BOUNDARY CONSTRUCTIONS OF PETALS AT THE WOLFF POINT IN THE PARABOLIC CASE

By

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Abstract. In the same spirit of the classical Leau–Fatou flower theorem, we prove the existence of a petal, with vertex at the Wolff point, for a holomorphic self-map f of the open unit disc $\Delta \subset \mathbb{C}$ of parabolic type. The result is obtained in the framework of two interesting dynamical situations which require different kinds of regularity of f at the Wolff point τ: f of non-automorphism type and $\Re e(f^{''}(\tau)) > 0$ or f injective of automorphism type, $f \in C^{3+\epsilon}(\tau)$ and $\Re e(f^{''}(\tau)) = 0$.

1 Introduction

Let f be a holomorphic map from the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ to itself without fixed points. For any $\sigma \in \partial\Delta$ and for any $R > 0$, let $O(\sigma, R) = \{z \in \Delta : \frac{|\sigma - z|^2}{1-|z|^2} < R\}$ be the horocycle with radius R and center σ. The classical Wolff Lemma (see, e.g., [1]) shows that there is a unique point $\tau \in \partial\Delta$ such that $f(O(\tau, R)) \subseteq O(\tau, R)$, for all $R > 0$, i.e., f maps each horocycle of center $\tau \in \partial\Delta$ and radius $R > 0$ into itself. Such a point τ is called the Wolff point of f.

For any $\sigma \in \partial\Delta$ and any $M > 1$, let $K(\sigma, M) = \{z \in \Delta : \frac{|\sigma - z|}{1-|z|} < M\}$ be the Stolz region of amplitude M and vertex σ. We say that δ is the non-tangential limit of f at σ, and write $K - \lim_{z\to\sigma} f = \delta$, if $f(z)$ tends to δ when z tends to σ within any Stolz region of vertex σ, for all $M > 1$. As a consequence of Wolff's Lemma, τ is the unique point in $\partial\Delta$ such that $K - \lim_{z\to\tau} f(z) = \tau$ and $K - \lim_{z\to\tau} |f'(z)| \leq 1$. Hence, from a dynamical point of view, the Wolff point is an "attracting fixed point." If τ is the Wolff point of f, we say that f is of parabolic type when $K - \lim_{z\to\tau} f'(z) = 1$.

In the classical case of germs of holomorphic maps at 0 which fix the point 0 and have modulus of the first derivative equal to 1 at the fixed point, a fruitful

[*]Partially supported by PRIN *Proprietà geometriche delle varietà reali e complesse.*

[†]Partially supported by GNSAGA of the Istituto Nazionale di Alta Matematica, Rome.

JOURNAL D'ANALYSE MATHÉMATIQUE, Vol. 104 (2008)
DOI 10.1007/s11854-008-0013-9

approach to explore the dynamics of the iterates is the construction of petals with vertex at 0 (see, e.g., [10], [13], [14], [17], [19], [21]).

In the case of a map $f \in Hol(\Delta, \Delta)$ without fixed points in Δ, the complete study of the local dynamics of f at its Wolff point τ is still an open problem in full generality when f is parabolic at τ. In fact, if f is parabolic at its Wolff point τ, the construction of petals cannot be performed by means of the same techniques used in the analogous classical case of an internal fixed point.

In this paper, we construct petals at the Wolff point for $f \in Hol(\Delta, \Delta)$ by mixing different techniques borrowed from the theory of linear fractional models due to Bourdon and Shapiro, [6], together with dynamical and topological arguments. Our main results are proved in Sections 3 and 4.

In Section 2, we (define and) investigate the existence of "attracting directions" at the Wolff point for a map $f \in Hol(\Delta, \Delta)$ in the parabolic case. We prove the existence of attracting directions in several interesting dynamical situations, depending on the values of the second and third derivative of f at its Wolff point.

