On the geometry of the quaternionic unit disc

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Abstract. In the space $\mathbb{H}$ of quaternions, the natural invariant geometry of the open unit disc $\Delta_{\mathbb{H}}$, diffeomorphic to the open half-space $\mathbb{H}^+$ via a Cayley-type transformation, has been investigated extensively. This was accomplished by constructing, in a natural geometrical manner, the quaternionic Poincaré distance on $\Delta_{\mathbb{H}}$ (and $\mathbb{H}^+$).

The open unit disc $\Delta_{\mathbb{H}}$ also inherits the complex Kobayashi distance when viewed as the open unit ball of $\mathbb{C}^2 \cong \mathbb{C} + \mathbb{C}j \cong \mathbb{H}$.

In this paper we give an original, very simple proof of the fact that there exists no isometry between the quaternionic Poincaré distance of $\Delta_{\mathbb{H}}$ and the Kobayashi distance inherited by $\Delta_{\mathbb{H}}$ as a domain of $\mathbb{C}^2$. This is in accordance with the well known consequence of the classification theorem for the non compact, rank 1, symmetric spaces.

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1. Introduction

Let $\mathbb{H}$ be the skew field of quaternions. Let $\Delta_{\mathbb{H}} = \{q \in \mathbb{H} : |q| < 1\}$ be the open unit disc and let $\mathbb{H}^+ = \{q \in \mathbb{H} : \Re e(q) > 0\}$ be the half-space, diffeomorphic via a Cayley-type transformation. The groups of Möbius transformations of $\Delta_{\mathbb{H}}$ and of $\mathbb{H}^+$ are the groups of all quaternionic, fractional, linear transformations which leave $\Delta_{\mathbb{H}}$ and $\mathbb{H}^+$ invariant, respectively. These groups are used in [4] to find a direct approach to a geometric definition of the analogue of the Poincaré distance (i.e. the real, hyperbolic distance) and differential metric in the quaternionic setting. These distances and differential metrics allowed the construction and study of the

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invariant quaternionic geometry of the classical hyperbolic domains $\Delta_{\mathbb{H}}$ and $\mathbb{H}^+$ of $\mathbb{H}$.

With respect to the standard basis $\{1, i, j, k\}$ of $\mathbb{H}$, the identification $\mathbb{H} \cong \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k \cong (\mathbb{R} + \mathbb{R}i) + (\mathbb{R} + \mathbb{R}i)j \cong \mathbb{C} + \mathbb{C}j$ leads to the identification $\Delta_{\mathbb{H}} \cong \Delta_{\mathbb{C}^2}$ between the open unit disc of $\mathbb{H}$ and the open unit ball $\Delta_{\mathbb{C}^2}$ of $\mathbb{C}^2$. Since the ball $\Delta_{\mathbb{C}^2}$ is naturally endowed with the Kobayashi distance and (metric), see e.g. [11, 22], it is natural to ask which is the relationship between the quaternionic Poincaré distance and the Kobayashi distance on $\Delta_{\mathbb{H}} \cong \Delta_{\mathbb{C}^2}$.

By means of the geometrical approach adopted in this paper, we are able to give an original, very simple proof of a deep result that is classically obtained as a consequence of the classification of non compact, rank 1, symmetric spaces (see, e.g., [9], [15]). This result states that:

Theorem 1.1. *There exists no isometry between the quaternionic Poincaré distance and the Kobayashi distance of* $\Delta_{\mathbb{H}} \cong \Delta_{\mathbb{C}^2}$.

