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On commuting polynomial automorphisms of $\mathbb{C}^k$, $k \geq 3$

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Abstract We characterize the polynomial automorphisms of $\mathbb{C}^3$, which commute with a *regular* automorphism. We use their meromorphic extension to $\mathbb{P}^3$ and consider their dynamics on the hyperplane at infinity. We conjecture the additional hypothesis under which the same characterization is true in all dimensions. We give a partial answer to a question of S. Smale that in our context can be formulated as follows: can any polynomial automorphism of $\mathbb{C}^k$ be the uniform limit on compact sets of polynomial automorphisms with trivial centralizer (i.e. $C(f) \simeq \mathbb{Z}$)?

Keywords Commuting polynomial automorphisms · Hénon maps · Indeterminacy sets · Green functions · Blow-up

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1 Introduction

Complex affine k-space, $\mathbb{C}^k$, is one of the basic objects in complex analysis and geometry. It seems quite hard to give an algebraic description of the group of (polynomial) automorphisms of $\mathbb{C}^k$, when $k \geq 3$. The group of polynomial automorphisms of $\mathbb{C}^k$, $Aut(\mathbb{C}^k)$, consists of bijective maps:

$$f : (z_1, \cdots, z_k) \in \mathbb{C}^k \to (f_1(z_1, \cdots, z_k), \cdots, f_k(z_1, \cdots, z_k)) \in \mathbb{C}^k$$

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where $f_1, \cdots, f_k \in \mathbb{C}[z_1, \cdots, z_k]$. When f is polynomial and bijective, then the inverse f^{-1} is a polynomial mapping.

In dimension 2, the algebraic structure of the group of polynomial automorphisms is well known. The result is due to Jung [10]; it was reproved in several different ways [17] and recently also in [14]. Jung's theorem asserts that the group $Aut(\mathbb{C}^2)$ is the amalgamated product of its subgroups $\mathcal{E}$ and $\mathcal{A}$ with respect to their intersection $\mathcal{AT}$, where the group $\mathcal{E}$ of *elementary* maps is:

$$\mathcal{E} = \{(z, w) \rightarrow (\alpha z + p(w), \beta w + \gamma) : \alpha, \beta, \gamma \in \mathbb{C}, \alpha\beta \neq 0, p \in \mathbb{C}[w]\}$$

the group $\mathcal{A}$ of *affine* maps is:

$$\mathcal{A} = \{(z, w) \rightarrow (a_1 z + b_1 w + c_1, a_2 z + b_2 w + c_2) : a_i, b_i, c_i \in \mathbb{C}, a_1 b_2 - a_2 b_1 \neq 0\}$$

and where $\mathcal{AT}$ denote the intersection $\mathcal{A} \cap \mathcal{E}$, i.e. the group of the automorphisms *affine and triangular*:

$$\mathcal{AT} = \{(z, w) \rightarrow (a_1 z + b_1 w + c_1, b_2 w + c_2) : a_1, b_i, c_i \in \mathbb{C}, a_1 b_2 \neq 0\}.$$

By this structure theorem, each automorphism $\varphi \in \left(Aut(\mathbb{C}^2) \setminus \mathcal{AT}\right)$ can be written as a composition of elementary and affine automorphisms. In 1989 Friedland and Milnor [8] proved that any polynomial automorphism of $\mathbb{C}^2$ is conjugated either to an elementary map or to a finite composition of Hénon maps h_j defined as follows

$$h_j(z, w) = (p_j(z) - a_j w, z), \quad a_j \in \mathbb{C},$$

where $deg(p_j) \geq 2$. We denote by $\mathcal{H}$ the semigroup generated by Hénon maps.

On the other hand, the algebraic structure of $Aut(\mathbb{C}^k)$, $k \geq 3$, is poorly understood even if the Nagata Conjecture has been recently proved [22,23]. Recently Shestakov and Umirbaev [22,23] have proved that tame and wild polynomial automorphisms of $\mathbb{C}^3$ are algorithmically recognizable. The following Nagata automorphism in $Aut(\mathbb{C}[x, y, z])$,

$$\begin{aligned}
\sigma(x) &= x + (x^2 - yz)z, \\
\sigma(y) &= y + 2(x^2 - yz)x + (x^2 - yz)^2 z, \\
\sigma(z) &= z
\end{aligned}$$

provides a candidate of such wild automorphisms.

We recall now some general facts. Let $z = (z_1, \cdots, z_k)$ be affine coordinates in $\mathbb{C}^k$ and let $[z : t] = [z_1 : \cdots : z_k : t]$ be corresponding homogeneous coordinates in $\mathbb{P}^k$, then the hyperplane at infinity Π_∞ has equation $\{t = 0\}$.

Each polynomial automorphism f of $\mathbb{C}^k$ can be considered as a birational map $\overline{f}$ of $\mathbb{P}^k$. We will denote, respectively, I_f^+ and I_f^- the indeterminacy subsets of $\overline{f}$ and of $\overline{f^{-1}}$. These are two analytic and algebraic subsets of complex codimension at least 2 in $\mathbb{P}^k$, contained in Π_∞. In the sequel we are going to write f instead of $\overline{f}$. In a point $p \in I_f^+$ it is possible to define the *blow-up* of f in p which is the set

$$\mathcal{B}_p^f = \bigcap_{\varepsilon > 0} \overline{f(\mathbb{B}(p, \varepsilon) \setminus I_f^+)}$$

In other words it is the fiber over p in the closure of the graph of f and it is an analytic subset of Π_∞ of dimension h with $1 \leq h \leq (k-1)$. We will say, [20], that f is an *algebraically stable* polynomial automorphism if and only if $\overline{f^n}(\{[z : 0]\} \setminus I_{f^n}^+)$ is not contained in I_f^+ for any $n > 0$; it follows $deg(f^n) = (deg f)^n$. Elements in $\mathcal{H}$ are algebraically stable. When f is algebraically stable, one can associate to f a Green function $G_f^+(z) = \lim_{n\to+\infty} \frac{1}{d^n} \log^+ |f^n(z)|$. If we define $T_f^+ = dd^c G_f^+$, one can show that T_f^+ is a non-zero, positive, closed, (1,1)-current, the so called Green current, [20].

Definition 1.1 [20] A polynomial automorphism is *regular* if $I_f^+ \cap I_f^- = \emptyset$.

Elements in $\mathcal{H}$ are regular. Observe that the notion of regular automorphism depends on the choice of coordinates because the coordinate changes allowed are all polynomial automorphisms of $\mathbb{C}^k$ and if we conjugate a regular automorphism by a polynomial automorphism, the action on the hyperplane at infinity is not under control, so the new automorphism is not necessarily regular.

A regular automorphism is in particular algebraically stable. When f is an algebraically stable polynomial automorphism of $\mathbb{C}^k$ we define inductively the following analytic sets X_f^j :

$$X_f^1 = \overline{f(\{[z:0]\} \setminus I_f^+)}, \ldots, X_f^{j+1} = \overline{f(X_f^j \setminus I_f^+)} \ldots$$

This sequence is decreasing and X_f^j is non-empty because f is algebraically stable. Hence it is stationary. Let X_f^+ be the corresponding limit set. Replacing f by an appropriate iterate, we can always assume that $X_f^+ = \overline{f(\{[z:0]\} \setminus I_f^+)}$. Analogously we construct X_f^- when f^{-1} is algebraically stable. Observe that X_f^+ and X_f^- are always contained in the hyperplane at infinity, Π_∞. We recall also another weaker notion of regularity, [9], that will be useful in the sequel.

Definition 1.2 A polynomial automorphism f of $\mathbb{C}^k$ is called *weakly regular* when $X_f^+ \cap I_f^+ = \emptyset$.

It follows from the definition, that a weakly regular automorphism is algebraically stable. Moreover X_f^+ is an attracting set for f, i.e. there exists an open neighborhood V of X_f^+ in $\mathbb{P}^k$, such that $f(V) \subset\subset V$ and $\cap_{j=1}^{+\infty} f^j(V) = X_f^+$.

