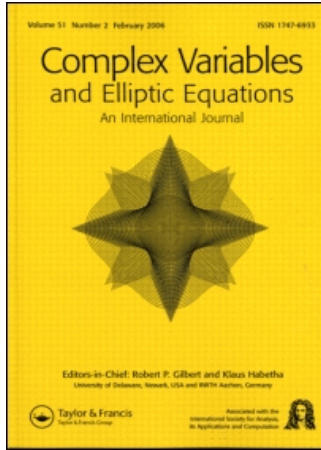


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## Commuting holomorphic maps and linear fractional models

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# Commuting Holomorphic Maps and Linear Fractional Models

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Let  $f$  be a holomorphic map of the open unit disc  $\Delta$  of  $\mathbb{C}$  into itself, having no fixed points in  $\Delta$  and Wolff point  $\tau \in \partial\Delta$ . In the open case in which  $f'(\tau) = 1$  we study the centralizer of  $f$ , i.e., the family  $G_f$  of all holomorphic maps of  $\Delta$  into itself which commute with  $f$  under composition. We prove that if the sequence of iterates  $\{f^n\}$  converges to  $\tau$  non tangentially, then  $G_f$  coincides with the set of all elements of the pseudo-iteration semigroup of  $f$  (in the sense of Cowen, see [5, 6]) whose Wolff point is  $\tau$ . In the same hypotheses we give a representation of the centralizer  $G_f$  in  $\text{Aut}(\Delta)$  or  $\text{Aut}(\mathbb{C})$ , study its main features and generalize a result due to Pranger ([15]).

**Keywords:** Iteration and composition of holomorphic maps; Linear fractional models; Centralizer of families of holomorphic maps; Wolff point and fundamental set

**AMS Subject Classification Numbers:** 30D05, 30C99, 32H50

## 1. INTRODUCTION

In this paper we give a contribution to the investigation of the connection between iteration theory and the study of sets of commuting holomorphic maps, in the open unit disc  $\Delta$  of  $\mathbb{C}$ . We will be concerned with the case of a holomorphic map of the open unit

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disc  $\Delta$  into itself,  $f \in \text{Hol}(\Delta, \Delta)$ , without fixed points in  $\Delta$ . For such a map  $f$  the Wolff Lemma states the existence of a unique point  $\tau \in \partial\Delta$ , called the Wolff point of  $f$ , such that each horocycle (*i.e.*, each Euclidean disc contained in  $\Delta$  and tangent to  $\partial\Delta$  at  $\tau$ ) is sent into itself. The Wolff Lemma and the Julia-Wolff-Carathéodory Theorem assert that the non-tangential derivative of such an  $f$  at  $\tau$  is a strictly positive real number smaller or equal than 1 ([1]). The Wolff point is fundamental in the study of the iterates of  $f$ : the Wolff-Denjoy theorem asserts that the iterates of  $f$  converge, uniformly on compact sets, to the Wolff point of  $f$  ([1, 20]).

Classical results point out that, in the case of a holomorphic map of the unit disc  $\Delta$  into itself having a fixed point in  $\Delta$ , the behaviour of the iterates of the map depends on the value of the derivative at the fixed point itself ([11, 16]). In a very similar way the value of the non-tangential derivative of  $f$  at its Wolff point  $\tau \in \partial\Delta$  is strictly connected with the behaviour of the iterates of  $f$ : if the value of  $f'(\tau)$  is strictly smaller than 1, then for any  $z \in \Delta$  the sequence of iterates  $\{f^n(z)\}_{n \in \mathbb{N}}$  converges to  $\tau$  non-tangentially and the behaviour of  $f$  has been widely investigated in recent years, also in connection with the study of the family of maps which commute with  $f$  with respect to composition ([2, 5, 9]). The case in which the value of  $f'(\tau)$  equals 1 is still open for many questions, in particular for what concerns the study of the family of maps which commute with  $f$  under composition. In this framework Cowen proves in his papers [5, 6] a fundamental theorem for the study of the geometry, and iteration theory, of a holomorphic map  $f \in \text{Hol}(\Delta, \Delta)$  with no fixed points in  $\Delta$ , and gives the nice and fertile definition of pseudo-iteration semigroup of such a map. Essentially, Cowen studies a boundary version of the Schroeder equation; he proves, among the other things, that (see Theorem [4]), given  $f \in \text{Hol}(\Delta, \Delta)$  with Wolff point  $\tau \in \partial\Delta$ , then there exist a holomorphic map  $\sigma_f \in \text{Hol}(\Delta, \Delta)$  (or  $\sigma_f \in \text{Hol}(\Delta, \mathbb{C})$ ) and a holomorphic automorphism  $\Phi_f$  of  $\Delta$  (or, respectively, of  $\mathbb{C}$ ), unique up to conjugation by automorphisms, such that

$$\sigma_f \circ f = \Phi_f \circ \sigma_f$$

The automorphism  $\Phi_f$  contains interesting informations on  $f$ , and allows the definition of pseudo-iteration semigroup of  $f$ : a map  $g$

belongs to the pseudo-iteration semigroup of  $f$  if there exists an automorphism  $\Psi_g$  of  $\Delta$  (or, respectively, of  $\mathbb{C}$ ) such that

$$\sigma_f \circ g = \Psi_g \circ \sigma_f$$

and that

$$\Phi_f \circ \Psi_g = \Psi_g \circ \Phi_f$$

The pseudo-iteration semigroup  $P_f$  of a map  $f$  turned out to be an interesting tool for the study of the geometric theory of holomorphic mappings and for iteration theory (see also [3, 5, 6, 13]) and the results obtained by Cowen will be widely used here.

In this paper we prove two main results. The first one concerns the study of the family  $G_f$  of all holomorphic maps from the unit disc  $\Delta$  into itself which commute under composition with a given  $f \in \text{Hol}(\Delta, \Delta)$  having no fixed points in  $\Delta$  and Wolff point  $\tau \in \partial\Delta$ . We prove that, in the open case in which  $f'(\tau) = 1$ , if for some  $z_0 \in \Delta$  (and hence for all  $z \in \Delta$ ) the sequence of iterates  $\{f^n(z_0)\}_{n \in \mathbb{N}}$  converges to  $\tau$  non-tangentially, then the family  $G_f$  coincides with the set of all elements of the pseudo-iteration semigroup  $P_f$  of  $f$  whose Wolff point is  $\tau$  (see theorem 6). This result is a nice improvement of a result obtained by Vlacci [8, 19].

The second main result of this paper is related to an interesting statement due to Pranger [15]. In the case in which  $f \in \text{Hol}(\Delta, \Delta)$  has  $0 \in \Delta$  as a fixed point, and if  $G_f$  denotes the family of all  $g \in \text{Hol}(\Delta, \Delta)$  such that  $f \circ g = g \circ f$ , let  $\lambda : G_f \rightarrow \overline{\Delta}$  denote the map  $g \mapsto g'(0)$ . Pranger [15], proves the following result:

**THEOREM 1** *Let  $\Gamma \subset \overline{\Delta}$  be a set with the following properties:*

- (a)  $\Gamma$  is a closed subset of  $\overline{\Delta}$ ;
- (b) 0 and 1 belong to  $\Gamma$  and  $\Gamma \cap \Delta \neq \{0\}$ ;
- (c)  $\Gamma$  is closed under multiplication;
- (d)  $\Gamma^*$ , the complement of  $\Gamma$ , is connected.

