A NATURAL EXTENSION OF THE YOUNG PARTITIONS LATTICE

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Abstract. Recently Andrews introduced the concept of signed partition: a *signed partition* is a finite sequence of integers $a_k, \dots, a_1, a_{-1}, \dots, a_{-l}$ such that $a_k \geq \dots \geq a_1 > 0 > a_{-1} \geq \dots \geq a_{-l}$. So far the signed partitions have been studied from an arithmetical point of view. In this paper we first generalize the concept of signed partition and we next use such a generalization to introduce a partial order on the set of all the signed partitions. Furthermore, we show that this order has many remarkable properties and that it generalizes the classical order on the Young lattice.

0. Introduction

In this paper we generalize the usual order of the classical Young lattice to the case of the integer signed partitions The concept of signed partition has been recently introduced by Andrews in [5] : a *signed partition* is a finite sequence of integers $a_k, \dots, a_1, a_{-1}, \dots, a_{-l}$ such that $a_k \geq \dots \geq a_1 > 0 > a_{-1} \geq \dots \geq a_{-l}$. Also Keith in [32] studies several interesting combinatorial and arithmetical properties of the signed partitions. However in both [5] and [32], the signed partitions are not studied from an order point of view and at present there are no studies in this direction. Therefore in this paper we introduce and study a natural generalization of the usual component-wise order on the integer partitions with positive summands. The partial order (denoted in the sequel by $\sqsubseteq$) that we obtain on the set (denoted in the sequel by P_*) of all the signed partitions is a lattice that contains the Young lattice $\mathbb{Y}$ as its sublattice. We briefly describe the content of this paper. In section 1 we recall some preliminary definitions that we use in all the other sections. In section 2, we introduce at first the set P of all the *generalized partitions* and a quasi-order $\sqsubseteq$ on it; next we define the lattice $(P_*, \sqsubseteq)$ of all the signed partitions as a quotient-poset of $(P, \sqsubseteq)$. We also show that P_* has some properties similar to those of the differential posets introduced by Stanley in [50], more in detail, we can consider P_* as a type of 0-differential poset without a minimum element. If $m \in \mathbb{Z}$, the set $Par(m)$ is an infinite subset of P_* that we can intuitively think as an horizontal axis. In section 3 we introduce the concept of 1-*covering sublattice* of P_* as a sublattice L that intersects each horizontal axis $Par(m)$, when the integer m runs between the sums made respectively on the summands of the minimum signed partition and the maximum signed partition of L. In section 4 we introduce the finite 1-covering sublattice $P(n, r)$ and we use such lattice to examine locally some order properties of P_* and we also exhibit a sublattice $P(n, d, r)$ of $P(n, r)$ that fails to be 1-covering and for it we determine (section 5) its covering relation identifying $P(n, d, r)$ with an appropriate discrete dynamical model with three evolution rules. In section 6 we define the *signed Young diagrams* and we use the

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lattice $P(n,r)$ as a natural environment to extend some classical combinatorial properties of the standard Young tableaux to the case of the signed partitions. The way to study a lattice of classical partitions as a discrete dynamical model having some particular evolution rules was implicit in [14], where Brylawski proposed a dynamical approach to study the lattice $L_B(m)$ of all the partitions of fixed positive integer m with the dominance order. However, the explicit theoretical association between integer partitions and discrete dynamical model begins in [25], where Goles and Kiwi introduced the Sand Piles Model $SPM(m)$. If m is a non negative integer, a configuration of $SPM(m)$ is represented by an ordered partition of m, i.e. a decreasing sequence $a = (a_1, ..., a_m)$ of non negative integers having sum m, and each positive entire unit is interpreted as a sand grain whose movement respects the following rule:

Rule 1 (vertical rule): *one grain can move from a column to the next one if the difference of height of these two columns is greater than or equal to 2.*

The Sand Piles Model (SPM) is a special case of a discrete dynamical system (see [2], [3], [4] for very recent studies on this topic). There are a lot of specializations and extensions of this model which have been introduced and studied under different names, different aspects and with different approaches. The SPM problem cames from the Self-Organized Criticality (SOC) problem introduced by Bak, Tang and Wiesenfield in [6]. The study of such systems have been developed in an algebraic context ([17], [19], [39], [52]), in a combinatorial games theory context ([8], [11], [12], [25], [27], [28]), in a graph theory context ([16], [40], [41]) and in the context of the cellular automata theory ([15], [26]). In the scope of the discrete dynamical systems, the Brylawski lattice can be interpreted then as a model $L_B(m)$, where the movement of a sand grain respects the previous Rule 1 and the following Rule 2:

Rule 2 (horizontal rule): *if a column containing $p+1$ grains, is followed by a sequence of columns containing p grains and next by one column containing $p-1$ grains, then one grain of the first column can slip to the last column.*

In [18], [23], [24], [28], [29], [31], [34], [35], [36], [37], [38], [45], [46], several dynamical models related to $SPM(n)$ have been studied. Almost all systems studied in the previous works have a linear topology and they have extended the classical models $SPM(n)$ and $L_B(n)$ to obtain more general models. An excellent survey on these topics is [30].