We study, in this paper, the equation $f \circ g = g \circ f$ for polynomial automorphisms of $\mathbb{C}^k$, $k \geq 3$, when f is regular. We have studied the case $k = 2$ in [1]. The previous paper [1] and the present one discuss in particular a question of Smale (see [21]). More precisely we will prove that each automorphism of $\mathbb{C}^2$ and $\mathbb{C}^3$ in an appropriate irreducible component of the space of polynomial automorphisms of degree d, is the limit with respect to the uniform convergence on compact sets, of a sequence of regular automorphisms with trivial centralizer.

In our setting f is a regular polynomial automorphism of $\mathbb{C}^3$ and we can always suppose that $\dim I_f^+ = 1$ and $\dim I_f^- = 0$, which implies that $I_f^+ = X_f^-$ is an irreducible curve and $I_f^- = X_f^+$ is one point, [20].

The Main Lemma that we are going to prove is the following:

Main Lemma 1.3 *Let f be a regular polynomial automorphism of $\mathbb{C}^3$. Let g be a polynomial automorphism of $\mathbb{C}^3$, not affine and let g be commuting with f, then, up to changing g with g^{-1} or with $(f \circ g^{-1})$ or with $(f \circ g)$, g is* weakly regular *with $I_g^+ = I_f^+$.*

Following [9], we recall that all the positive iterates of a weakly regular automorphism are still weakly regular automorphisms.

The main result of the paper is:

Theorem 1.4 *Let f, g be two polynomial automorphisms of $\mathbb{C}^3$, respectively of degree d_1 and d_2. Suppose that f is regular, that g is weakly regular with $I_f^+ = I_g^+$, $X_g^+ = I_f^-$ and that $f \circ g = g \circ f$. Then*

$$\textit{there exist } m_0, n_0 \in \mathbb{N} \ \ s.t. \ \ d_1^{m_0} = d_2^{n_0} \tag{1.1}$$

and there exists an affine automorphism h s.t. $f^{m_0} = g^{n_0} \circ h$ with $h^q = Id$, for a certain $q \in \mathbb{N}$.

Corollary 1.5 *In the hypothesis of the Theorem 1.4, there exist $n, m \in \mathbb{N}$ s.t. $f^n = g^m$.*

We want to point out that if, in Main Lemma 1.3 we have substitute g with g^{-1}, or $(f \circ g^{-1})$ or $(f \circ g)$, we still have this final Corollary 1.5, even if with different exponents; in any of the three substitutions of g suggested by the Main Lemma 1.3, we arrive to the same conclusion of Corollary 1.5, with g the starting one.

Main Theorem 1.1 *Let f be a regular polynomial automorphism of $\mathbb{C}^3$ and let g be commuting with f. Then there exist $n, m \in \mathbb{Z}$ s.t. $f^n = g^m$.*

This means that $C(f) \simeq \mathbb{Z} \rtimes \mathbb{Z}_q$, for a certain $q \in \mathbb{N}$; more specifically, the affine centralizer of f, $C_{\mathcal{A}}(f)$, is identified with $\mathbb{Z}_q$ and f^n with $\mathbb{Z}$, via $f^n \to n \in \mathbb{Z}$.

2 Study of the blow-up of f

In this preliminary paragraph we are going to list some properties of the blow-up of a regular polynomial automorphism f and to show how the blows-up of f change under g, assuming $f \circ g = g \circ f$. We have seen that if f is a regular polynomial automorphism of $\mathbb{C}^3$, we can assume that I_f^- is a point and I_f^+ is a curve. We then have:

Proposition 2.1 1. $\mathcal{B}_{I_f^-}^{f^{-1}} = \Pi_\infty$;

2. *for each $\alpha \in I_f^+$, $\mathcal{B}_\alpha^f = \overline{\{q \neq I_f^- \,|\, f^{-1}(q) = \alpha\}} \ni I_f^-$;*
3.

$$\overline{\bigcup_{\alpha \in I_f^+} \mathcal{B}_\alpha^f} = \Pi_\infty. \tag{2.1}$$

Proof 1. Let $\{q_n\}_{n \in \mathbb{N}}$ be an arbitrary sequence of points in $\mathbb{C}^3$ s.t. $q_n \to q_\infty \in (\Pi_\infty \setminus I_f^+)$ where q_∞ is an arbitrary point in $(\Pi_\infty \setminus I_f^+)$. Then $\{f(q_n)\}_{n \in \mathbb{N}}$ is a well defined sequence of points p_n s.t. $p_n \to I_f^-$, because $f(q_\infty) = I_f^-$. Moreover

$$\begin{aligned} &f^{-1}(p_n) = q_n \to q_\infty \\ &p_n \to I_f^- \end{aligned} \tag{2.2}$$

hence $q_\infty \in f^{-1}(I_f^-)$ in the sense of blow-up.

These two facts imply that $\forall q_\infty \in (\Pi_\infty \setminus I_f^+)$, $q_\infty \in \mathcal{B}_{I_f^-}^{f^{-1}} = \bigcap_{\varepsilon > 0} \overline{f^{-1}(\mathbb{B}(I_f^-, \varepsilon) \setminus I_f^-)}$.

2. We start proving that the inverse image under f^{-1} of α is contained in $\mathcal{B}_\alpha^f$. Let q be s.t. $f^{-1}(q) = \alpha$. Let $\zeta_n \in \mathbb{C}^3$ s.t. $\zeta_n \to q$, then $\zeta_n = f(f^{-1}(\zeta_n))$ hence there exists a sequence $\varepsilon_n = f^{-1}(\zeta_n) \to \alpha$ s.t. $f(\varepsilon_n) \to q$ hence $q \in \mathcal{B}_\alpha^f$.

Conversely if $q \in \mathcal{B}_\alpha^f \setminus I_f^-$, there exists a sequence $\zeta_n \to \alpha$ s.t. $f(\zeta_n) \to q$. Hence $f^{-1}(f(\zeta_n)) = \zeta_n \to f^{-1}(q)$ and therefore $f^{-1}(q) = \alpha$.
3. Let $\zeta_n \to q \neq I_f^-$ and s.t. $f^{-1}(\zeta_n) \to \alpha \in I_f^+$. Hence $\zeta_n = f(f^{-1}(\zeta_n)) \to q$. Therefore there exists a sequence $\xi_n \to \alpha$ s.t. $f(\xi_n) \to q$ and so $q \in \mathcal{B}_\alpha^f$. Hence $\bigcup_{\alpha \in I_f^+} \mathcal{B}_\alpha^f \supset (\Pi_\infty \setminus I_f^-)$ and taking the closure we have that $\overline{\bigcup_{\alpha \in I_f^+} \mathcal{B}_\alpha^f} = \Pi_\infty$. □

An example of how a map g, which commutes with f regular, acts on the blows-up of f, is the following proposition:

Proposition 2.2 *Suppose that f, g are two commuting polynomial automorphisms of $\mathbb{C}^3$. Let f be a regular automorphism and suppose g not affine. Then for each $\alpha \in I_f^+ \setminus I_g^+$ we have that $g(\mathcal{B}_\alpha^f \setminus I_g^+) \subseteq \mathcal{B}_{g(\alpha)}^f$.*

Proof We start recalling that $g(I_f^+ \setminus I_g^+) \subset I_f^+$ because I_f^+ is a f^{-1}−invariant set and g commutes with f^{-1}. For each $q \in \mathcal{B}_\alpha^f$ there exists a sequence $\xi_n \to \alpha$ s.t. $f(\xi_n) \to q$ and $g(f(\xi_n)) = f \circ g(\xi_n)$. Then for $n \to +\infty$, if $q \in \mathcal{B}_\alpha^f \setminus I_g^+$, $g(q) = f \circ g(\alpha)$. Hence $g(\mathcal{B}_\alpha^f \setminus I_g^+) \subseteq \mathcal{B}_{g(\alpha)}^f$. □

