $\mathbb{G}(\mathcal{I})$ has been called *granular partition lattice* of $\mathcal{I}$ (see [16]), by recalling the granular interpretation given by Yao in [76]. The Hasse diagram of the granular partition lattice for our car example is represented in Fig. 3.

However, we can note that in all the above diagrams the user loses the information concerning the value set of $\mathcal{C}$. On the other hand, it is also clear that all the lattices $\mathbb{P}_{ind}(\mathcal{C})$, $\mathbb{M}(\mathcal{C})$ and $\mathbb{G}(\mathcal{C})$ induce an order isomorphic lattice structure $\mathbb{S}(\mathcal{C})$ on the set $ISUB(\mathcal{C}) := \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4\}$ with the following partial order $\sqsubseteq$:

$$\mathcal{C}_k \sqsubseteq \mathcal{C}_j :\Longleftrightarrow Att_{\mathcal{C}_k} \subseteq^* Att_{\mathcal{C}_j}. \quad (1)$$

The diagram of $\mathbb{S}(\mathcal{C})$ is drawn in Fig. 4.

In general, for any $\mathcal{I}$, by using the diagram of the lattice $\mathbb{S}(\mathcal{I})$, the user has three types of information: the blocks of any indiscernibility partition, the maximum partitioners and the corresponding values common to any indiscernibility block. It is clear that, with respect to the lattices $\mathbb{P}_{ind}(\mathcal{I})$, $\mathbb{M}(\mathcal{I})$ and $\mathbb{G}(\mathcal{I})$, the lattice $\mathbb{S}(\mathcal{I})$ provides to the user a greater amount of information. In general, we will call the lattice $\mathbb{S}(\mathcal{I})$ *indiscernibility sub-table lattice* of $\mathcal{I}$.

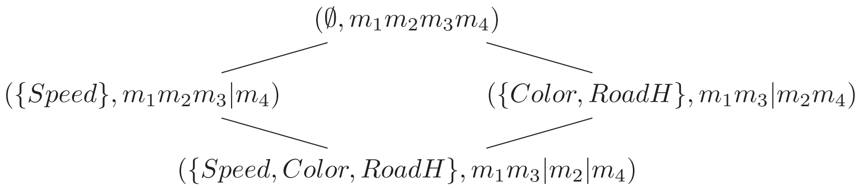

Fig. 3. Diagram of the lattice $\mathbb{G}(\mathcal{C})$.

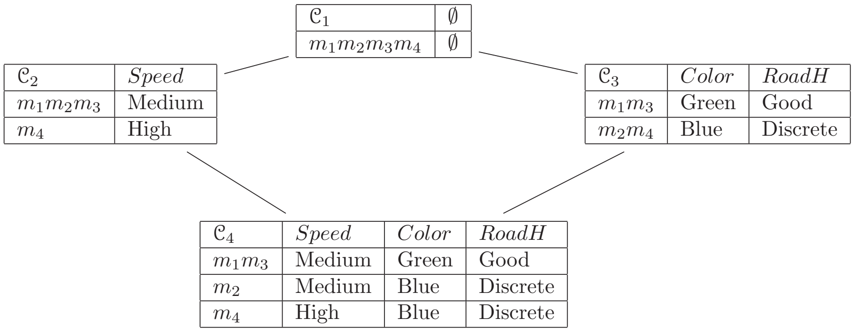

Fig. 4. Diagram of the lattice $\mathbb{S}(\mathcal{C})$.

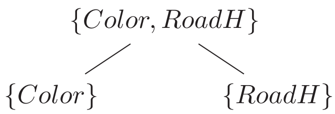

Fig. 5. Some inclusions between attribute subsets.

Nevertheless, the usage of the indiscernibility sub-table lattice also involves for the user a loss of information concerning the set inclusion relation among all the attribute subsets that induce the same indiscernibility partition. For example, in reference to the car information table, we know that the attribute subsets $\{Color\}$, $\{RoadH\}$ and $\{Color, RoadH\}$ generate the same indiscernibility partition $m_1m_3|m_2m_4$, but in none diagram of the above lattice structures the user can see the inclusion relation given in Fig. 5.

This remark induces us to introduce a further type of lattice structure to represent also this new type of information. In order to do this, we first introduce the following equivalence relation $\approx$ between attribute subsets of any knowledge representation system $\mathcal{I}$: if $A, A' \subseteq Att$ and $\pi(A)$, $\pi(A')$ are their corresponding indiscernibility partitions induced on U, we set

$$A \approx A' :\iff \pi(A) = \pi(A').$$

Let $[A]_\approx$ be the equivalence class of A with respect to $\approx$. We say that $\approx$ is the *indistinguishability relation* of $\mathcal{I}$ and $[A]_\approx$ the *indistinguishability class* (or, equivalently, the *indistinguishability granule*) of A. Then the maximum partitioner $M(A) := Max(\pi(A))$ coincides with the maximum member of the poset $([A]_\approx, \subseteq)$, named *local indistinguishability poset* of A. It is easy to see that we can use the whole equivalence class $[A]_\approx$ when we represent the diagram of $\mathbb{M}(\mathcal{I})$ instead of using only its maximum element $M(A)$ (see [16] for details). Furthermore, we can consider the equivalence class $[A]_\approx$ as a type of *macro-granule* within which we can also represent the *micro-granular inclusions* among the members of $[A]_\approx$. We denote this lattice by $\mathbb{I}(\mathcal{I})$ and we call it the *indistinguishability granular lattice* of $\mathcal{I}$. In Fig. 6 we provide a representation of the diagram of $\mathbb{I}(\mathcal{C})$.

At this point it is clear that although all the previous lattice structures are order isomorphic among them, anyone of these representations provide a different and useful point of view by means of which investigate the granular structure of all the indiscernibility partitions induced from an information table. Then the basic purpose of this work consists in the prosecution of the study of these order structures started in [16].

Let $\min([A]_\approx)$ be the family of all minimal members of $[A]_\approx$, that we call *minimal partitioners* of $\mathcal{I}$, and

$$MINP(\mathcal{I}) := \bigcup_{A \in MAXP(\mathcal{I})} \min([A]_\approx).$$

In reference to Fig. 6, it results that

$$MINP(\mathcal{C}) = \{\emptyset, \{Speed\}, \{Color\}, \{RoadHolding\}, \{Speed, Color\}, \{Speed, RoadHolding\}\}.$$

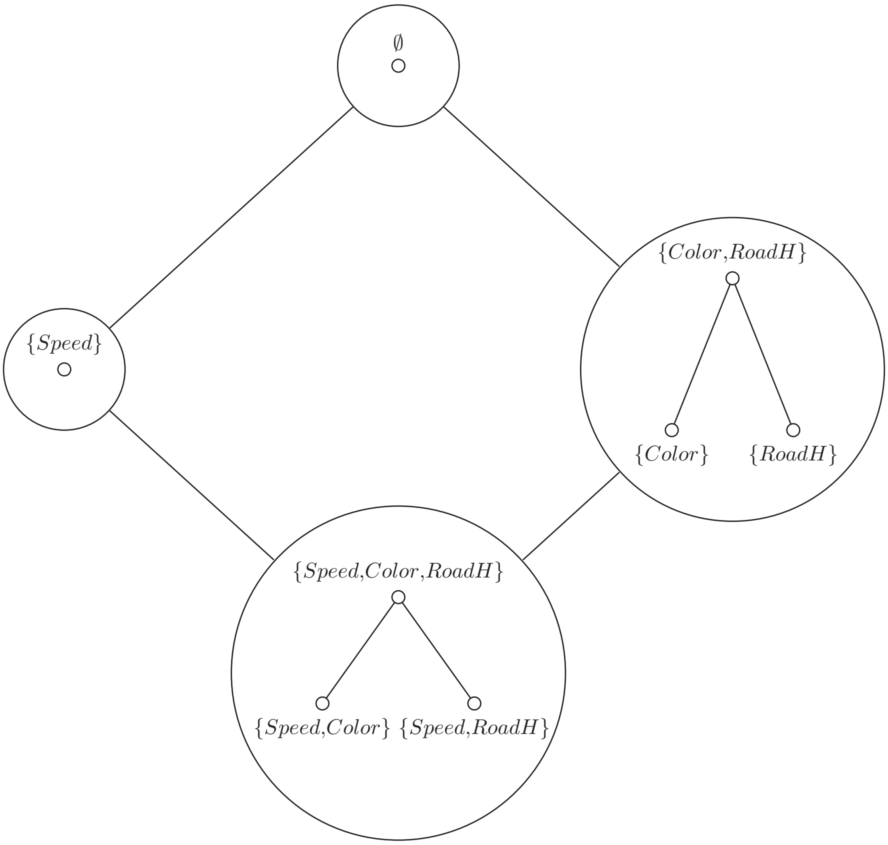

Fig. 6. Diagram of the lattice $\mathbb{I}(\mathcal{C})$.

As we see, if $A \in MINP(\mathcal{C})$, then any of its subsets does. From a mathematical outlook, this means that $MINP(\mathcal{C})$ is an *abstract simplicial complex* and this is true in general (see Theorem 5.6). Moreover, (*iii*) of Theorem 5.15 tells us that $MINP(\mathcal{I})$ behaves as the independent sets of a matroid, whereas the elements of $MXMN(\mathcal{I}) := \max MINP(\mathcal{I})$ are slated to be the *bases* of this matroid (see [47] for further notions on matroid theory). Nevertheless, in general $MINP(\mathcal{I})$ fails to be a matroid. In our specific example, however, this happens. Hence, if we substitute the attribute Color with RoadHolding, the information induced by {*Speed*, *Color*} and {*Speed*, *RoadHolding*} is the same and, furthermore, continue to be minimal in the indistinguishabillity class of *Att*. In general, to say that $MINP(\mathcal{I})$ is a matroid means that whenever we consider two different minimal partitioners of $\mathcal{I}$, the exchange of an attribute of the former with an attribute of the latter, gives rise to another subset of $MINP(\mathcal{I})$. In the particular context of $\mathcal{C}$, we also observe that applying the previous argument to reducts, we obtain again a reduct. At this point, a question arises: how $MINP(\mathcal{I})$ is related to $MAXP(\mathcal{I})$? The definition of the *max-min function* ψ and of the corresponding set operator $\hat{\psi}$ (see (67)) enables us to find the closed sets of the matroid. Usually, they do not coincide with $MAXP(\mathcal{I})$ but, in this example at issue, it is easy to verify that $\hat{\psi}(A) = M(A)$ for any $A \subseteq Att$. So $MAXP(\mathcal{C})$ coincides with the closed family of the matroid $MINP(\mathcal{C})$ and the map $A \mapsto M(A)$ is exactly its closure operator. It is then interesting to find sufficient conditions ensuring that $\hat{\psi}(A) = M(A)$ for any $A \subseteq Att$ and to verify when $MINP(\mathcal{I})$ is a matroid, though the latter is a difficult task going beyond the scopes of this paper. From an intuitive outlook, the aforementioned mathematical properties, i.e. to be an abstract simplicial complex or a matroid, are strictly linked to the variation of knowledge under deletion or exchange of attributes and to the optimization of the variables to take into account when we extract some knowledge from a table.

1.3. Micro and macro granular structures of the indistinguishability

Taking into account that the granular paradigm [45,46,49,50,62,69,70,74] is permeating much of the new interpretation of information structures (in rough set theory [35,36,48,69,72,75,80,81], generalized rough set theory [3–5,34], data mining [20–22,32,33,37,51,71], interactive computing [54,55], machine learning [73], computational cognitive science [29], formal concept analysis [26,68,77,78], mathematical morphology [57–61], graph theory [7–15,52], hypergraph theory [6,79], matroid theory [23,24,28,30,31,38–40,63–67,82], lattice theory [25]), it seems appropriate to discuss also our above models $\mathbb{I}(\mathcal{I})$ within the context of this new stimulating research paradigm. Therefore we will call the lattices $\mathbb{P}_{ind}(\mathcal{I})$ and $\mathbb{M}(\mathcal{I})$ the *macro* *granular order structures* (induced by the indiscernibility) of $\mathcal{I}$. The motivation of this terminology is that we are not interested in an operation of zoom-in [76] for any node of the aforementioned lattices, but simply to a global vision that takes into account only the order relation between the nodes. In this interpretation we don't see what happens inside each node. In contrast with this macro granular interpretation, we will call the lattices $\mathbb{S}(\mathcal{I})$ and $\mathbb{I}(\mathcal{I})$ the *micro granular order structures* (induced by the indiscernibility) of $\mathcal{I}$. In this case we are interested to both the order relation between the granules and the inner structure of each granule. For the lattice $\mathbb{S}(\mathcal{I})$, each granule is a sub-information table of $\mathcal{I}$, and this sub-table can be examined by means of all the classical investigation tools used in RST and granular computing GrC-IT. On the other hand, a granule of the lattice $\mathbb{I}(\mathcal{I})$ contains all the attribute subsets inducing the same indiscernibility partition, and these subsets have a natural poset structure with respect to the usual set inclusion. Therefore these posets can be studied with the usual techniques derived from the order theory. Obviously, in both the cases of $\mathbb{S}(\mathcal{I})$ and $\mathbb{I}(\mathcal{I})$ a relevant question is to understand *if*, and *how*, the micro granular behaviors of the nodes of these lattices *interact* with their corresponding macro order structure. To this regard, we can establish an imaginative analogy with the *solar system model*: a macro granular order structure correspond to the solar system, whereas any indistinguishability granule $[A]_{\approx}$ consists of a planet (the maximum member $M(A)$) and all its satellites (the members of $[A]_{\approx}\setminus\{M(A)\}$). In this paper we begin to investigate the interactions between the macro and micro granularity of the above models and we provide the following three types of results as a consequence of our study.

- *Micro granular results*: results concerning the properties of the indistinguishability granules. For example, these granules are union-closed families and blocks of a set partition of the power set $\mathcal{P}(Att)$.
- *Macro granular results*: results concerning the macro granular order structures. For example, these are complete lattices and the maximum partitioner family is intersection-closed.
- *Micro-macro granular results*: results that concern the possible interactions between the inner structure of any indistinguishability granule and the macro granular order structures. To this regard see for example the *global-local regularity* property that we discuss in Subsection 1.4.