1. Definitions

For all the classic properties concerning posets and lattices we refer to [20], [47], [48]. For all the results and definitions concerning the discrete dynamical models of integer partitions we refer to [30]. In this section we recall only some definitions and fundamental results.
A *quasi-order* on a set X is a binary reflexive and transitive relation on X. If $(X, \leq)$ is a poset, the *dual* of X is the poset $(X, \leq^*)$, where $x \leq^* y$ iff $y \leq x$, for all $x, y \in X$. We say that X is a *self-dual* poset if the identity is an isomorphism between $(X, \leq)$ and $(X, \leq^*)$.
If X is a finite lattice, then X has a minimum element and a maximum element, that we denote respectively by $\hat{0}$ and $\hat{1}$ if X is clear from the context. If X is a poset, a *rank function* of X is a function $\rho : X \to \mathbb{N}$ such that $\rho(x) = 0$ for some minimal element x of X (if there exists at least one minimal element) and if y covers x then $\rho(y) = \rho(x) + 1$.

A *ranked poset* is a couple (X, ρ), where ρ is a rank function of X. If (X, ρ) is a ranked poset such that $\rho(x) = 0$ for all minimal elements x of X and ρ takes the same value on all maximal elements of X then ρ is the unique rank function of X with these properties (in this case we say that such map ρ is *the* rank function) and we call (X, ρ) a *graded poset*. In particular, if X is a finite distributive lattice, then X is graded (see [47]). If (X, ρ) is a ranked poset, the number $rank(X) := \sup\{\rho(x) : x \in X\}$ is called *rank* of X; the subset $N_i(X) := \{x \in X : \rho(x) = i\}$ is called i-th *rank level* of X and the number $W_i(X) := |N_i(X)|$ is called i-th *Whitney number* of X, for $i = 0, 1, \dots, \rho(X)$. We say that X has the *Sperner property* if some rank level of X is an antichain of X having maximal length. The set X is called a k-*Sperner poset* if none union of k antichains has cardinality greater than the union of the k biggest rank levels (therefore, P has the Sperner property iff X is a 1-Sperner poset). We say that X is a *strongly Sperner poset* if X is a k-Sperner poset for all the values of k. Let $x_i := W_i(X)$ and $n = \rho(X)$. X is called *rank-symmetric* if $x_i = x_{n-i}$ for $i = 0, 1, \dots, n$, and *rank-unimodal* if there exists $j \in \{0, 1, \dots, n\}$ such that $x_0 \leq x_1 \leq x_2 \leq \dots \leq x_j \geq x_{j+1} \geq x_{j+2} \geq \dots \geq x_n$. Therefore, if X is both rank-symmetric and rank-unimodal, then $x_0 \leq x_1 \leq x_2 \leq \dots \leq x_m \geq x_{m+1} \geq x_{m+2} \geq \dots \geq x_n$ if $n = 2m$, and $x_0 \leq x_1 \leq x_2 \leq \dots \leq x_m = x_{m+1} \geq x_{m+2} \geq \dots \geq x_n$ if $n = 2m + 1$. A graded poset X is called *Peck poset* if it is rank-symmetric, rank-unimodal and strongly Sperner.
If $(X, \leq)$ and $(Y, \leq')$ are two posets, a map $\phi : X \to Y$ is said *order-reversing* if $x \leq x'$ implies $\phi(x') \leq' \phi(x)$ for $x, x' \in X$. If $(X, \leq)$ is a poset and $x, y \in X$, we set $[x, y]_X = \{u \in X : x \leq u \leq y\}$. If the subset $[x, y]_X$ is finite for all $x, y \in X$, the poset $(X, \leq)$ is said *locally finite*.

2. The Signed Partitions Lattice

The next definition generalizes the concept of signed partition introduced in [5] and studied in [32] from an arithmetical point of view. The aim of such a generalization is to define a partial order on the set of all the signed partitions which will be an extension of the classical order on the Young lattice.

Definition 2.1. *Let q and p be two non-negative integers. A generalized partition (briefly a g-partition) w with balance (q, p) is a finite sequence of integers $a_q, \dots, a_1, b_1, \dots, b_p$, called parts of w, such that $a_q \geq \dots \geq a_1 \geq 0 \geq b_1 \geq \dots \geq b_p$.*