3 Characterization of commuting polynomial automorphisms of $\mathbb{C}^k$, $k \geq 3$

Lemma 3.1 *Suppose that f, g are two commuting polynomial automorphisms of $\mathbb{C}^3$. Let f be a regular automorphism and suppose g not affine. Then:*

1. *I_g^+ cannot contain both I_f^+ and I_f^-;*
2. *if I_g^- doesn't contain I_f^-, then $I_{f \circ g^{-1}}^+ \subset I_f^+$;*
3. *if $I_g^+ \supseteq I_f^+$, then $g(\Pi_\infty \setminus I_g^+) = I_f^-$.*

Proof

1. We have that $f(\Pi_\infty \setminus I_f^+) = I_f^- = X_f^+$ and $f^{-1}(\Pi_\infty \setminus I_f^-) = X_f^- = I_f^+$ and by hypothesis $I_f^- \cap I_f^+ = \emptyset$. Consider now $X_g^1 = g(\Pi_\infty \setminus I_g^+)$: it is a connected and irreducible set (because it is the image via a holomorphic map of an analytic, connected and irreducible set). Hence $g(\Pi_\infty \setminus I_g^+)$ cannot be contained in both I_f^+ and I_f^- because it is connected and irreducible and I_f^+, I_f^- are disjoint.
   If $X_g^1 \not\subset I_f^+$ then $deg(f \circ g) = deg(f) \times deg(g)$, by Proposition 1.4.3. of [20]; $f \circ g = g \circ f$ and therefore $deg(g \circ f) = deg(g) \times deg(f)$ which implies that $I_f^- \not\subset I_g^+$.
   If $X_g^1 \subset I_f^+$ then by the previous observation $X_g^1 \not\subset I_f^-$ and hence $deg(f^{-1} \circ g) = deg(f^{-1}) \times deg(g)$ and by $f^{-1} \circ g = g \circ f^{-1}$ it follows that $deg(g \circ f^{-1}) = deg(g) \times deg(f^{-1})$, which means that $I_f^+ \not\subset I_g^+$.
   Therefore either $I_f^- \not\subset I_g^+$ or $I_f^+ \not\subset I_g^+$. In any case I_g^+ cannot contain both I_f^+ and I_f^-.
2. If I_g^- doesn't contain I_f^-, (which is a point) then the indeterminacy set of $(f \circ g^{-1})$ is contained in I_f^+. Indeed if $p \notin I_f^+$, let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence of points tending to p

when $n \to +\infty$; then $\lim_{n\to+\infty} f \circ g^{-1}(x_n) = \lim_{n\to+\infty} g^{-1} \circ f(x_n) = g^{-1}(I_f^-)$. In view of the fact that $g^{-1}(I_f^-)$ is a well defined point, we have that $p \notin I^+_{f\circ g^{-1}}$. Hence up to changing g with $f \circ g^{-1}$, which still commutes with f, and that for this reason we can continue to call g, we have proved that $I_g^+ \subset I_f^+$.

3. If $I_g^+ \supseteq I_f^+$, then $deg(g \circ f^{-1}) < deg(g) \times deg(f^{-1})$.
   Hence also $deg(f^{-1} \circ g) < deg(f^{-1}) \times deg(g)$.
   Therefore $g(\Pi_\infty \setminus I_g^+) \subset I_f^-$, but I_f^- is a point so $g(\Pi_\infty \setminus I_g^+) = I_f^-$.

□

Lemma 3.2 *Suppose that f, g are two commuting polynomial automorphisms of $\mathbb{C}^3$. Assume f be a regular automorphism and g not affine. Then either*

1. *$deg(f \circ g) < deg(f) \times deg(g)$, (i.e. $I_f^- \in I_g^+$ and $X_g^1 \subset I_f^+$),*

or

2. *$deg(f \circ g) = deg(f) \times deg(g)$ (i.e. $I_f^- \in X_g^1$).*

Proof Consider $f \circ g = g \circ f$ and recall that I_f^- is a point. Then take $g \circ f(\Pi_\infty \setminus (I_f^+ \cup I_g^+))$. We recall that $f(\Pi_\infty \setminus (I_f^+ \cup I_g^+)) = I_f^-$; indeed $f^{-1} \circ f = t^d \cdot Id$, where $deg(f) = d$ and this implies that $f^{-1} \circ f(\{t = 0\} \setminus (I_f^+ \cup I_g^+)) = (0 : \cdots : 0) \in \mathbb{C}^4$, that is $f(\Pi_\infty \setminus (I_f^+ \cup I_g^+)) = I_f^-$, [20]. Then

$$g \circ f(\Pi_\infty \setminus (I_f^+ \cup I_g^+)) = g(I_f^-) \begin{array}{l} \nearrow (i) \quad either\ I_f^- \in I_g^+ \\ \searrow (ii) \quad or \quad\quad g \circ f(\Pi_\infty \setminus (I_f^+ \cup I_g^+)) \subset X_g^1 \subset I_g^-. \end{array}$$

Now consider $f \circ g(\Pi_\infty \setminus (I_f^+ \cup I_g^+))$.

$$f \circ g(\Pi_\infty \setminus (I_f^+ \cup I_g^+)) \begin{array}{l} \nearrow (iii) \quad either\ g(\Pi_\infty \setminus (I_f^+ \cup I_g^+)) \subset I_f^+ \\ \searrow (iv) \quad or \quad\quad f \circ g(\Pi_\infty \setminus (I_f^+ \cup I_g^+)) = I_f^-. \end{array}$$

We observe that $(i) \Leftrightarrow (iii)$ and it corresponds to case 1., i.e. $deg(f \circ g) < deg(f) \times deg(g)$, i.e. to $I_f^- \in I_g^+$ and $X_g^1 \subset I_f^+$. On the other hand $(ii) \Leftrightarrow (iv)$: it corresponds to Case 2., i.e. $deg(f \circ g) = deg(f) \times deg(g)$, i.e. it corresponds to $I_f^- \in X_g^1 \subset I_g^-$. Indeed $f \circ g = g \circ f$ and if and only if at one side of the equality the first map has image contained in the indeterminacy set of the second one, the same has to happen at the other side of the equality. In this case $deg(f \circ g) < deg(f) \times deg(g)$ because there are cancellations in the projective coordinates of $f \circ g(z)$, as proved in [7].

In particular, it cannot happen that $I_f^- \subset (X_g^1 \cap I_g^+)$, because in this case we would have $I_f^+ \supseteq X_g^1 \ni I_f^-$ and this is impossible because f is regular.

The same proof applied with g^{-1} instead of g, gives that either

1. $I_f^- \in X^1_{g^{-1}}$

or

2. $I_f^- \in I_g^-$ and $X^1_{g^{-1}} \subset I_f^+$.

□

Lemma 3.3 *Assume f is a regular automorphism of $\mathbb{C}^3$. Let g be a polynomial automorphism of degree greater or equal than 2, such that $f \circ g = g \circ f$. Then either*

1. $I_f^+ \subseteq I_g^+$,

or

2. $I_f^+ \subseteq I_g^-$.