1.4. Interaction between (object) indiscernibility and (attribute) indistinguishability

Let Ω be an arbitrary non-empty set. Usually, in mathematics the elements of Ω do not have a well-specified nature. In an abstract context, we can imagine the elements of Ω as a *potential attribute set Att* on some universe of objects *U*. Usually, in GrC-IT, the main emphasis is placed on the family of all the indiscernibility relations $\equiv_A$, that induce on the universe set a corresponding set of indiscernibility partitions $\pi(A)$. On the other hand, we can also consider the above equivalence relation $\approx$ on $\mathcal{P}(Att)$ as a type of relation that *induces indiscernibility* between attribute subsets. In other terms, we can use the universe set *U* simply as an intermediate tool to work on attribute subsets. From this perspective, it is convenient to give an appropriate name to the equivalence relation $\approx$, in order to study the formal properties induced from an information table by means of the attribute subset families described in the previous parts of this introductory section. We say therefore that $\approx$ is the *indistinguishability relation* on $\mathcal{I}$ and $[A]_{\approx}$ the *indistinguishability class* of *A*.

Then, two attribute subsets *A* and *A′* are indistinguishable when they induce the same set partition on the universe set *U*. The important fact is that the equivalence relation $\approx$ induces a very rich mathematical structure on the power set $\mathcal{P}(Att)$, and the richness of this structure is a consequence of the way in which the family of all the indiscernibility partitions $\pi(A)$ are interrelated with each other.

For example, one of the more interesting property that we obtain by studying the indistinguishability relation is what we call *global-local regularity*, briefly (GLR), and that can be expressed as follows:

(GLR) $M(A) \subsetneq M(A') \Rightarrow Y \nsubseteq X$ for any $X \in [A]_{\approx}$ and $Y \in [A']_{\approx}$.

The global-local regularity property tell us that the inclusion between the maximum elements of any two indistinguishability classes, preserves the same type of inclusion between any two members of these classes. In other terms, we can say that the *macro*-inclusion relations between the maximum partitioners of $\mathcal{I}$ have a direct influence also on the *micro*-inclusion relations between the attribute subsets of their corresponding indistinguishability classes.

Let us note that the global-local regularity is a property which has its basic foundation in the interesting fact that we can consider the order structure $\mathbb{I}(\mathcal{I})$ as a *lattice of posets*, i.e. a lattice whose nodes are the posets $([A]_{\approx}, \subseteq)$, when *A* runs in $\mathcal{P}(\mathcal{I})$. We will call these partially ordered sets *local indistinguishability posets*.

By starting from $\mathbb{I}(\mathcal{I})$, we want to analyze how the family of all minimal partitioners $MINP(\mathcal{I})$ behaves. In this paper we establish several properties of $MINP(\mathcal{I})$ and we will show that this attribute subset family plays a central role in determining new relations between classical and new notions of GrC-IT.

A basic relation that relates $MINP(\mathcal{I})$ to the relative reduct family $RED(A)$ of any attribute subset *A* of $\mathcal{I}$ is the following (Theorem 5.4):

$$MINP(\mathcal{I}) \supseteq \bigcup_{B \in RED(A)} \mathcal{P}(B). \tag{2}$$

In general, the complete determination of the reduct family $RED(\mathcal{I})$ of $\mathcal{I}$ is not an easy task, because the reducts of $\mathcal{I}$ are exactly the minimal transversals of the subset family $ESS(\mathcal{I})$ (see [9]) of all the attribute subsets *A* such that $\pi(Att \setminus A) \neq$

$\pi(Att)$ and $\pi(Att \setminus A') = \pi(Att)$ for any $A' \subsetneq A$ (the family $ESS(\mathcal{I})$ was introduced in [9] and its members are called *essential subsets* of $\mathcal{I}$).

Returning now to the above discussion concerning the notion of indistinguishability, the relevant aspect is that in a very general situation, where we have simply an information table $\mathcal{I}$, we can construct several attribute subset families, $MAXP(\mathcal{I})$, $MINP(\mathcal{I})$, $RED(\mathcal{I})$, $ESS(\mathcal{I})$ (and others that we introduce in the next sections), that have not trivial links between them and interesting formal properties. The basic tools for such an investigation are the above discussed subset families and their related structures that will be the main object of study in this paper.

1.5. Content of the paper

We now describe briefly the content of any section of the paper. In Section 2 we recall the basic notions and fix the notations used in the remaining part of the paper. Furthermore, we relativize the classical notions of discernibility matrix, essential and reduct to any attribute subset and associate to any knowledge representation system three hypergraphic structures, i.e. the discernibility hypergraph, the essential hypergraph and the reduct hypergraph. In Section 3 we investigate in details the structures of three macro granular order structures induced by the indiscernibility relation of $\mathcal{I}$. In order to fulfill this aim, we focus our attention to the indistinguishability relation definable on the power set of *Att* and see how it gives rise to a maximum partitioner lattice. Therefore, in particular, we analyze the behavior of the indiscernibility partitions induced by all the maximum partitioners. In Section 4 we study knowledge representation systems from a micro granular perspective, by introducing two new lattice structures isomorphic to the indiscernibility partition lattice. In particular, these mathematical objects are defined by means of the indistinguishability relation and allow us to associate two canonical families to any knowledge representation system: the maximum partitioners $MAXP(\mathcal{I})$ (just studied in Section 3) and the minimum partitioners $MINP(\mathcal{I})$. In Section 5, our intent consists in studying in details the family $MINP(\mathcal{I})$, providing all the main properties it satisfies, such as the global-local regularity (see Proposition 5.3) and the inheritance (see Theorem 5.6). Finally, we analyze the strict relation occurring between the minimum partitioners and the relative reducts of any attribute subset.

2. Basics and recalls

If X is a set, we denote by $\mathcal{P}(X)$ the power set of X and by $|X|$ the cardinality of X. If X is a finite set and $\mathcal{F}$ is a family of subsets of X having all the same cardinality, we say that $\mathcal{F}$ has *uniform cardinality*, and we denote by $||\mathcal{F}||$ the common cardinality of all members of $\mathcal{F}$. A pair $(X, \mathcal{F})$ is an *abstract simplicial complex* (see [27]) if $\emptyset \in \mathcal{F}$ and whenever $Y \in \mathcal{F}$ and $Z \subseteq Y$, then $Z \in \mathcal{F}$.

A *hypergraph* (see [1]) is a pair $H = (V(H), E(H))$, where $V(H) = \{v_1, \dots, v_n\}$ is an arbitrary finite set and $E(H) = \{Y_1, \dots, Y_m\}$ is a non-empty family of subsets $Y_1, \dots, Y_m$ of $V(H)$. The elements $v_1, \dots, v_n$ are called vertices of H and the subsets $Y_1, \dots, Y_m$ are called hyperedges of H. A *hypergraph on* $V(H)$ is a hypergraph having $V(H)$ as vertex set.

2.1. Posets

A *partially ordered set* (abbreviated *poset*) is a pair $P = (X, \leq)$, where X is a set and $\leq$ is a binary relation on X that is reflexive, antisymmetric and transitive. If $P = (X, \leq)$ is a partially ordered set and $x, y \in X$, we also write $x < y$ if $x \leq y$ and $x \neq y$. If x, y are two distinct elements of X, we say that y *covers* x, denoted by $x \lessdot y$ (or, equivalently, by $y \gtrdot x$), if $x \leq y$ and there is no element $z \in X$ such that $x < z < y$. If $x \in X$ we set $[x|P\uparrow] := \{z \in X\colon x \lessdot z\}$ and $[x|P\downarrow] := \{y \in X\colon y \lessdot x\}$. We call the elements of $[x|P\uparrow]$ *covers* of x and the elements of $[x|P\downarrow]$ *co-covers* of x. By the covering relation, we can provide a graphical representation of the poset, the so called *Hasse diagram* of P (see [2]): draw a small circle for any element of P and a segment connecting x to y whenever x covers y. An element $x \in X$ is called *minimal* in P if $z \leq x$ implies $z = x$, and in a similar way one defines a *maximal* element in P. If there is an element $\hat{0}_X \in X$ such that $\hat{0}_X \leq x$ then $\hat{0}_X$ is unique and it is called the *minimum* of P. Analogously the *maximum* of P, usually denoted by $\hat{1}_X$ is defined (if it exists). We call *upper bound* of a subset X of P an element $y \in P$ such that $x \leq y$ for any $x \in X$. We call *least upper bound* the minimum of all the upper bounds. The notions of *lower bound* and *greatest lower bound* are dual. We call *lattice* a poset any two of whose elements has both the least upper bound that the greatest lower bound. A lattice is *complete* when each of its subsets X has a least upper bound and greatest lower bound in the lattice.

A poset $P = (X_1, \leq_1)$ is said *isomorphic* to another poset $P_2 = (X_2, \leq_2)$ if there exists a bijective map $\phi\colon X_1 \to X_2$ such that $x \leq_1 y \Leftrightarrow \phi(x) \leq_2 \phi(y)$, for all $x, y \in X_1$. The *dual poset* of P is the poset $P^* := (X, \leq^*)$, where $\leq^*$ is the partial order on X defined by $x \leq^* y\colon \Leftrightarrow y \leq x$, for all $x, y \in X$. A poset P is called *self-dual* if P is isomorphic to its dual poset P^*. For further details on posets and lattices, see [2].

2.2. Set partitions

We now recall the basic notions of set partitions (for further details, see [56]). If X is an arbitrary set and π is a set partition on X, we usually denote by $\{B_i\colon i \in I\}$ the block family of π. If $x \in X$, we denote by $\pi(x)$ the block of π which contains the element x. If $Y \subseteq X$ and $Y \subseteq B_i$, for some index $i \in I$, we say that Y is a *sub-block* of π and we write $Y \preccurlyeq \pi$. When

X is finite we use the standard notation $\pi := B_1|\ldots|B_{|\pi|}$, where $|\pi|$ is the number of distinct blocks of π. We denote by $\pi(X)$ the set of all set-partitions of X. It is well known that on the set $\pi(X)$ we can consider a partial order $\preceq$ defined as follows: if $\pi, \pi' \in \pi(X)$, then

$$\pi \preceq \pi' :\Longleftrightarrow (\forall B \in \pi)\,(\exists B' \in \pi') \,:\, B \subseteq B' \tag{3}$$

A useful and equivalent characterization of the partial order given in (3) is the following:

$$\pi \preceq \pi' \Longleftrightarrow (\forall x \in X)\,(\pi(x) \subseteq \pi'(x)) \tag{4}$$

We will write $\pi \prec \pi'$ when $\pi \preceq \pi'$ and $\pi \neq \pi'$.

The pair $\mathbb{P}(X) = (\pi(X), \preceq)$ is a complete lattice which is called *partition lattice* of the set X. We now recall the basic facts about the meet and the join of this lattice.

Let $\pi_1 = A_1|\ldots|A_m$ and $\pi_2 = B_1|\ldots|B_n$ be two partitions on the same finite universe X, i.e., $\pi_1, \pi_2 \in \pi(X)$, we firstly set

$$\mathcal{S}_{\pi_1,\pi_2} := \{C \subseteq X : \text{ if } x \in C, \text{ then } \pi_1(x) \subseteq C \text{ and } \pi_2(x) \subseteq C\}$$

Then the join of π_1 and π_2, denoted by $\pi_1 \wedge \pi_2$, is the set partition of X whose block set is given by

$$\pi_1 \wedge \pi_2 := \{A_i \cap B_j : i = 1, \ldots m;\ j = 1, \ldots, n\}. \tag{5}$$

On the other hand, the more simple way to describe the join of π_1 and π_2, denoted by $\pi_1 \vee \pi_2$, is the following:

$$\pi_1 \vee \pi_2 := C_1|\ldots|C_k, \tag{6}$$

where $C_1, \ldots, C_k$ are the minimal elements of the set family $\mathcal{S}_{\pi_1,\pi_2}$ with respect to the inclusion.

Example 2.1. Let us consider $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and let $\pi_1 = \{x_1, x_2\}|\{x_3\}|\{x_4, x_5\}|\{x_6\}$ and $\pi_2 = \{x_1, x_3\}|\{x_2\}|\{x_4\}|\{x_5\}|\{x_6\}$ be two set partitions of X. Then $\pi_1 \wedge \pi_2$ is the partition

$$\pi_1 \wedge \pi_2 = \{x_1\}|\{x_2\}|\{x_3\}|\{x_4\}|\{x_5\}|\{x_6\}.$$

The family $\mathcal{S}_{\pi_1,\pi_2}$ is equal to:

$$\mathcal{S}_{\pi_1,\pi_2} = \{\{x_1, x_2, x_3\}, \{x_1, x_2, x_3, x_4, x_5\}, \{x_1, x_2, x_3, x_6\}, \{x_4, x_5\}, \{x_4, x_5, x_6\}, \{x_6\}\}.$$

Then the meet of π_1 and π_2 is the partition

$$\pi_1 \vee \pi_2 = \{x_1, x_2, x_3\}|\{x_4, x_5\}|\{x_6\}.$$

2.3. Knowledge representation system and indiscernibility

In this section we provide a generalization to the classical notion of information table theory (see [41–44] for further details).

Definition 2.2. A *knowledge representation system* is a structure $\mathcal{I} = \langle U, Att, Val- \rangle$, where U is a non-empty set called *universe set* and $Att = \{a_i : i \in I\}$ is a set of attributes a_i: $U \to Val$. We call the map F: $U \times Att \to Val$, defined by

$$F(u, a_i) := a_i(u), \tag{7}$$

the *information map* of $\mathcal{I}$. If Val is a set with two elements (usually denoted by 0 and 1), we say that $\mathcal{I}$ is a *Boolean knowledge representation system*. We call the elements of U *objects* of $\mathcal{I}$, the elements of Att *attributes* of $\mathcal{I}$ and those of Val *values* of $\mathcal{I}$. When both the sets $U = \{u_1, \ldots, u_m\}$ and $Att = \{a_1, \ldots, a_n\}$ are finite, we denote by $T[\mathcal{I}]$ the $m \times n$ rectangular table having on the ith row the element u_i, on the jth column the attribute a_j and the value $a_j(u_i)$ in the place (i, j). We call $T[\mathcal{I}]$ the *information table* of $\mathcal{I}$. However, in the finite case we identify $\mathcal{I}$ with $T[\mathcal{I}]$ and we call $\mathcal{I}$ *information table*.