In Section 3, we consider parabolic non-automorphism type maps $f \in Hol(\Delta, \Delta)$ (see, e.g., [6]) such that the sequence $\{f^n(z)\}_{n \in \mathbb{N}}$ of iterates of f converges non-tangentially to the Wolff point $\tau \in \partial\Delta$ (for any $z \in \Delta$). We also require that $f^{''}(\tau) = K - \lim_{z \to \tau} f''(z) \neq 0$ (which implies $\Re e f^{''}(\tau) > 0$). For these maps, we construct an "attracting invariant boundary petal" (see Theorem 3.2) with vertex at the Wolff point τ and consequently describe the local dynamics at τ. Our result here is inspired by the Leau–Fatou construction [15] of attracting and repulsive directions and petals.

In Section 4, we consider parabolic automorphism type maps $f \in Hol(\Delta, \Delta)$ (see, e.g., [6]). We also require that f be injective in Δ, that it have sufficient boundary regularity at τ ($C^{3+\epsilon}(\Delta \cup \{\tau\}) = C^{3+\epsilon}(\tau)$), and that $\Re e(f^{''}(\tau)) = 0$; necessarily, $f^{''}(\tau) \neq 0$. In the spirit of the Bourdon–Shapiro construction of the linear fractional model [6], we prove that there exists an "invariant attracting petal" for f at its Wolff point τ.

The existence of petals at the Wolff point τ for a map $f \in Hol(\Delta, \Delta)$ of parabolic type remains an open problem when $f^{''}(\tau) = 0$ and (in general) if no regularity is assumed for f at τ. We plan to study this problem and related issues in a forthcoming paper.

2 Attracting and repulsive directions at the Wolff point

Let $f \in Hol(\Delta, \Delta)$ have 1 as Wolff point. Suppose that $K - \lim_{z \to 1} f'(z) = 1$, i.e., that 1 is a parabolic fixed point. The behaviour of the orbits of the iterates

$\{f^m(z)\}_{m\in\mathbb{N}}$ of f depends on f only, and not on the choice of z in Δ. Specifically they may

(1) converge *tangentially* to 1;

or

(2) converge *non-tangentially* to 1.

Suppose that the K-derivative of f of order $(n+1)$ exists and is different from zero at 1. Suppose as well that for all $0 \leq j < n$, the K-derivative of f of order $(j+1)$ is zero at 1. By a theorem of Burns and Krantz [7], if $f \neq id_\Delta$, it follows that $n \leq 2$, that is, $n = 1$ or $n = 2$. In the easy case in which f extends to a germ of a holomorphic map at 1, f has at most 2 attracting directions and at most 2 repulsive directions (in the classical sense [15]); in this case, the dynamics in a neighbourhood of 1 is well-known (see, e.g., [15]). Otherwise, if f does not extend to a germ of holomorphic map at 1, let S_α be the Stolz region $K(1, M)$ of vertex 1 and angular opening α at 1. Suppose that f admits a K-Taylor series expansion at 1, i.e., that for all $0 < \alpha < \pi$ and for all $z \in S_\alpha$,

$$f_\alpha(z) = f_{|S_\alpha}(z) = 1 + (z-1) + a(z-1)^{n+1} + o_\alpha(z-1)^{n+1}, \tag{2.1}$$

where $n = 1$ or $n = 2$, $a = K - \lim_{z\to 1} f^{(n+1)}(z)/(n+1)!$, and $o_\alpha(z-1)^{n+1}$ is such that $o_\alpha(z-1)^{n+1}/(z-1)^{n+1} \to 0$ when $z \to 1$ within S_α. Setting $z - 1 = w$, we obtain

$$f(w+1) = 1 + w + a(w)^{n+1} + o_\alpha(w)^{n+1}, \tag{2.2}$$

where a does not depend on $\alpha \in (0, \pi)$.

By adapting the fundamental ideas of Milnor [15] to the case of a boundary point, we now define the **attracting directions** and the **repulsive directions** at $1 \in \overline{f(\Delta)}$.

Definition 2.1. A vector $v \in \{w \in \mathbb{C} : (w+1) \in \Delta\}$ defines an **attracting direction** at 1 if av^n is real and negative.

Definition 2.2. A vector $v \in \{w \in \mathbb{C} : (w+1) \in \Delta\}$ defines a **repulsive direction** at 1 if av^n is real and positive.