Proof By Lemma 3.2 applied first to f and g, and then to f and g^{-1}, we can suppose that $I_f^- \in (I_g^+ \setminus X_g^1) \cap (X_{g^{-1}}^1 \setminus I_g^-)$ (hence automatically I_g^+ doesn't contain I_f^+, by Lemma 3.1, point 1). In view of the fact that $f(\Pi_\infty \setminus (I_f^+ \cup I_g^+)) = I_f^-$, that I_f^- is a point and by $f \circ g = g \circ f$, we have that $f \circ g(\Pi_\infty \setminus (I_f^+ \cup I_g^+)) \equiv \mathcal{B}_{I_f^-}^g$; indeed

$$f \circ g(\Pi_\infty \setminus (I_f^+ \cup I_g^+)) = g \circ f(\Pi_\infty \setminus (I_f^+ \cup I_g^+)). \tag{3.1}$$

and $f(\Pi_\infty \setminus (I_f^+ \cup I_g^+)) = I_f^-$. Then either

(1) $\mathcal{B}_{I_f^-}^g = \Pi_\infty$ and in this case, in view of equation (3.1) and (2.1), we have that $g(\Pi_\infty \setminus (I_f^+ \cup I_g^+)) = I_f^+$. Since g commutes with f^{-1} and I_f^+ is $(f^{-1})-$invariant, we have that $g(I_f^+ \setminus I_g^+) \subset I_f^+$ and hence $g(\Pi_\infty \setminus I_g^+) = I_f^+$; therefore $I_f^+ \subset I_g^-$; or

(2) $\mathcal{B}_{I_f^-}^g$ is a curve. In this case $g(\Pi_\infty \setminus (I_f^+ \cup I_g^+)) = \beta$, where β is a point in I_f^+. Then we have two possibilities for $g(\Pi_\infty \setminus I_g^+)$: either it is equal to all I_f^+ (and in this case we are done as at the previous point) or it is a point of I_f^+. In this last case the image point has to be the same point β because $g(\Pi_\infty \setminus I_g^+)$ is connected. Moreover $\mathcal{B}_\beta^{g^{-1}} = \Pi_\infty = f^{-1} \circ g^{-1}(\Pi_\infty \setminus I_f^-)$ because $f^{-1} \circ g^{-1}(\Pi_\infty \setminus I_f^-) = g^{-1} \circ f^{-1}(\Pi_\infty \setminus I_f^-) = g^{-1}(I_f^+) \supseteq g^{-1}(\beta)$. This implies that $g^{-1}(\Pi_\infty \setminus (I_f^-) = I_f^-$, i.e. that $I_f^- = X_{g^{-1}}^1$ and this is a contradiction because $X_{g^{-1}}^1 \ni I_f^-$ but they are different. If they were equal we would have $\mathcal{B}_{I_f^-}^g = \Pi_\infty$.

In conclusion, in the case $I_f^- \in (I_g^+ \setminus X_g^1) \cap (X_{g^{-1}}^1 \setminus I_g^-)$ we have $I_f^+ \subset I_g^-$. □

These three first Lemmas are necessary in order to prove that if g commutes with a regular f, up to changing g with another element of the centralizer of f, g is weakly regular with $I_g^+ = I_f^+$.

Main Lemma 3.4 *Let f be a regular automorphism of $\mathbb{C}^3$. Let g be a polynomial automorphism of $\mathbb{C}^3$, not affine and let g be commuting with f, then, up to changing g with g^{-1} or with $(f \circ g^{-1})$ or with $(f \circ g)$, g is* weakly regular *with $I_g^+ = I_f^+$.*

Proof We want to show that one of the four among g, g^{-1}, $(f \circ g^{-1})$ and $(f \circ g)$ is weakly regular. We have already seen that in the case $I_f^- \in (I_g^+ \setminus X_g^1) \cap (X_{g^{-1}}^1 \setminus I_g^-)$, then $I_f^+ \subset I_g^-$.

If $I_g^- = I_f^+$, then g^{-1} is weakly regular because $X_g^- = I_f^-$, by Lemma 3.1 point 3. Analogously in the case $I_f^- \in (I_g^- \setminus X_{g^{-1}}^1) \cap (X_g^1 \setminus I_g^+)$, we have $I_f^+ \subset I_g^+$.

If $I_g^+ = I_f^+$, then g is weakly regular because $X_g^+ = I_f^-$ by Lemma 3.1 point 3.

On the other hand, if neither $I_g^- \not\subseteq I_f^+$ nor $I_g^+ \not\subseteq I_f^+$, we can consider either $(f \circ g^{-1})$ or $(f \circ g)$ instead of g^{-1} or g, depending on the position of I_f^-. Indeed by Lemma 3.3 and by Lemma 3.1 point 1, we can suppose that either $I_f^- \notin I_g^-$ or $I_f^- \notin I_g^+$.

Suppose for example that $I_f^- \notin I_g^-$, then $I_{f \circ g^{-1}}^+ \subseteq I_f^+$ by Lemma 3.1 point 2, and $X_{f \circ g^{-1}}^+ = I_f^-$ since $f \circ g^{-1} = g^{-1} \circ f$ and $f(\Pi_\infty \setminus I_f^+) = I_f^- \notin I_g^-$. Hence $(f \circ g^{-1})$ is weakly regular.

Analogously if $I_f^- \notin I_g^+$, we have $(f \circ g)$ is weakly regular. Therefore if either g or g^{-1} is not weakly regular, then either $(f \circ g^{-1})$ or $(f \circ g)$ is weakly regular, depending on the position of I_f^-, because I_f^+ and I_f^- are disjoint. □

Theorem 3.5 *Let $f, g \in Aut(\mathbb{C}^3)$, s.t. $f \circ g = g \circ f$. Suppose $d_1 = deg(f)$, and $d_2 = deg(g) \geq 2$. Let f be a regular automorphism and g a weakly regular automorphism with $I_g^+ = I_f^+$ and $X_g^+ = I_f^-$. Then the Green functions of f and g coincide.*

Proof In view of the fact that U_g^+ is f and f^{-1}−invariant and U_f^+ is g and g^{-1}-invariant, it holds $U_g^+ = U_f^+$.

Indeed we can choose a neighborhood B of $I_f^- = X_g^+$ s.t. $B \subset U_f^+ \cap U_g^+$, $g(B) \subset B$ and also $f(B) \subset B$, since the point $X_g^+ = I_f^-$ is an attracting point for both f and g. Consider $x \in U_f^+$, then for n sufficiently large $f^n(x) \in B$ and $g \circ f^n(x) \in B$, for the g-invariance of B. Hence $f^n \circ g(x) \in B \subseteq U_f^+$ and applying f^{-n} to both sides we obtain that $g(x) \in U_f^+$. Therefore U_f^+ is g-invariant.

In the same way, we can prove that $f^{-1}(U_g^+) \subseteq U_g^+$, because $U_g^+ = \bigcup_{n\in\mathbb{N}} g^{-n}(f^{-1}(B)) = \bigcup_{n\in\mathbb{N}} g^{-n}(B)$. Now $B \subseteq U_g^+$, hence, for what we have already proved, $f^{-n}(B) \subseteq U_g^+$ but $\bigcup_{n\in\mathbb{N}} f^{-n}(B) = U_f^+$, therefore $U_f^+ \subseteq U_g^+$. Analogously we prove that $U_g^+ \subseteq U_f^+$.

Consider now the sequence of maps:

$$H_n = \frac{G_f^+ \circ g^n}{d_2^n} \geq 0$$

for all $n \in \mathbb{N}$. We have that each $H_n \leq G_f^+$ because H_n and G_f^+ satisfy the same functional equation:

$$H_n \circ f = d_1 \cdot H_n$$

and G_f^+ is the largest solution [20] of this equation among the p.s.h. functions bounded by $\log^+|z| + O(1)$ at infinity. Then, in a neighborhood of $X_f^+ = I_f^-$, outside K_f^+, it holds:

$$\log^+|z| + c_2 \leq G_f^+(z) \leq \log^+|z| + c_1, \tag{3.2}$$

because f is regular.

By hypothesis and by Lemma 3.1 point 3, $X_g^+ = I_f^-$ and $U_f^+ = U_g^+$, therefore, composing with $g^n(z)$, we have:

$$\frac{\log^+|g^n(z_1, z_2, z_3)| + c_2}{d_2^n} \leq \frac{G_f^+ \circ g^n}{d_2^n} \leq \frac{\log^+|g^n(z_1, z_2, z_3)| + c_1}{d_2^n}. \tag{3.3}$$

Then we have that the first and the last term of the sequence of inequalities in (3.3) tend to $G_g^+(z_1, z_2, z_3)$.

Therefore $\lim\limits_{n\to+\infty} \dfrac{G_f^+ \circ g^n}{d_2^n} = G_g^+$ on $U_f^+ = U_g^+$.

On all $\mathbb{C}^3$, we have

$$0 \leq H_n \leq G_f^+. \tag{3.4}$$

Hence on U_f^+, $\lim_{n\to+\infty} H_n = G_g^+ \leq G_f^+$.