Remark 2.3. It is clear that if we start with any set U_i, Att and any map F: $U \times Att \to Val$ then we can set

$$a_i(u) := F(u, a_i) \tag{8}$$

Therefore a knowledge representation system can be equivalently described as a structure

$$\mathcal{I} := \langle U, Att, F, Val \rangle, \tag{9}$$

where U, Att and Val are given sets and F is a map from $U \times Att \to Val$. We will use the form (9) to describe a knowledge representation system.

In what follows we assume that $\mathcal{I} = \langle U, Att, F, Val- \rangle$ is a given knowledge representation system.

Definition 2.4. If $A \subseteq Att$ we consider the following binary relation $\equiv_A$ on the universe set U: if $u, u' \in U$ then

$$u \equiv_A u' :\Longleftrightarrow F(u, a) = F(u', a), \forall a \in A. \tag{10}$$

The binary relation $\equiv_A$ is an equivalence relation on U that is called *A-indiscernibility relation* on $\mathcal{I}$ [41]. If $u \in U$, we denote by $[u]_A$ the equivalence class of u with respect to $\equiv_A$. We also set

$$\pi(A) := \{[u]_A : u \in U\}. \tag{11}$$

We call $\pi(A)$ the *A-indiscernibility partition* of the knowledge representation system $\mathcal{I}$. If $B \subseteq U$ is such that $B = [u]_A$, for some $u \in U$, we say that B is an *A-indiscernibility block* of $\mathcal{I}$. We also set

$$\Pi_{ind}(\mathcal{I}) := \{\pi(A) : A \subseteq Att\}. \tag{12}$$

2.4. Indiscernibility structures of $\mathcal{I}$

In this subsection we introduce some set operator notations that generalize the basic notion of *discernibility matrix* [53] for a knowledge representation system.

Let $Z \subseteq U$ and $A \subseteq Att$. Then we can consider the map $\Delta_{Z,A} : Z \times Z \to \mathcal{P}(A)$ defined by

$$\Delta_{Z,A}(u, u') := \{a \in A : F(u, a) \neq F(u', a)\}, \tag{13}$$

for any $u, u' \in Z$. In particular, we set $\Delta := \Delta_{U, Att}$ and $\Delta_A := \Delta_{U, A}$. Therefore we have

$$\Delta_{Z,A}(u, u') = \Delta_{Z,Att}(u, u') \cap A, \tag{14}$$

for any $u, u' \in Z$.

Definition 2.5. We say that the hypergraph with vertex set A and hyperedge set

$$DISC(Z, A) := \{B \subseteq A : \exists u, u' \in Z \, s.t. \, F(u, b) = F(u', b) \ \forall b \in B\},$$

is the (Z, A)-*discernibility hypergraph*. In particular, we set $DISC(\mathcal{I}) := DISC(U, Att)$ and $DISC(A) := DISC(U, A)$, and we call $\mathfrak{D}(\mathcal{I}) := (Att, DISC(\mathcal{I}))$ the *discernibility hypergraph* of $\mathcal{I}$.

Remark 2.6.

(i) Let us note that

$$DISC(Z, A) = \{\Delta_{Z,A}(u, u') : u, u' \in Z \text{ and } \Delta_{Z,A}(u, u') \neq \emptyset\}.$$

(ii) If both the sets $Z = \{u_1, \dots, u_m\}$ and $A = \{a_1, \dots, a_n\}$ are finite, the $m \times m$ table having the objects $u_1, \dots, u_m$ on its rows and on its columns and the attribute subset $\Delta_{Z,A}(u_i, u_j)$ in the (u_i, u_j)-entry is the usual discernibility matrix $\Delta[\mathcal{I}]$ of $\mathcal{I}$ [53].

The following result relates the subsets $\Delta_{Z,A}(u, u')$ to the indiscernibility relation.

Proposition 2.7. *Let $D \subseteq A$ and $u, u' \in Z$. Then:*

(i) $D = \Delta_{Z,A}(u, u') \Longrightarrow u \equiv_{A \setminus D} u'$;
(ii) $u \equiv_{A \setminus D} u' \Rightarrow \Delta_{Z,A}(u, u') \subseteq D$;
(iii) *Let $C \subseteq A$. Then* $\Delta_{Z,A}(u, u') \cap C = \emptyset \Longleftrightarrow u \equiv_C u'$.

Proof.

(i) Let $D = \Delta_{Z,A}(u, u')$ and let $a \in A \setminus D$. By definition of $\Delta_{Z,A}(u, u')$ it holds $F(u, a) = F(u', a)$. Thus $u \equiv_{A \setminus D} u'$;
(ii) Let $u \equiv_{A \setminus D} u'$ and let $a \in A \setminus D$. By definition of the relation $\equiv_{A \setminus D}$ we have $F(u, a) = F(u', a)$, so $a \notin \Delta_{Z,A}(u, u')$. It follows $A \setminus D \subseteq A \setminus \Delta_{Z,A}(u, u')$ and equivalently $\Delta_{Z,A}(u, u') \subseteq D$;
(iii) Let $C \subseteq A$. Let us assume that $\Delta_{Z,A}(u, u') \cap C = \emptyset$ and let $c \in C$. Then $c \notin \Delta_{Z,A}(u, u')$, so $F(u, c) = F(u', c)$. This proves that $u \equiv_C u'$. Let us suppose now that $u \equiv_C u'$. Then, by definition of $\equiv_C$, it follows that $\Delta_{Z,A}(u, u') \cap C = \emptyset$. The proposition is thus proved. □

In GrC-IT there are two well studied investigation notions: the *core* and the *reducts* of a knowledge representation system [41].

Definition 2.8 [41]**.** Let $A \subseteq Att$. An attribute $c \in A$ is said *indispensable* if $\pi(A) \neq \pi(A \setminus \{c\})$. The subset of all indispensable attributes of A is called the *core* of A and it is denoted by $CORE(A)$. In particular, we set $CORE(\mathcal{I}) := CORE(Att)$.

Definition 2.9 [41]**.** A subset $C \subseteq A$ is called a *reduct* of A if:

(i) $\pi(A) = \pi(C)$;
(ii) $\pi(A) \neq \pi(D)$ for all $D \subsetneq C$.

We denote by $RED(A)$ the family of all reducts of A and we set $RED(\mathcal{I}) := RED(Att)$. The members of $RED(\mathcal{I})$ are usually called *reducts* of $\mathcal{I}$. We call the hypergraph

$$\mathfrak{R}(A) := (A, RED(A))$$

reduct hypergraph of A and the hypergraph $\mathfrak{R}(\mathcal{I}) := \mathfrak{R}(Att)$ *reduct hypergraph* of $\mathcal{I}$.

Core and reducts are linked each other by the following result.

Table 3
Mobile phone information table.

$\mathcal{T}$	*RAM*	*Memory*	*Color*	*BatteryLife*
u_1	Insufficient	Small	Black	Very Long
u_2	Sufficient	Middle	Blue	Short
u_3	Insufficient	Small	Black	Long
u_4	Sufficient	Big	Grey	Short
u_5	Excellent	Middle	Blue	Long

Proposition 2.10 [42]**.** $CORE(\mathcal{I}) := \bigcap\{C : C \in RED(\mathcal{I})\}$.

The notion of relative core has been generalized in [9] in the following way.

Definition 2.11. Let $A \subseteq Att$. We say that a subset $C \subseteq A$ is *A-essential* if:

(i) $\pi(A \setminus C) \neq \pi(A)$;
(ii) $\pi(A \setminus D) = \pi(A)$ for all $D \subsetneqq C$.

We denote by $ESS(A)$ the family of all A-essential subsets of A and we set $ESS(\mathcal{I}) := ESS(Att)$. We call the members of $ESS(\mathcal{I})$ *essentials* of $\mathcal{I}$. We call the hypergraph

$$\mathfrak{E}(A) := (A, ESS(A))$$

essential hypergraph of A and *essential hypergraph* of $\mathcal{I}$, the hypergraph $\mathfrak{E}(\mathcal{I}) := \mathfrak{E}(Att)$.

In the next result we characterize the A-essentials as the minimal elements of the discernibility hypergraph.

Theorem 2.12. $ESS(A) = \min(DISC(A))$.

Proof. The proof of Theorem 4.11 in [9] can be easily adapted for the more general statement of this theorem. □

The classical notion of *transversal* for a finite hypergraph [1] can be also given for an arbitrary hypergraph.

Definition 2.13. Let H be a hypergraph with vertex set $V(H)$ and hyperedge set $E(H)$. We say that a subset $Y \subseteq V(H)$ is a *transversal* of H if $Y \cap A \neq \emptyset$ for each non-empty hyperedge $A \in E(H)$. We say that a transversal A of H is *minimal* if no proper subset of A is a transversal of H. We denote by $Tr(H)$ the family of all minimal transversals of H. We call the hypergraph $H^{tr} := (V(H), Tr(H))$ *transversal* hypergraph of H.

In literature the *hypergraph transversal problem* for a *finite* hypergraph H is the problem of generating all the elements of $Tr(H)$. In general, this is an important mathematical problem which has many applications in mathematics and in computer science [18].

Remark 2.14. It is clear that $Tr(DISC(A)) = Tr(ESS(A))$ by Theorem 2.12.

The next result shows that the reducts of A are exactly the minimal transversals of the essential hypergraph $ESS(A)$.

Theorem 2.15. *Let $B \subseteq A$. Then:*

(i) $\pi(A) = \pi(B)$ *if and only if* $B \in Tr(DISC(A))$.
(i) $RED(A) = Tr(DISC(A)) = Tr(ESS(A))$.

Proof. The proof of Theorem 4.20 in [9] can be easily adapted for the more general statement of this theorem. □

The three hypergraphs $\mathfrak{D}(\mathcal{I})$, $\mathfrak{R}(\mathcal{I})$ and $\mathfrak{E}(\mathcal{I})$ are interrelated between them by means of Theorem 2.12 and Theorem 2.15. These structures are all defined by using the classical indiscernibility relations $\equiv_A$, for any $A \subseteq Att$, therefore we use the following terminology.

Definition 2.16. We call $\mathfrak{D}(\mathcal{I})$, $\mathfrak{R}(\mathcal{I})$ and $\mathfrak{E}(\mathcal{I})$ *indiscernibility hypergraphic structures* of $\mathcal{I}$.

In order to provide an appropriate interpretation for all results that we establish in this paper, we introduce the following example.

Example 2.17. Suppose that an individual has to buy a new mobile phone. He is interested in the following attributes: RAM, memory, color and battery life and suppose he can choose among five models u_1, u_2, u_3, u_4, u_5. Hence, we can modelize the situation occurring by means of the information table $\mathcal{T} = \langle U, Att, F, Val- \rangle$ given in Table 3, where $U := \{u_1, u_2, u_3, u_4, u_5\}$ is the mobile set and $Att := \{RAM, Memory, Color, BatteryLife\}$. In the following, we indicate by R the RAM, by M the memory, by C the color and by B the battery life.

Now, we represent in Table 4, the discernibility matrix $\Delta[\mathcal{T}]$ of the knowledge representation system $\mathcal{T}$.

Table 4
Discernibility matrix $\Delta[\mathcal{T}]$.

	u_1	u_2	u_3	u_4	u_5
u_1	∅	RMCB	B	RMCB	RMCB
u_2	*	∅	RMCB	MC	RB
u_3	*	*	∅	RMCB	RMC
u_4	*	*	*	∅	RMCB
u_5	*	*	*	*	∅

Table 5
Discernibility matrix $\Delta_A[\mathcal{T}]$.

	u_1	u_2	u_3	u_4	u_5
u_1	∅	RMC	∅	RMC	RMC
u_2	*	∅	RMC	MC	R
u_3	*	*	∅	RMC	RMC
u_4	*	*	*	∅	RMC
u_5	*	*	*	*	∅

Hence, we have that

$$DISC(\mathcal{T}) = \{B, RB, MC, RMC, RMCB\}.$$

By Theorem 2.12, we have

$$ESS(\mathcal{T}) = \{B, MC\}$$

while, by Theorem 2.15, we have

$$RED(\mathcal{T}) = \{BC, MB\} \quad \text{and} \quad CORE(\mathcal{T}) = \{B\}.$$

Let us now fix $A = \{R, M, C\}$. In Fig. 5, we represent the discernibility matrix $\Delta_A[\mathcal{T}]$ restricted to A.
Hence, we have that

$$DISC(A) = \{R, MC, RMC\}.$$

By Theorem 2.12, we have

$$ESS(A) = \{R, MC\}$$

while, by Theorem 2.15, we have

$$RED(A) = \{RC, RM\} \quad \text{and} \quad CORE(A) = \{R\}.$$

This means that the attribute subset $\{R, C\}$ and $\{R, M\}$ provide the same information given by $\{R, M, C\}$. To be more specific, if we take $\{R, M\}$, we can avoid to consider the attribute Color in order to distinguish two models of mobile phones, being the other attributes enough. Furthermore, if we restrict our global knowledge on the attributes of A, then R provides a fundamental information in order to deduce which models have the same characteristics with respect to A. Finally, note that $\pi(\{R, C\}) = \pi(\{R, M, C\})$ but $\pi(\{M, C\}) \neq \pi(\{R, M, C\})$: roughly speaking, we must delete both M and C in order to lose some information or, equivalently, to make more difficult the choice of a mobile phone.

3. The two macro granular order structures of $\mathcal{I}$

In this section we investigate the three macro granular order structures induced by the indiscernibility relations of $\mathcal{I}$. We set

$$\mathbb{P}(\mathcal{I}) := (\pi(U), \preceq) \tag{15}$$

and

$$\Pi_{ind}(\mathcal{I}) := \{\pi(A) : A \subseteq Att\}. \tag{16}$$

Since $\Pi_{ind}(\mathcal{I})$ is a subset of $\pi(U)$, we can consider on $\Pi_{ind}(\mathcal{I})$ the induced partial order $\preceq$ from the previous partition lattice of $\mathcal{I}$. We set therefore

$$\mathbb{P}_{ind}(\mathcal{I}) := (\Pi_{ind}(\mathcal{I}), \preceq) \tag{17}$$

In this way $\mathbb{P}_{ind}(\mathcal{I})$ becomes a sub-poset of $\mathbb{P}(\mathcal{I})$. According to Yao [76], the order structure $\mathbb{P}_{ind}(\mathcal{I})$ “can be used to develop a partition model of databases”.

Definition 3.1. We call the partially ordered set $\mathbb{P}_{ind}(\mathcal{I})$ *indiscernibility partition poset* of the knowledge representation system $\mathcal{I}$.