A g-*partition* w is a g-partition having balance (q, p), for some non-negative integers q and p. We shall denote by P the set of all the g-partitions. We call $a_q, \dots, a_1$ the *non-negative parts of* w and $b_1, \dots, b_p$ the *non-positive parts of* w; also, we call *positive parts of* w the integers a_i with $a_i > 0$ and *negative parts of* w the integers b_j with $b_j < 0$. We assume that $q = 0$ [$p = 0$] iff there are no non-negative [non-positive] parts of w; in particular, it results that $p = 0$ and $q = 0$ iff w is the empty g-partition. In the sequel, to describe a generic g-partition w we shall use two possible notations : $w = a_q \dots a_1 | b_1 \dots b_p$ or, if it is not necessary to distinguish which parts of a g-partition w are non-negative integers and which are non-positive integers, we simply write $w = l_1 \cdots l_n$, without specifying the sign of l_i's. We denote by $(|)$ the empty g-partition. However, in all the numerical examples and also in the graphical representation of the Hasse diagrams, we omit the minus sign for all the parts $b_1, \dots, b_p$. This means, for example, that we write $w = 44000|0113$ instead of $w = 44000|0(-1)(-1)(-3)$. If $w = a_q \dots a_1 | b_1 \dots b_p$ we set $w_+ = a_q \dots a_1|$ and $w_- = |b_1 \dots b_p$, moreover we shall call *maximal part of* w and *minimal part of* w respectively the non-negative integers a_q and b_p and we shall set $M^+(w) = a_q$, $M^-(w) = |b_p|$. If m is an integer such that $m = a_q + \cdots + a_1 + b_1 + \cdots + b_p$, we say that w is a *g-partition of the integer* m and we shall write $w \vdash m$. If w is a g-partition, we denote by $\{w \geq 0\}$ $[\{w > 0\}]$ the multi-set of all the non-negative [positive] parts of w and by $\{w \leq 0\}$ $[\{w < 0\}]$ the multi-set of the non-positive [negative] parts of w. We denote respectively with $|w|_{\geq}$, $|w|_{\leq}$, $|w|_{>}$, $|w|_{<}$ the cardinality of $\{w \geq 0\}$, $\{w \leq 0\}$, $\{w > 0\}$, $\{w < 0\}$. We call the ordered couple $(|w|_{>}, |w|_{<})$ the *signature* of w and we note that $(|w|_{\geq}, |w|_{\leq})$ is exactly the balance of w . Finally, we set $||w|| = |w|_{>} + |w|_{<}$. For example, if $w = 444221000|011333$, then $\{w \geq 0\} = \{4^3, 2^2, 1^1, 0^3\}$, $\{w \leq 0\} = \{0^1, -1^2, -3^3\}$, $\{w > 0\} = \{4^3, 2^2, 1^1\}$, $\{w < 0\} = \{-1^2, -3^3\}$, w has balance $(|w|_{\geq}, |w|_{\leq}) = (9, 6)$, signature $(|w|_{>}, |w|_{<}) = (6, 5)$ and $||w|| = 11$.

Remark 2.2.
We shall consider equals two g-partitions $w = a_q \dots a_1|b_1 \dots b_p$ *and* $w' = a'_{q'} \dots a'_1|b'_1 \dots b'_{p'}$ *(and we shall write* $w = w'$*) if and only if* $q = q'$, $p = p'$ *and* $a_i = a'_i$, $b_j = b'_j$ *for* $i = 1, \dots, q$ *and* $j = 1, \dots, p$. *Therefore, for example, in our context, the two g-partitions* $555000|11$ *and* $55500|011$ *are considered different.*

If $w = a_q \dots a_{q-t+1} a_{q-t} \dots a_1 | b_1 \dots b_{p-s} b_{p-s+1} \dots b_p$ is a g-partition having signature (t, s) and balance (q, p), where $a_q \geq \cdots \geq a_{q-t+1} > 0 > b_{p-s+1} \geq \cdots \geq b_p$, $a_{q-t} = \ldots = a_1 = 0$ and $b_1 = \ldots = b_{p-s} = 0$, then we write $w = a_q \dots a_{q-t+1} 0_{q-t} | 0_{p-s} b_{p-s+1} \dots b_p$ and $w_* = a_q \dots a_{q-t+1} | b_{p-s+1} \dots b_p$. We call w_* the *reduced* g-partition of w. If W is a subset of P, we set $W_* = \{w_* : w \in W\}$. We note that a signed partition, as introduced in [5], is exactly the reduced g-partition of some g-partition, therefore the set of all the signed partitions is exactly P_*. Let us note that $\{w > 0\} = \{w_* > 0\}$ and $\{w < 0\} = \{w_* < 0\}$. If U is a subset of g-partitions, we say that U is *uniform* if all the g-partitions in U have the same balance; in particular, if all the g-partitions in U have balance (q, p), we also say that U is (q,p)-*uniform*. If $v_1, \dots, v_k$ are g-partitions, we say that they are *uniform* [(q,p)-*uniform*] if the subset $U = \{v_1, \dots, v_k\}$ is uniform [(q,p)-*uniform*]. If F is a finite subset of g-partitions, we define a way to make uniform all the g-partitions in F: we set $q_F = \max\{|v|_{\geq} : v \in F\}$, $p_F = \max\{|v|_{\leq} : v \in F\}$ and if $v = a_q \dots a_1|b_1 \dots b_p \in F$, $\overline{v}^F = a_q \dots a_1 0_{q_F - q} | 0_{p_F - p} b_1 \dots b_p$ and $\overline{F} = \{\overline{v}^F : v \in F\}$. Then $\overline{F}$ is (q_F, p_F)-uniform and $|\overline{F}| \leq |F|$. If F is uniform we note that $\overline{v}^F = v$ for each $v \in F$, hence $\overline{F} = F$. We call $\overline{F}$ the *uniform closure of* F. When F is clear from the context we simply write $\overline{v}$ instead of $\overline{v}^F$. In particular, if v and w are two g-partitions, when we write $\overline{v}$ and $\overline{w}$ without further specification we always mean $\overline{v}^F$ and $\overline{w}^F$, where $F = \{v, w\}$. We observe that if v and w are two uniform g-partitions then $\overline{v} = v$ and $\overline{w} = w$, moreover let us also note that if $w \in F$ we have $\{w > 0\} = \{\overline{w}^F > 0\}$ and $\{w < 0\} = \{\overline{w}^F < 0\}$.