On the other hand, on K_f^+, $G_f^+ = 0$ hence also $\lim_{n\to+\infty} H_n$ exists and it is equal to zero by (3.4).

In conclusion:

1. $G_g^+ \leq G_f^+$ on $U_f^+ = U_g^+$;
2. $G_g^+ = G_f^+ = 0$ on $K_f^+ = \mathcal{K}_g^+$, where $\mathcal{K}_g^+ = (U_g^+)^c$.

Changing f regular with g weakly regular, the inequalities (3.2) still hold (see [9]), and we obtain $G_f^+ \leq G_g^+$ on $U_f^+ = U_g^+$. Finally $G_f^+ = G_g^+$ on all $\mathbb{C}^3$. □

We need to describe the group of affine automorphisms commuting with f.

Proposition 3.6 *Let $C_{\mathcal{A}}(f)$ be the group of affine automorphisms of $\mathbb{C}^3$ which commute with f. If f is regular, then $C_{\mathcal{A}}(f)$ is a finite cyclic subgroup of $\mathcal{A}$.*

Proof Recall, [20] p. 132, that a regular biholomorphism f has infinitely many distinct periodic orbits (this follows from Bézout theorem), and that no subvariety of dimension greater than or equal to 1 is periodic.

First we want to prove that all the periodic points of f cannot lie on the same complex linear subspace. Suppose on the contrary that there exists a complex linear subspace L of dimension 1 or 2, s.t. $\bigcup_{n\in\mathbb{Z}} Fix(f^n) \subset L$ (indeed a periodic point for f is a periodic point also for f^{-1} of the same period). Of course $L \not\subseteq \Pi_\infty$ and L is at the same time f-invariant and f^{-1}-invariant. Let $V = L \cap \Pi_\infty$, then V has to be contained into I_f^-, because $f(I_f^-) = I_f^-$, and it has also to be contained into I_f^+ because $f^{-1}(I_f^+) = I_f^+$. Since $I_f^+ \cap I_f^- = \emptyset$, this is a contradiction.

If h is affine and $f \circ h = h \circ f$, then, for all $N \in \mathbb{N}$, h induces a permutation on $Fix(f^N) = \{$periodic points of order less or equal to N for f$\}$. So we have a group homomorphism φ from $C_{\mathcal{A}}(f)$ into the group Σ_N of the permutations of the points of $Fix(f^N)$:

$$\varphi : C_{\mathcal{A}}(f) \to \Sigma_N$$

If N is large enough, the points of $Fix(f^N)$ do not lie on the same subspace and hence φ is injective. Hence $C_{\mathcal{A}}(f)$ is a finite group of a suitable order q.

To prove the cyclicity of $C_{\mathcal{A}}(f)$, we prove that:

(1) $C_{\mathcal{A}}(f)$ is abelian;
(2) the eigenvalues of the linear part of each affine automorphism $h \in C_{\mathcal{A}}(f)$ are roots of unity of the same order;
(3) for all $h_1, h_2 \in C_{\mathcal{A}}(f)$ of the same order q, there exist $n_0, m_0 \in \mathbb{N}$ s.t. $h_1^{n_0} = h_2$ and $h_2^{m_0} = h_1$.

In order to prove (1), we recall that if $h \circ f = f \circ h$, then $h(I_f^-) = I_f^-$ and $h(I_f^+) = I_f^+$. Then, up to conjugation, we can assume that $I_f^- = [0 : 0 : z_3 : 0]$. In these coordinates:

$$h([x_1 : x_2 : x_3 : t]) = [a_1x_1 + b_1x_2 + d_1t : a_2x_1 + b_2x_2 + d_2t : a_3x_1 + b_3x_2 + c_3x_3 + d_3t : t].$$

$$A_1 = \begin{pmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

has to be diagonalizable (with eigenvalues which are roots of unity) because it satisfies $A_1^n = Id$. Hence up to conjugation with an affine map which preserves I_f^+ and I_f^-, we can suppose that A_1 is diagonal. Consider now the commutator $[h_1, h_2]$ of two maps $h_1, h_2 \in C_{\mathcal{A}}(f)$, then its linear part in $\mathbb{C}^3$ is the identity 3×3 matrix, because the linear part of each of them is diagonal. But $[h_1, h_2]$ cannot be a translation of $\mathbb{C}^3$ because $C_{\mathcal{A}}(f)$ is a finite group. Hence the only possibility is $[h_1, h_2] = Id$.

Since $C_{\mathcal{A}}(f)$ is abelian, it follows that all the elements in $C_{\mathcal{A}}(f)$ have a common fixed point, hence, up to conjugation, we can suppose that they are all rotations fixing the origin, therefore they are of type $(\alpha x_1, \beta x_2, \gamma x_3)$.

In order to prove (2), we recall that, since the order of the group $C_{\mathcal{A}}(f)$ is q, then for all $h \in C_{\mathcal{A}}(f)$ there exists $k \in \mathbb{N}$ which divides q s.t. $h^k = Id$. This means that $\alpha^k = \beta^k = \gamma^k = 1$, and the eigenvalues of h are k-roots of unity. But suppose that they have different orders, then there exists a $n \in \mathbb{N}$ which divides k s.t. h^n is the identity in one component but not in the other one. Suppose that $\alpha^n = 1$ and $\beta^n, \gamma^n \neq 1$. This means that all the points $(x_1, 0, 0)$ are fixed by h^n. Since for all $m \in \mathbb{Z}$, f^m commutes with h^n, the line $\{x_2 = x_3 = 0\}$ is invariant for all f^m, with $m \in \mathbb{Z}$. For the invariance of the line $\{x_2 = x_3 = 0\}$ by f and by f^{-1}, it follows that $V = \{x_2 = x_3 = 0\} \cap L_\infty$ has to be contained into I_f^+ and at the same time into I_f^-, but this contradicts the regularity of f.

The assertion in (3) follows directly from (1) and (2) : since the order q of the rotation is exactly the common order of its eigenvalues, there exist a $n_0 \in \mathbb{N}$ s.t. $h_1^{n_0} \circ h_2^{-1}$ has an eigenvalue equal to 1. But $h_1^{n_0} \circ h_2^{-1}$ is still an element in $C_{\mathcal{A}}(f)$ and hence its three eigenvalues have the same order; this implies also that the second and the third eigenvalue have to be equal to 1 and $h_1^{n_0} = h_2$.

The cyclicity of the group $C_{\mathcal{A}}(f)$ follows from (1), (2), (3). If h_0 is one of the elements of $C_{\mathcal{A}}(f)$ of maximal order $s \leq q$, then $\langle h_0 \rangle = C_{\mathcal{A}}(f)$. Indeed for each $h \in C_{\mathcal{A}}(f)$, the order of h has to be a divisor of the maximal order s; hence there exists an element in $\langle h_0 \rangle$, h_0^r, which has the same order of h, but, by (3), h is a power of h_0^r and so $h \in \langle h_0 \rangle$. In conclusion $C_{\mathcal{A}}(f)$ is isomorphic to $\mathbb{Z}_q$. □

The proof of the Theorem 3.7 below uses the same technique that the author has used in dimension 2, in the proof of Theorem 1.5., see [1].

Theorem 3.7 *Let f, g be two automorphisms of $\mathbb{C}^3$, respectively of degree d_1 and d_2. Suppose that f is regular, that g is weakly regular with $I_f^+ = I_g^+$, $X_g^+ = I_f^-$ and that $f \circ g = g \circ f$. Then*

$$\textit{there exist} \quad m_0, n_0 \in \mathbb{N} \quad s.t. \ d_1^{m_0} = d_2^{n_0} \tag{3.5}$$

and there exists an affine automorphism h s.t. $f^{m_0} = g^{n_0} \circ h$ with $h^q = Id$, for a certain $q \in \mathbb{N}$.

Proof **(a)** First of all, we want to prove that, for all $m, n \in \mathbb{N}$ one and only one of the four following cases can occur:

(i) d_2^n divides d_1^m;
(ii) d_1^m divides d_2^n;
(iii) d_2^n divides d_1^{m+1};
(iv) d_1^{m-1} divides d_2^n.