If we ignore the terms of higher order in the K-Taylor series (2.2) we have

$$(f(v+1) - (v+1)) \approx av^n \cdot v.$$

Hence the attracting directions (resp., the repulsive directions) are such that the vector $(f(v+1) - (v+1)) \approx av^n \cdot v$ is a real negative (resp., positive) multiple of v.

According to the structure of the K-Taylor series expansion of f at its Wolff point 1, we obtain the following direct results on the existence or non-existence of attracting directions.

Lemma 2.3. *If $n = 2$ and $a < 0$, then there exists exactly one attracting direction at 1 and $\{f^n(z)\}_{n\in\mathbb{N}}$ converges non-tangentially to 1.*

Proof. To prove that the real axis is an attracting direction it suffices to notice that $ar^2 < 0$, if r is real, $-1 \leq r < 0$ (i.e., $0 \leq r + 1 < 1$). □

Lemma 2.4. *If $n = 1$ and $\Re e(a) = 0$, then there is no attracting direction at 1 and $\{f^n(z)\}_{n\in\mathbb{N}}$ converges tangentially to 1.*

Proof. If there exists an attracting direction v at 1, then av would be real and negative. By hypothesis, $\Re e(a) = 0$; hence it is not possible for av to be real and negative for all $v \in \{w \in \mathbb{C} : w + 1 \in \Delta\}$. □

Lemma 2.5. *If $n = 1$ and $\Re e(a) > 0$, then there exists exactly one attracting direction at 1 and $\{f^n(z)\}_{n\in\mathbb{N}}$ converges non-tangentially to 1.*

Proof. Take v such that $arg(v) = \pi - arg(a)$. □

Thanks to the above Lemmas, it is possible to prove the existence or non-existence of attracting directions in several interesting dynamical situations. A few of these situations which appear to be particularly significant are presented here.

Following [6] and [8], suppose that $f \in Hol(\Delta, \Delta)$ and 1 is its Wolff point we say that $f \in C^k(1)$ if the j-th derivative $f^{(j)}$ extends continuously to $\Delta \cup \{1\}$ for all $j = 1, \ldots, k$. In this case, f admits a Taylor expansion

$$f(z) = f(1) + f'(1)(z-1) + \cdots + \frac{1}{k!} f^{(k)}(1)(z-1)^k + \Gamma(z),$$

where $z \in \Delta$ and $\Gamma(z) = o(|z-1|^k)$. Moreover, for $\epsilon > 0$, we say that $f \in C^{k+\epsilon}(1)$ if $f \in C^k(1)$ and $\Gamma(z) = O(|z-1|^{k+\epsilon})$.

Theorem 2.6 ([8]). *Let $f \in Hol(\Delta, \Delta)$ and let $\rho : [0, 1[\to \Delta$ be a continuous curve that tends to 1 non-tangentially. If*

$$\lim_{t\to 1^-} \frac{f(\rho(t)) - \rho(t)}{(\rho(t) - 1)^3} = l \tag{2.3}$$

for some $l \in \mathbb{C}$, then $l \in \mathbb{R}^-$. Moreover, f is the identity map if and only if $l = 0$. Furthermore, if $f \in C^3(1)$, then $f(1) = 1$, $f'(1) = 1$, $f''(1) = 0$, and $f'''(1) = 6l$ if and only if (2.3) *holds.*

When the above theorem applies and f is not the identity, Lemma 2.3 states the existence of exactly one attracting direction in 1.

Bourdon and Shapiro [6] give the following classification of all parabolic maps by means of their second derivative at the Wolff point.

Theorem 2.7. *Let f be a holomorphic self-map of Δ of parabolic type with Wolff point 1 and suppose that $f \in C^2(1)$.*

Then

(a) $\Re e(f''(1)) \geq 0$;

(b) *if $f''(1) = 0$ or $\Re e(f''(1)) > 0$, then f is of parabolic non-automorphism type;*

(c) *if, on the contrary, $f''(1) \neq 0$, $\Re e(f''(1)) = 0$ and $f \in C^{3+\epsilon}(1)$, then f is of parabolic automorphism-type.*

In case (c), Lemma 2.4 states the non-existence of any attracting direction at 1. On the other hand, in case (b), Lemma 2.5 proves the existence of exactly one attracting direction at the boundary fixed point.