Remark 3.2. (*i*): Let us note here that $\hat{1}_{\mathbb{P}_{ind}(\mathcal{I})}$ always coincides with $\hat{1}_{\mathbb{P}(\mathcal{I})}$ since $\pi(\emptyset) = U = \hat{1}_{\mathcal{I}}$, whereas (in general) $\hat{0}_{\mathbb{P}_{ind}(\mathcal{I})}$ can be different with respect to the minimum $\hat{0}_{\mathbb{P}(\mathcal{I})}$. Thus, in general, $\mathbb{P}_{ind}(\mathcal{I})$ is not a sub-lattice of $\mathbb{P}(\mathcal{I})$.

(*ii*): Theorem 6.10 of [16] provides a criterion to establish whether a given partition belongs or not to $\mathbb{P}_{ind}(\mathcal{I})$.

We now introduce an equivalence relation on attribute subsets of a knowledge representation system and show how it is related to the partition lattice of a knowledge representation system through the notion of maximum partitioner.

If A and B are two attribute subsets of $\mathcal{I}$ we set

$$A \approx B :\Longleftrightarrow \pi(A) = \pi(B). \tag{18}$$

Then the binary relation $\approx$ is an equivalence relation on $\mathcal{P}(Att)$, and we denote by $\pi_{\approx}(\mathcal{I})$ the set partition on $\mathcal{P}(Att)$ induced by $\approx$. We also set

$$[A]_{\approx} := \{B \subseteq Att : A \approx B\}. \tag{19}$$

Definition 3.3. We call:

- the equivalence relation $\approx$ *indistinguishability relation* of $\mathcal{I}$;
- the equivalence class $[A]_{\approx}$ *indistinguishability class*, or also *indistinguishability granule*, of $A \subseteq Att$.

The following result provides several indispensable characterizations concerning the indistinguishability classes for any knowledge representation system.

Definition 3.4.

- If $A \subseteq Att$, we call the attribute subset $M(A) = \bigcup\{B : B \in [A]_{\approx}\}$ the *co-maximal* of A in $\mathcal{I}$.
- If $\pi \in \Pi_{ind}(\mathcal{I})$ is such that $\pi = \pi(A)$, for some $A \subseteq Att$, we say that $M(A)$ is the *maximum partitioner* of π, and we set $Max(\pi) := M(A)$. Therefore, with this notation, we have that $M(A) = Max(\pi(A))$.

Proposition 3.5 [16]**.** *Let $A \subseteq Att$. Then:*

(i) *$M(A)$ is the unique subset in $[A]_{\approx}$ such that $B \subseteq M(A)$, for all $B \in [A]_{\approx}$, .*

(ii) *If $A' \subseteq Att$ then*

$$\pi(A) \preceq \pi(A') \Longleftrightarrow M(A') \subseteq M(A) \tag{20}$$

and

$$\pi(A) \prec \pi(A') \Longleftrightarrow M(A') \subsetneq M(A). \tag{21}$$

(iii) *Let $A' \subseteq Att$ such that $A \subseteq A'$. Then $M(A) \subseteq M(A')$.*

We provide another characterization for the maximum partitioners that will be useful later.

Proposition 3.6. *Let $A \subseteq Att$. Then*

$$M(A) = \bigcap\{B : B \in MAXP(\mathcal{I}),\ A \subseteq B\}. \tag{22}$$

Proof. By (*iii*) of Proposition 3.5, it's clear that $M(A) \subseteq B$ for any maximum partitioner $B \supseteq A$. Vice versa, the intersection in the right side of (22) is contained in $M(A)$. Hence (22) holds. □

Definition 3.7. The collection of all maximum partitioners is denoted as

$$MAXP(\mathcal{I}) := \{M(A) : A \subseteq Att\} = \{Max(\pi) : \pi \in \Pi_{ind}(\mathcal{I})\}, \tag{23}$$

and we also introduce the poset

$$\mathbb{M}(\mathcal{I}) := (MAXP(\mathcal{I}), \subseteq^*), \tag{24}$$

where $\subseteq^*$ is the dual inclusion order.

The following result has been implicitly given in the proof of Theorem 3.6 of [16].

Proposition 3.8 [16]**.** *The posets $\mathbb{M}(\mathcal{I})$ and $\mathbb{P}_{ind}(\mathcal{I})$ are two complete lattices isomorphic between them.*

By virtue of the previous result, it is worthwhile to introduce the following terminology.

Definition 3.9. We call $\mathbb{P}_{ind}(\mathcal{I})$ the *indiscernibility partition lattice* of $\mathcal{I}$ and $\mathbb{M}(\mathcal{I})$ the *maximum partitioner lattice* of $\mathcal{I}$.

The general form of the join and of the meet of a family of maximum partitioners is now given.

Proposition 3.10. *Let $\{A_j : j \in J\} \subseteq MAXP(\mathcal{I})$ a family of maximum partitioners of $\mathcal{I}$. Then:*

(i) *the join of $\{A_j : j \in J\}$ in $\mathbb{M}(\mathcal{I})$ is $A := \bigcap_{j \in J} A_j$.*

(ii) *the meet of $\{A_j : j \in J\}$ in $\mathbb{M}(\mathcal{I})$ is $B := M(\bigcup_{j \in J} A_j)$.*

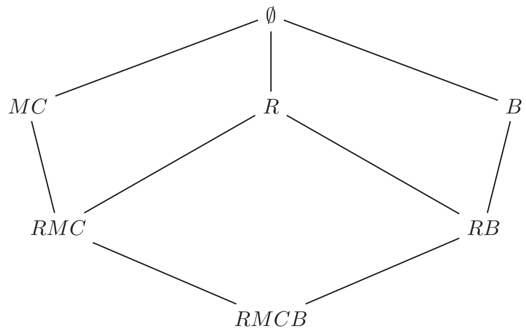

Fig. 7. Diagram of $\mathbb{M}(\mathcal{T})$.

Corollary 3.11. *The join and the meet in the lattice* $\mathbb{P}_{ind}(\mathcal{I})$ *are obtained as follows. Let* $\{\pi_j : j \in J\} \subseteq \Pi_{ind}(\mathcal{I})$ *and set* $A_j := f(\pi_j) = Max(\pi_j)$*, for all* $j \in J$*. Then:*

(i) *the join of* $\{\pi_j : j \in J\}$ *in* $\mathbb{P}_{ind}(\mathcal{I})$ *is the partition* $\pi(A)$*, where* $A := \bigcap_{j\in J} A_j$*.*
(i) *the meet of* $\{\pi_j : j \in J\}$ *in* $\mathbb{P}_{ind}(\mathcal{I})$ *is the partition* $\pi(B)$*, where* $B := M(\bigcup_{j\in J} A_j)$*.*

In the next example, we represent the maximum partitioner lattice of a specific information table.

Example 3.12. In reference to Example 2.17, we have:

$$MAXP(\mathcal{T}) = \{\emptyset, R, B, MC, RB, RMC, RMCB\}.$$

In Fig. 7, we represent its maximum partitioner lattice $\mathbb{M}(\mathcal{T})$.

In the next result we provide a first basic link between relative reducts and minimal members of the equivalence class $[A]_{\approx}$, for $A \subseteq Att$.

Proposition 3.13. *Let* $A \subseteq Att$*. Then*

$$RED(A) = \{X : X \in \min([A]_{\approx}),\ X \subseteq A\}. \tag{25}$$

Proof. Let $C \in RED(A)$. Clearly, $C \subseteq A$. By Definition 2.9 we have then $\pi(C) = \pi(A)$, therefore $C \in [A]_{\approx}$. On the other hand, if $D \in [A]_{\approx}$ is such that $D \subsetneq C$, the identity $\pi(D) = \pi(A)$ is in contrast with the condition (*ii*) of Definition 2.9. Hence C is necessarily minimal in the subset family $[A]_{\approx}$. Hence $RED(A) \subseteq \{X: X \in \min([A]_{\approx}), X \subseteq A\}$. Conversely, let $C \in \min([A]_{\approx})$ such that $C \subseteq A$. Then $\pi(C) = \pi(A)$. Let now $C' \subsetneq C$. Since C is minimal in $[A]_{\approx}$, it follows that $C' \notin [A]_{\approx}$, so that $\pi(C') \neq \pi(A)$. This shows that the attribute subset C satisfies both the conditions of Definition 2.9, i.e. $C \in RED(A)$. □

For the maximum partitioners we have the following stronger results.

Theorem 3.14. *Let* $A \in MAXP(\mathcal{I})$*. Then:*

$$RED(A) = \min([A]_{\approx}). \tag{26}$$

Proof. Since A is a maximum partitioner, the subset family $\{X: X \in \min([A]_{\approx}), X \subseteq A\}$ in (25) coincides with $\min([A]_{\approx})$. The thesis follows then by Proposition 3.13. □

Corollary 3.15. *We have:*

$$RED(\mathcal{I}) = \min([Att]_{\approx}). \tag{27}$$

Proof. It is a direct consequence of Theorem 3.14 when we take $A = Att$. □

Example 3.16. In reference to the information table $\mathcal{T}$ given in Example 2.17, we have

$$RED(RCB) = \{BC\}$$

and

$$RED(RMC) = \{RM, RC\}.$$

We now see how to relativize the maximum partitioner lattice to any attribute subset. If $A \subseteq Att$ we set now

$$MAXP(A) := \{B \cap A : B \in MAXP(\mathcal{I})\}, \tag{28}$$

so that $MAXP(\mathcal{I}) = MAXP(Att)$.

Definition 3.17. We call any element of *MAXP*(*A*) *A-relative maximum partitioner* of $\mathcal{I}$.

We now give a characterization for the family $MAXP(A)$.

Proposition 3.18. *Let $A \in MAXP(\mathcal{I})$. Then*

$$MAXP(A) = \{B \in MAXP(\mathcal{I}) : B \subseteq A\} = \{B \in MAXP(\mathcal{I}) : A \subseteq^* B\}.$$

Proof. Let $B \in MAXP(A)$. Then there exists $C \in MAXP(\mathcal{I})$ such that $B = A \cap C$. Therefore, $B \subseteq A$ and $B \in MAXP(\mathcal{I})$ because is set-theoretic intersection closed. On the other hand, let $B \in MAXP(\mathcal{I})$ and $B \subseteq A$. Then $B = A \cap B \in MAXP(A)$. □

Example 3.19. In reference to the information table $\mathcal{T}$ given in Example 2.17, we have:

$$MAXP(RCB) = \{\emptyset, R, B, C, RC, RB, RCB\}$$

and

$$MAXP(RMC) = \{\emptyset, R, B, MC, RMC\}.$$

In the next result we show that we can use the structure $MAXP(A)$ to provide an alternative characterization of the relative core of A.

Theorem 3.20. *Let $A \subseteq Att$. Then*

(i) $CORE(A) = \{a \in A : A \setminus \{a\} \in MAXP(A)\}$.
(ii) $CORE(M(A)) \subseteq A$.

Proof. (i): Let $a \in CORE(A)$. We must show that $A \setminus \{a\} \in MAXP(A)$, i.e. there exists $B \in MAXP(\mathcal{I})$ such that $A \setminus \{a\} = A \cap B$. Let us assume by absurd that $A \setminus \{a\} \neq A \cap B$ for any $B \in MAXP(\mathcal{I})$. Since $\pi(A) \neq \pi(A \setminus \{a\})$, then $A \setminus \{a\} \not\approx A$. Let B be the maximum partitioner of $[A \setminus \{a\}]_{\approx}$. Clearly, $A \setminus \{a\} \subseteq B$ and we have $A \setminus \{a\} \subseteq A \cap B$ therefore, by our assumption, it follows that $A \setminus \{a\} \subsetneq A \cap B$. Hence $A \cap B = A$, i.e. $A \subseteq B$. It follows that $\pi(B) = \pi(A \setminus \{a\}) \preceq \pi(A) \preceq \pi(A \setminus \{a\}) = \pi(B)$, i.e. $\pi(A) = \pi(B)$, that is equivalent to say that $A \approx B \approx A \setminus \{a\}$, absurd. Thus $A \setminus \{a\} \in MAXP(A)$.

Conversely, let $a \in A$ such that $A \setminus \{a\} \in MAXP(A)$. Hence, there exists $B \in MAXP(\mathcal{I})$ such that $A \setminus \{a\} = B \cap A$. If $B = A \setminus \{a\}$, then $A \setminus \{a\} \in MAXP(\mathcal{I})$. In this case, by maximality of $A \setminus \{a\}$, then $A \notin [A \setminus \{a\}]_{\approx}$, so $\pi(A) \neq \pi(A \setminus \{a\})$, therefore $a \in MAXP(A) \subseteq CORE(A)$ and the claim is proved. Otherwise, let $A \setminus \{a\} \subsetneq B$. Clearly, it results that $a \notin B$. Furthermore, it results $\pi(B) \preceq \pi(A \setminus \{a\})$. Suppose by contradiction that $\pi(A) = \pi(A \setminus \{a\})$. Then, we have $\pi(B) \preceq \pi(A)$, so

$$u \equiv_B u' \Rightarrow u \equiv_A u' \Rightarrow F(u, a) = F(u', a) \Rightarrow u \equiv_{B \cup \{a\}} u'.$$

In other terms, we have shown that $\pi(B) \preceq \pi(B \cup \{a\})$. Nevertheless, we also have $\pi(B \cup \{a\}) \preceq \pi(B)$, i.e. $\pi(B) = \pi(B \cup \{a\})$. This contradicts the maximality of B, absurd. Thus, $a \in CORE(A)$.

(ii): Let $a \in CORE(M(A))$ and suppose by contradiction that $a \notin A$. Therefore we have $A \subseteq M(A) \setminus \{a\} \subseteq M(A)$ and, hence, that $\pi(M(A)) \preceq \pi(M(A) \setminus \{a\}) \preceq \pi(A)$. Since $\pi(M(A)) = \pi(A)$, we conclude that $\pi(M(A) \setminus \{a\}) = \pi(M(A))$, contradicting the fact that $a \in CORE(A)$. □

Example 3.21. By referring to the information table given in Example 2.17, let us fix $A = \{R, M, C\}$. It is easy to see that the only $a \in A$ such that $A \setminus \{a\} \in MAXP(\mathcal{I})$ is R, hence we deduce that $CORE(A) = \{R\}$. Thus the attribute RAM is fundamental to discover the mobile phones with the same qualities whenever we would evaluate only RAM, Memory and Color.