*Then, there is a map  $f \in \text{Hol}(\Delta, \Delta)$  such that  $f(0) = 0$ ,  $0 < |f'(0)| < 1$ ,  $f$  is locally univalent and  $\Gamma = \lambda(G_f)$ .*

*Conversely, given a map  $f \in \text{Hol}(\Delta, \Delta)$  such that  $f(0) = 0$ ,  $0 < |f'(0)| < 1$ ,  $f$  is locally injective, then  $\lambda(G_f)$  satisfies properties (a) ... (d).*

In the same spirit in which the Julia-Wolff-Carathéodory Theorem and the Wolff Lemma (see Section 2) can be viewed as boundary versions of the Schwarz Lemma, we prove a (partial) boundary version of the above theorem of Pranger. Let  $G_f$  still denote the set of all holomorphic maps which commute with a given  $f \in \text{Hol}(\Delta, \Delta)$  with no fixed points in  $\Delta$ , and let  $\lambda: G_f \rightarrow \mathbb{A}$  be the map which associates to  $g \in G_f$  the automorphism  $\Psi_g$  of  $\Delta$  (or, respectively, of  $\mathbb{C}$ ) which appears in the definition of pseudo-iteration semigroup of  $f$ . We prove the following (see Section 4):

**THEOREM 2** *Let  $f \in \text{Hol}(\Delta, \Delta)$  be such that its Wolff point  $\tau$  is equal to 1. Suppose that there exists  $z_0 \in \Delta$  such that  $\{f^n(z_0)\}_{n \in \mathbb{N}}$  converges to 1 non-tangentially. Then the set  $\lambda(G_f)$  satisfies the following properties:*

- (a)  $\lambda(G_f)$  is a subset of  $\text{Aut}(\Delta)$  (or, respectively, of  $\text{Aut}(\mathbb{C})$ ) topologically closed;
- (b) the identity map  $I \in \lambda(G_f)$ ;
- (c)  $\lambda(G_f)$  is closed with respect to composition;
- (d) the complement of  $\lambda(G_f)$  in  $\text{Aut}(\Delta)$  (or, respectively, in  $\text{Aut}(\mathbb{C})$ ) is connected.

Section 2 contains some fundamental preliminary results and definitions, which are essential to understand the setting in which the paper is located, and which are used extensively.

Section 3 contains the new result which completely describes the family of all holomorphic maps which commute with a given  $f \in \text{Hol}(\Delta, \Delta)$  having no fixed points in  $\Delta$ , and whose iterates converge non-tangentially to the Wolff point  $\tau \in \partial\Delta$ . As we have already mentioned this description is made in terms of the pseudo-iteration semigroup  $P_f$  of  $f$ . Some consequences are also presented.

In Section 4 the boundary version of the above mentioned theorem due to Pranger (see Theorem 8) is presented, together with some related results.

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## 2. PRELIMINARIES AND A FEW BASIC RESULTS

In this section we will present some preliminary results which will be extensively used in the rest of the paper. Some of them are quite

classical results and some other are recently discovered statements, all concerned with the role played by fixed points of holomorphic maps of the unit disc into itself (including the case of boundary fixed points) in the theory of iteration and in connection with the study of families of commuting holomorphic functions. It is in the theory of iteration that the horocycles play the role of “Poincarè discs at the boundary of  $\Delta$ ”: let  $\tau \in \partial\Delta$ ; then for all  $R > 0$  the open disc of  $\Delta$  tangent to  $\partial\Delta$  at  $\tau$  defined as

$$E(\tau, R) = \left\{ z \in \Delta : \frac{|\tau - z|^2}{1 - |z|^2} < R \right\}$$

is called the *horocycle* of center  $\tau$  and radius  $R$ .

The following, well known, Wolff Lemma (see e.g. [1,16]), guarantees the existence of a fixed point at the boundary for a function  $f \in \text{Hol}(\Delta, \Delta)$  having no fixed points in  $\Delta$ . It also plays the role of a boundary Schwarz Lemma.

LEMMA 1 (Wolff) *Let  $f \in \text{Hol}(\Delta, \Delta)$  be without fixed points. Then there is a unique  $\tau \in \partial\Delta$  such that for all  $z \in \Delta$*

$$\frac{|\tau - f(z)|^2}{1 - |f(z)|^2} \leq \frac{|\tau - z|^2}{1 - |z|^2} \tag{1}$$

that is

$$\forall R > 0 \quad f(E(\tau, R)) \subseteq E(\tau, R), \tag{2}$$

where  $E(\tau, R)$  is the horocycle of center  $\tau$  and radius  $R > 0$ . Moreover, the equality (1) holds at one point (and hence at all points) if and only if  $f$  is a (parabolic) automorphism of  $\Delta$  leaving  $\tau$  fixed.

If  $f \in \text{Hol}(\Delta, \Delta)$  has a fixed point in  $\Delta$  (and  $f \neq id$ ) then we denote this fixed point by  $\tau(f)$ . Otherwise,  $\tau(f)$  denotes the point constructed in Lemma 1. In both cases  $\tau(f)$  is called the *Wolff point* of  $f$ .

DEFINITION 1 Take  $\sigma \in \partial\Delta$  and  $M > 1$ . The set

$$K(\sigma, M) = \left\{ z \in \Delta : \frac{|\sigma - z|}{1 - |z|} < M \right\} \tag{3}$$

is called Stolz region  $K(\sigma, M)$  of vertex  $\sigma$  and amplitude  $M$ .

The Stolz region  $K(\sigma, M)$  is an “angular region” of  $\Delta$  with vertex at  $\sigma$  and opening  $\alpha < \pi$ . Stolz regions are necessary to give the following

**DEFINITION 2** Let  $f : \Delta \rightarrow \overline{\mathbb{C}}$  be a function. We say that  $c$  is the non-tangential limit (or angular limit) of  $f$  in  $\sigma \in \partial\Delta$  if  $f(z) \rightarrow c$  as  $z$  tends to  $\sigma$  within  $K(\sigma, M)$ , for all  $M > 1$ . We shall write

$$K - \lim_{z \rightarrow \sigma} f(z) = c.$$

The definition of non-tangential limit is used in (see [1]).

**THEOREM 3 (Julia-Wolff-Carathéodory)** Let  $f \in \text{Hol}(\Delta, \Delta)$  and let  $\tau, \sigma$  be any two points in  $\partial\Delta$ , then

$$K - \lim_{z \rightarrow \sigma} \frac{\tau - f(z)}{\sigma - z} \quad (4)$$

exists. If the above non-tangential limit is finite, then

$$K - \lim_{z \rightarrow \sigma} f(z) = \tau \quad (5)$$

and there exists  $r \in \mathbb{R}$ ,  $r \geq 0$ , such that

$$K - \lim_{z \rightarrow \sigma} f'(z) = K - \lim_{z \rightarrow \sigma} \frac{\tau - f(z)}{\sigma - z} = \bar{\sigma}\tau r \quad (6)$$

In particular, if  $\tau = \sigma$  then the non-tangential limit  $r$  of  $f'$  at  $\sigma$  is a strictly positive real number.

We say that  $\tau \in \partial\Delta$  is a *fixed point of  $f$  on the boundary of  $\Delta$* , if

$$K - \lim_{z \rightarrow \tau} f(z) = \tau;$$

at the same time we call *derivative of  $f$  at a fixed point  $\tau$  on the boundary of  $\Delta$*  the value of

$$K - \lim_{z \rightarrow \tau} f'(z).$$

As a consequence of the Julia-Wolff-Carathéodory theorem, it follows that if  $f \in \text{Hol}(\Delta, \Delta)$  has no fixed points then the value of the derivative  $f'(\tau)$  at the Wolff point  $\tau$  of  $f$  ( $\tau \in \partial\Delta$ ) is real and such that  $0 < f'(\tau) \leq 1$  (see, e.g. [1]).

**DEFINITION 3** Let  $f$  be a holomorphic function mapping the unit disc  $\Delta$  into itself. The functions  $f^n = f \circ f^{n-1}$  defined inductively for the

natural numbers (where  $f^1 = f$ ) will be called the “natural iterates” of  $f$ .

The two following lemmas give some description of the behaviour of the iterates of  $f$  (see, e.g. [5]).