3 The Case of a map f of parabolic non-automorphism type

In this section, we present a construction of a boundary petal which resembles the classical one given by Milnor [15] for holomorphic germs.

We adapt here the classical definition of petals to the case of a boundary point.

Definition 3.1. Suppose $f \in Hol(\Delta, \Delta)$ has no fixed points, and let 1 be the Wolff point of f. A connected open set $P \subset \Delta$ with $1 \in \overline{P}$ is an **attracting petal** for f at 1 if

$$f(\overline{P}) \subset P \cup \{1\}$$

and

$$\bigcap_{k \geq 0} f^k(\overline{P}) = \{1\}.$$

Theorem 3.2. *Let f be of parabolic non-automorphism type with Wolff point 1. If $K - \lim_{z \to 1} f''(z) = a \neq 0$, i.e., if $n = 1$ in the K-Taylor series expansion of f at 1, and if $\Re e(a) > 0$* (see [6])*, then there exists an attracting petal P at 1.*

Proof. Consider the transformation $k(z) = w = -1/a(z-1) = 1/a(1-z)$ with inverse $k^{-1}(w) = z = 1 - 1/(aw) = (aw-1)/wa$. The conformal transformation k is such that

$$\begin{aligned} k(0) &= \frac{1}{a} = \frac{\Re e(a) - i\Im m(a)}{(\Re e(a))^2 + (\Im m(a))^2}, \\ k(1) &= \infty, \\ k(\infty) &= 0, \end{aligned}$$

$$k(i) = \frac{1}{a(1-i)} = \frac{(\Re e(a) + \Im m(a)) + i(\Re e(a) - \Im m(a))}{2|a|^2},$$
$$k(-i) = \frac{1}{a(1+i)} = \frac{(\Re e(a) - \Im m(a)) - i(\Re e(a) + \Im m(a))}{2|a|^2},$$
$$k(-1) = \frac{1}{2a} = \frac{\Re e(a) - i\Im m(a)}{2|a|^2}.$$

Hence the unit circle is mapped by k into the line c which passes through $k(i)$, $k(-i)$ and $k(-1)$, which has a slope equal to $\tan\beta = \cot a = \Re e(a)/\Im m(a)$. The real axis is sent into the line r through $k(0)$ and $k(-1)$, which has a slope equal to $-\tan a$. The imaginary axis is mapped into the circle passing through 0, $k(0)$, $k(i)$ and $k(-i)$. Therefore, Δ is mapped by k onto the right half plane Π_c^+ determined by the line c. Any open angular region A_α in Δ with vertex 1 and amplitude $0 < \alpha < \pi$ symmetric with respect to the real axis, is mapped by k onto an open angular region $k(A_\alpha) = B_\alpha$ of vertex 0 (in the $w-$plane) which intersects Π_c^+. Let us set $B_\alpha \cap \Pi_c^+ = E_\alpha$. With the notation established in Section 2, we have for all $\alpha \in (0,\pi)$, $\tilde{f}_{|E_\alpha} = k \circ f_\alpha \circ k^{-1}(w) = w + 1 + o_\alpha(1)$, where $o_\alpha(1)$ tends to 0 when w tends to ∞ within E_α.

Let $\alpha \in (0,\pi)$ be such that $\alpha/2 > \arg a > 0$ and set $\gamma = (\alpha/2 - \arg a)$. Then there exists (small) ϵ_α such that $0 < \sin(\epsilon_\alpha) < \sin(\gamma)$ and r_{ϵ_α} such that if $|w| > r_{\epsilon_\alpha}$ and $w \in E_\alpha$, then $|\tilde{f}(w) - w - 1| \leq \sin(\epsilon_\alpha) < \sin(\gamma)$. Hence the **slope** s of the vector which joins w and $\tilde{f}(w)$ satisfies

$$|s| < \tan(\epsilon_\alpha) < \tan(\gamma), \qquad \text{for all } w \in E_\alpha, |w| > r_{\epsilon_\alpha}. \tag{3.1}$$