By means of $MAXP(\mathcal{I})$ we can introduce two set operators that mimic respectively the extent and the intent operators of FCA [19]. Let $Z \subseteq U$ and $A \subseteq Att$. We set

$$\Gamma^{\uparrow}(Z) := \{a \in Att \ : \ \forall\, z, z' \in Z, F(z, a) = F(z', a)\}, \tag{29}$$

and

$$\Gamma^{\downarrow}(A) := \{z \in U \ : \ \forall\, a, a' \in A, F(z, a) = F(z, a')\}. \tag{30}$$

Let us observe that $\Gamma^{\uparrow} : \mathcal{P}(U) \to \mathcal{P}(Att)$ and $\Gamma^{\downarrow} : \mathcal{P}(Att) \to \mathcal{P}(U)$.

By analogy with the extent and the intent operators, if $Z \subseteq U$ and $A \subseteq Att$, we can define the following composition:

$$\Gamma^{\downarrow\uparrow}(A) := \Gamma^{\uparrow}(\Gamma^{\downarrow}(A)) \tag{31}$$

and, consequently, the following restricted maps:

$$\gamma^{\uparrow\downarrow} := \Gamma^{\uparrow\downarrow}_{|MAXP(\mathcal{I})} \tag{32}$$

We obtain then the following immediate result

Proposition 3.22. *$\gamma^{\uparrow\downarrow}$ is an inclusion-preserving set operator from $MAXP(\mathcal{I})$ to $MAXP(\mathcal{I})$.*

Proof. Straightforward. □

We now illustrate the interpretative meaning of the operator $\gamma^{\downarrow\uparrow}$ in a concrete case and we postpone a thorough investigation of its formal properties to forthcoming papers.

Table 6
The incidence matrix of the student hypergraph H.

	A_L	G_E	A_N	C_S
Adam	1	1	0	0
Bill	0	0	1	1
Carol	1	0	1	0
Dana	0	1	1	1
Eve	1	0	1	1

Example 3.23. Let Adam, Bill, Carol, Dana and Eve be 5 students chosen in a math class and suppose that we want to classify their attitudes according to the exams they passed. Let Algebra, Geometry, Analysis and Computer Science be the courses they attended during the first semester. Let $A_L = \{Adam, Carol, Eve\}$, $G_E = \{Adam, Dana\}$, $A_N = \{Bill, Eve, Dana, Carol\}$, $C_S = \{Bill, Eve, Dana\}$ be the sets of students that passed respectively Algebra, Geometry, Analysis and Computer Science. Then we can consider the hypergraph H having the five students as vertices and A_L, G_E, A_N, C_S as hyperedges. Let $\hat{H}$ be the Boolean information table given by the incidence matrix $Inc(H)$ (see [1]). In Fig. 6 we represent $Inc(H)$.

We have then the following indistinguishability classes of the information table $\hat{H}$:

$$[\emptyset]_\approx = \{\emptyset\},\ [A_L]_\approx = \{A_L\},\ [G_E]_\approx = \{G_E\},\ [A_N]_\approx = \{A_N\},\ [C_S]_\approx = \{C_S\},$$

$$[A_LA_N]_\approx = \{A_LA_N\},\ [A_LC_S]_\approx = \{A_LC_S\},\ [G_EA_N]_\approx = \{G_EA_N\},\ [A_NC_S]_\approx = \{A_NC_S\},$$

$$[A_LA_NC_S]_\approx = \{A_LA_NC_S\},\ [A_LG_EA_N]_\approx = \{A_LG_EA_N, A_LG_E\},\ [G_EA_NC_S]_\approx = \{G_EA_NC_S, G_EC_S\},$$

$$[A_LG_EA_NC_S]_\approx = \{A_LG_EA_NC_S, A_LG_EC_S\}.$$

Therefore the maximum partitioners of $\hat{H}$ are

$$\emptyset, A_L, G_E, A_N, C_S, A_LA_N, A_LC_S, G_EA_N, A_NC_S, A_LA_NC_S, A_LG_EA_N, G_EA_NC_S, A_LG_EA_NC_S.$$

We now interpret the role assumed by the maps $\Gamma^\uparrow$ and $\Gamma^\downarrow$. Let us fix the subset of students $Z = \{Bill, Dana\}$. Then, by (29), it is immediate to see that $\Gamma^\uparrow(Z) = \{A_L, A_N, C_S\}$. By referring to Table 6, we note that $\Gamma^\uparrow(Z)$ consists of all exams whose result had been the same for both Bill and Dana. Hence, in general, $\Gamma^\uparrow(Z)$ is the biggest subset of exams where all the students of Z achieved the same result.

On the other hand, let us fix a subject subset $A = \{A_L, A_N, C_S\}$. Then, by (30), it is immediate to see that $\Gamma^\downarrow(A) = \{Eve\}$. In fact, as we can see by taking the transpose of the matrix in Table 6, Eve is the only student who passed all the three exams of the set A. Hence, we deduce that $\Gamma^\downarrow(A)$ is the biggest subset of students that achieved the same result in all the exams of the set A.

We now provide an interpretation to $\gamma^{\downarrow\uparrow}$. Let $A = \{A_L, A_N\}$. Since $\Gamma^\downarrow(A) = \{Bill, Eve\}$ and $\Gamma^\uparrow(\{Bill, Eve\}) = \{A_L, G_E, A_N\}$, it follows immediately that $\gamma^{\downarrow\uparrow}(A) = \{A_L, G_E, A_N\}$. In other terms, Bill and Eve achieved the same outcome in Algebra and Analysis, but these are not the only exams in which it is true; in fact, neither of them have passed Geometry, hence we must add Geometry to the list of exams in which they have the same result. Thus, we can see $\gamma^{\downarrow\uparrow}(A)$ as the full list of exams whose outcome has been the same for both Bill and Eve. In general, whenever we fix a list of exams A, we found the biggest set of students achieving the same result in each of them. Afterwards, we have found the set $\gamma^{\downarrow\uparrow}(A)$ of all exams in which these students obtained the same outcome.

4. Local indistinguishability posets as micro-granules of a macro-order structure

In this section we formally introduce the notions of indiscernibility subtable lattice and of indistinguishability granular lattice for an information table $\mathcal{I}$, which we have already discussed informally in the introductory section. Our aim consists in defining some lattice structures that at the macro-granular level provide the same information of the indiscernibility partition lattice but, on the other hand, give more detailed information at a micro-granular level. As a matter of fact, let us consider the maximum partitioner lattice provided in Fig. 7. We observe that the attribute subset $\{R, M, C\}$ gives rise to a certain partition of the object set U but, a priori, it is impossible to argue if another attribute subset, with a smaller number of elements, can yield the same partition. In other terms, maximum partitioner lattice takes into account the variations of information but does not tell us if there are redundant elements whose deletion does not cause a loss of information. Hence, an investigation at a micro-granular level is necessary . This leads us to introduce the family $MINP(\mathcal{I})$, defined in Eq. (40). In our context, the family $MINP(\mathcal{I})$ assumes a primary role. In fact, it contains all the minimal elements of any indistinguishability class or, in equivalent interpretative terms, it contains all the minimal attribute subsets providing any determined degree of information.

Let therefore $\mathcal{I}$ be an information table. Let $MAXP(\mathcal{I}) = \{C_1, \dots, C_k\}$ and let $B_{1j}, \dots, B_{p_j j}$ be the distinct blocks of the indiscernibility partition of $\pi(C_j)$ for $1 \le j \le k$. Let $C_j = \{c_{1j}, \dots, c_{q_j j}\}$. Fix $1 \le j \le k$. For any $1 \le i \le p_j$ and $1 \le k \le q_j$ we define the following map

$$\phi_j(B_{ij}, c_{kj}) := F(u, c_{kj}), \tag{33}$$

where u is any element of B_{ij}. This map is well defined because, by definition of indiscernibility, the information map F is constant on the elements belonging to the same indiscernibility block.

Definition 4.1. We denote by $S_j(\mathcal{I})$ the information table having objects $B_{1j}, \dots, B_{p_j j}$, attributes $c_{1j}, \dots, c_{q_j j}$ and information map ϕ_j given in (33). We call $S_j(\mathcal{I})$ the *j^{th}-indiscernibility sub-table* of $\mathcal{I}$.

We set now

$$ISUB(\mathcal{I}) := \{S_1(\mathcal{I}), \dots, S_k(\mathcal{I})\}, \tag{34}$$

and

$$PART(\mathcal{I}) := \{[C_1]_\approx, \dots, [C_k]_\approx\}. \tag{35}$$

Since the elements of $PART(\mathcal{I})$ are also a set partition of the power set $\mathcal{P}(Att)$, we also use the following set partition notation:

$$\Lambda := [C_1]_\approx | \dots | [C_k]_\approx. \tag{36}$$

At this point we introduce the following partial orders $\sqsubseteq$ and $\sqsubseteq'$, respectively on the sets $ISUB(\mathcal{I})$ and $PART(\mathcal{I})$.

If $S_j(\mathcal{I}), S_l(\mathcal{I}) \in ISUB(\mathcal{I})$ and $[C_l]_\approx, [C_j]_\approx \in PART(\mathcal{I})$ we set

$$S_l(\mathcal{I}) \sqsubseteq S_j(\mathcal{I}) :\Longleftrightarrow C_l \subseteq^* C_j \Longleftrightarrow: [C_l]_\approx \sqsubseteq' [C_j]_\approx \tag{37}$$

and

$$\mathbb{S}(\mathcal{I}) := (ISUB(\mathcal{I}), \sqsubseteq), \tag{38}$$

$$\mathbb{I}(\mathcal{I}) := (PART(\mathcal{I}), \sqsubseteq'). \tag{39}$$

We have then the following immediate result:

Theorem 4.2. *$\mathbb{S}(\mathcal{I})$ and $\mathbb{I}(\mathcal{I})$ are two lattices that are both order isomorphic to the maximum partitioner lattice $\mathbb{M}(\mathcal{I})$.*

Proof. The thesis follows immediately from the definition of the sets $ISUB(\mathcal{I})$, $PART(\mathcal{I})$ and by (37). □

By virtue of the previous result, we introduce the following terminology.

Definition 4.3. We call $\mathbb{S}(\mathcal{I})$ the *indiscernibility sub-table lattice* of $\mathcal{I}$ and $\mathbb{I}(\mathcal{I})$ the *indistinguishability lattice* of $\mathcal{I}$.

Example 4.4. In reference to the example of car information table $\mathcal{C}$ discussed in the introductory section, we have that $MAXP(\mathcal{C}) = \{C_1, C_2, C_3, C_4\}$, where $C_1 = \emptyset$, $C_2 = \{Speed\}$, $C_3 = \{Color, RoadH\}$ and $C_4 = \{Speed, Color, RoadH\}$. Moreover, $S_j(\mathcal{C}) = \mathcal{C}_j$, for $j = 1, 2, 3, 4$, and $ISUB(\mathcal{C}) = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4\}$. The diagram of the indiscernibility sub-table lattice $\mathbb{S}(\mathcal{C})$ is represented in Fig. 4. We discuss now some potentialities concerning the interpretation of the lattice $\mathbb{S}(\mathcal{C})$. Let us suppose that a user is interested to the car color blue. In this case he takes a look only at the nodes $\mathcal{C}_3$ and $\mathcal{C}_4$ in the diagram of Fig. 4, where *Color* Blue only appears. On the other hand, since the node $\mathcal{C}_4$ is smaller than $\mathcal{C}_3$ with respect to the corresponding partial order, the user can deduce that the examination of the node $\mathcal{C}_4$ is more meaningful for him than the examination of the node $\mathcal{C}_3$. In fact, in the sub-table $\mathcal{C}_3$ the cars m_2 and m_4 are indiscernible between them with respect to the color value Blue and the attribute *RoadH*, whereas this does not happen in $\mathcal{C}_4$, inasmuch as the presence of the new attribute *Speed* provides a discernibility of m_2 and m_4 with respect to this property. Hence, for a practical use of the diagram of the indiscernibility sub-table lattice, a user can first select a node N where an attribute value appears that is of interest for him. Next he finds the other nodes that contain this value and that are located in the down-set of N. Obviously, in order to find a precise algorithm that provides a uniquely determined node choice for the user, it is necessary to have other information. However, in a first approximation, we can say that a simple heuristic technique is the following : *find some nodes of your interest and move down along the chains in the down sets of these nodes. In this region you have more probability to find cases more meaningful for your requests*.

In some sense, we can "navigate" our data with the advantage, with respect to a simple graphical data representation, to have a richer mathematical structure behind. In this way, we also have the possibility to use all the RST tools in order to extract knowledge.

For the indistinguishability granular lattice of $\mathcal{C}$, we have that

$$PART(\mathcal{C}) = \{[C_1]_\approx, [C_2]_\approx, [C_3]_\approx, [C_4]_\approx\},$$

where

$$[C_1]_{\approx} = \{\emptyset\}, [C_2]_{\approx} = \{\{Speed\}\}, [C_3]_{\approx} = \{\{Color\}, \{RoadH\}, \{Color, RoadH\}\},$$

$$[C_4]_{\approx} = \{\{Speed, Color\}, \{Speed, RoadH\}, \{Speed, Color, RoadH\}\},$$

and

$$\Lambda_{\mathcal{C}} = \{\emptyset\}|\{Speed\}|\{Color\}\{RoadH\}\{Color, RoadH\}|\{Speed, Color\}\{Speed, RoadH\}\{Speed, Color, RoadH\}$$

The diagram of the indistinguishability lattice $\mathbb{I}(\mathcal{C})$ is given in Fig. 6. When the user observes this diagram, he can see immediately what is the inner inclusion relations in each *granule* $[C_i]_{\approx}$. In each one of such granules there are all the attribute subsets that determine a same indiscernibility partition of the universe set. For example, if our user chooses to examine the node $\mathcal{C}_4$, on the basis of the above discussion, it is possible that he is not interested to examine also the attribute values concerning the attribute *RoadH*. Therefore he takes under consideration only the sub-tables having the attributes *Speed* and *Color*. From an observation of the minimum node in the diagram of Fig. 6, he sees that the attribute subset {*Speed*, *Color*} is more down than {*Speed*, *Color*, *RoadH*} and that, however, it provides the same indiscernibility partition than the latter. In other terms, a more detailed inner vision of the inclusion relation in each granule could provide interesting information for a potential user.