Notations and terminology are those used in [4]. The elements of $\mathbb{H}$ will be denoted by $q = x_0 + ix_1 + jx_2 + kx_3$, where the x_l are real, and i, j, k, are imaginary units (i.e. their square equals -1) such that $ij = -ji = k$, $jk = -kj = i$, and $ki = -ik = j$. We will denote by $\mathbb{S}^3_{\mathbb{H}}$ the sphere of quaternions of unitary modulus $\{q \in \mathbb{H} : |q| = 1\}$ and by $\mathbb{S}$ the unit sphere of purely imaginary quaternions, i.e. $\mathbb{S} = \{q = ix_1 + jx_2 + kx_3 : x_1^2 + x_2^2 + x_3^2 = 1\}$. Notice that if $I \in \mathbb{S}$, then $I^2 = -1$; for this reason the elements of $\mathbb{S}$ are called imaginary units. We will also use the fact that for any non-real quaternion $q \in \mathbb{H} \backslash \mathbb{R}$, there exist, and are unique, $x, y \in \mathbb{R}$ with $y > 0$, and $I \in \mathbb{S}$ such that $q = x + yI$.

The paper is organized as follows. In section 2 we survey and illustrate the construction and the main features of the quaternionic Poincaré distance and differential metric of $\Delta_{\mathbb{H}}$ and describe the geometrical character of the group of quaternionic Möbius transformations. In Section 3 we briefly recall the definitions of the Kobayashi distance and differential metric, and the structure theorem for complex Möbius transformations in the case of the unit ball of $\mathbb{C}^n$. We then conclude by giving the announced new proof of Theorem 1.1.

2. Basics of quaternionic invariant geometry

In [4] the authors found an original geometrical approach to the study of the invariant geometry of the unit open disc of $\mathbb{H}$. The study of the relevant groups of matrices with quaternionic entries is based on the definition of the Dieudonné determinant of a quaternionic 2×2 matrix, which is recovered in a very natural way in the same paper [4]:

Definition 2.1. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a 2×2 matrix with quaternionic entries, then the *(Dieudonné) determinant* of A is defined to be the non negative real number

$$det_{\mathbb{H}}(A) = \sqrt{|a|^2|d|^2 + |c|^2|b|^2 - 2\Re e(c\overline{a}b\overline{d})}. \tag{2.1}$$

This definition (luckily!) agrees with the one given in [3, 8, 12, 21], and allows us to set

$$\mathbb{G} = \{g(q) = (aq+b)(cq+d)^{-1} : a, b, c, d \in \mathbb{H},\ g \text{ invertible }\},$$

$$GL(2,\mathbb{H}) = \{A \quad 2\times 2 \quad \text{matrix with quaternionic entries} : det_{\mathbb{H}}(A) \neq 0\}$$

and to present the following result for $\mathbb{H}$, already established in [31]:

Theorem 2.2. *The set $\mathbb{G}$ of all quaternionic, fractional, linear transformations is a group with respect to composition. The map*

$$\Phi : A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto L_A(q) = (aq+b)\cdot(cq+d)^{-1} \tag{2.2}$$

is a group homomorphism of $GL(2,\mathbb{H})$ onto $\mathbb{G}$ whose kernel is the center of $GL(2,\mathbb{H})$, that is the subgroup

$$\left\{ \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} : t \in \mathbb{R}\backslash\{0\} \right\}.$$

For a detailed, modern proof of this theorem, and for bibliographical references, we refer the reader to [4]. The structure-theorem of the complex, fractional, linear transformations can be extended to the quaternionic environment:

Proposition 2.3. *The group $\mathbb{G}$ is generated by all the similarities, $L(q) = aq + b$ $(a, b \in \mathbb{H}, a \neq 0)$ and the inversion $R(q) = q^{-1}$. Moreover, all the elements of $\mathbb{G}$ turn out to be conformal.*

The basic geometrical ingredient to construct the quaternionic Poincaré distance (and metric) is the quaternionic cross-ratio: indeed the generalizations of the cross-ratio to higher dimensions in $\mathbb{R}^n$ play a crucial role in conformal geometry. In fact L. Ahlfors, while studying the conformal structure of $\mathbb{R}^n$, has given in [2] three different definitions of the cross-ratio of 4 points of $\mathbb{R}^n$. The one that we adopt here is the one given in [4], that specializes to the quaternionic case the definition given by C. Cao and P.L. Waterman in [5], and that is new with respect to the ones given by Ahlfors. This definition of cross-ratio has the peculiar feature that the quaternionic, fractional, linear transformations act on it transforming its value by (quaternionic) conjugation. In particular it turns out that