LEMMA 2 *Suppose that a map  $f \in \text{Hol}(\Delta, \Delta)$  has its Wolff point  $\tau(f)$  on the boundary of  $\Delta$ . If*

$$K - \lim_{z \rightarrow \tau} f'(z) < 1,$$

*then for any  $z \in \Delta$  the sequence of iterates  $\{f^n(z)\}_{n \in \mathbb{N}}$  converges to  $\tau$  non-tangentially, that is, converges within  $K(\tau, M)$ , for some  $M > 1$ .*

LEMMA 3 *Suppose that a map  $f \in \text{Hol}(\Delta, \Delta)$  has its Wolff point  $\tau(f)$  on the boundary of  $\Delta$ . If for some  $z_0$  in  $\Delta$  the sequence of iterates  $\{f^n(z_0)\}_{n \in \mathbb{N}}$  converges to  $\tau$  non-tangentially, then for any compact set  $K$  in  $\Delta$ , the sequence of iterates  $\{f^n(K)\}_{n \in \mathbb{N}}$  converges to  $\tau$  non-tangentially. In particular for any  $z \in \Delta$ , the sequence of iterates  $\{f^n(z)\}_{n \in \mathbb{N}}$  converges to  $\tau$  non-tangentially.*

If  $f, g \in \text{Hol}(\Delta, \Delta)$  commute under composition, then it is well known that  $f, g$  have the same fixed point in  $\Delta$  or the same Wolff point in  $\partial\Delta$ , unless both of them are hyperbolic automorphisms of  $\Delta$  (see, e.g. [1, 4]). If  $f, g \in \text{Hol}(\Delta, \Delta)$  are commuting holomorphic maps, then a connection between the value of their derivatives at a common fixed point  $\tau \in \overline{\Delta}$  is presented in the following:

PROPOSITION 1 *If  $f$  and  $g$  commute and  $\tau$  is their common Wolff point, then*

- (1) *if  $f'(\tau) = 0$ , then  $g'(\tau) = 0$ ;*
- (2) *if  $0 < |f'(\tau)| < 1$ , then  $0 < |g'(\tau)| < 1$ ;*
- (3) *if  $f'(\tau) = 1$ , then  $g'(\tau) = 1$ .*

Now we will state the “main theorem” of Cowen and give the definition of pseudo-iteration semigroup of a map  $f \in \text{Hol}(\Delta, \Delta)$ . We need the following:

DEFINITION 4 *An open, connected, simply connected subset  $V$  of  $\Delta$  is called a fundamental set for  $f \in \text{Hol}(\Delta, \Delta)$  if  $f(V) \subset V$  and if for any compact set  $K$  in  $\Delta$ , there is a positive integer  $n$  so that  $f^n(K) \subset V$ .*



The “fundamental set” of a map  $f$  is a set of points “near” the Wolff point  $\tau(f)$  “small enough” that  $f$  is “well behaved” on it, and “large enough” that  $f^n(z)$  belongs eventually to this set.

**THEOREM 4 (Cowen)** *Let  $f \in \text{Hol}(\Delta, \Delta)$  be neither a constant map nor an automorphism of  $\Delta$ . Let  $\tau$  be the Wolff point of  $f$  and suppose that  $f'(\tau) \neq 0$ . Then there exists a fundamental set  $V_f$  for  $f$  in  $\Delta$  on which  $f$  is injective.*

*Furthermore there also exist:*

- (1) a domain  $\Omega$  which is either the complex plane  $\mathbb{C}$  or the unit disc  $\Delta$ ,
- (2) a linear fractional transformation  $\varphi$  mapping  $\Omega$  onto  $\Omega$ ,
- (3) an analytic map  $\sigma_f$  mapping  $\Delta$  into  $\Omega$ , such that
  - (i)  $\sigma_f$  is univalent on  $V_f$ ,
  - (ii)  $\sigma_f(V)$  is a fundamental set for  $\varphi$  in  $\Omega$ ,
  - (iii)  $\sigma_f \circ f = \varphi \circ \sigma_f$ .

*Finally,  $\varphi$  is unique up to conjugation under linear fractional transformations mapping  $\Omega$  onto  $\Omega$ , and the maps  $\varphi$  and  $\sigma_f$  depend only on  $f$  and not on the choice of the fundamental set  $V_f$ .*

It has been proved, [5], that the map  $\varphi$  and the set  $\Omega$  in the statement of the above theorem, fall into four essentially different cases:

- (1)  $\Omega = \mathbb{C}$ ,  $\sigma_f(\tau) = 0$ ,  $\varphi(z) = sz$ ,  $0 < |s| < 1$ ;
- (2)  $\Omega = \Delta$ ,  $\sigma_f(\tau) = 1$ ,  $\varphi(z) = (((1+s)z+1-s)/((1-s)z+1+s))$ ,  $0 < s < 1$ ;
- (3)  $\Omega = \mathbb{C}$ ,  $\sigma_f(\tau) = \infty$ ,  $\varphi(z) = z+1$ ;
- (4)  $\Omega = \Delta$ ,  $\sigma_f(\tau) = 1$ ,  $\varphi(z) = ((1 \pm 2i)z - 1)/(z - 1 \pm 2i)$ .

Following Cowen, [5, 6], we will now define the pseudo-iteration semigroup of a map  $f \in \text{Hol}(\Delta, \Delta)$ .

**DEFINITION 5** *Let  $f, g$  be holomorphic maps of  $\Delta$  into  $\Delta$ . Let  $\tau$  be the Wolff point of  $f$  and suppose that  $f'(\tau) \neq 0$ . Let  $V_f, \Omega, \sigma_f$  and  $\varphi$  be as in theorem (4), relative to  $f$ . We say that  $g$  “is in the pseudo-iteration semigroup of  $f$ ” ( $g \in P_f$ ), if there exists a linear fractional transformation  $\psi$  which commutes with  $\varphi$ , such that  $\sigma_f \circ g = \psi \circ \sigma_f$ .*

The following results relate the study of the pseudo-iteration semigroup of a map  $f \in \text{Hol}(\Delta, \Delta)$  with the study of the family of all maps  $g \in \text{Hol}(\Delta, \Delta)$  which commute with  $f$  under composition.

**THEOREM 5** *Let  $f, g \in \text{Hol}(\Delta, \Delta)$  be neither constants nor automorphisms of  $\Delta$ , and let  $f \circ g = g \circ f$ . Let  $\tau \in \partial\Delta$  be the (common) Wolff point of  $f$  and  $g$ . If there exist  $z_0$  and  $w_0$  in  $\Delta$  so that  $g^n(z_0) \rightarrow \tau$  and  $f^n(w_0) \rightarrow \tau$  non-tangentially, then  $g$  is in the pseudo-iteration semigroup of  $f$ .*

**PROPOSITION 2** *Let  $f$  be a holomorphic map of  $\Delta$  into  $\Delta$  neither constant nor an automorphism of  $\Delta$  and let  $\tau \in \partial\Delta$  be its Wolff point. Let  $g$  be in the pseudo-iteration semigroup of  $f$ . Then  $f$  and  $g$  commute if and only if there is an open set  $U$  in  $\Delta$  such that  $g(U)$  and  $g(f(U))$  are contained in the fundamental set  $V_f$  of  $f$ .*

If the sequence of all iterates of a point  $z_0 \in \Delta$  under  $f$ ,  $\{f^n(z_0)\}_{n \in \mathbb{N}}$ , has a non-tangential behaviour, then the fundamental set  $V_f$  of  $f$ , which is constructed in Theorem 4, has nice geometric properties, which help much to understand the geometric structure of the pseudo-iteration semigroup. Before stating the geometric results contained in Propositions 3 and 4, we need a few definitions:

**DEFINITION 6** Let  $\sigma \in \partial\Delta$  and let  $T$  be the line containing the diameter of  $\bar{\Delta}$  passing through  $\sigma$ . An angular sector of vertex  $\sigma$  and opening  $\theta$  in  $\Delta$ , is the intersection of  $\Delta$  with the open angle having vertex in  $\sigma$ , bisectrix line  $T$  and opening  $\theta$ .

A small angular sector of vertex  $\sigma$  and opening  $\theta$  in  $\Delta$ , is the intersection of an angular sector of vertex  $\sigma$  and opening  $\theta$  in  $\Delta$  with any open, Euclidean disc of positive radius centered at  $\sigma$ .

**LEMMA 4** *Let  $f \in \text{Hol}(\Delta, \Delta)$  be without fixed points and with Wolff point  $\tau \in \partial\Delta$ . Then  $f$  sends Stolz regions of vertex  $\tau$  into Stolz regions of vertex  $\tau$ .*

*Proof* Using the inequality (1) stated in the Wolff Lemma, we have:

$$\frac{|\tau - f(z)|}{1 - |f(z)|} \leq \frac{|\tau - z|}{1 - |z|} \cdot \frac{1 + |f(z)|}{1 + |z|} \cdot \frac{|\tau - z|}{|\tau - f(z)|} \quad (7)$$

Away from the Wolff point  $\tau$ , the fraction

$$\frac{|\tau - z|}{|\tau - f(z)|}$$


---

is bounded from above; on the other hand, if we suppose that  $z$  belongs to some Stolz region  $K(\tau, M)$  and  $z$  is “near”  $\tau$ , then

$$\frac{|\tau - z|}{|\tau - f(z)|} < \frac{1}{|f'(\tau)|} + \varepsilon$$

because

$$K - \lim_{z \rightarrow \tau} \frac{|\tau - z|}{|\tau - f(z)|} = K - \lim_{z \rightarrow \tau} \frac{1}{|f'(z)|} = \frac{1}{|f'(\tau)|} \geq 1 \quad (8)$$

(by the Julia-Wolff-Carathéodory theorem,  $0 < f'(\tau) \leq 1$ ). Moreover

$$\frac{1}{2} < \frac{1 + |f(z)|}{1 + |z|} < 2$$

Hence, the right-hand member of inequality (7) is bounded from above, if we suppose that  $z$  belongs to some Stolz region  $K(\tau, M)$ ; so we have proved that  $f$  sends Stolz regions of vertex  $\tau$  into Stolz regions of vertex  $\tau$ .  $\blacksquare$

**DEFINITION 7** A set  $V \subseteq \Delta$  is said to be angular in  $\tau \in \partial\Delta$  if it contains small angular sectors of vertex  $\tau$  and opening  $\theta$ , for all  $\theta < \pi$ .