Hence, in general, $\mathbb{S}(\mathcal{I})$ is a lattice whose nodes are the sub-tables of $\mathcal{I}$ that are associated to all the distinct indiscernibility partitions of $\mathcal{I}$, whereas $\mathbb{I}(\mathcal{I})$ is a lattice whose nodes are the posets $([C_j]_{\approx}, \subseteq)$, where C_j is a maximum partitioner of $\mathcal{I}$.

Definition 4.5. We call $([C_1]_{\approx}, \subseteq), \ldots, ([C_k]_{\approx}, \subseteq)$ the *local indistinguishability posets* of $\mathcal{I}$.

Let us note that, in general, a local indistinguishability poset of an information table $\mathcal{I}$ is not a lattice: although it has a maximum element (that is the corresponding maximum partitioner), it has not a minimum element. However, by Theorem 3.14 we know that the minimal elements of the local indistinguishability poset $[C_i]_{\approx}$ are exactly the relative reducts of $RED(C_i)$.

Example 4.6. In reference to the information table $\mathcal{T}$ given in Example 2.17, in Fig. 8 we represent the diagram $\mathbb{I}(\mathcal{T})$.

We observe that there are seven local indistinguishability posets:

$$([\{\emptyset\}]_{\approx}, \subseteq) = (\emptyset, \subseteq)$$

$$([\{MC\}]_{\approx}, \subseteq) = (\{M, C, MC\}, \subseteq)$$

$$([\{R\}]_{\approx}, \subseteq) = (R \subseteq)$$

$$([\{B\}]_{\approx}, \subseteq) = (B \subseteq)$$

$$([\{RMC\}]_{\approx}, \subseteq) = (\{RM, RC, RMC\}, \subseteq)$$

$$([\{RB\}]_{\approx}, \subseteq) = (RB, \subseteq)$$

$$([\{RMCB\}]_{\approx}, \subseteq) = (\{CB, MB, RMB, MCB, RCB, RMCB\} \subseteq).$$

By taking inspiration from the notion of indispensable attribute given by Pawlak (see [41]), we now introduce a new class of attributes preserving an indistinguishability granule. In other terms, we are formulating a concept similar to Pawlak's indispensability notion in the case of the indistinguishability relation.

Definition 4.7. Let $A \subseteq Att$. We say that:

- an attribute $a \in A$ is *A-indistinguishability indispensable* if there exists $B \in [A]_{\approx}$ such that $B \setminus \{a\} \notin [A]_{\approx}$. We denote by $K(A)$ the set of all A-indistinguishability indispensable attributes and we call it the *indistinguishability kernel* of A.
- an attribute $a \in A$ is *A-indistinguishability granular preserving* if $a \notin K(A)$ and we denote by $K^c(A) := A \setminus K(A)$ the set of all A-indistinguishability granular preserving attributes. We call it the *co-indistinguishability kernel* of A.

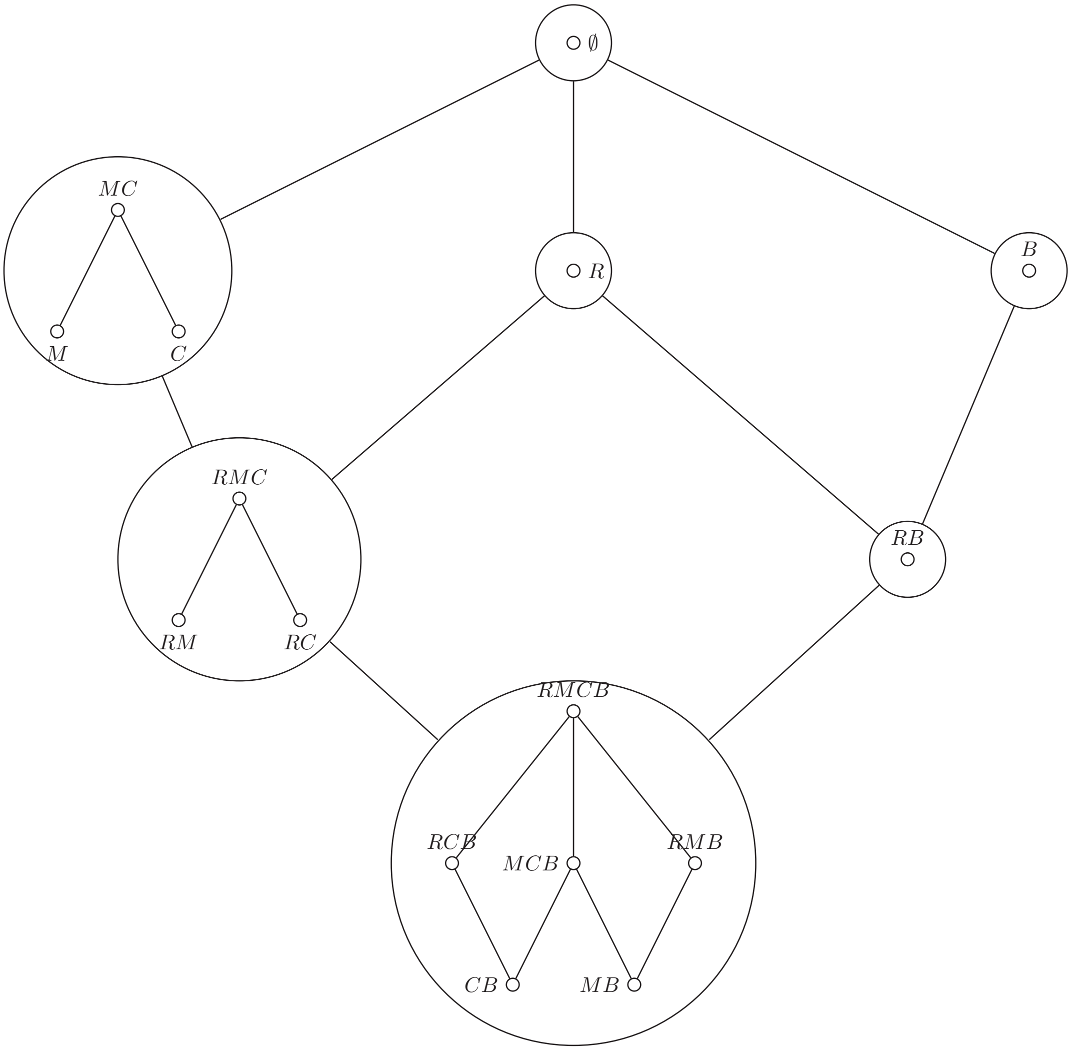

Fig. 8. Diagram of $\mathbb{I}(\mathcal{T})$.

Table 7
Information table $\mathcal{J}$.

$\mathcal{J}$	1	2	3	4	5
u_1	1	0	0	1	0
u_2	0	1	0	0	1
u_3	0	0	0	0	1
u_4	1	0	0	1	0
u_5	1	0	0	0	1

Example 4.8. Let us consider the information table $\mathcal{J}$ given in Table 7:
Let $A = \{1, 3, 4, 5\}$. Then $K(A) = \{1, 4, 5\}$ and $K^c(A) = \{3\}$.

We set now

$$MINP(\mathcal{I}) := \bigcup \{\min([A]_{\approx}) : A \in MAXP(\mathcal{I})\}. \tag{40}$$

This attribute set family can be considered as a dual version of $MAXP(\mathcal{I})$. In the next section we study in details the family $MINP(\mathcal{I})$ and we will show that it plays a fundamental role in our investigations.

In the next result we establish some basic properties concerning the notions introduced in Definition 4.7.

Theorem 4.9. *The following conditions hold.*

(i) *Let $A \in MAXP(\mathcal{I})$. Then $a \in K(A)$ if and only if there exists $C \in MAXP(A)$ such that $a \notin C$ and $M(C \cup \{a\}) = A$.*

(ii) *Let* $A \in MAXP(\mathcal{I})$*. Then* $K^c(A) = \bigcap\{B : B \in [A|\mathbb{M}(\mathcal{I}) \uparrow]\}$.
(iii) *If* $A \subseteq Att$*, then* $CORE(A) \subseteq K(A)$.
(iv) *If* $A \subseteq A'$ *and* $A \approx A'$*, then* $K(A) \subseteq K(A')$.
(v) *If* $A \in MINP(\mathcal{I})$*, then* $K(A) = A$.
(vi) *If* $A \subseteq Att$*, then* $A \approx K(M(A))$.
(vii) *If* $A \subseteq Att$*, then* $M(A) = M(K(M(A)))$.

Proof.

(i): Let $a \in K(A_{\mathcal{I}})$. By definition of $K(A)$ there exists $B \in [A]_{\approx}$ such that

$$B \setminus \{a\} \not\approx A. \tag{41}$$

Since $B \approx A$, by (41) we have that $a \in B$. Let $C := M(B \setminus \{a\})$. We first show that $a \notin C$. In fact, let us assume by absurd that $a \in C$. In this case $B \subseteq C$ because $B \setminus \{a\} \subseteq M(B \setminus \{a\}) = C$. Therefore $A = M(B) \subseteq M(C) = C$ because $A, C \in MAXP(\mathcal{I})$ and $B \in [A]_{\approx}$, and this implies that

$$\pi(C) \preceq \pi(A). \tag{42}$$

On the other hand, we also have

$$\pi(A) = \pi(B) \preceq \pi(B \setminus \{a\}) = \pi(C). \tag{43}$$

Then, by (42) and (43) we deduce $\pi(A) = \pi(C)$, that is $A \approx C$, and this implies (by definition of C) $A \approx B \setminus \{a\}$, that is in contrast with (41). Hence $a \notin C$. Let us observe now that

$$C = M(B \setminus \{a\}) \subseteq M(B) = M(A) = A, \tag{44}$$

because $A \in MAXP(\mathcal{I})$. By (44) we have then $C \in MAXP(A)$. Finally, since $B \setminus \{a\} \subseteq C$, we have $B \subseteq C \cup \{a\} \subseteq A$, so that

$$\pi(A) \preceq \pi(C \cup \{a\}) \preceq \pi(B),$$

and this implies $\pi(C \cup \{a\}) = \pi(A)$ because $\pi(A) = \pi(B)$. Hence $M(C \cup \{a\}) = M(A) = A$ because $A \in MAXP(\mathcal{I})$. This proves the first implication.

For the other implication, we assume that there exists $C \in MAXP(A)$ such that $a \notin C$ and $M(C \cup \{a\}) = A$. Then $a \in A$ and we set $B := C \cup \{a\}$. Therefore $B \in [A]_{\approx}$ and $B \setminus \{a\} = C$. Let us note that $C \notin [A]_{\approx}$. In fact, if $C \approx A$ then $C = A$ because $C \in MAXP(A)$ and $A \in MAXP(\mathcal{I})$, that is in contrast with the conditions $a \in A$ and $a \notin C$. Hence we obtain an attribute subset $B \in [A]_{\approx}$ such that $B \setminus \{a\} \notin [A]_{\approx}$ and $a \in A$, that is $a \in K(A)$.

(ii): Let $a \in \bigcap\{B : B \in [A|\mathbb{M}(\mathcal{I}) \uparrow]\}$ and suppose by contradiction that $a \in K(A)$. By (i), there exists $B \in MAXP(A)$ such that $a \notin B$ and $M(B \cup \{a\}) = A$. We clearly have that $B \notin [A|\mathbb{M}(\mathcal{I}) \uparrow]$. Therefore, there exists $C \in MAXP(A)$ such that $B \subsetneqq C \subsetneqq A$. Hence

$$\pi(A) \preceq \pi(C). \tag{45}$$

We claim that $a \notin C$. In fact, if $a \in C$, we would have $B \cup \{a\} \subseteq C$, hence

$$\pi(C) \preceq \pi(B \cup \{a\}) = \pi(A). \tag{46}$$

Thus, by (45) and (46), $\pi(A) = \pi(C)$ and, so, $C = A$, contradicting our assumption on C. Proceeding in this way, we will find an attribute subset $D \in [A|\mathbb{M}(\mathcal{I}) \uparrow]$ not containing a, absurd. Thus $a \in K^c(A)$.

Conversely, let $a \in K^c(A)$. Suppose by contradiction that there exists $B \in [A|\mathbb{M}(\mathcal{I}) \uparrow]$ such that $a \notin B$. Hence $C := B \cup \{a\} \approx A$, so we have found an element $C \in [A]_{\approx}$ such that $C \setminus \{a\} \not\approx A$, i.e. $a \in K(A)$, absurd.

(iii): Let $a \in CORE(A)$. Hence $\pi(A) \neq \pi(A \setminus \{a\})$, i.e. $a \in K(A)$.

(iv): Let $a \in K(A)$. Hence, for any $B \approx A$, it results that $B \setminus \{a\} \approx A$. Therefore, $a \in K(A')$.

(v): Since $A \in MINP(\mathcal{I})$, we have that $\pi(A) \neq \pi(A \setminus \{a\})$ for any $a \in A$, hence $K(A) = A$.

(vi): Set $A' = K(M(A)))$. We have that $A' \subseteq M(A)$. Let $B \in \min([A]_{\approx})$, then $B \subseteq M(A)$, hence, by part (iv), we have that $K(B) \subseteq A'$. Since $B \in MINP(\mathcal{I})$, by part (v) $K(B) = B$, so $B \subseteq A'$. We conclude that

$$B \subseteq A' \subseteq M(A).$$

Therefore, we have

$$\pi(M(A)) = \pi(A) \preceq \pi(A') \preceq \pi(B) = \pi(A),$$

so $\pi(A) = \pi(A')$ and $A \approx A' = K(M(A))$.

(vii): It follows directly by part (vi). □

Example 4.10. Again with reference to the information table $\mathcal{T}$ given in Example 2.17, let $A = Att = \{R, M, C, B\}$. We have that:

$$K(A) = \{M, C, B\}.$$

Let us note that $CORE(A) = \{B\} \subsetneq K(A)$. Furthermore

$$K^c(A) = \{R\} = \{R, M, C\} \cap \{R, B\}.$$

In other terms, the deletion of R preserves the information provided by any attribute subset in the indistinguishability class $[A]_{\approx}$. On the other hand, since M, C and B are attributes belonging to the minimal subsets of $[A]_{\approx}$, their deletion involves a loss of information.