(b) Then we will prove that in all the four cases (i), (ii), (iii) and (iv) there exist $n_0, m_0 \in \mathbb{N}$ s.t. $d_1^{m_0} = d_2^{n_0}$.

(a) Given $n, m \in \mathbb{N}$, consider $h = g^{-n} \circ f^m$ which is a polynomial automorphism of $\mathbb{C}^3$ which commutes with a regular one, i.e. f; hence by Lemma 1.3, h is affine or h is weakly regular or h^{-1} or $(f \circ h^{-1})$ or $(f \circ h)$ is weakly regular with the same I^+ of f.

Suppose h affine. Then $deg(h) = 1$ and there is no hypersurface mapped by h into $I^+_{g^n}$. Hence, from $f^m = g^n \circ h$, it follows that $deg(f^m) = deg(g^n \circ h) = deg(g^n) \times deg(h)$; then $deg(f^m) = deg(g^n)$ and $d_1^m = d_2^n$.

Suppose h is weakly regular of degree δ. Let H be the Green function of h, i.e. $H = \lim_{n \to +\infty} \frac{1}{\delta^n} \log^+(|h^n(z_1, z_2, z_3)|)$. Then $H(h) = \delta \cdot H$.

On the other hand, by hypothesis and by Theorem 3.5, $G_f^+ = G_g^+ = G^+$ hence

$$G^+(f^m \circ g^{-n}) = d_1^m \times G^+(g^{-n}) = \frac{d_1^m}{d_2^n} G^+ \tag{3.6}$$

But h commutes with f which is regular, therefore they have the same Green function by Theorem 3.5.

Hence $H = G^+$ and $\delta = \left(\frac{d_1^m}{d_2^n}\right)$.

But $\delta \in \mathbb{N}$ because $\delta = deg(h)$, hence $\frac{d_1^m}{d_2^n} \in \mathbb{N}$ and d_2^n divides d_1^m and we have proved (i).

Suppose h^{-1} weakly regular, then repeating the same argument with $h := g^n \circ f^{-m}$, we have that d_1^m divides d_2^n, and we have proved (ii).

Analogously if $f \circ h$ is weakly regular, then we have (iii) and if $(f \circ h^{-1})$ is weakly regular then we have (iv).

(b) Now we want to prove the second main point, we mean that there exist n_0, m_0 s.t. $d_1^{m_0} = d_2^{n_0}$.

Suppose on the contrary that for all $n, m \in \mathbb{N}$, $d_1^m \neq d_2^n$. Then $\frac{d_1^m}{d_2^n} \neq 1$, and $\log(\frac{d_1^m}{d_2^n}) \neq 0$, then $\alpha = \frac{\log(d_2)}{\log(d_1)}$ is irrational.

So, for every $\varepsilon > 0$, there is $n, m \in \mathbb{N}$ s.t.

$$\left|\alpha - \frac{m}{n}\right| < \frac{\varepsilon}{n}.$$

Multiplying both the sides of the inequality by $\frac{n}{m}$, we obtain:

$$\left|\frac{\log(d_2^n)}{\log(d_1^m)} - 1\right| < \frac{\varepsilon}{m},$$

which is equivalent to:

$$|\log(d_2^n) - \log(d_1^m)| = \left|\log\left(\frac{d_2^n}{d_1^m}\right)\right| < \frac{\varepsilon}{m} \cdot m \cdot \log(d_1) = \varepsilon \log(d_1).$$

If d_1^m divides d_2^n, i.e. (ii), and they are different, then $\frac{d_2^n}{d_1^m}$ is an integer greater or equal to 2, hence:

$$\log(2) \leq \left|\log\left(\frac{d_2^n}{d_1^m}\right)\right| < \varepsilon \times \log(d_1),$$

a contradiction, if ε is sufficiently small.

If d_2^n divides d_1^m, i.e. (i), and they are different, then $\frac{d_1^m}{d_2^n}$ is an integer greater or equal to 2, hence:

$$\log(2) \leq \left|\log\left(\frac{d_1^m}{d_2^n}\right)\right| < \varepsilon \times \log(d_1),$$

a contradiction, if ε is sufficiently small.

If d_2^n divides d_1^{m+1}, i.e. (iii), and they are different, then $\frac{d_1^{m+1}}{d_2^n}$ is an integer greater or equal to 2, hence:

$$\log(\frac{2}{d_1}) \leq \left|\log\left(\frac{d_1^{m+1}}{d_2^n} \times \frac{1}{d_1}\right)\right| < \varepsilon \times \log(d_1),$$

a contradiction, if ε is sufficiently small.

If d_1^{m-1} divides d_2^n, i.e. (iv), and they are different, then $\frac{d_2^n}{d_1^{m-1}}$ is an integer greater or equal to 2, hence:

$$\log(\frac{2}{d_1}) \leq \left|\log\left(\frac{d_2^n}{d_1^{m-1}} \times \frac{1}{d_1}\right)\right| < \varepsilon \times \log(d_1),$$

a contradiction, if ε is sufficiently small.

Consider now:
$h := f^{m_0} \circ g^{-n_0}$, then h is affine or weakly regular with $I_h^+ = I_f^+$, up to changing h with its inverse or with $(f \circ h^{-1})$ or with $(f \circ h)$. If h were not affine and it were weakly regular, then by $f^{m_0} = h \circ g^{n_0} = g^{n_0} \circ h$ we would obtain $deg(f^{m_0}) = deg(h) \times deg(g^{n_0})$ because $I_h^+ = I_f^+$ which is disjoint from X_g^+ by hypothesis. This means that $d_1^{m_0} = deg(h) \times d_2^{n_0} = deg(h) \times d_1^{m_0}$ which implies $deg(h) = 1$ contradicting the assumption.

If h^{-1} were weakly regular then $h^{-1} = g^{n_0} \circ f^{-m_0}$, then $h^{-1} \circ f^{m_0} = g^{n_0}$ but $I_f^- \notin I_h^- = I_f^+$ hence $deg(h^{-1}) \times d_1^{m_0} = d_2^{n_0}$, which implies that $deg(h^{-1}) = 1$, which is a contradiction.

If $f \circ h$ were weakly regular then $f \circ h = g^{-n_0} \circ f^{m_0+1}$, i.e. $f \circ h \circ g^{n_0} = f^{m_0+1}$ but $I_f^- = X_g^+ \notin I_{f\circ h}^+ = I_f^+$ hence $deg(f \circ h) \cdot d_2^{n_0} = d_1^{m_0+1}$, which implies that $deg(f \circ h) = d_1$, but it is also equal to $deg(h) \times deg(f)$ by $I_{f\circ h}^+ = I_f^+$ and this contradicts h not affine.

If $f \circ h^{-1}$ were weakly regular then $f \circ h^{-1} = g^{n_0} \circ f^{-m_0+1}$, i.e. $f \circ h^{-1} \circ f^{m_0-1} = g^{n_0}$ but $I_f^- \notin I_{f\circ h^{-1}}^+ = I_f^+$, hence $deg(f \circ h^{-1}) \times d_1^{m_0-1} = d_2^{n_0}$, which implies that $deg(f \circ h^{-1}) = d_1$, but it is also equal to $deg(f) \times deg(h^{-1})$ by $I_{f\circ h^{-1}}^+ = I_f^+$ and this contradicts that h^{-1} is not affine. □

Corollary 3.8 *In the hypothesis of the Theorem 1.4, there exist $n, m \in \mathbb{N}$ s.t. $f^n = g^m$.*

As a consequence of Corollary 3.8, we have that even if we substitute g with g^{-1} or $(f \circ g)$ or $(f \circ g^{-1})$, in order to have a weakly regular map, the starting g is also weakly regular, a posteriori, because a certain positive iterate of it is regular.