If $u = l_1 \cdots l_n$ and $u' = l'_1 \cdots l'_n$ are two uniform g-partitions in P, we define:

$$u \leqslant u' \Longleftrightarrow l_i \leq l'_i \tag{1}$$

for all $i \in \{1, \dots, n\}$.
We define then on P the following binary relation: if $v, w \in P$, we set

$$v \sqsubseteq w \Longleftrightarrow \overline{v} \leqslant \overline{w} \tag{2}$$

In particular, if v and w are uniform, then $v \sqsubseteq w$ iff $v \leqslant w$. With the notation $v \sqsubset w$ we mean $v \sqsubseteq w$ with $v \neq w$.

Proposition 2.3. *$\sqsubseteq$ is a quasi-order on the set P; moreover, if $v, w \in P$ the following conditions are equivalent:*
(i) $v \sqsubseteq w$;
(ii) $\overline{v}^F \leqslant \overline{w}^F$ for some particular finite subset F of P that contains v and w;
(iii) $\overline{v}^F \leqslant \overline{w}^F$ for each finite subset F of P that contains v and w;
(iv) $v_* \sqsubseteq w_*$.

Proof. The proof is straightforward. □

If $v, w \in P$, we set

$$v \sim w \Longleftrightarrow \{v > 0\} = \{w > 0\} \text{ and } \{v < 0\} = \{w < 0\} \tag{3}$$

Then $\sim$ is an equivalence relation on P and it results that

$$v \sim w \Longleftrightarrow v \sqsubseteq w \text{ and } w \sqsubseteq v \tag{4}$$

Proposition 2.4. *(i) If v and w are two g-partitions, then $v = w$ iff v and w are uniform and $v \sim w$.*
(ii) If F is a finite subset of P such that $v, w \in F$, then $v \sim w$ iff $\overline{v}^F = \overline{w}^F$.
(iii) If F is a finite subset of P such that $v \in F$ then $v \sim \overline{v}^F$.
(iv) If v is a g-partition then $v \sim v_*$.

Proof. The proof is straightforward. □

By (4) it follows that $\sim$ is exactly the equivalence relation on P induced by the quasi-order $\sqsubseteq$, therefore if $\mathcal{F}$ is any subset of P, we can consider on the quotient set $\mathcal{F}/\sim$ the usual partial order induced by $\sqsubseteq$, that we here denote by $\sqsubseteq'$. We recall that $\sqsubseteq'$ is defined as follows: if $Z, Z' \in \mathcal{F}/\sim$ then

$$Z \sqsubseteq' Z' \Longleftrightarrow v \sqsubseteq v' \tag{5}$$

for any/all $v, w \in \mathcal{F}$ such that $v \in Z$ and $v' \in Z'$.
If $w \in \mathcal{F}$, in some case we set $[w]_{\sim}^{\mathcal{F}} = \{v \in \mathcal{F} : v \sim w\}$, that is the equivalence class of w in $\mathcal{F}/\sim$.

Remark 2.5. *If $\mathcal{F} \subseteq \mathcal{H} \subseteq P$ we can consider $\mathcal{F}/\sim$ as a subset of $\mathcal{H}/\sim$ through the identification of $[v]_{\sim}^{\mathcal{F}}$ with $[v]_{\sim}^{\mathcal{H}}$, for each $v \in \mathcal{F}$. Therefore, if $\mathcal{F} \subseteq \mathcal{H} \subseteq P$ we always can assume that $(\mathcal{F}/\sim, \sqsubseteq')$ is a sub-poset of $(\mathcal{H}/\sim, \sqsubseteq')$.*

Let us observe that we can choose in (5) the representatives of the correspondent equivalence classes in $\mathcal{F}/\sim$ in several ways, depending on the choice of the subset $\mathcal{F}$ of P on which we take the quotient. In particular, if $v \in \mathcal{F}$, by Proposition 2.4 (iv) we can always choose v_* as a representative for the equivalence class $[v]_{\sim}^{\mathcal{F}}$. In such a way we identify the quotient set $\mathcal{F}/\sim$ with the subset $\mathcal{F}_*$ of P, and we shall write $\mathcal{F}/\sim \equiv \mathcal{F}_*$. However, when $\mathcal{F}$ is a finite subset of P, we have also another possibility. In fact, if $\mathcal{F}$ is finite and $v \in \mathcal{F}$, by Proposition 2.4 (iii) we can also choose $\overline{v}^{\mathcal{F}}$ as a representative for $[v]_{\sim}^{\mathcal{F}}$. In such a way we identify $\mathcal{F}/\sim$ with the subset $\overline{\mathcal{F}}$ of P, and we shall write $\mathcal{F}/\sim \equiv \overline{\mathcal{F}}$ (and $\mathcal{F}/\sim \equiv \mathcal{F}$ if $\mathcal{F}$ is

finite and uniform). Therefore, if $\mathcal{F} \subseteq P$, Z, Z' are any two equivalence classes in $\mathcal{F}/\sim$ and v, v' are any two elements in $\mathcal{F}$ such that $v \in Z$, $v' \in Z'$, we must distinguish two cases:
(**F1**) $\mathcal{F}/\sim \equiv \mathcal{F}_*$. In this case the order (5) will have the following equivalent form (by Proposition 2.3 (iv)):

$$Z \sqsubseteq' Z' \Longleftrightarrow v_* \sqsubseteq v'_* \tag{6}$$

(**F2**) $\mathcal{F}/\sim \equiv \overline{\mathcal{F}}$. In this case the order (5) will have the following equivalent form (by Proposition 2.3 (ii)):

$$Z \sqsubseteq' Z' \Longleftrightarrow \overline{v}^{\mathcal{F}} \sqsubseteq \overline{v'}^{\mathcal{F}} \tag{7}$$