5. New indiscernibility hypergraphic structures of $\mathcal{I}$

In the previous two sections we provided the two possible standpoints (micro and macro) to interpret the information given by any table. In this section, we focus our attention to the interrelation between these two perspectives. To be more specific, as we said in the introductory section, we will prove the property of global-local regularity thanks to which we can transfer the partial order from the macro-granular structure to the micro-granular one. Furthermore, we show that the hypergraph $MINP(\mathcal{I})$ is an abstract simplicial complex (see Theorem 5.6). Finally, we characterize $MINP(\mathcal{I})$ by means of the relative reducts (see Theorem 5.9) and this will be the starting point for the investigation of some matroidal properties through a max-min function.

5.1. The minimal partitioners of an information table

In Definition 2.16 we called the hypergraphs $\mathfrak{D}(\mathcal{I})$, $\mathfrak{N}(\mathcal{I})$ and $\mathfrak{C}(\mathcal{I})$ indiscernibility hypergraphic structures of $\mathcal{I}$ because their construction depends on the indiscernibility relation. Since indistinguishability relation $\approx$ on the power set $\mathcal{P}(Att)$ allows us to define the maximum partitioner hypergraph $\mathfrak{M}(\mathcal{I})$, we now use $\mathfrak{M}(\mathcal{I})$ as a basic structure to build several other induced indiscernibility hypergraphs. The introduction of these new structures and the investigation of the interrelations between them is the basic topic of the remaining part of this paper.

In what follows, let $MAXP(\mathcal{I}) = \{C_i : i \in I\}$.

By (40) we have

$$MINP(\mathcal{I}) := \bigcup_{i \in I} \min([C_i]_{\approx}) = \bigcup_{i \in I} RED(C_i). \tag{47}$$

Moreover we also set

$$MINP^c(\mathcal{I}) := \mathcal{P}(Att) \setminus MINP(\mathcal{I}), \tag{48}$$

$$MXMN(\mathcal{I}) := \max(MINP(\mathcal{I})) \tag{49}$$

and

$$MNMN^c(\mathcal{I}) := \min(MINP^c(\mathcal{I})). \tag{50}$$

Definition 5.1. We call:

- the hypergraph $\mathfrak{M}(\mathcal{I}) := (Att, MAXP(\mathcal{I}))$ *maximum partitioner hypergraph* of $\mathcal{I}$
- the map $M : A \in \mathcal{P}(Att) \mapsto M(A) \in \mathcal{P}(Att)$ *maximum partitioner operator* of $\mathcal{I}$
- any member of $MINP(\mathcal{I})$ *minimal partitioner* of $\mathcal{I}$;
- the hypergraph $\mathfrak{m}(\mathcal{I}) := (Att, MINP(\mathcal{I}))$ *minimal partitioner hypergraph* of $\mathcal{I}$;
- any member of $MINP^c(\mathcal{I})$ *co-minimal partitioner* of $\mathcal{I}$;
- the hypergraph $\mathfrak{m}^c(\mathcal{I}) := (Att, MINP^c(\mathcal{I}))$ *co-minimal partitioner hypergraph* of $\mathcal{I}$;
- any member of $MXMN(\mathcal{I})$ *max-minimal partitioner* of $\mathcal{I}$;
- the hypergraph $\max(\mathfrak{m}(\mathcal{I})) := (Att, MXMN(\mathcal{I}))$ *max-minimal hypergraph* of $\mathcal{I}$;
- any member of $MNMN^c(\mathcal{I})$ *min-co-minimal partitioner* of $\mathcal{I}$;
- the hypergraph $\min(\mathfrak{m}^c(\mathcal{I})) := (Att, MNMN^c(\mathcal{I}))$ *min-co-minimal hypergraph* of $\mathcal{I}$.

Hence a minimal partitioner of $\mathcal{I}$ is the relative reduct of some maximum partitioner of $\mathcal{I}$.

Example 5.2. Again in reference to the information table $\mathcal{J}$ introduced in Example 4.8, it results that (we write the members of $MINP(\mathcal{J})$ and $MINP^c(\mathcal{J})$ as strings):

$$MINP(\mathcal{J}) = \{\emptyset, 1, 2, 4, 5, 12, 14, 15, 24, 25, 124, 125\}, \tag{51}$$

$$MINP^c(\mathcal{J}) = \{3, 13, 23, 34, 35, 45, 123, 134, 135, 234, 245, 345, 1234, 1235, 1245, 2345, 12345\}, \tag{52}$$

so that

$$MXMN(\mathcal{J}) = \{124, 125\} \tag{53}$$

and

$$MNMN^c(\mathcal{J}) = \{3\}. \tag{54}$$

Let us note that for the information table $\mathcal{J}$ we also have that

$$RED(\mathcal{J}) = \{124, 125\}.$$

In the next result, we prove the property of global-local regularity discussed in the introductory section.

Proposition 5.3. *Let $C_i \subsetneq C_j$. Then, if $X \in [C_i]_\approx$ and $Y \in [C_j]_\approx$ we have $Y \not\subseteq X$.*

Proof. Since C_i and C_j are two distinct maximum partitioners such that $C_i \subsetneq C_j$, we have $\pi(C_j) \prec \pi(C_i)$. Now, by absurd, let $X \in [C_i]_\approx$ and $Y \in [C_j]_\approx$ such that $Y \subseteq X$. Then $\pi(C_i) = \pi(X) \preceq \pi(Y) = \pi(C_j)$, that is a contradiction. □

In the following result we establish a more deep link between minimal partitioners and reducts.

Theorem 5.4. *The following conditions hold:*

(i) *Let $A \subseteq Att$ and $B \in RED(A)$ and $C \subseteq B$. Then $\pi(C \backslash \{x\}) \neq \pi(C)$ for any $x \in C$.*
(ii) *We have that*

$$MINP(\mathcal{I}) \supseteq \bigcup_{B \in RED(A)} \mathcal{P}(B), \ \forall A \subseteq Att. \tag{55}$$

Proof.

(i): Let us suppose by absurd that there exists an element $x \in C$ such that $\pi(C \setminus \{x\}) = \pi(C)$. Then, since $C \backslash \{x\} \subseteq B \backslash \{x\}$, we have

$$\pi(B \setminus \{x\}) \preceq \pi(C \setminus \{x\}) = \pi(C). \tag{56}$$

Therefore, if $u, u' \in U$ by (56) it follows that

$$u \equiv_{B \backslash \{x\}} u' \Rightarrow u \equiv_C u' \Rightarrow F(u, x) = F(u', x) \Rightarrow u \equiv_B u',$$

therefore

$$\pi(B \setminus \{x\}) \preceq \pi(B). \tag{57}$$

On the other hand, since we also have $\pi(B) \preceq \pi(B \backslash \{x\})$, by (57) we deduce that $\pi(B \setminus \{x\}) = \pi(B)$, that is in contrast with the hypothesis that $B \in RED(A)$. This concludes the proof of part (i).

(ii): Let $C \in \bigcup_{B \in RED(A)} \mathcal{P}(B)$. Then there is some reduct $B \in RED(A)$ such that $C \subseteq B$. Let $j \in I$ such that $C \in [C_j]_\approx$. Let us assume, by absurd, that $C \notin \min([C_j]_\approx)$. Then there exists some $C' \in [C_j]_\approx$ such that $C' \subsetneq C$. Let $x \in C \backslash C'$. Then

$$C' \subseteq C \setminus \{x\} \subseteq C,$$

and therefore

$$\pi(C) \preceq \pi(C \setminus \{x\}) \preceq \pi(C'). \tag{58}$$

Since $C, C' \in [C_j]_\approx$, we have $\pi(C) = \pi(C')$, therefore, by (58) we deduce that

$$\pi(C) = \pi(C \setminus \{x\}). \tag{59}$$

But the identity in (59) is in contrast with part (i), because $C \subseteq B$ and $B \in RED(A)$. Hence $C \in \min([C_j]_\approx) \subseteq MINP(\mathcal{I})$. This concludes the proof of part (ii) □.

Corollary 5.5. *We have that*

$$MINP(\mathcal{I}) \supseteq \bigcup_{A \in RED(\mathcal{I})} \mathcal{P}(A) = \{X \in \mathcal{P}(\Omega) : X \subseteq A, \ A \in RED(\mathcal{I})\}. \tag{60}$$

Proof. The right member of (60) is a particular case of the right member of (55) when $A = Att$. □

In next fundamental result, we will show that $MINP(\mathcal{I})$ is an abstract simplicial complex.

Theorem 5.6. *If $C \in MINP(\mathcal{I})$ and $K \subseteq C$, then $K \in MINP(\mathcal{I})$.*

Proof. Let $C \in MINP(\mathcal{I})$. Then there exists $B \in MAXP(\mathcal{I})$ such that $C \in \min([B]_\approx)$. Let $K \subsetneq C$, then there exists $B' \in MAXP(\mathcal{I})$ such that $K \in [B']_\approx$. Suppose by contradiction that $K \notin \min([B']_\approx)$, then there exists $K' \subsetneq K$ such that $K' \in \min([B']_\approx)$. Hence, there is $x \in K \backslash K'$. We deduce that

$$K' \subseteq K \setminus \{x\} \subseteq K$$

i.e.

$$\pi(K) \preceq \pi(K \setminus \{x\}) \preceq \pi(K').$$

But since $\pi(K) = \pi(K') = \pi(B)$, we conclude that $\pi(K) = \pi(K \setminus \{x\})$, contradicting Theorem 5.4. Hence $K \in MINP(\mathcal{I})$. □

Clearly, the notion of minimal partitioner can be relativized to any $A \subseteq Att$. As a matter of fact, we set

$$MINP(A) := \{X : X \subseteq A,\ X \in MINP(\mathcal{I})\}. \tag{61}$$

and

$$MXMN(A) := \max(MINP(A)). \tag{62}$$

In particular we have $MINP(Att) = MINP(\mathcal{I})$ and $MXMN(Att) = MXMN(\mathcal{I})$.

In the next proposition we establish the basic links between the reducts of A and its minimal partitioners.

Proposition 5.7. *Let $A \subseteq Att$. Then:*

$$RED(A) \subseteq MXMN(A). \tag{63}$$

Proof. Let $X \in RED(A)$, then $X \subseteq A$ and $X \in \min([A]_{\approx}) \subseteq MINP(\mathcal{I})$ by Proposition 3.13. This proves that $X \in MINP(A)$. Suppose by contradiction that there exists $Y \in MINP(A)$ such that $X \subsetneq Y$. Then $\pi(Y) \preceq \pi(X) = \pi(A)$. Furthermore, we have that $Y \in MINP(\mathcal{I})$, hence there exists $B \in MAXP(\mathcal{I})$ such that $Y \in \min([B]_{\approx})$. Thus $\pi(Y) = \pi(B)$ and, since $Y \subseteq A$, it results that $\pi(A) \preceq \pi(Y) = \pi(B)$. Hence $\pi(A) = \pi(B)$, i.e. $A = B$. Therefore, $Y \in \min([A]_{\approx})$ and, by (25), we conclude that $Y \in RED(A)$, absurd since it contains X. So $X \in MXMN(A)$. □

Eq. (63)cannot be reversed, as we see in next example.

Example 5.8. In reference to Example 2.17, let $A = \{R, C, B\}$. By Fig. 8, we observe that $RED(A) = \{\{C, B\}\}$, while $MXMN(A) = \{\{C, B\}, \{R, C\}, \{R, B\}\}$. Let us observe that if we consider $\{C, B\}$ and exchange C with R, we lose some information. By classical results of matroid theory (see Lemma 3.1 of [47]), it is easy to verify that $MINP(A)$ is a matroid. Nevertheless, the exchange property does not ensure that we swap reducts each other. Thus, we conclude that the exchange property preserves maximality for the minimal partitioner of A, but not necessarily the condition of being a reduct.

By means of Proposition 5.7, we obtain the following characterization of $MINP(\mathcal{I})$.

Theorem 5.9. *We have that*

$$MINP(\mathcal{I}) = \{A \subseteq Att : RED(A) = \{A\}\}. \tag{64}$$

Proof. Let $A \in MINP(\mathcal{I})$, then $A \in \min([A]_{\approx})$. Moreover, the unique subset satisfying both conditions of Definition 2.9 is exactly A, that is $RED(A) = \{A\}$.

Vice versa, suppose that $RED(A) = \{A\}$. Then, by (63) we have that $A \in MXMN(A)$, so by (61) we have that $A \in MINP(\mathcal{I})$. □

5.2. The max-min function for information tables

We introduce now a non-negative integer value function on $\mathcal{P}(Att)$, which is useful in order to provide new results concerning the indiscernibility hypergraphic structures.

For any attribute subset $A \subseteq Att$ we set

$$\psi(A) := \max\{|X| : X \in MINP(A)\}. \tag{65}$$

So that we obtain the function

$$\psi : A \in \mathcal{P}(Att) \mapsto \psi(A) \in \mathbb{N} \cup \{0\}.$$

Definition 5.10. We call ψ the *max-min function* of $\mathcal{I}$.

We now provide an example of max-min function.

Example 5.11. In reference to the information table $\mathcal{J}$ of the Example 4.8, we have that (we write the attribute subsets as strings): $\psi(3) = \psi(\emptyset) = 0$, $\psi(13) = \psi(1) = 1$, $\psi(23) = \psi(2) = 1$, $\psi(345) = \psi(34) = \psi(35) = \psi(45) = \psi(4) = \psi(5) = 1$, $\psi(345) = \psi(34) = \psi(35) = \psi(45) = \psi(4) = \psi(5) = 1$, $\psi(123) = \psi(12) = 2$, $\psi(1345) = \psi(134) = \psi(135) = \psi(145) = \psi(14) = \psi(15) = 2$, $\psi(2345) = \psi(234) = \psi(235) = \psi(245) = \psi(24) = \psi(25) = 2$, $\psi(12345) = \psi(1234) = \psi(1235) = \psi(1245) = \psi(124) = \psi(125) = 3$.

Proposition 5.12. *The function ψ satisfies the following properties:*

(i) $\psi(\emptyset) = 0$;
(ii) *If $A \subseteq B$, then $\psi(A) \leq \psi(B)$;*

(iii) $0 \leq \psi(A) \leq |A|$ *for any* $A \subseteq Att$;
(iv) $\psi(A) \leq \psi(A \cup \{x\}) \leq \psi(A) + 1$.

Proof. All these properties are direct consequences of (65). □

Definition 5.13. Let $C \subseteq Att$. We say that C is a ψ*-incremental subset* if

$$\psi(C \cup \{x\}) := \psi(C) + 1,$$

for all $x \in Att \setminus C$.