**DEFINITION 8** Let  $f \in \text{Hol}(\Delta, \Delta)$ . A set  $U$  is said to be  $f$ -absorbing if for any compact set  $K$  in  $\Delta$  there exists  $n_0 \in \mathbb{N}$  such that  $f^n(K) \subset U$ , for all  $n \in \mathbb{N}$ ,  $n > n_0$ .

The proof of the following proposition, which gives a nice geometric property of the fundamental set of a map  $f \in \text{Hol}(\Delta, \Delta)$ , can be found in [5].

**PROPOSITION 3** *Let  $f \in \text{Hol}(\Delta, \Delta)$  be neither a constant map nor an automorphism of  $\Delta$  and let the Wolff point  $\tau(f)$  belong to the boundary of  $\Delta$ . If, for some point  $z_0$  of  $\Delta$ , the sequence  $\{f^n(z_0)\}_{n \in \mathbb{N}}$  converges to  $\tau(f)$  non-tangentially, then any fundamental set  $V$  of  $f$  is angular at  $\tau(f)$ .*

At this point, we will state the Noshiro Lemma, an elementary result (see e.g. [5]) which, in particular, will be used later as a powerful tool to guarantee injectivity of a map  $f \in \text{Hol}(\Delta, \Delta)$  on convex subsets of the fundamental set  $V_f$ .

LEMMA 5 (Noshiro) *If  $U$  is a convex open subset of the plane  $\mathbb{C}$ ,  $f$  is holomorphic on  $U$  and  $\operatorname{Re} f'(z) > 0$  for all  $z \in U$ , then  $f$  is injective on  $U$ .*

To end this section, we will present the following result (whose proof is sketched in [19]) which states a very nice geometric property of holomorphic maps at their Wolff point. This property, which is interesting for itself, will be important in the sequel of the paper.

PROPOSITION 4 *Let  $f \in \operatorname{Hol}(\Delta, \Delta)$  be such that  $\tau \in \partial\Delta$  is its Wolff point. Let  $W$  be a subset of  $\Delta$  which is angular at  $\tau$ . Then  $f(W)$  is also angular at  $\tau$ .*

*Proof* Indeed, fixed  $\alpha < \pi$ , consider in  $\Delta$  the angular sector  $S_\alpha$  of vertex  $\tau$  and opening  $\alpha$ . Take  $\varepsilon > 0$  and define  $r_1^{\alpha+\varepsilon}$  and  $r_2^{\alpha+\varepsilon}$  the two sides of the angular sector  $S_{\alpha+\varepsilon}$  of opening  $\alpha+\varepsilon$  at  $\tau$ . The angle at  $\tau$  between  $f(r_1^{\alpha+\varepsilon})$  and  $f(r_2^{\alpha+\varepsilon})$  has amplitude  $\alpha+\varepsilon$  ( $f$  is isogonal at  $\tau \in \partial\Delta$ , i.e., the angles between curves starting in  $\tau$  are preserved, see, e.g. [14]). Since  $W$  is angular, it is possible to find a horocycle  $O_1$  of center  $\tau$  such that  $O_1 \cap S_{\alpha+\varepsilon} \subset W$  and  $f$  is injective on  $\overline{O_1 \cap S_{\alpha+\varepsilon}}$  (by the Noshiro Lemma). Now, the boundary of the region  $O_1 \cap S_{\alpha+\varepsilon}$  consists of two portions of the two sides of  $S_{\alpha+\varepsilon}$ -let us call the two portions  $l_1^{\alpha+\varepsilon}$  and  $l_2^{\alpha+\varepsilon}$ -and of a portion of the boundary of  $O_1$  denoted by  $l_3$ . The map  $f$  (being injective on it) transform the region  $O_1 \cap S_{\alpha+\varepsilon}$  onto the region whose boundary is  $f(l_1^{\alpha+\varepsilon}) \cup f(l_2^{\alpha+\varepsilon}) \cup f(l_3)$ . Since, as we have noticed, the angle between  $f(l_1^{\alpha+\varepsilon})$  and  $f(l_2^{\alpha+\varepsilon})$  at  $\tau$  is  $\alpha+\varepsilon$ , then  $f(O_1 \cap S_{\alpha+\varepsilon})$  contains a small angular sector of vertex  $\tau$  and opening  $\alpha$ . Since  $\alpha < \pi$  can be chosen arbitrarily, then the assertion follows. ■

### 3. PSEUDO-ITERATION SEMIGROUPS AND COMMUTING HOLOMORPHIC MAPS

This section contains a new result which describes completely the family of all holomorphic maps which commute with a given  $f \in \operatorname{Hol}(\Delta, \Delta)$  having no fixed points in  $\Delta$  and whose iterates converge non-tangentially to the Wolff point  $\tau \in \partial\Delta$ . The proof of this result is based on the following Lemmas.

LEMMA 6 *Let  $f, g \in \text{Hol}(\Delta, \Delta)$  be such that  $\tau(f) = \tau(g) = 1$ . Suppose that  $\exists z_0 \in \Delta$  such that  $f^n(z_0) \rightarrow 1$  non-tangentially.*

*If  $V_f$  is a fundamental set for  $f$  (see Definition 4), then there exists a set  $\tilde{V}_f$  with the following properties:*

- (1)  $\tilde{V}_f$  is open and simply connected;
- (2)  $\tilde{V}_f \subset V_f$ ;
- (3)  $\tilde{V}_f$  is angular at 1;
- (4)  $g(\tilde{V}_f) \subset V_f$ ;
- (5)  $\text{Reg}'(z) > 0, \forall z \in \tilde{V}_f$ .

*Proof* In order to construct  $\tilde{V}_f$ , we choose an arbitrary  $\alpha < \pi$  and let  $S_\alpha$  be an angular sector of vertex 1 and opening  $\alpha$ . Thanks to Proposition 3, the set  $V_f$  is angular at 1. Then there exists a horocycle  $O_\alpha$  of center 1 and positive radius such that  $(S_\alpha \cap O_\alpha) \subset V_f$ . Moreover, since  $0 < g'(1) = K - \lim_{z \rightarrow 1} g'(z) \leq 1$  (i.e., since  $g'$  is  $K$ -continuous) the horocycle  $O_\alpha$  can be chosen in such a way that  $\text{Reg}'(z) > 0$  for all  $z$  in  $S_\alpha \cap O_\alpha$ . By Lemma 4, we can find  $\beta < \pi, \beta = p(\alpha)$  depending on  $\alpha$  and  $g$ , such that  $g(S_\alpha) \subset S_{p(\alpha)}$ .