Proposition 2.4. *Let $\mathcal{CR}(q_1, q_2, q_3, q_4) := (q_1 - q_3)(q_1 - q_4)^{-1}(q_2 - q_4)(q_2 - q_3)^{-1}$ be the cross-ratio of the four quaternions q_1, q_2, q_3, q_4. When the cross ratio of four quaternions is real, then it is invariant under the action of all quaternionic, fractional, linear transformations.*

The above result has a great deal of interest in view of the fact (already proven in [5] in Clifford algebra setting) that

Proposition 2.5. *Four pairwise distinct points q_1,q_2,q_3,$q_4 \in \mathbb{H}$ lie on a same (one-dimensional) circle or straight line if, and only if, their cross-ratio is real.*

Adopting the point of view due to A.F. Möbius one defines (for $i = 3, 2, 1$, respectively) the families $\mathcal{F}_i = \mathcal{S}_i \cup \mathcal{P}_i$, where $\mathcal{S}_i$ is the family of all $i-$(real) dimensional spheres and $\mathcal{P}_i$ is the family of all $i-$(real) dimensional affine subspaces of $\mathbb{H}$. With an approach which is diverse from the one used by Wilker in [31], the authors showed in [4] that

Theorem 2.6. *The group $\mathbb{G}$ of all quaternionic, fractional, linear transformations maps elements of $\mathcal{F}_i$ onto elements of $\mathcal{F}_i$, for $i = 3, 2, 1$.*

The above result, and the use of the point of view of C. L. Siegel for the homologous problem in the complex case (see [28], [7]), led to the following, genuine geometrical approach to the definition of the quaternionic Poincaré distance on $\Delta_{\mathbb{H}}$ (often simply called Poincaré distance). Set the *non-Euclidean line* through two points $q_1, q_2 \in \Delta_{\mathbb{H}}$ to be the unique circle, or diameter, containing the two points and intersecting $\partial\Delta_{\mathbb{H}}$ orthogonally in the two *ends* q_3, q_4, and give the following

Definition 2.7. The *(quaternionic) Poincaré distance* of $\Delta_{\mathbb{H}}$ is defined as

$$\delta_{\Delta_{\mathbb{H}}}(q_1, q_2) = \frac{1}{2}\log(\mathcal{CR}(q_1, q_2, q_3, q_4)) \tag{2.3}$$

where the q_3 and q_4 are the two ends of the non-Euclidean line through q_1 and q_2, and the four points are arranged cyclically on the non-Euclidean line through q_1 and q_2.

It is very easy to see that

Proposition 2.8. *On each complex plane $L_I = \mathbb{R} + I\mathbb{R}$ (for any imaginary unit $I \in \mathbb{S}$) the quaternionic Poincaré distance coincides with the classical Poincaré distance of $\Delta_I = \Delta_{\mathbb{H}} \cap L_I$.*

The group of *Möbius transformations* is defined as the subgroup $\mathbb{M}$ of $\mathbb{G}$ whose elements map $\Delta_{\mathbb{H}}$ onto itself. It is natural to study how the quaternionic Poincaré distance behaves under the action of the elements of $\mathbb{M}$. To do this, we need to know the structure of the group of Möbius transformations of $\Delta_{\mathbb{H}}$, whose study is performed, for example, in [6], and completed in [4], in terms of the (classical) group $Sp(1,1)$. The group $Sp(1,1)$ is defined (see, e.g., [13]) as

$$Sp(1,1) = \left\{A \in GL(2,\mathbb{H}) \;:\; {}^t\overline{A}HA = H\right\} \tag{2.4}$$

where $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and it can be written equivalently as (see, e.g., [6])

$$Sp(1,1) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : |a| = |d|, \quad |b| = |c|, \quad |a|^2 - |c|^2 = 1, \quad \overline{a}b = \overline{c}d, \quad a\overline{c} = b\overline{d} \right\}.$$