We denote by $INCR(\psi)$ the family of all ψ-incremental subsets. Moreover, if $A \subseteq Att$, we denote by $INCR(A|\psi)$ the family of all ψ-incremental subsets that contain the attribute subset A.

Example 5.14. In reference to the information table $\mathcal{T}$ of the Example 2.17, we have that: $\psi(\emptyset) = 0$, $\psi(R) = 1$, $\psi(B) = 1$, $\psi(MC) = \psi(M) = \psi(C) = 1$, $\psi(RB) = 2$, $\psi(RMC) = \psi(RM) = \psi(RC) = 2$, $\psi(RMCB) = \psi(RCB) = \psi(MCB) = \psi(RMB) = \psi(CB) = \psi(MB) = 2$. Moreover, it is easy to see that

$$INCR(\psi) = \{\emptyset, \{R\}, \{B\}\}.$$

Let us consider $\{R\}$. To say that $\{R\} \in INCR(\psi)$ means that, whenever we add some other attribute to $\{R\}$, we improve the distinction between the various models of mobile.

In the next result we give three new different characterizations for the minimum partitioner family of $\mathcal{I}$.

Theorem 5.15. *The following conditions hold:*

(i) $MINP(\mathcal{I}) = \{A \subseteq Att : C \not\subseteq A\ \forall C \in MNMN^c(\mathcal{I})\}$;
(ii) $MINP(\mathcal{I}) = \{A \subseteq Att :\ \psi(A) = |A|\}$;
(iii) $MINP(\mathcal{I}) = \{A \subseteq Att :\ a \notin M(A \setminus \{a\})\ \forall a \in A\}$.

Proof. (i): Let $A \in MINP(\mathcal{I})$. Then, by Theorem 5.6 any subset of A belongs to $MINP(\mathcal{I})$ hence A cannot contain any element of $MNMN^c(\mathcal{I})$. So $MINP(\mathcal{I}) \subseteq \{A \subseteq Att : C \not\subseteq A\, \forall C \in MNMN^c(\mathcal{I})\}$. On the other hand let $A \subseteq Att$ such that $C \not\subseteq A$ for any $C \in MNMN^c(\mathcal{I})$. Suppose by contradiction that $A \in MINP^c(\mathcal{I})$. Then we have that $A \in \min(MINP^c(\mathcal{I}))$ or it contains an attribute subset $C \in \min(MINP^c(\mathcal{I}))$. In both cases, we are contradicting our assumption on A, hence $A \in MINP(\mathcal{I})$ and (i) has been shown.

(ii): Let $A \in MINP(\mathcal{I})$. Since $\psi(A) = \{|X| : X \subseteq A,\ X \in MINP(\mathcal{I})\}$, it's clear that $\psi(A) = |A|$, thus $MINP(\mathcal{I}) \subseteq \{A \subseteq Att : \psi(A) = |A|\}$. Conversely, let $A \subseteq Att$ such that $\psi(A) = |A|$ and suppose by contradiction that there exists $B \subsetneq A$ such that $\pi(B) = \pi(A)$ and $B \in MINP(\mathcal{I})$. Hence, we should have $|B| \leq \psi(A) < |A|$, contradicting our assumption. So (ii) has been shown.

(iii): Let $A \in MINP(\mathcal{I})$ and $a \in A$. Then $A \setminus \{a\} \notin [A]_{\approx}$, so $M(A \setminus \{a\}) \notin [A]_{\approx}$. Since $A \setminus \{a\} \subsetneq A$, we have $\pi(A) \preceq \pi(A \setminus \{a\})$. Moreover, suppose by contradiction that $a \in M(A \setminus \{a\})$. Then $A \subseteq M(A \setminus \{a\})$ and, by (22), $M(A) \subseteq M(A \setminus \{a\})$. By ($ii$) of Proposition 3.5, we have $\pi(A \setminus \{a\}) \preceq \pi(A)$, so $\pi(A \setminus \{a\}) = \pi(A)$ and $A \setminus \{a\} \in [A]_{\approx}$, absurd. So $a \notin M(A \setminus \{a\})$ and $MINP(\mathcal{I}) \subseteq \{A \subseteq Att : a \notin M(A \setminus \{a\})\ \forall a \in A\}$. On the other hand, let $A \subseteq Att$ such that $a \notin M(A \setminus \{a\})$ for any $a \in A$. Suppose by contradiction that there exists $B \subsetneq A$ such that $\pi(A) = \pi(B)$. Then there exists $a \in A$ such that $B \subseteq A \setminus \{a\} \subseteq A$. This implies that $\pi(A) \preceq \pi(A \setminus \{a\}) \preceq \pi(B)$, i.e. $\pi(A \setminus \{a\}) = \pi(A)$. In other terms, we have $M(A \setminus \{a\}) = M(A) \supseteq A$, contradiction. So ($iii$) follows. □

We introduce now an operator $\hat{\psi} : \mathcal{P}(Att) \to \mathcal{P}(Att)$ as follows.
If $A \subseteq Att$ and $c \in Att$ we set

$$c \vdash_{\psi} A : \Longleftrightarrow\ \psi(A \cup \{c\}) = \psi(A), \tag{66}$$

and

$$\hat{\psi}(A) := \{x \in Att : x \vdash_{\psi} A\}. \tag{67}$$

Example 5.16. In reference to the information table $\mathcal{J}$ of the Example 4.8, it is easy to verify that we have $\hat{\psi}(A) = M(A)$ for any attribute subset A.

However the identity $\hat{\psi}(A) = M(A)$ does not hold in general, as we can see in the next example.

Example 5.17. Let $\mathcal{I}$ be the following information table:

	1	2	3	4	5
u_1	0	0	1	2	1
u_2	1	0	2	2	0
u_3	2	1	0	1	1
u_4	1	1	2	0	2
u_5	1	0	0	0	2
u_6	0	1	2	2	0

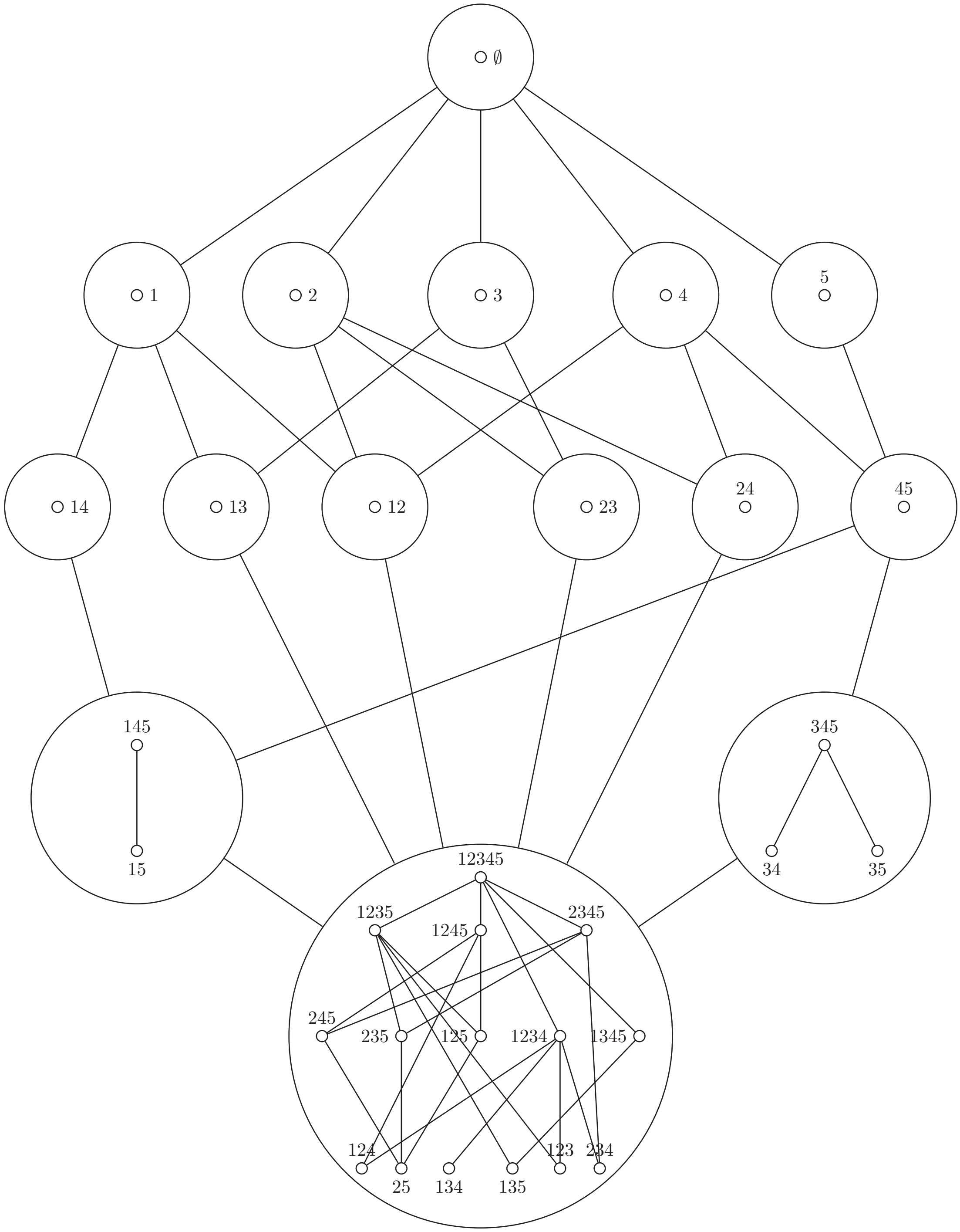

Fig. 9. Diagram of $\mathbb{I}(\mathcal{I})$.

Let $A = \{1, 2\}$. It is immediate to see that $\psi(\{1, 2\}) = \psi(\{1, 2, 5\}) = 2$ and that $5 \notin M(\{1, 2\}) = \{1, 2\}$. Hence $M(A) \neq \hat{\psi}(A)$.

The previous example enables us to introduce a new class of attribute subsets.

Definition 5.18. Let $A \in \mathcal{P}(Att)$. We say that

- A is *reduct uniform* if *RED*(A) has uniform cardinality.

- A is *maxp-reduct uniform* if $RED(M(A))$ has uniform cardinality.

Example 5.19. Let $A = \{1, 2, 3, 4\}$. We observe that $RED(A) = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}\}$, hence A is reduct uniform. Nevertheless, as we can observe in Fig. 9, $M(A) = Att$ but $RED(\mathcal{I}) = RED(Att)$ has not uniform cardinality, hence A is not maxp-reduct uniform.

However the particular cases described in Example 5.16 and in Example 5.19 induce us to investigate in more detail the link between the dependency operator and the maximum partitioner operator. In the next definition we introduce a first class of information tables for which the identity $\hat{\psi}(A) = M(A)$ holds for any $A \subseteq Att$.

Definition 5.20. We say that an information table $\mathcal{I}$ is *maxp-reduct uniform* if the following two conditions are satisfied:

(i) any $A \in \mathcal{P}(Att)$ is maxp-reduct uniform;
(ii) if $A, B \in \mathcal{P}(Att)$, $A \subsetneq B$ and $A \not\approx B$, then $||RED(M(A))|| < ||RED(M(B))||$.

Example 5.21. It is easy to verify that the information table $\mathcal{J}$ of the Example 4.8 is maxp-reduct uniform.

In the next result we will show that for any maxp-reduct uniform information table the operators M and $\hat{\psi}$ coincide.

Theorem 5.22. *Let $\mathcal{I}$ be a maxp-reduct uniform information table. Then:*

(i) $\hat{\psi}(A) = M(A)$ *for any* $A \in \mathcal{P}(Att)$*;*
(ii) $MAXP(\mathcal{I}) = INCR(\psi)$.

Proof.

(i): Let $A \subseteq Att$ and $a \in \hat{\psi}(A)$. Then $\psi(A) = \psi(A \cup \{a\})$. Suppose by contradiction that $a \notin M(A)$, then $A \cup \{a\} \notin [A]_{\approx}$. By (*ii*) of Definition 5.20, we have that $||RED(A)|| < ||RED(A \cup \{a\})||$, so $\psi(A \cup \{a\}) > \psi(A)$, absurd. On the other hand, let $a \in M(A)$. Then, by (*ii*) of Proposition 5.12, we have that

$$\psi(A) \le \psi(A \cup \{a\}) \le \psi(M(A)).$$

But since $||RED(A)|| = ||RED(M(A))||$ and $A \cup \{a\} \in [A]_{\approx}$, we have that $\psi(A) = \psi(A \cup \{a\})$, so $a \in \hat{\psi}(A)$.

(ii): Let $A \in MAXP(\mathcal{I})$, hence by (*ii*) of Definition 5.20, for any $a \in Att \backslash A$, we have that $\psi(A \cup \{a\}) = \psi(A) + 1$, so $A \in INCR(\psi)$. Conversely, let $A \in INCR(\psi)$. Then $\psi(A \cup \{a\} = \psi(A) + 1$ for any $a \in Att \backslash A$, hence $a \notin \hat{\psi}(A) = M(A)$. Thus $A \in MAXP(\mathcal{I})$ and the thesis follows. □

6. Conclusions

In this paper we have continued the study started in [16], where the maximum partitioner family $MAXP(\mathcal{I})$ of any knowledge representation system $\mathcal{I}$ has been introduced. To be more precise, in this work we established new formal properties of the hypergraph $(Att, MAXP(\mathcal{I}))$. Furthermore, the family $MAXP(\mathcal{I})$ enables us to define several new attribute subset families, that we called *indiscernibility hypergraphic structures* of $\mathcal{I}$. Our basic aim was to strengthen a granular interpretation in the investigation of any knowledge representation system. In fact, we showed as the indiscernibility hypergraphs of $\mathcal{I}$ lead towards the construction and the interaction of macro-granular and micro-granular structures, that are both strongly related to many classical notions of GrC-IT. We also discussed the basic properties of the above structures on two concrete cases, in order to provide a basis for better interpreting the role of these hypergraphs. We close this concluding section noticing that a knowledge representation system $\mathcal{I}$ is an ubiquitous structure, both in information science and mathematics. Surprisingly, the original Pawlak practical necessity have helped us to develop interesting and complex micro-macro granular structures whose mathematical richness and interpretative potentiality arise from the absolute simplicity of a data table. Thus, GrC-IT could play a major role both in all applications based on data and in the development of new mathematical theories.

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