Now, again by Proposition 4, in correspondence of the angular sector  $S_{p(\alpha)}$  there exists a horocycle  $O_{p(\alpha)} \subset O_\alpha$  such that  $S_{p(\alpha)} \cap O_{p(\alpha)} \subset V_f$  and that  $\text{Reg}'(z) > 0$  for all  $z$  in  $S_\alpha \cap O_{p(\alpha)}$ . Let us now define

$$\tilde{V}_f := \bigcup_{\alpha < \pi} (S_\alpha \cap O_{p(\alpha)})$$

We obtain:

- (1)  $\tilde{V}_f$  is open and simply connected because it is the union of open angular sectors with not empty intersection;
- (2)  $\tilde{V}_f \subset V_f$  because  $\forall \alpha, S_\alpha \cap O_{p(\alpha)} \subseteq S_\alpha \cap O_\alpha \subset V_f$ ;
- (3)  $\tilde{V}_f$  is angular at 1;
- (4) Since for any horocycle  $O_{p(\alpha)}$ , centered at the Wolff point 1, we have  $g(O_{p(\alpha)}) \subset O_{p(\alpha)}$  then

$$\begin{aligned} g(\tilde{V}_f) &= \bigcup_{\alpha < \pi} g(S_\alpha \cap O_{p(\alpha)}) \subset \bigcup_{\alpha < \pi} (g(S_\alpha) \cap g(O_{p(\alpha)})) \\ &\subset \bigcup_{\alpha < \pi} S_{p(\alpha)} \cap O_{p(\alpha)} \subset V_f; \end{aligned}$$

(5)  $\text{Reg}'(z) > 0$ , for all  $z \in \tilde{V}_f$  because

$$\tilde{V}_f = \bigcup_{\alpha < \pi} (S_\alpha \cap O_{p(\alpha)})$$

and  $\text{Reg}'(z) > 0$  on  $S_\alpha \cap O_{p(\alpha)}$ , for all  $\alpha < \pi$ . ■

The following Lemma will permit us to use the result of Noshiro (Lemma 5) to obtain a set  $U$ , contained in the above encountered set  $\tilde{V}_f$ ,  $U$  open and angular at 1 and on which the function  $g$  will be injective. Such a set  $U$  will play a key role in the sequel.

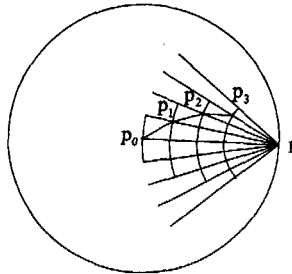
**LEMMA 7** *Let  $W$  be an open set, angular in  $1 \in \partial\Delta$ . Then there exists  $U \subset W$ ,  $U$  open, angular at 1 and convex.*

*Proof* Let  $\alpha_n = \pi - (2/n)$ ,  $n \in \mathbb{N}$ , and let  $S_n$  be the interior of an open angle contained in  $\Delta$  with vertex in 1, of opening  $\alpha_n$  and symmetric with respect to the ray  $\overline{01}$ . Since  $W$  is angular in 1, there exists a small open disc  $C_n$ , with center 1 and radius smaller than 1, such that  $S_n \cap C_n \subset W$ ,  $\forall n \in \mathbb{N}$ . Without loss of generality, one can take  $C_{n+1} \subset C_n$ , that means that the sequence  $(r_n)_{n \in \mathbb{N}}$  of radii of  $(C_n)_{n \in \mathbb{N}}$  is a decreasing sequence. Finally let  $\Gamma = \bigcup_{n \in \mathbb{N}} (S_n \cap C_n) \subset W$ .

Consider now the sequence of points:

$$p_0 = \partial C_1 \cap (-1, 1)$$

$$p_n = (\partial S_n \cap \partial C_{n+1} \cap H^+) \quad (n > 0)$$



where  $(-1, 1)$  is the open interval bounded by  $-1$  and  $1$  on the real axis and where  $H^+$  is the open upper half-plane of  $\mathbb{C}$ .

We can also suppose that the sequence of the open discs  $\{C_n\}_{n \in \mathbb{N}}$  is chosen in a such a way that, for all  $n \in \mathbb{N}$ ,  $n \geq 0$ ,  $p_{n+2}$  lies on the same

side of the point 1 with respect to the straight line connecting  $p_n$  and  $p_{n+1}$ . Notice that the straight line containing  $p_n$  and  $p_{n+1}$  never contains the point 1 since  $W$  is angular at 1. We pass now to consider the set of points  $P_\infty = \{p_n\}_{n \in \mathbb{N}} \cup \{\overline{p_n}\}_{n \in \mathbb{N}} \cup \{1\}$  consisting of the sequence  $\{p_n\}_{n \in \mathbb{N}}$ , of the sequence of the conjugates  $\{\overline{p_n}\}_{n \in \mathbb{N}}$  of  $\{p_n\}_{n \in \mathbb{N}}$  and of the point 1. Let  $U$  be the interior of the convex hull of  $P_\infty$ . The set  $U$  is such that:

- (1) the interior of  $U$  is not empty, since the segment  $\overline{p_n p_{n+1}}$  is never radial;
- (2)  $U \subset \Gamma$  and hence  $U \subset W$ ;
- (3)  $U$  is angular at 1. Indeed, let  $S \subset \Delta$  be an open angular sector with vertex 1 and amplitude strictly smaller than  $\pi$ . Then there exists  $n \in \mathbb{N}$  such that  $S \subseteq S_n$  and, by construction, the open disc  $C$  of center 1 and radius  $d(p_n, 1)$  is so that  $S_n \cap C \subset U$ . In conclusion, the small angular sector  $S \cap C$  is contained in  $U$ .
- (4)  $U$  is convex and in particular it is simply connected. ■

We will point out a nice connection between the property of being angular and that of being  $f$ -absorbing, for a subset of the unit disc  $\Delta$ .

**LEMMA 8** *Let  $f \in \text{Hol}(\Delta, \Delta)$  be without fixed points and suppose that there exists  $z_0 \in \Delta$  such that  $f^n(z_0) \rightarrow \tau(f)$  (the Wolff point of  $f$ ) non-tangentially. Then any subset  $W$  of  $\Delta$  which is angular at  $\tau(f)$  is also  $f$ -absorbing.*

*Proof* We want to prove that: for any  $K$  compact set in  $\Delta$ , there exists  $m \in \mathbb{N}$  such that  $f^m(K) \subset W$ .

From Lemma 3,  $f^n(K) \rightarrow \tau(f)$  non-tangentially, i.e.,  $f^n(K) \rightarrow \tau(f)$  inside a Stolz angle  $S$ . Therefore for all  $\varepsilon > 0$  there exists  $\bar{n}_\varepsilon$  such that for all  $n > \bar{n}_\varepsilon$ , we have  $f^n(K) \subset O_\varepsilon \cap S$ , where  $O_\varepsilon$  is the horocycle with center  $\tau(f)$  and radius  $\varepsilon$ . Since  $W$  is angular at  $\tau(f)$ , then for  $\varepsilon$  sufficiently small,  $f^n(K) \subset S \cap O_\varepsilon \subset W$  for all  $n \geq \bar{n}_\varepsilon$ . ■

We are now ready to prove the main result of this section.

**THEOREM 6** *Let  $f \in \text{Hol}(\Delta, \Delta)$  be such that its Wolff point  $\tau(f) = 1$  and suppose there exists  $z_0 \in \Delta$  such that  $f^n(z_0) \rightarrow 1$  non-tangentially. Let  $g \in \text{Hol}(\Delta, \Delta)$  with  $\tau(g) = 1$ . Then  $f \circ g = g \circ f$  if and only if  $g$  belongs to the pseudo-iteration semigroup  $P_f$  of  $f$ .*

*Proof* By Proposition 2, and since  $f'(1) \neq 0$ , if  $g \in P_f$ , then  $g$  commutes with  $f$  if and only if there exists  $U$  open in  $\Delta$  such that  $g(U)$  and  $g(f(U))$  are contained in  $V_f$ , fundamental set of  $f$ .

Let  $S_\alpha \subseteq \Delta$  be an arbitrary open angular sector of opening  $\alpha < \pi$  and vertex 1; by Lemma 4, there exist  $\beta, \gamma, \delta$  such that

$$\begin{aligned} g(S_\alpha) &\subset S_\beta & \beta < \pi \\ f(S_\alpha) &\subset S_\gamma & \gamma < \pi \\ g(S_\gamma) &\subset S_\delta & \delta < \pi \end{aligned}$$

Since for a  $z_0 \in \Delta$ , the iteration sequence  $\{f^n(z_0)\}_{n \in \mathbb{N}}$  converges to 1 non-tangentially, the fundamental set of  $f$ ,  $V_f$ , is angular at 1 (see Proposition 3).

Therefore there exists  $O_\beta$ , horocycle of center 1, such that  $S_\beta \cap O_\beta \subset V_f$  and there exists  $O_\delta$ , horocycle of center 1, such that  $S_\delta \cap O_\delta \subset V_f$ .