We now construct an attracting region for ∞ inside E_α in the $w-$plane. Set

$$P_\alpha = \left\{ w = u + iv : |w| > r_{\epsilon_\alpha}, w \in E_\alpha, u > d_\alpha - \frac{|v|}{\tan(2\epsilon_\alpha)} \right\},$$

where the constant d_α is so large that $|w| > r_{\epsilon_\alpha}$, for all $w \in P_\alpha$, and the two lines $u = d_\alpha - |v|/(\tan 2\epsilon_\alpha)$ intersect c in two points belonging to Π_c^+. We define the **attracting petal** for f as

$$P = \bigcup_{2\arg a < \alpha < \pi} k^{-1}(P_\alpha).$$

Put

$$\widetilde{P} = \bigcup_{2\arg a < \alpha < \pi} P_\alpha.$$

Now

$$f\Big(\bigcup_{2\arg a < \alpha < \pi} k^{-1}(P_\alpha) \Big) \subset \bigcup_{2\arg a < \alpha < \pi} k^{-1}(P_\alpha).$$

if and only if

$$\tilde{f}\Big(\bigcup_{2\arg a<\alpha<\pi} P_\alpha\Big) \subset \bigcup_{2\arg a<\alpha<\pi} P_\alpha.$$

The proof of the f-invariance of P coincides with the proof of the $\widetilde{f}$-invariance of $\widetilde{P}$. We proceed as follows: if $w \in \widetilde{P}$, then there exists α with $2\arg a < \alpha < \pi$ such that $w \in P_\alpha$. On P_α, we have $\tilde{f}(w) = w + 1 + o_\alpha(1)$, where $|w| > r_{\epsilon_\alpha}$, and $u > d_\alpha - \frac{|v|}{\tan(2\epsilon_\alpha)}$. We have to prove that $\widetilde{f}(w) \in P_\alpha$. The inequality $|\tilde{f}(w)| > r_{\epsilon_\alpha}$ follows from

$$|\tilde{f}(w) - w - 1| < \sin(\epsilon_\alpha),$$

which implies

$$|w+1| - |\tilde{f}(w)| < |(w+1) - \tilde{f}(w)| < \sin(\epsilon_{(\alpha)})$$

and hence

$$|\tilde{f}(w)| > |w+1| - \sin(\epsilon_\alpha).$$

Taking ϵ_α sufficiently small, we see that (3.1) implies the $\widetilde{f}$-invariance of $\widetilde{P}$. □

4 The case of a map $f \in C^{3+\epsilon}(1)$ of parabolic automorphism type

Suppose that

(1) $f \in C^{3+\epsilon}(1)$;
(2) f is injective on Δ;
(3) $f'(1) = 1$;
(4) $f''(1) \neq 0$.

Under these hypotheses, it follows from Theorem 2.7 that $\Re e(f''(1)) \geq 0$; moreover, $\Re e(f''(1)) = 0$ implies that f is of parabolic automorphism type.