Hence, taking into account the conventions established in (6) and (7), we can always think that on each quotient set $\mathcal{F}/\sim$ the partial order $\sqsubseteq'$ is identified with $\sqsubseteq$. Therefore, in the case (**F1**) we identify the poset $(\mathcal{F}/\sim, \sqsubseteq')$ with $(\mathcal{F}_*, \sqsubseteq)$; in the case (**F2**) we identify the poset $(\mathcal{F}/\sim, \sqsubseteq')$ with $(\overline{\mathcal{F}}, \sqsubseteq)$. In particular, if $\mathcal{F}$ is a finite uniform subset of P then $(\overline{\mathcal{F}}, \sqsubseteq)$ coincides with $(\mathcal{F}, \sqsubseteq)$, and in (7) we have that $\overline{v}^{\mathcal{F}} = v$ and $\overline{v'}^{\mathcal{F}} = v'$.
In the sequel, if $\mathcal{F}$ coincides with P then we take always $P/\sim \equiv P_*$. However, if $\mathcal{F}$ coincides with some finite uniform subset of P, then $\mathcal{F}/\sim \equiv \mathcal{F}$ if we write the g-partitions in $\mathcal{F}$ with all their zeros, otherwise $\mathcal{F}/\sim \equiv \mathcal{F}_*$. The question is not only formal when the subset $\mathcal{F}$ is a finite uniform subset of P. In fact, in this case, if we take $\mathcal{F}/\sim \equiv \mathcal{F}$, then $\mathcal{F}$ is not a subset of P_*, hence using such an identification we cannot consider $\mathcal{F}/\sim$ as a sub-poset of P_*. Therefore, if we want to consider $\mathcal{F}/\sim$ as a sub-poset of P_*, then it is necessary to take $\mathcal{F}/\sim \equiv \mathcal{F}_*$. On the other side, if we want to isolate some specific properties of $\mathcal{F}/\sim$, for example some covering properties of its elements, then it will be convenient to take $\mathcal{F}/\sim \equiv \mathcal{F}$, in order to work with uniform g-partitions. To solve the question without loading the notations, in the sequel if $\mathcal{F}$ is a finite uniform sub-poset of P, we shall write simply $(\mathcal{F}, \sqsubseteq)$ or $(\mathcal{F}_*, \sqsubseteq)$ instead of $(\mathcal{F}/\sim, \sqsubseteq')$, depending on the context. In particular, we shall always consider $(\mathcal{F}, \sqsubseteq)$ as sub-poset of $(P_*, \sqsubseteq)$. If m is an arbitrary integer, we set $Par(m) = \{w \in P_* : w \vdash m\}$ and we call $Par(m)$ the *horizontal axis in P_* of height m*. An *horizontal axis in P_** is an horizontal axis $Par(m)$, for some integer m. In particular, we call *x-axis of P_** the horizontal axis $Par(0)$. We call *horizontal* some type of orders that we shall consider on the horizontal axes of P_*.

Definition 2.6. *We call $(P_*, \sqsubseteq)$ the signed partitions poset.*

If $u = a_q \dots a_1 | b_1 \dots b_p$ and $u' = a'_q \dots a'_1 | b'_1 \dots b'_p$ are two uniform g-partitions, we set:

$$u \bigtriangleup u' = \min\{a_q, a'_q\} \dots \min\{a_1, a'_1\} | \min\{b_1, b'_1\} \dots \min\{b_p, b'_p\}$$

and

$$u \bigtriangledown u' = \max\{a_q, a'_q\} \dots \max\{a_1, a'_1\} | \max\{b_1, b'_1\} \dots \min\{b_p, b'_p\}$$

The next definition describes a type of subset of g-partitions which shall permit us to find several distributive sub-lattices of $(P/\sim, \sqsubseteq') \equiv (P_*, \sqsubseteq)$.

Definition 2.7. *We say that a subset $\mathcal{F} \subseteq P$ is lattice-inductive if for each finite subset $F \subseteq \mathcal{F}$ it results that:*
(i) $\overline{F} \subseteq \mathcal{F}$;
(ii) if $v, w \in F$, then $\overline{v}^F \bigtriangleup \overline{w}^F \in \mathcal{F}$ and $\overline{v}^F \bigtriangledown \overline{w}^F \in \mathcal{F}$.

Let us note that obviously P is lattice-inductive. The relevance of the lattice-inductive subsets of P is established in the following result.