The use of the group $Sp(1,1)$ allowed us to rephrase and complete a result of [6] as follows:

Theorem 2.9. *The quaternionic, fractional, linear transformation defined by formula $g(q) = (aq+b)(cq+d)^{-1}$ is a Möbius transformation of $\Delta_{\mathbb{H}}$ if and only if $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Sp(1,1)$. Moreover the map*

$$\phi : Sp(1,1) \to \mathbb{M}$$
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto L_A(q) = (aq+b)\cdot(cq+d)^{-1} \tag{2.5}$$

is a group homomorphism whose kernel is the center of $Sp(1,1)$, that is the subgroup

$$\left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

For the purposes of this paper, we need the following characterization of the quaternionic Möbius transformations, which closely resembles the classical representation of the complex Möbius transformations:

Theorem 2.10. *Each quaternionic Möbius transformation of the form*

$$g(q) = (aq+b)\cdot(cq+d)^{-1} \in \mathbb{M}$$

can be written uniquely as:

$$g(q) = \alpha(q-q_0)(1-\overline{q_0}q)^{-1}\beta^{-1} \tag{2.6}$$

where $q_0 = -a^{-1}b \in \Delta_{\mathbb{H}}$ and where $\alpha = \dfrac{a}{|a|} \in \partial\Delta_{\mathbb{H}}$, $\beta = \dfrac{d}{|d|} \in \partial\Delta_{\mathbb{H}}$.

A detailed proof of the structure result given in Theorem 2.10 can be found in [4]. This structure result, that was also stated without proof in [14], is different from the one given in a more general setting in [2].

Remark 2.11. Thanks to Lemma 2.3, the Möbius transformations are conformal (see also [29], [19], [23], [24], [26], [25], [27], [10]). Since any Möbius transformation maps $\mathbb{S}^3_{\mathbb{H}}$ onto itself, the Möbius transformations map non-Euclidean lines of $\Delta_{\mathbb{H}}$ onto non-Euclidean lines of $\Delta_{\mathbb{H}}$ in view of Theorem 2.6.

Combining Propositions 2.4, 2.5 and Remark 2.11, one gets the following result, whose statement is implicit in the work of Wilker [31] (see also [16]), and urges a comparison with the complex case.

Proposition 2.12. *The Poincaré distance of $\Delta_{\mathbb{H}}$ is invariant under the action of the group of all Möbius transformations $\mathbb{M}$ and of the map $q \mapsto \overline{q}$.*

It is now possible to mimic the definition of the classical, complex Poincaré differential metric to set the length of the vector $\tau \in \mathbb{H}$ for the *(quaternionic) Poincaré metric* at $q \in \Delta_{\mathbb{H}}$ to be the number

$$\langle \tau \rangle_q = \frac{|\tau|}{1-|q|^2}. \tag{2.7}$$

The following results, see [4], will be used in the sequel:

Theorem 2.13. *All the elements of the group $\mathbb{M}$ of Möbius transformations of $\Delta_{\mathbb{H}}$, as well as the map $q \mapsto \overline{q}$, leave the quaternionic Poincaré differential metric invariant.*

Proposition 2.14. *The quaternionic Poincaré distance δ_Δ of the unit disc $\Delta_{\mathbb{H}}$ is the integrated distance of the quaternionic Poincaré differential metric of $\Delta_{\mathbb{H}}$.*

3. Poincaré and Kobayashi distances on the quaternionic unit disc

In this section, we will first of all briefly recall the classical definitions of Kobayashi distance and differential metric for the open unit ball $\Delta_{\mathbb{C}^n}$ of the space $\mathbb{C}^n$. To help the reader to follow the proof of the main result, we will also recall the structure of the group of all complex Möbius transformations of $\Delta_{\mathbb{C}^n}$.

After a result due to L. Lempert, [18], the definition of the Kobayashi distance for a convex set of $\mathbb{C}^n$ can be given in the following simple fashion (for the classical general definition see, e.g. [11, 22]).