Let $\Phi = C \circ f \circ C^{-1}$, where C is the **Cayley transformation**

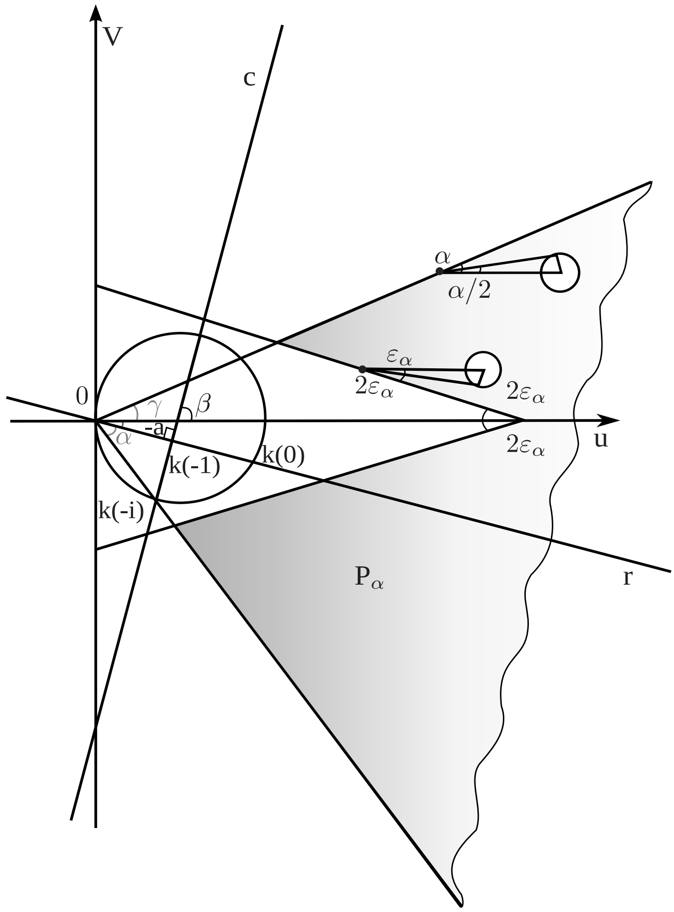
$$C : \mathbb{H}_{rg} \to \Delta$$

$$C^{-1} : \Delta \to \mathbb{H}_{rg}$$

where $\mathbb{H}_{rg} = \{w \in \mathbb{C} : \Re e(w) > 0\}$. Then, as is proved in [6],

$$\Phi(w) = w + a + \frac{b}{w+1} + \Gamma(w+1)$$

with $a = f''(1) \neq 0, b = f''(1)^2 - \frac{2}{3}f'''(1) = -\frac{2}{3}(Sf)(1)$, where Sf is the Schwarzian derivative of f and $|\Gamma(w+1)| = o\left(1/|w+1|^{1+\epsilon}\right)$ when $w \to \infty$. Let us consider



the linear fractional model for Φ in the sense of Bourdon–Shapiro [6], i.e., let us consider the two maps (φ, ν) such that $\nu \circ \Phi(z) = \varphi \circ \nu(z) = \nu(z) + a$, where $0 \neq a = f''(1) < \infty$. From $\Phi(\infty) = \infty$ it follows that $\nu(\infty) = \nu(\infty) + a$, and so also $\nu(\infty) = \infty$. The model (φ, ν) has the following properties:

(1) $\nu(w) = w - \frac{b}{a}\log(1 + w) + B(w)$, where B is a holomorphic, bounded map on $\mathbb{H}_{rg}$, continuous on $\overline{\mathbb{H}_{rg}}$, and where $\nu : \mathbb{H}_{rg} \to \mathbb{H}_{rg}$ is injective on $\mathbb{H}_{rg}$;

(2) $\varphi(w) = w + a$.

Assume that f satisfies the conditions stated at the beginning of this section. Then we have

Lemma 4.1. *Suppose* $\Re e(f''(1)) = 0$ *and let* Π *be an arbitrary halfplane contained in* $\mathbb{H}_{rg} = \{w : \Re e(w) > 0\}$. *Then* $\nu(\Pi)$ *contains a halfplane.*

Proof. Observe that $\Re e(f''(1)) = 0$ implies that f is of parabolic-automorphism type and that the sequence $\{f^n(z_0)\}_{n\in\mathbb{N}}$ converges tangentially to 1. For $r > 0$, let $O(1, r)$ denote the horocycle of center 1 and radius r. The Cayley transformation $C : \Delta \to \mathbb{H}_{rg}$ maps any such horocycle onto a halfplane. Hence to prove Lemma 4.1 it suffices to prove that in the linear fractional model (ψ, σ) for f, for all $\rho > 0$, there exists $\epsilon > 0$ such that $O(1, \epsilon) \subset \sigma(O(1, \rho))$. By hypothesis, $\Re e(a) = 0$, which implies that $b \geq 0$, as is proved in [6], and hence $-b/a \in i\mathbb{R}$.