Proposition 3.9 *If there exists $q \in \mathbb{N}$ s.t. f^q is a regular automorphism, then f is a weakly regular automorphism.*

Proof f^q regular implies that $I^+_{f^q} \cap I^-_{f^q} = \emptyset$. We show that it is not possible that $I^+_f \cap X^+_f \neq \emptyset$, because $I^+_f \subset I^+_{f^q} \subseteq \bigcup_{l=0}^{q-1} f^{-l}(I^+_f)$ and $X^+_f \subset X^+_{f^q} = I^-_{f^q}$. Hence, if it were $(X^+_f \cap I^+_f)$ not empty, then $(I^+_{f^q} \cap I^-_{f^q})$ would be not empty, which is a contradiction. □

This final Theorem 3.7 implies also that, among the non-affine automorphisms, only weakly regular automorphisms with a weakly regular inverse can commute with a regular automorphism in $\mathbb{C}^3$. Indeed the fact that $g^m = f^n$, forces also that $g^{-m} = f^{-n}$ which means that g^{-m} is regular and g^{-1} is weakly regular by Proposition 3.9. This is a quite restrictive condition on the centralizer of a regular f, because not all weakly regular automorphisms k have a weakly regular inverse: in this case indeed it holds $d_+ = d_-^2$ where $d_- = deg(k^{-1})$ and $d_+ = deg(k)$.

If in the Theorem 3.7 we were obliged to do one of the three substitutions of g suggested by Main Lemma 3.4, we would still have the relation (3.5) between the degrees of f and of the starting g since, if we substitute g with g^{-1}, the degree of g^{-1} is a square root of $deg(g)$; if we substitute g with $(f \circ g)$, then $deg(f \circ g) = deg(f) \times deg(g)$ and finally if we substitute g with $(f \circ g^{-1})$ we have $deg(f \circ g^{-1}) = deg(f) \times deg(g^{-1})$; these two final assertions both follows by Main Lemma 3.4.

In [21], S. Smale asks the following question: let M be a compact manifold, is each diffeomorphism of M approximated by diffeomorphisms which commute only with their iterates?

We are now ready to answer to a variant of the question, in a positive way, for polynomial automorphisms of $\mathbb{C}^2$ and partially for polynomial automorphisms of $\mathbb{C}^3$ in appropriate irreducible components, i.e. in the irreducible components with al least one regular automorphism, of the affine algebraic variety of polynomial automorphisms of degree at least d, $\mathcal{A}_d$. Indeed if f is a polynomial automorphism of degree d, in [20] has been proved that $deg(f^{-1}) \leq d^{k-1}$, which means that, from $(f \circ f^{-1}) = Id$, $\mathcal{A}_d$ is an analytic and algebraic set, see [20,8].

Each polynomial automorphism of degree d of $\mathbb{C}^k$, $k \geq 2$, which is in a connected component with regular maps, is the uniform limit on compact sets of regular automorphisms of $\mathbb{C}^k$ of the same degree d. Indeed, if we fix the degree, the regular automorphisms are a Zariski open set in $\mathcal{A}_d$ because $I^+ \cap I^- = \emptyset$ is an open condition; hence they are dense in the irreducible components which contains one.

The $\mathbb{C}^2-$setting was well understood by Friedland and Milnor, [8], because they proved that $Aut(\mathbb{C}^2) = \mathcal{A} \cup G[2] \cup G[3] \cup G[2,2] \cup G[4] \cup G[5] \cup \ldots$; that is $Aut(\mathbb{C}^2)$ is a disjoint union of smooth Zariski open subsets of algebraic varieties, where $G[d_1, d_2, \cdots, d_m]$ consists of all automorphisms g of $\mathbb{C}^2$ such that some representative of the double coset $A \circ g \circ A$ can be written as a reduced word of the form $e_m \circ a_{m-1} \circ e_{m-1} \circ \cdots \circ a_2 \circ e_2 \circ a_1 \circ e_1$ with $deg(e_i) = d_i$. The sequence $(d_1, \cdots, d_m)$ is called the polydegree of g; if $g \in G[d_1, d_2, \cdots, d_m]$ then $deg(g) = d_1 \cdot d_2 \cdots d_m$. If two polynomial automorphisms are conjugated, then they have the same polydegree $(d_1, d_2, \cdots, d_m)$. We recall that it is possible to construct the following fibration (see [8]):

$$\begin{array}{lll} \pi : G[d_1, \cdots, d_m] & \to & \mathbb{P}^1 \times \mathbb{P}^1 \\ a' \circ f \circ a & \mapsto & (a^{-1}\mathcal{AT}, a'\mathcal{AT}) \end{array}$$

where a, a' are arbitrary affine maps, where $\mathbb{P}^1$ will be identified with the coset space $\dfrac{\mathcal{A}}{\mathcal{AT}}$ and where the fiber $F[d_1, \cdots d_m]$ is parametrized by an open subset of $\mathbb{P}^{d_1+d_2+\cdots d_m+4}$.

Proposition 3.10 *In each $G[d_1, \cdots, d_m]$ there is a dense set of regular automorphisms with trivial centralizer, i.e. the regular automorphisms of $G[d_1, \cdots, d_m]$ are not the empty set.*

Proof In each $G[d_1, \cdots, d_m]$ we eliminate, for all $q > 1$, the set $G^q[d_1, \cdots d_m] = \{f$ regular $C_{\mathcal{A}}(f) \simeq \mathbb{Z}_q$, with $q > 1\}$. $G^q[d_1, \cdots, d_m]$ is a Zariski closed set because $G^q[d_1, \cdots, d_m] = \Pi \circ \Psi^{-1}(1)$ where $\Psi : G[d_1, \cdots, d_m] \times \mathrm{A}ff \to G[d_1, \cdots, d_m]$ is the algebraic map $(g, a) \to g \circ a \circ g^{-1} \circ a^{-1}$, and Π is the projection on the first factor which is still algebraic. Hence getting away $\bigcup_{q>1} G^q[d_1, \cdots, d_m]$ we have eliminated a countable union of Zariski closed sets. In the complement there is at least one regular automorphism of type $h_\varepsilon(z, w) = (z^{d_1 \cdot \cdots \cdot d_m} + \varepsilon z^n - w, z)$ with n coprime with $d_1 \times d_2 \cdot \cdots \cdot d_m$ (if $d_1 \times d_2 \cdot \cdots \cdot d_m \geq 3$) and $C_{\mathcal{A}}(h_\varepsilon) = Id$; it is an easy computation because the affine maps have to be diagonal. In case $d = 2$ for $\mathbb{C}^2$, the regular polynomial automorphism $f(z, w) = (w, w^2 - az + c)$ is an example in $G[2]$ of an automorphism with $C_{\mathcal{A}}(f) = Id$. □

The $\mathbb{C}^3$-setting is not so clear but we can consider the Fornaess–Wu's classification for polynomial automorphisms of degree 2, up to affine conjugation, [5]:
$Aut(\mathbb{C}^3)_2 = H_1 \cup H_2 \cdots H_5 \cup (\mathcal{A} \cup \mathcal{E})$ where

$$
\begin{aligned}
H_1(x, y, z) &= (P(x, z) + ay, Q(z) + x, cz + d)\\
H_2(x, y, z) &= (P(y, z) + ax, Q(y) + bz, y)\\
H_3(x, y, z) &= (P(x, z) + ay, Q(x) + z, x)\\
H_4(x, y, z) &= (P(x, y) + az, Q(y) + x, y)\\
H_5(x, y, z) &= (P(x, y) + az, Q(x) + by, x)
\end{aligned}
$$

Only in H_4 and H_5 there are regular automorphisms; H_4 and H_5 are two disjoint open sets of an appropriate $\mathbb{C}^N$ and they are invariant under affine conjugation by $PGL(4, \mathbb{C})$. A simple calculation proves that the affine centralizer of the normal forms H_4 and H_5 is trivial: an affine map which commute with H_4 or H_5 has to be diagonal and equal to the identity.