Definition 3.1. Let $\mathbb{D}$ be the open, unit disc of $\mathbb{C}$ and $\delta_{\mathbb{D}}$ the Poincaré distance of $\mathbb{D}$. The *Kobayashi distance* between any two points z_1 and z_2 of the open unit ball $\Delta_{\mathbb{C}^n} \subset \mathbb{C}^n$ is defined as

$$\begin{aligned}&k_{\Delta_{\mathbb{C}^n}}(z_1, z_2)\\ &= \inf_{\zeta_1,\zeta_2\in\mathbb{D}}\{\delta_{\mathbb{D}}(\zeta_1,\zeta_2) \mid \exists f : \mathbb{D} \to \Delta_{\mathbb{C}^n} \text{ holomorphic, with } f(\zeta_1) = z_1,\ \ f(\zeta_2) = z_2\}\end{aligned}$$

The *Kobayashi differential metric* at a point $z \in \Delta_{\mathbb{C}^n}$ is defined, for all $w \in \mathbb{C}^n$, by

$$\begin{aligned}&\gamma_{\Delta_{\mathbb{C}^n}}(z; w)\\ &= \inf_{\zeta\in\mathbb{D}}\{\frac{|\tau|}{1-|\zeta|^2} \ \mid \exists f : \mathbb{D} \to \Delta_{\mathbb{C}^n} \ \text{ holomorphic, with } \ f(\zeta) = z,\ \ df_\zeta(\tau) = w\}.\end{aligned}$$

It turns out that, to explicitly compute the distance $k_{\Delta_{\mathbb{C}^n}}(z_1, z_2)$, it is enough to consider the complex line L_{z_1,z_2} of $\mathbb{C}^n$ that contains the two points z_1, z_2 of $\Delta_{\mathbb{C}^n}$, intersect it with the ball $\Delta_{\mathbb{C}^n}$ and measure the Poincaré distance between z_1 and z_2 in the complex disc $L_{z_1,z_2} \cap \Delta_{\mathbb{C}^n}$. This observation will be useful in the sequel.

Both the Kobayashi distance and the Kobayashi differential metric are invariant, in the obvious sense, under the action of the group of Möbius transformations of $\Delta_{\mathbb{C}^n}$, whose structure we are going to describe (see, e.g. [22]).

Denote by $\langle\cdot,\cdot\rangle$ the classical Hermitian inner product of $\mathbb{C}^n$. For $z_0 \in \Delta_{\mathbb{C}^n}$, let P_{z_0} be the orthogonal projection of $\mathbb{C}^n$ onto the subspace $[z_0]$ spanned by z_0, i.e. let

$$P_0(z) = 0 \quad \text{if} \quad z_0 = 0, \quad \text{and} \quad P_{z_0}(z) = \frac{\langle z\ , z_0\rangle}{\langle z_0, z_0\rangle}z_0 \quad \text{if } z_0 \neq 0. \tag{3.1}$$

Set $Q_{z_0} = I - P_{z_0}$ to be the projection of $\mathbb{C}^n$ onto the orthogonal complement of $[z_0]$. For $s_{z_0} = (1 - |z_0|^2)^{\frac{1}{2}}$ define the map $\varphi_{z_0} : \mathbb{C}^n \setminus \{z : \langle z\ , z_0\rangle = 1\} \to \mathbb{C}^n$ as

$$\varphi_{z_0}(z) = \frac{z_0 - P_{z_0}(z) - s_{z_0} Q_{z_0}(z)}{1 - \langle z\ , z_0\rangle}. \tag{3.2}$$

Since $\{z : \langle z\ , z_0\rangle = 1\} \cap \Delta_{\mathbb{C}^n} = \emptyset$, the map φ_{z_0} defines a holomorphic map from $\Delta_{\mathbb{C}^n}$ to $\mathbb{C}^n$, which turns out to be a holomorphic automorphism of $\Delta_{\mathbb{C}^n}$. Now, if $\mathbb{U}(n) = \{U \in GL(n, \mathbb{C}) : U^t U = I\}$ denotes the unitary group of $\mathbb{C}^n$, then for $U \in \mathbb{U}(n)$ and $z_0 \in \Delta_{\mathbb{C}^n}$ we put

$$M_{U,z_0}(z) = U\varphi_{z_0}(z) = U\frac{z_0 - P_{z_0}(z) - s_{z_0} Q_{z_0}(z)}{1 - \langle z\ , z_0\rangle}. \tag{3.3}$$