Recall that

$$\nu(w) = w - \frac{b}{a}\log(1 + w) + B(w)$$

and

$$\Re e(\nu(w)) = \Re e(w) - \Im m(-b/a)\Im m[\log(1 + w)] + \Re e(B(w)).$$

Since

$$\log(1 + w) = \log(|1 + w|) + i \cdot arg(1 + w),$$

we have

$$\Im m[\log(1 + w)] = arg(1 + w),$$

so that

$$\Re e[\nu(w)] = \Re e(w) - \Im m(-b/a)arg(1 + w) + \Re e(B(w)).$$

Fix $x_0 > 0$ and let $\Pi_{x_0} = \{z \in \mathbb{C} : \Re e(z) > x_0\}$ be the corresponding halfplane. We want to prove that

$$\sup_{w\in\partial\Pi_{x_0}} \{\Re e(\nu(w))\} < +\infty.$$

Now, as is proved in [6] Theorem 4.12 b), $\Re e(B(w))$ is bounded. Hence there exists $K > 0$ such that for all $w \in \mathbb{H}_{rg}$, $|\Re e(B(w))| < K$, i.e., $-K < \Re e(B(w)) < +K$.

If $w \in \partial\Pi_{x_0}$, then when $\Im m(w)$ tends to $+\infty$, we have $x_0 - \Im m(-b/a)arg(1+w)$ approaches $x_0 - \Im m(-b/a)\pi/2 < +\infty$. In other words, for all $\epsilon > 0$, there exists $M > 0$ such that for all $w \in \partial\Pi_{x_0}$ with $\Im m(w) > M$, the following inequalities hold:

$$x_0 - \Im m(-b/a)\pi/2 - \epsilon < x_0 - \Im m(-b/a)arg(1 + w) < x_0 - \Im m(-b/a)\pi/2 + \epsilon$$
$$x_0 - \Im m(-b/a)\pi/2 - \epsilon - K < \Re e(\nu(w)) < x_0 - \Im m(-b/a)\pi/2 + \epsilon + K.$$

In the case in which $w \in \partial\Pi_{x_0}$ and $\Im m(w)$ tends to $-\infty$, the proof is analogous.

What we have proved up to now is that $\nu_{|\{\Re e(w)=x_0\}}$ is a curve with bounded real part. The map ν is $1 : 1$ on Δ and hence sends Jordan domains (in particular horocycles) into Jordan domains. It also sends a simply connected and connected region into another region with the same properties. Now

$$\Im m(\nu(w)) = \Im m(w) + \Im m(-b/a)\log(|1 + w|) + \Im m(B(w)).$$

Therefore (see [6]),

$$\nu(\infty) = \infty;$$

and since

$$\lim_{w\to\infty} \frac{\nu(w)}{w} = \lim_{w\to\infty} 1 - \frac{b}{a}\frac{\log(1+w)}{w} + \frac{B(w)}{w} = 1,$$

we have also

$$\nu'(\infty) = 1.$$

Therefore, ∞ is the Wolff point for ν and hence

$$\nu(O(1,\rho)) \subset O(1,\rho)$$

for all $\rho > 0$. □

Under the hypotheses of the above lemma, we have $\Re e(a) = 0$ and the model (φ, ν) for Φ is such that $\varphi(w) = w + a$ with $a \in i\mathbb{R}$.

Theorem 4.2. *If f, ν, ϕ, and C are the maps defined in the setting of this section, if f satisfies hypotheses (1), (2), (3) and (4), and if $\Re e(f''(1)) = 0$, then there exists a constant $k_0 > 0$ such that any arbitrary vertical line $L_k = \{w : \Re e(w) = k > k_0\}$ is an invariant curve for φ. Therefore, if k is sufficiently large, $\nu^{-1}(L_k) = L'_k$ is a totally invariant curve for Φ and $C^{-1}\left(\nu^{-1}(L_k)\right)$ is a totally invariant curve for f.*

For $k > 0$, let us set $\Pi_k^+ = \{w \in \mathbb{C} : \Re e(w) > k\}$ and $P_k = \nu^{-1}(\Pi_k^+)$.

Corollary 4.3. *If f, ν, ϕ, and C are the maps defined in the setting of this section, if f satisfies hypotheses (1), (2), (3) and (4), and if $\Re e(f''(1)) = 0$, then there exists a constant $k_0 > 0$ such that for any $k > k_0$, the set P_k is a totally invariant region for f.*

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(Received June 6, 2006)