In general, let  $r(O)$  be the radius of the horocycle  $O$ , and let  $r(O_\eta) = \min\{r(O_\beta), r(O_\delta)\}$ . Then define  $U = O_\eta \cap S_\alpha$ . The Wolff lemma yields  $g(O_\eta) \subset O_\eta, f(O_\eta) \subset O_\eta$  and therefore:

$$g(U) = g(S_\alpha \cap O_\eta) \subset g(S_\alpha) \cap g(O_\eta) \subset S_\beta \cap O_\eta \subset S_\beta \cap O_\beta \subset V_f$$

and also

$$\begin{aligned} g(f(U)) &= g(f(S_\alpha \cap O_\eta)) \subset g(f(S_\alpha) \cap f(O_\eta)) \subset g(S_\gamma \cap O_\eta) \subset \\ &\subset g(S_\gamma) \cap g(O_\eta) \subset S_\delta \cap O_\eta \subset S_\delta \cap O_\delta \subset V_f. \end{aligned}$$

Therefore  $g$  commutes with  $f$  under composition.

On the other hand, suppose that  $f \circ g = g \circ f$ .

Let  $V_f, \Omega, \sigma_f$  and  $\varphi$  be the same as in Theorem 4 (Cowen). By using Proposition 3, we can find a fundamental set  $V_f$  of  $f$  that is angular at 1. Let  $U$  be obtained from  $V_f$ , like in Lemma 7. We want to prove now that  $\sigma_f(U)$  is  $\varphi$ -absorbing. Since  $\sigma_f(V_f)$  is fundamental for  $\varphi$ , given any compact set  $K \subset \Omega$  there exists  $m_0 \in \mathbb{N}$  such that

$$\varphi^m(K) \subset \sigma_f(V_f)$$

for all  $m > m_0$ . Being now  $\sigma_f$  injective on  $V_f$  we obtain, if  $\sigma_f^{-1} := (\sigma_f|_{V_f})^{-1}$ :

$$\sigma_f^{-1}(\varphi^m(K)) \subset V_f$$



for all  $m > m_0$ , and since  $U$  is  $f$ -absorbing by Lemma 8, there exists  $k_0 \in \mathbb{N}$  such that

$$f^k(\sigma_f^{-1}(\varphi^m(K))) \subset U$$

for all  $k > k_0$ . Hence

$$\sigma_f(f^k(\sigma_f^{-1}(\varphi^m(K)))) = \varphi^{m+k}(K) \in \sigma_f(U)$$

for all  $k, m$  such that  $m+k > m_0+k_0$ .

Taking into account that  $\sigma_f(U)$  is  $\varphi$ -absorbing, we are now ready to define:

$$\begin{aligned} \psi : \Omega &\rightarrow \Omega \\ w &\mapsto (\varphi^{-1})^n(\sigma_f(g(\sigma_f^{-1}(\varphi^n(w)))))) \end{aligned}$$

where  $n$  is sufficiently large such that  $\varphi^n(w) \in \sigma_f(U)$ .

The map  $\psi$  is well-defined, because if  $\varphi^n(w) \in \sigma_f(U)$  and if  $p \in \mathbb{N}$ , then:

$$\begin{aligned} (\varphi^{-1})^{n+p}(\sigma_f(g(\sigma_f^{-1}(\varphi^{n+p}(w)))))) &= \\ &= (\varphi^{-1})^{n+p}(\sigma_f(g(f^p(\sigma_f^{-1}(\varphi^n(w)))))) = \\ &= (\varphi^{-1})^{n+p}(\sigma_f(f^p(g(\sigma_f^{-1}(\varphi^n(w)))))) = \\ &= (\varphi^{-1})^n(\sigma_f(g(\sigma_f^{-1}(\varphi^n(w))))). \end{aligned}$$

Moreover, since  $\sigma_f$  is injective on  $V_f$  and  $g$  is injective on  $U$  then  $\psi$  is injective on  $\Omega$ . It remains to prove that  $\psi$  is also surjective on  $\Omega$ . To this aim it is now necessary to prove that the set  $\sigma_f(g(U))$  is also  $\varphi$ -absorbing. The procedure is similar to the one used above for  $\sigma_f(U)$ . Notice at first that, by Proposition 4, since  $U$  is angular at 1, so is  $g(U)$ . Lemma 8 implies now that  $g(U)$  is  $f$ -absorbing. Consider that for any compact set  $K \subset \Omega$ , there exists  $n_1 \in \mathbb{N}$  such that

$$\varphi^n(K) \subset \sigma_f(V_f)$$

for all  $n > n_1$  (since  $\sigma_f(V_f)$  is fundamental for  $\varphi$  on  $\Omega$ )

As usual this is to say that for all  $n > n_1$

$$\sigma_f^{-1}(\varphi^n(K)) \subset V_f$$

Since  $g(U)$  is  $f$ -absorbing, there exists  $k_2 \in \mathbb{N}$  such that for all  $k > k_2$

$$f^k(\sigma_f^{-1}(\varphi^n(K))) \subset g(U)$$

and then

$$\sigma_f(f^k(\sigma_f^{-1}(\varphi^n(K)))) = \varphi^{k+n}(K) \subset \sigma_f(g(U))$$

for all  $k+n$  such that  $k > k_2$ ,  $n > n_1$ ; hence  $\sigma_f(g(U))$  is  $\varphi$ -absorbing. The map  $\psi$  is surjective on  $\Omega$  if for all  $z \in \Omega$  there exists  $w \in \Omega$  such that  $\psi(w) = z$ , i.e.,  $(\varphi^{-1})^n(\sigma_f(g(\sigma_f^{-1}(\varphi^n(w)))))) = z$ .

Let  $z \in \Omega$  be given. Then, for  $n$  sufficiently large, since  $\sigma_f(g(U))$  is  $\varphi$ -absorbing, set  $z_1 = \varphi^n(z) \in \sigma_f(g(U))$  and consider  $(\sigma_f \circ g \circ \sigma_f^{-1})^{-1}$  on  $\sigma_f(g(U))$ .

Then, since  $g$  is injective on  $U$ , we can define  $g^{-1} := (g|_U)^{-1}$  and set  $z_2 = (\sigma_f \circ g \circ \sigma_f^{-1})^{-1}(z_1) \in \sigma_f(U)$ . Finally, take  $w = \varphi^{-n}(z_2) \in \Omega$ . It is straightforward to prove now that  $\psi(w) = z$ , i.e., that  $\psi$  is surjective.

Hence  $\psi$  is a linear transformation from  $\Omega$  onto  $\Omega$ .

By following the construction, we have  $\psi \circ \varphi \circ \psi^{-1} = \varphi$  and

$$\begin{aligned} \psi \circ \sigma_f &= (\varphi^{-1})^n \circ \sigma_f \circ g \circ \sigma_f^{-1} \circ \varphi^n \circ \sigma_f \\ &= (\varphi^{-1})^n \circ \sigma_f \circ g \circ f^n = (\varphi^{-1})^n \circ \varphi^n \circ \sigma_f \circ g = \sigma_f \circ g \end{aligned}$$

And so  $g \in P_f$ . ■

As an immediate consequence, we obtain a result proved directly by Cowen in [5, 6].