Since the elements of $\mathbb{U}(n)$ are $\mathbb{C}$-linear automorphisms of $\Delta_{\mathbb{C}^n}$, the set

$$\mathcal{M} = \{M_{U,z_0} : U \in \mathbb{U}(n),\ z_0 \in \Delta_{\mathbb{C}^n}\} \tag{3.4}$$

consists of holomorphic automorphisms of $\Delta_{\mathbb{C}^n}$, and is called the set of *Möbius transformations of* $\Delta_{\mathbb{C}^n}$. The transitivity of the group $\mathcal{M}$ and a direct application of the n-dimensional Schwarz Lemma lead to the following well known result (see, e.g. [22]):

Theorem 3.2. *The group* $\mathrm{Aut}(\Delta_{\mathbb{C}^n})$ *of all holomorphic automorphisms of the open unit ball* $\Delta_{\mathbb{C}^n}$ *of* $\mathbb{C}^n$ *coincides with the group* $\mathcal{M}$ *of all Möbius transformations of* $\Delta_{\mathbb{C}^n}$*. Moreover, the elements of* $\mathcal{M}$ *are isometries for the Kobayashi distance* $k_{\Delta_{\mathbb{C}^n}}$ *and for the Kobayashi differential metric* $\gamma_{\Delta_{\mathbb{C}^n}}$*.*

One of the important properties of the Kobayashi distance is the following:

Proposition 3.3. *The Kobayashi distance* $k_{\Delta_{\mathbb{C}^n}}$ *of the unit ball* $\Delta_{\mathbb{C}^n}$ *is the integrated distance of the Kobayashi differential metric* $\gamma_{\Delta_{\mathbb{C}^n}}$*.*

Let us now consider the isomorphism $\mathbb{H} \cong \mathbb{C} + \mathbb{C}j$, which yields the identification $\Delta_{\mathbb{H}} \cong \Delta_{\mathbb{C}^2} =: \Delta$ between the open unit disc of $\mathbb{H}$ and the open unit ball of $\mathbb{C}^2$. Now that we have given a direct, geometrical definition of the Poincaré distance δ_Δ of $\Delta_{\mathbb{H}}$, the natural question arises to find a direct proof of the fact that there exists no isometry between δ_Δ and the Kobayashi distance k_Δ. To find such a proof we begin with the following remark, whose justification can be found, for example, in [11, 22]:

Remark 3.4. Both the Poincaré distance δ_Δ and the Kobayashi distance k_Δ have the property that

$$\delta_\Delta(0, q) = k_\Delta(0, q) = \delta_{\mathbb{D}}(0, |q|)$$

for all $q \in \Delta$. Moreover, the Poincaré differential metric of $\Delta_{\mathbb{H}}$ defined in (2.7) and the Kobayashi differential metric γ_Δ of $\Delta_{\mathbb{C}^2}$ both coincide with the Euclidean differential metric at the origin of the open unit disc of $\mathbb{H}$.

With this in mind, we will prove the following technical result:

Lemma 3.5. *If there exists an isometry $f : \Delta \to \Delta$ between the Kobayashi distance k_Δ and the Poincaré distance δ_Δ, then the identity function of Δ is an isometry between k_Δ and δ_Δ, and hence $k_\Delta \equiv \delta_\Delta$.*