Proposition 2.8. *Let $\mathcal{F}$ be a lattice-inductive subset of P. Then $(\mathcal{F}/\sim, \sqsubseteq')$ is a distributive lattice.*

Proof. To prove that $(\mathcal{F}/\sim)$ is a lattice, we take two equivalence classes $[v]_\sim^{\mathcal{F}}$ and $[w]_\sim^{\mathcal{F}}$ in $(\mathcal{F}/\sim)$ and we define the following operations: $[v]_\sim^{\mathcal{F}} \wedge [w]_\sim^{\mathcal{F}} = [\overline{v} \bigtriangleup \overline{w}]_\sim^{\mathcal{F}}$ and $[v]_\sim^{\mathcal{F}} \vee [w]_\sim^{\mathcal{F}} = [\overline{v} \bigtriangledown \overline{w}]_\sim^{\mathcal{F}}$. It easy to see that the operations $\wedge$ and $\vee$ are well defined because they do not depend on the choice of representatives in the respective equivalence classes and that $[v]_\sim^{\mathcal{F}} \wedge [w]_\sim^{\mathcal{F}} \sqsubseteq' [v]_\sim^{\mathcal{F}}$, $[v]_\sim^{\mathcal{F}} \wedge [w]_\sim^{\mathcal{F}} \sqsubseteq' [w]_\sim^{\mathcal{F}}$. Let now $[z]_\sim \in \mathcal{F}/\sim$ such that $[z]_\sim^{\mathcal{F}} \sqsubseteq' [v]_\sim^{\mathcal{F}}$ and $[z]_\sim^{\mathcal{F}} \sqsubseteq' [w]_\sim^{\mathcal{F}}$. We set $F = \{v, w, z\}$. Then, $([z]_\sim^{\mathcal{F}} \sqsubseteq' [v]_\sim^{\mathcal{F}}$ and $[z]_\sim^{\mathcal{F}} \sqsubseteq' [w]_\sim^{\mathcal{F}}) \iff$ (by definition of $\sqsubseteq'$) ($z \sqsubseteq v$ and $z \sqsubseteq w$) $\iff$ (by Proposition 2.3 (iii)) $\overline{z}^F \leqslant \overline{v}^F$ and $\overline{z}^F \leqslant \overline{w}^F \iff$ (by definition of $\leqslant$ and of $\bigtriangleup$) $\overline{z}^F \leqslant \overline{v}^F \bigtriangleup \overline{w}^F \iff$ (since the g-partitions are uniform) $\overline{z}^F \sqsubseteq \overline{v}^F \bigtriangleup \overline{w}^F$ $\iff$ (by definition of $\sqsubseteq'$ and by Proposition 2.4 (iii)) $[z]_\sim^{\mathcal{F}} = [\overline{z}^F]_\sim^{\mathcal{F}} \sqsubseteq' [\overline{v}^F \bigtriangleup \overline{w}^F]_\sim^{\mathcal{F}}$. Now, since $[\overline{v}^F \bigtriangleup \overline{w}^F]_\sim^{\mathcal{F}} =$ (by definition of $\wedge$) $[\overline{v}^F]^{\mathcal{F}} \wedge [\overline{w}^F]^{\mathcal{F}} =$ (by Proposition 2.4 (iii)) $[v]_\sim^{\mathcal{F}} \wedge [w]_\sim^{\mathcal{F}}$, it follows that $[z]_\sim^{\mathcal{F}} \sqsubseteq' [v]_\sim^{\mathcal{F}} \wedge [w]_\sim^{\mathcal{F}}$. This proves that the operation $\wedge$ defines effectively the inf in $(\mathcal{F}/\sim, \sqsubseteq')$. In the same way we can proceed for the operation $\vee$ in the sup-case. Finally, let us note that the distributivity holds because the operations $\bigtriangleup$ and $\bigtriangledown$ are defined on the components of the uniform g-partitions. $\square$

Since P is obviously lattice-inductive, it follows that:

Corollary 2.9. $(P/\sim, \sqsubseteq') \equiv (P_*, \sqsubseteq)$ *is a distributive lattice.*

When $\mathcal{F}$ is finite and uniform, it is simple to verify if $\mathcal{F}$ is lattice-inductive:

Proposition 2.10. *Let $\mathcal{F}$ be a finite uniform subset of P, then $\mathcal{F}$ is lattice-inductive iff whenever $v, w \in \mathcal{F}$ also $v \bigtriangleup w \in \mathcal{F}$ and $v \bigtriangledown w \in \mathcal{F}$.*

We denote now by P^+ the subset of P of all the g-partitions having balance $(q, 0)$ and with P^- the subset of P of all the g-partitions having balance $(0, p)$. It is easy to see that P^+ and P^- are both lattice-inductive. Since they are infinite, we identify the lattices $(P^+/\sim, \sqsubseteq')$ and $(P^-/\sim, \sqsubseteq')$ respectively with $(P_*^+, \sqsubseteq)$ and $(P_*^-, \sqsubseteq)$. We denote by $\mathbb{Y}$ the classical Young lattice of the integer partition and by $\mathbb{Y}^*$ its dual lattice. It is clear that $(P_*^+, \sqsubseteq) \cong \mathbb{Y}$, $(P_*^-, \sqsubseteq) \cong \mathbb{Y}^*$ and $(P_*, \sqsubseteq) \cong \mathbb{Y} \times \mathbb{Y}^*$.

3. Finite 1-Covering Sub-Lattices of P_*

In our work, a relevant role is played by a class of sub-posets of $(P_*, \sqsubseteq)$ that we have called 1-covering posets, therefore in this section we define such posets and we prove some useful properties for such class.

Definition 3.1. *If $(U, \sqsubseteq)$ is an induced sub-poset of the quasi-poset $(P, \sqsubseteq)$, we say that U is 1-covering if whenever w and w' are two g-partitions in U with $\overline{w} = l_1 \cdots l_n$ and $\overline{w'} = l_1' \cdots l_n'$, it results that w' covers w in U iff $\overline{w'}$ and $\overline{w}$ differ in exactly one place k and in this place, it holds $l_k' = l_k + 1$.*

In particular, let us note that if U is uniform, then $w = \overline{w}$ and $w' = \overline{w'}$. Moreover, we also observe that P_* is 1-covering.