**COROLLARY 1** *Let  $f \in \text{Hol}(\Delta, \Delta)$  such that  $\tau(f) = 1$  and  $f'(1) < 1$ . Let  $g \in \text{Hol}(\Delta, \Delta)$  with  $\tau(g) = 1$ .*

*Then  $g \circ f = f \circ g$  if and only if  $g$  belongs to the pseudo-iteration semigroup  $P_f$  of  $f$ .*

#### 4. A BOUNDARY GENERALIZATION OF A THEOREM OF PRANGER

The assertion of Theorem 6 can be restated as follows:

**THEOREM 7** *Let  $f \in \text{Hol}(\Delta, \Delta)$  be such that its Wolff point  $\tau(f)$  is equal to 1, not an hyperbolic automorphism, and let  $P_f$  denote as usual the pseudo-iteration semigroup of  $f$ . Define:*

$$G_1 = \{g \in \text{Hol}(\Delta, \Delta) : f \circ g = g \circ f\}$$

and

$$G_2 = \{g \in \text{Hol}(\Delta, \Delta) : \tau(g) = 1, g \in P_f\}.$$

*If there exists  $z_0 \in \Delta$  such that  $f^n(z_0) \rightarrow 1$  non-tangentially, then  $G_1 = G_2$ .*

In the hypotheses in which  $f \in \text{Hol}(\Delta, \Delta)$  is such that  $\{f^n(z_0)\}$  converges to the Wolff point  $\tau(f) = 1$  non-tangentially, by applying Theorem 4 (Cowen) one proves that, up to conjugation under linear fractional transformations,  $\Omega, \varphi, \sigma_f$  are such that

- (a)  $\Omega = \mathbb{C}, \varphi(z) = z + 1, f'(\tau(f)) = 1;$
- (b)  $\Omega = \Delta, \varphi(z) = (((1+s)z + (1-s))/((1-s)z + (1+s))), f'(\tau(f)) < 1,$   
 $0 < s < 1.$

Case (a) or (b) occurs according as  $f'(1) = 1$  or  $f'(1) < 1$ . In the hypotheses of Theorem 6, we will now consider  $\Omega, \varphi, \sigma_f$  relative to  $f$ , fixed. Since  $G_1 = G_2 = G_f$ , given  $g \in G_f$  there exists a linear fractional transformation  $\Phi_g$  belonging to  $\text{Aut}(\mathbb{C})$  or to  $\text{Aut}(\Delta)$  such that:

$$\Phi_g \circ \varphi = \varphi \circ \Phi_g$$

and that

$$\sigma_f \circ g = \Phi_g \circ \sigma_f$$

If  $\mathbb{A}$  is set to be

$$\mathbb{A} = \begin{cases} \text{Aut}(\mathbb{C}) & \text{in case (a)} \\ \text{Aut}(\Delta) & \text{in case (b)} \end{cases} \quad (9)$$

then the map

$$\begin{aligned} \lambda: G_f &\rightarrow \mathbb{A} \\ g &\mapsto \Phi_g \end{aligned} \quad (10)$$

is well defined both in case (a) and in case (b); in fact, given  $g$ , it is easy to see that  $\Phi_g$  is unique as defined. The construction given above for  $\lambda$  recalls the construction given by Pranger in his paper [15], in the case in which the Wolff point  $\tau(f)$  of  $f$  belongs to the interior of the unit disc. And in fact in the same spirit in which the Julia-Wolff-Carathéodory Theorem and the Wolff Lemma are boundary versions of the Schwarz's Lemma, we are now ready to prove a boundary version of a theorem of Pranger [15].

**THEOREM 8** *Let  $f \in \text{Hol}(\Delta, \Delta)$  be such that  $\tau(f) = 1$  and not an hyperbolic automorphism. Suppose there exists  $z_0 \in \Delta$  such that  $f^n(z_0) \rightarrow 1$  non-tangentially. Let  $\mathbb{A}$  be defined as in (9) and let  $\lambda: G_f \rightarrow \mathbb{A}$  be the map defined in (10) which associates to each  $g \in G_f$  the automorphism  $\Phi_g$ . Set  $\lambda(G_f) = \Gamma \subset \mathbb{A}$ .*

*Then the following properties hold:*

- (1)  $\Gamma$  is a subset of  $\mathbb{A}$  topologically closed;
- (2) the identity map  $I \in \Gamma$ ;
- (3)  $\Gamma$  is closed with respect to composition;
- (4) the complement of  $\Gamma$ ,  $\Gamma^*$ , is connected.

*Proof* We start by proving (3) (see also [7]): taken two maps  $h, k \in G_f$ , we have, by definition of  $G_f$ :

$$\begin{aligned} f \circ h &= h \circ f \\ f \circ k &= k \circ f \end{aligned}$$

It is straightforward that also  $(h \circ k)$  commutes with  $f$ , i.e.:

$$f \circ (h \circ k) = (h \circ k) \circ f$$

Hence  $(h \circ k) \in G_f$  and  $G_f$  is closed with respect to composition.

Therefore, it makes sense to consider  $\Phi_{h \circ k} \in \lambda(G_f)$ . Now, by definition of pseudo-iteration semigroup, it follows that:

$$\sigma_f \circ (h \circ k) = \Phi_{h \circ k} \circ \sigma_f$$

Since  $h \in G_f$ , then

$$\sigma_f \circ (h \circ k) = (\sigma_f \circ h) \circ k = (\Phi_h \circ \sigma_f) \circ k$$

and since also  $k \in G_f$ , then

$$(\Phi_h \circ \sigma_f) \circ k = \Phi_h \circ (\sigma_f \circ k) = \Phi_h \circ \Phi_k \circ \sigma_f = (\Phi_h \circ \Phi_k) \circ \sigma_f$$

We conclude that

$$\Phi_{h \circ k} = \Phi_h \circ \Phi_k$$

*i.e.:*

$$\lambda(h \circ k) = \lambda(h) \circ \lambda(k)$$

Hence  $\Gamma$  is closed with respect to composition.

It is immediate to prove (2). Let  $g = I$ ; then  $g \in G_f$  because  $I$  commutes with  $f$ , yielding  $\Phi_g = I$  and  $I \in \lambda(G_f)$ .

We continue by proving (1), (see also [7]). The set  $\Gamma = \lambda(G_f)$  is closed if given  $\{\Phi_j\}_{j \in \mathbb{N}} \in \Gamma$  such that it converges uniformly on compact sets to  $\Phi$ , then  $\Phi \in \lambda(G_f) = \Gamma$ .

Given  $\Phi_j \in \lambda(G_f)$ , there exists  $g_j \in G_f$  such that  $\lambda(g_j) = \Phi_j$ , for all  $j \in \mathbb{N}$ . By definition of pseudo-iteration semigroup, we have

$$\forall j \in \mathbb{N}, \quad \sigma_f \circ g_j = \Phi_j \circ \sigma_f$$

Take the limit for  $j \rightarrow \infty$  on each side of the equality and find

$$\lim_{j \rightarrow \infty} \sigma_f \circ g_j = \lim_{j \rightarrow \infty} \Phi_j \circ \sigma_f = \Phi \circ \sigma_f$$

We have that  $\{g_j\}_{j \in \mathbb{N}}$  is a normal family of holomorphic maps from  $\Delta$  to  $\Delta$ , and then, by the Montel's theorem, we can extract a convergent sub-sequence of maps:  $\{g_{j_k}\}_{k \in \mathbb{N}} \rightarrow \tilde{g}$ .

Hence

$$\{g_{j_k}\} \rightarrow \tilde{g} \text{ for } k \rightarrow \infty$$

It follows that

$$\lim_{j \rightarrow \infty} (\sigma_f \circ g_j) = \lim_{k \rightarrow \infty} (\sigma_f \circ g_{j_k}) = \sigma_f \circ \tilde{g}$$

Furthermore

$$\lim_{k \rightarrow \infty} \Phi_{j_k} \circ \sigma_f = \Phi \circ \sigma_f$$

Therefore

$$\lim_{k \rightarrow \infty} (\sigma_f \circ g_{jk}) = \lim_{k \rightarrow \infty} (\Phi_{jk} \circ \sigma_f)$$

*i.e.*,

$$\sigma_f \circ \tilde{g} = \Phi \circ \sigma_f.$$

Moreover, since

$$\varphi \circ \Phi_j = \Phi_j \circ \varphi, \quad \forall j \in \mathbb{N}$$

take the limit for  $j \rightarrow \infty$ , and obtain

$$\lim_{j \rightarrow \infty} \varphi \circ \Phi_j = \lim_{j \rightarrow \infty} \Phi_j \circ \varphi,$$

*i.e.*,

$$\varphi \circ \Phi = \Phi \circ \varphi$$

Hence

$$\begin{aligned} & \tilde{g} \in G_f \\ (\text{i.e., } & \tilde{g} \in P_f \text{ and } \tau(\tilde{g}) = 1) \end{aligned}$$

and

$$\Phi = \lambda(\tilde{g}) \in \Gamma = \lambda(G_f).$$

Now we prove (4). In our hypotheses on  $f$ , and if  $f'(1) < 1$ , then (see [5]) the automorphism  $\varphi$  of  $\Delta$  associated to  $f$  is, up to conjugation, of the type:

$$\varphi(z) = \frac{[(1+s)z + (1-s)]}{[(1-s)z + (1+s)]} \quad 0 < s < 1.$$

and its matrix representation in  $SU(1, 1)$  is:

$$B_\varphi = \begin{pmatrix} \frac{1+s}{\sqrt{4s}} & \frac{1-s}{\sqrt{4s}} \\ \frac{1-s}{\sqrt{4s}} & \frac{1+s}{\sqrt{4s}} \end{pmatrix}$$

Now consider the set of all matrices associated to the automorphisms of  $\lambda(G_f)$  and denote it by the same symbol  $\lambda(G_f)$ .