Proof. If f is the identity function of Δ, then there is nothing to prove. Otherwise, let $M \in \mathbb{M}$ be a quaternionic, Möbius transformation of Δ such that $M(f(0)) = 0$. By Proposition 2.12, the function $M \circ f$ is an isometry between k_Δ and δ_Δ that fixes 0. If we identify $\mathbb{H}$ with $\mathbb{R}^4$, then Remark 3.4, together with Propositions 2.14 and 3.3, yield that the real differential $d(M \circ f)_0$ is an orthogonal matrix. Now the geometrical definition of the Poincaré distance δ_Δ given in Definition 2.7 makes it clear that any orthogonal transformation of Δ is a δ_Δ-isometry together with its inverse. Therefore the function $F = d(M \circ f)_0^{-1} \circ M \circ f : \Delta \to \Delta$ is an isometry between k_Δ and δ_Δ, whose differential dF_0 is the identity function. Since the geodesic curves of both k_Δ and δ_Δ passing through 0 are the diameters of Δ, then, in view of Remark 3.4, the isometry F itself is the identity map. □

Given any two points $q_1, q_2 \in \Delta$ there exist a quaternionic Möbius transformation M of $\Delta_{\mathbb{H}}$ and a complex Möbius transformation ϕ of $\Delta_{\mathbb{C}^2}$ such that $M(q_1) = 0 = \phi(q_1)$. Now M and ϕ leave invariant, respectively, δ_Δ and k_Δ, and we want to investigate the relation between $|M(q_2)|$ and $|\phi(q_2)|$. Consider $q_1 = \alpha = \alpha + 0j$ and $q_2 = \beta j = 0 + \beta j$ with $\alpha, \beta \in \mathbb{C}$. Using Theorem 2.10, choose $M \in \mathbb{M}$ to be the quaternionic Möbius transformation of $\Delta_{\mathbb{H}}$

$$M(q) = (q - \alpha)(1 - \overline{\alpha}q)^{-1}$$

and consider the complex Möbius transformation (of $\Delta_{\mathbb{C}^2}$) $\phi_{(\alpha,0)} \in \mathcal{M}$ defined by

$$\phi_{(\alpha,0)}(z, w) = \frac{(\alpha, 0) - (z, 0) - (1 - |\alpha|^2)^{1/2}(0, w)}{1 - z\overline{\alpha}}.$$

We get

$$|M(\beta j)|^2 = |(\beta j - \alpha)(1 - \overline{\alpha}\beta j)^{-1}|^2 = \frac{(|\beta|^2 + |\alpha|^2)}{(1 + |\alpha|^2|\beta|^2)} \tag{3.5}$$

and

$$|\phi_{(\alpha,0)}(0, \beta)|^2 = |(\alpha, -(1 - |\alpha|^2)^{1/2}\beta)|^2 = |\alpha|^2 + (1 - |\alpha|^2)|\beta|^2. \tag{3.6}$$

Since the equality among (3.5) and (3.6) does not hold in general (due to the identity principle for real polynomials), remark 3.4 leads to the following

Lemma 3.6. *The identity map of Δ is not an isometry between the Kobayashi distance k_Δ and the Poincaré distance δ_Δ. In particular k_Δ and δ_Δ do not coincide.*

As a direct corollary of the last two lemmas, and in accordance with a classical consequence of the classification of non compact, rank 1, symmetric spaces (see, e.g., [9], [15]), we can now state the following

Theorem 3.7. *There exists no isometry between the quaternionic Poincaré distance and the Kobayashi distance of $\Delta_{\mathbb{H}} \cong \Delta_{\mathbb{C}^2}$.*

It is very easy to prove that, for any $I \in \mathbb{S}$, the Poincaré distance and the Kobayashi distance coincide on the subsets of Δ of type $\Delta_I = \Delta \cap L_I$, where $L_I = \{x + yI : x, y \in \mathbb{R}\}$. Thanks to our geometrical approach, it is also direct to verify that all the real, sectional curvatures of the quaternionic, Poincaré differential metric at 0 (and hence by homogeneity at all points of Δ) coincide with a same negative constant. As it is known, this is not the case for the Kobayashi differential metric of $\Delta_{\mathbb{C}^2}$ (see, e.g., [17]), for which only the holomorphic, sectional curvatures at all points coincide with a same negative constant.

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