We define now the function $\vartheta : P \to \mathbb{Z}$ such that

$$\vartheta(a_q \dots a_1 | b_1 \dots b_p) := a_q + \cdots a_1 + b_1 \cdots + b_p \tag{8}$$

Proposition 3.2. *Let $(U, \sqsubseteq)$ be a finite sub-lattice with minimum $\hat{0}$ of the quasi-poset $(P, \sqsubseteq)$ and let $\rho : U \to \mathbb{N}$ such that $\rho(w) := \vartheta(w) - \vartheta(\hat{0})$ for each $w \in U$. Then U is 1-covering if and only if ρ is the rank function of U. Moreover, in this case, $N_k(U) = \{w \in U : w \vdash k + \vartheta(\hat{0})\}$ for $k = 0, 1, \dots, rank(U)$.*

Proof. Let w, w' be two g-partitions in U such that $\overline{w} = l_1 \cdots l_n$ and $\overline{w'} = l_1' \cdots l_n'$. By definition of ρ we have $\rho(\hat{0}) = 0$ and by definition of $\overline{w}$ and $\overline{w'}$, it results that $\rho(w) = \rho(\overline{w})$ and $\rho(w') = \rho(\overline{w'})$. If U is 1-covering and w' covers w, by definition of 1-covering $\overline{w}$ and $\overline{w'}$ differ between them only in exactly one place $k \in \{1, \cdots, n\}$, where $l_k' = l_k + 1$. Therefore $\rho(w') = \rho(\overline{w'}) = l_1' + \cdots + l_n' = l_1 \cdots + l_n + 1 = \rho(\overline{w}) + 1 = \rho(w) + 1$. Hence ρ is the rank function of U.
If ρ is the rank function of U and w' covers w, then $\rho(w') = \rho(w) + 1$, i.e. $\rho(\overline{w'}) = \rho(\overline{w}) + 1$, that, in components, becomes

$$l_1' + \cdots + l_n' = 1 + l_1 + \cdots + l_n. \tag{9}$$

Since $w \sqsubseteq w'$, we also have $l_1 \leq l_1', \dots, l_n \leq l_n'$, and this is compatible with (9) if and only if there exists $k \in \{1, \dots, n\}$ such that $l_k' = l_k + 1$ and $l_i' = l_i$ if $i \neq k$, because the components are integer numbers. On the other hand, if there exists $k \in \{1, \dots, n\}$ such that $l_k' = l_k + 1$ and $l_i' = l_i$ if $i \in \{1, \dots, n\}$ and $i \neq k$, we must to show that w' covers w in U. Let us suppose on the contrary that there exists $w'' \in U$ such that $w \sqsubset w'' \sqsubset w'$. To compare between them the three g-partitions w, w' and w'', we must consider $\overline{w}$, $\overline{w'}$ and $\overline{w''}$ with respect to $F = \{w, w', w''\}$. Let $\overline{w} = m_1 \dots m_h$, $\overline{w'} = m_1' \dots m_h'$, $\overline{w''} = m_1'' \dots m_h''$. It is clear then that it must be $h \geq n$ and the possible parts m_i and m_i' that appear different from l_i and l_i' can only be zeros, therefore there exists $j \in \{1, \dots, h\}$ such that $m_j' = m_j + 1$ and $m_i' = m_i$ if $i \in \{1, \dots, h\}$ and $i \neq j$. Since $w \sqsubset w'' \sqsubset w'$, it is also $\overline{w} \sqsubset \overline{w''} \sqsubset \overline{w'}$, therefore it must be necessarily $m_j < m_j'' < m_j'$, that contradicts $m_j' = m_j + 1$.
Finally, if $k \in \{0, 1, \dots, rank(U)\}$, then $N_k(U) = \{w \in U : \rho(w) = k\} = \{w \in U : \vartheta(w) = k + \vartheta(\hat{0})\}$, and the equality $\vartheta(w) = k + \vartheta(\hat{0})$ is equivalent to $w \vdash k + \vartheta(\hat{0})$. □

We show now that the lattice $(P_*, \sqsubseteq)$ has two similar properties to the class of the differential posets. The class of differential posets was introduced by Stanley [50] and further studied in [21], [22], [33], [43], [44], [51]. Since P_*^+ coincides with the classical Young lattice, it is a 1-differential lattice, i.e. if $x \in P_*^+$ covers k elements then $(k+1)$ elements cover x. By symmetric reasons, P_*^- is a 1-differential lattice in the opposite sense (-1-differential lattice), i.e. if $y \in P_*^-$ covers k elements then $(k-1)$ elements cover y : indeed if you draw P_*^+ and P_*^- the first appears as a cone with a bottom vertex and the second as a cone with an above vertex.