By definition of  $\lambda(G_f)$ , its elements commute with  $B_\varphi$ . In general,  $\lambda(G_f)$  is a subset of the group  $\mathcal{C}(B_\varphi)$  of all matrices which commute with  $B_\varphi$  and it turns out (see also [7]):

$$\mathcal{C}(B_\varphi) = \left\{ \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \mid \alpha, \beta \in \mathbb{R}, \alpha^2 - \beta^2 = 1 \right\}$$

More precisely,  $\lambda(G_f)$  is a subset of  $\mathcal{C}^+(B_\varphi)$ , the semigroup of all the matrices of  $\mathcal{C}(B_\varphi)$  which represent automorphisms of  $\Delta$  commuting with  $B_\varphi$  and with Wolff point equal to 1. After simple computations:

$$\mathcal{C}^+(B_\varphi) = \left\{ \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \mid \alpha, \beta \in \mathbb{R}, \alpha > 0, \beta \geq 0, \alpha^2 - \beta^2 = 1 \right\}$$

Now,  $\mathcal{C}^+(B_\varphi) \subset SU(1, 1)$  is a real manifold of dimension 1, and  $SU(1, 1)$  is a real connected manifold of dimension 3. Hence the complement of  $\lambda(G_f)$  has, at least, dimension two and consequently it is connected.

Instead, if  $f'(1) = 1$ , the automorphism of  $\mathbb{C}$  associated to  $f$  is:

$$\varphi(z) = z + 1.$$

In this case recall that

$$\begin{aligned} \text{Aut}(\mathbb{C}) &= \{z \mapsto \alpha z + \beta, \alpha \in \mathbb{C}^*, \beta \in \mathbb{C}\} \\ \text{Aut}(\mathbb{C}) &\hookrightarrow \text{Aut}(\mathbb{P}^1(\mathbb{C})) \subset SL(2, \mathbb{C}) \end{aligned}$$

Hence

$$\text{Aut}(\mathbb{C}) = \{z \mapsto \alpha z + \beta\} \longleftrightarrow \left\{ \begin{pmatrix} \alpha^{1/2} & \beta\alpha^{-1/2} \\ 0 & \alpha^{-1/2} \end{pmatrix} \right\} \subset SL(2, \mathbb{C})$$

and, up to conjugation, the matrix which represent  $\varphi$  is in this case:

$$A_\varphi = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Again, all the matrices associated to the automorphisms of  $\mathbb{C}$  which are in  $\lambda(G_f)$ , commute with  $A_\varphi$ . In general,  $\lambda(G_f)$  is a subset of the

group  $\mathcal{C}(A_\varphi)$  of all matrices which commute with  $A_\varphi$ , and it turns out that:

$$\mathcal{C}(A_\varphi) = \left\{ \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}, \beta \in \mathbb{C} \right\}$$

Observe that  $\mathcal{C}(A_\varphi) \subset \text{Aut}(\mathbb{C})$  is a real manifold of dimension 2, but  $\text{Aut}(\mathbb{C})$  is a real connected manifold of dimension 4; the complement of  $\lambda(G_f)$  has at least dimension 2 and so it is connected. ■

It is worthwhile noticing that, while proving Theorem 8, we have also proved that  $\lambda: G_f \rightarrow \mathbb{A}$  is a homomorphism of semigroups.

**COROLLARY 2** *Let  $f \in \text{Hol}(\Delta, \Delta)$  be such that  $\tau(f) = 1$ . Suppose that  $f'(1) < 1$ . Then, with the same notations as in Theorem 8,  $\lambda(G_f) = \Gamma$  is a subset of the group of the Moebius transformations of  $\Delta$  and:*

- (1)  $\Gamma$  is topologically closed and conjugated to a semigroup contained in  $\mathcal{C}^+(A_\varphi)$  (see [10]);
- (2) the identity map  $I \in \Gamma$ ;
- (3)  $\Gamma$  is closed with respect to the composition;
- (4) the complement of  $\Gamma$ ,  $\Gamma^*$  is connected.

If  $\mathbb{A}$  denotes the group  $\text{Aut}(\Delta)$  or the group  $\text{Aut}(\mathbb{C})$ , (according to the hypotheses on  $f$ ), then we can prove the following result on  $\lambda$ :

**PROPOSITION 5** *Let  $f \in \text{Hol}(\Delta, \Delta)$  be such that  $\tau(f) = 1$ . Suppose there exists  $z_0 \in \Delta$  such that  $f^n(z_0) \rightarrow 1$  non-tangentially. Let  $\mathbb{A}$  be defined as in (9) and let  $\lambda: G_f \rightarrow \mathbb{A}$  be defined as in (10).*

*Then the map  $\lambda: G_f \rightarrow \lambda(G_f) \subset \mathbb{A}$  is injective.*

*Proof* Let  $\Phi \in \lambda(G_f)$  and suppose that there exist  $h, g \in G_f$  such that  $\lambda(h) = \lambda(g) = \Phi$ . Let  $S_\alpha$  be an arbitrary open angular sector of opening  $\alpha < \pi$  and vertex 1. Then, by Lemma 4, the functions  $g, h$  send Stolz angles of vertex  $\tau$  in Stolz angles of vertex  $\tau$  (at least locally near  $\tau$ ).

Hence, there exist  $\beta < \pi, \gamma < \pi$  such that:

$$\begin{aligned} g(S_\alpha) &\subset S_\beta \quad \text{locally near } \tau \\ h(S_\alpha) &\subset S_\gamma \quad \text{locally near } \tau \end{aligned}$$



Since by hypothesis, there exists  $z_0 \in \Delta$  such that  $f^n(z_0) \rightarrow \tau$ , then  $V_f$  is angular at 1 and hence there exist horocycles  $O_\beta$  and  $O_\gamma$  such that:

$$\begin{aligned} S_\beta \cap O_\beta &\subset V_f \\ S_\gamma \cap O_\gamma &\subset V_f \end{aligned}$$

Let  $r(O_\delta) = \min\{r(O_\beta), r(O_\gamma)\}$ ; then:

$$\begin{aligned} S_\beta \cap O_\delta &\subset V_f \\ S_\gamma \cap O_\delta &\subset V_f \end{aligned}$$

Now, by the Wolff lemma:

$$\begin{aligned} g(S_\alpha \cap O_\delta) &\subset S_\beta \cap O_\delta \subset V_f \\ h(S_\alpha \cap O_\delta) &\subset S_\gamma \cap O_\delta \subset V_f \end{aligned}$$

and hence the open set  $U = S_\alpha \cap O_\delta \subset \Delta$  is such that:

$$\begin{aligned} g(U) &\subset V_f \\ h(U) &\subset V_f \end{aligned}$$

The map  $\sigma_f$  is invertible on  $V_f$  and so is on  $g(U)$  and  $h(U)$ .

Since

$$\begin{aligned} h, g &\in G_f \\ \lambda(g) &= \lambda(h) = \Phi, \end{aligned}$$

we have

$$\begin{aligned} \Phi \circ \sigma_f &= \sigma_f \circ g \\ \Phi \circ \sigma_f &= \sigma_f \circ h \end{aligned}$$

In particular

$$\begin{aligned} \Phi(\sigma_f(U)) &= \sigma_f(g(U)) \subset \sigma_f(V_f) \\ \Phi(\sigma_f(U)) &= \sigma_f(h(U)) \subset \sigma_f(V_f) \end{aligned}$$

Therefore  $\sigma_f$  is invertible on  $\Phi(\sigma_f(U)) \subset \sigma_f(V_f)$  and by setting, as usual

$$\sigma_f^{-1} := \left( \sigma_{f|_{V_f}} \right)^{-1}$$

we obtain:

$$(\sigma_f^{-1} \circ \Phi \circ \sigma_f) = g \text{ on } U$$

$$(\sigma_f^{-1} \circ \Phi \circ \sigma_f) = h \text{ on } U$$

Then  $g=h$  on the open set  $U$  and therefore, by the analytic continuation principle,  $g=h$  on  $\Delta$  and  $\lambda$  is injective. ■

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