4 Examples

1. The example that we are going to describe prove in a very simple situation that if f is not regular then the centralizer of f can be non trivial and also not countable. Choose h a Hénon map of $\mathbb{C}^2$. Consider the following polynomial automorphism of $\mathbb{C}^3$:

$$H(z_1, z_2, z_3) = (p(z_1) - az_2, z_1, z_3) = (h(z_1, z_2), z_3)$$

Then we are going to characterize the group of all polynomial automorphisms F of $\mathbb{C}^3$ which commute with $H : C(H) = \{F \in Aut(\mathbb{C}^3) \mid F \circ H = H \circ F\}$.

Observe that H is not regular on $\mathbb{C}^3$; indeed it is weakly regular as its inverse and $X_H^+ \cap X_H^- = \emptyset$.

In view of the fact that $F \circ H = H \circ F$, we want to prove that $F_3(z_1, z_2, z_3) = F_3(z_3)$, i.e. that the third component of F depends only on the third variable. By the commutation property and since $H_3(z_1, z_2, z_3) = z_3$, by the equation $H^n \circ F = F \circ H^n$, which holds $\forall n \in \mathbb{N}$, we have:

$$F_3(z_1, z_2, z_3) = F_3(h^n(z_1, z_2), z_3)$$

Therefore F_3 is constant on each orbit of H, which means that F_3 is constant on the $\partial K_h^+ \subset \mathbb{C}^2$. Hence $\partial K_h^+ \subset \{F_3 = constant\}$, then F_3 is constant on all $\mathbb{C}^2$ because ∂K_h^+ cannot be contained in an algebraic hypersurface.

Hence $F(z_1, z_2, z_3) = (F_1(z_1, z_2, z_3), F_2(z_1, z_2, z_3), F_3(z_3))$.

Now we fix z_3 and then the following 2 cases appear, see [1]:

(A) case (F_1, F_2) affine, which means $(F_1, F_2) = (\alpha(z_3)z_1 + \gamma(z_3), \beta(z_3)z_2 + \delta(z_3))$.

Then it is easy to see that in order to have an automorphism it is necessary that α and β do not depend on z_3 and that $F_3(z_3) = az_3 + b$, with a, b arbitrary complex number, $a \neq 0$.

(B) case (F_1, F_2) regular, which means that, by [1], $(F_1, F_2)^{n(z_3)} = h^m \circ a$ where a is an affine map of $\mathbb{C}^2$. When we fix z_3, (F_1, F_2) becomes a regular automorphism of $\mathbb{C}^2$, therefore it cannot be part of a 1-parameter group of automorphisms of $\mathbb{C}^2$, following [3]. Hence $n(z_3)$ is independent of z_3. In this case we have that $F = (h(z_1, z_2), F_3(z_3))$ where $F_3(z_3) = \gamma z_3 + \delta$, because F has to be an automorphism of $\mathbb{C}^3$, and with γ, δ any complex number and $\gamma \neq 0$.

In conclusion we find that, even if the dynamic of H is essentially of Hénon type, $C(H)$ is no more countable because H_3 is the projection on z_3 and this implies that every affine map in z_3 commutes with it.

2. Now we present an example of a regular automorphism f of $\mathbb{C}^4$ and of an automorphism g which commutes with f but s.t. neither I_g^+ nor I_g^- are equal to I_f^+, and there is no iterate of g equal to an iterate of f. Let

$$f = (h_1(z_1, z_2), h_2(z_3, z_4))$$

where h_1, h_2 are two Hénon maps of $\mathbb{C}^2$ of the same degree d and $C_{\mathcal{A}}(h_1)$ is non trivial. Let

$$g = (A(z_1, z_2), h_2^2(z_3, z_4))$$

s.t. A is an affine automorphism of $\mathbb{C}^2$, different from the identity s. t. $A \circ h_1 = h_1 \circ A$.

This proves that the characterization $C(f) = \mathbb{Z} \rtimes \mathbb{Z}_q$ for a regular automorphism of $\mathbb{C}^4$ doesn't work, because $\forall n, m \in \mathbb{Z},\ f^n \neq g^m$. Analogously we can prove that it fails in all dimensions strictly greater than 4.

3. An example in which Lemma 3.3 fails in $\mathbb{C}^5$ is the following:

$$f(x_1, x_2, x_3, x_4, x_5) = (x_3, x_2^2 + ax_1, x_3^2 + ax_2, x_4^2 + x_5, x_4)$$

The map f is regular and it commutes with

$$g(x_1, x_2, x_3, x_4, x_5) = (x_3, x_2^2 + ax_1, x_3^2 + ax_2, bx_4, cx_5)$$

but neither $I_g^+ = I_f^+$ nor $I_g^- = I_f^+$. Indeed $I_f^+ = [x_1 : 0 : 0 : 0 : x_5 : 0]$ and $I_f^- = [0 : x_2 : x_3 : x_4 : 0 : 0]$. On the other hand $I_g^+ = [x_1 : 0 : 0 : x_4 : x_5 : 0]$ and $I_g^- = [0 : x_2 : x_3 : x_4 : x_5 : 0]$ and no iterate of g is equal to an iterate of f.

The first three components of f are the second iterates of the following shift-like map of $\mathbb{C}^3$:

$$h(x_1, x_2, x_3) = (x_2, x_3, x_2^2 + ax_1).$$

The automorphism $h^2(x_1, x_2, x_3)$ is indeed regular. The last two components of f are a Hénon map of $\mathbb{C}^2$.

In view of these last two examples we formulate the following Conjecture.

Conjecture: If f is a regular automorphism of $\mathbb{C}^k$ with $k \geq 2$ such that $\pi \circ f$ is not an automorphism of $\mathbb{C}^j$ with $j < k$ for any π projection on any combination of j variables of $\mathbb{C}^k$, then $C(f)$ is isomorphic to $\mathbb{Z} \rtimes \mathbb{Z}_q$, for some integer q. On the contrary if f is the direct product of regular automorphisms f_{i_j}, for $j = 1, \cdots, s$ of some $\mathbb{C}^{i_j}$ with $i_j < k$ then $C(f)$ is isomorphic to the product of s copies of $\mathbb{Z} \rtimes \mathbb{Z}_q$.

4. In $\mathbb{C}^4$:

$$f(x_1, x_2, x_3, x_4) = (h_1(x_1, x_2), x_3, x_4)$$

where h_1 is a Hénon map of $\mathbb{C}^2$ and

$$g = (x_1, x_2, h_2(x_3, x_4))$$

where h_2 in another Hénon map of $\mathbb{C}^2$. They commute but none of them is an iterate of the other. None of them is regular but both are weakly regular.

5. Consider the real flow of the following nilpotent derivation, [13,18]:

$$D(x) = -2y,$$
$$D(y) = z,$$
$$D(z) = 0.$$

It is the following 1-parameter group:

$$\sigma_\lambda(x, y, z) = (x + \lambda(x^2 - yz)z, y + 2\lambda(x^2 - yz)x + \lambda^2(x^2 - yz)^2 z, z)$$

For $\lambda = 1$ we obtain the Nagata automorphism σ_1 of $\mathbb{C}^3$. In view of the fact that the Nagata automorphism is the 1-time map of a 1-parameter group, we have $(\sigma_1)^{-1} = \sigma_{-1}$ and $\sigma_{\lambda+\mu} = \sigma_{\mu+\lambda} = \sigma_\lambda \circ \sigma_\mu$. Hence for each $\lambda \in \mathbb{R} \setminus \mathbb{Z}$, σ_λ commutes with σ_1 and it is not an iterate of σ_1 or of its inverse. Hence $\{\sigma_\lambda\}_{\lambda \in \mathbb{R}}$ is contained in the centralizer of the Nagata automorphism and this is an example of a non-trivial centralizer. This doesn't contradict our Theorem 1.4 because, for each $\lambda \in \mathbb{R}$, neither σ_λ nor $(\sigma_\lambda)^{-1}$ is conjugate to weakly regular and in particular none of them is conjugate to regular. An other way of saying this is:

Proposition 4.1 *Regular automorphisms on $\mathbb{C}^3$ are never contained in a 1-parameter family of automorphisms.*

It has already been proved in dimension 2 by Buzzard, [3].

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