Proposition 3.3. *If $x \in P_*$ is an element that covers exactly k elements of P_*, then there are exactly k elements of P_* which cover x.*

Proof. If $x \in P_*$ covers exactly k elements, this implies that l elements are covered by x in the non-negative part of x and $(k-l)$ are covered by x in the non-positive part. Since P_*^+ is 1-differential and P_*^- is 1-differential in the opposite direction, we have that there are $(l+1)$

elements that cover x because they cover the non- negative part of x, and $(k-l-1)$ elements that cover x because they cover the non-positive part of x. Hence exactly k elements cover it. In other words, the covering operation on P_*, by definition, coincides either with the covering operation on P_*^+ or with the covering operation on P_*^-. $\square$

Proposition 3.4. *If x and y are different elements of P_*, and there are k elements of P_* covered at the same time by x and y, then there are exactly k elements of P_* that cover both x and y.*

Proof. Let $x = a_q^x \dots a_1^x | b_1^x \dots b_p^x$, $y = a_q^y \dots a_1^y | b_1^y \dots b_p^y$ and $z = a_q^z \dots a_1^z | b_1^z \dots b_p^z$ be three uniformized elements of P_*. We assume that x and y both cover z in the non-negative part. Because P_* is 1-covering, there exist $i > j \geq 1$ such that x and z differ only in place i, where $a_i^x = a_i^z + 1$ and such that y and z differ only in place j, where $a_j^y = a_j^z + 1$. Then the element $z' := a_q^z \dots a_{i-1}^z a_i^x a_{i+1}^z \dots a_{j+1}^z a_j^y a_{j-1}^z \dots a_1^z | b_1^z \dots b_p^z$ covers x and y; in other words, there is a $(1:1)$ correspondence between the elements covered by x and y in the non-negative part and the elements that cover both x and y in the non-negative part. The same thing happens if x and y both cover z in the non-positive part. Suppose now that x covers z in the non-negative part and that y covers z in the non-positive part, then the element that has the same non-negative part of x and the same non-positive part of y covers both x, y. Hence there is also a $(1:1)$ correspondence between the elements covered by x in the non-negative part and by y in the non-positive part and the elements that cover x in the non-positive part and y in the non-negative part. $\square$

4. The Lattice $P(n,r)$

In order to study locally the lattice P_*, in the sequel we call *filling chain of* P_* a numerable sequence $\{X_k\}_{k\geq 0}$ of finite sublattices of P_* such that $X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$ and $\bigcup_{k\geq 0} X_k = P_*$. If $\{X_k\}_{k\geq 0}$ and $\{Y_k\}_{k\geq 0}$ are two filling chains of P_*, we say that $\{Y_k\}_{k\geq 0}$ is a *refinement* of $\{X_k\}_{k\geq 0}$ if $Y_k \subseteq X_k$, for each $k \geq 0$. In this section we introduce the finite 1-covering sub-lattice $P(n,r)$ of $(P_*, \sqsubseteq)$ that will be used to investigate locally the structure of P_*. We prove several symmetry properties for $P(n,r)$. Using this type of sub-lattices we provide a filling chain of P_* that we call *square chain* of P_*. If n and r are two fixed integers such that $0 \leq r \leq n$, we set:

$$P(n,r) = \{w \in P : |w|_{\geq} = r, |w|_{\leq} = n-r, M^+(w) \leq r, M^-(w) \leq n-r\} \tag{10}$$

Furthermore, if d is an integer with $0 \leq d \leq n$, we also set:

$$P(n,d,r) = \{w \in P(n,r) : ||w|| = d\} \tag{11}$$

Sometime we call (n,r)-*partition* an element of $P(n,r)$. We recall now the definition of the classical lattice $L(m,n)$. If m is a non-negative integer, the set $L(m,n)$ is the set of all the usual partitions with at most m parts and with largest part at most n. Then $L(m,n)$ is a sublattice of $\mathbb{Y}$ that has very remarkable properties. Such lattice was introduced by Stanley in [49], who showed that $L(m,n)$ is peck. Then it is immediate to observe that $P(n,r)$ and $P(n,d,r)$ are both uniform, lattice-inductive and that

$$P(n,r) \cong L(r,r) \times L(n-r,n-r)^* \tag{12}$$

We recall now the concept of involution poset (see [1] and [13] for some recent studies on such class of posets). An *involution poset* (IP) is a poset $(X, \leq, c)$ with a unary operation $c : x \in X \mapsto x^c \in X$, such that:

$I1)$ $(x^c)^c = x$, for all $x \in X$;
$I2)$ if $x, y \in X$ and if $x \leq y$, then $y^c \leq x^c$.

The map c is called *complementation* of X and x^c the *complement* of x. Let us observe that if X is an involution poset, by $I1)$ follows that c is bijective and by $I1)$ and $I2)$ it holds that if $x, y \in X$ are such that $x < y$, then $y^c < x^c$. If $(X, \leq, c)$ is an involution poset and if $Z \subseteq X$, we will set $Z^c = \{z^c : z \in Z\}$. We note that if X is an involution poset then X is a self-dual poset because from $I1)$ and $I2)$ it follows that if $x, y \in X$ we have that $x \leq y$, iff $y^c \leq x^c$, and this is equivalent to say that the complementation is an isomorphism between X and its dual poset X^*. Now, if $w = a_r \dots a_1|b_1 \dots b_{n-r} \in P(n, r)$, we set $w^c = (r - a_1)\dots(r - a_r)|(|b_{n-r}| - (n - r))\dots(|b_1| - (n - r))$ and let us note that w^c is still a g-partition in $P(n, r)$, therefore we can define a unary operation $c : P(n, r) \to P(n, r)$ such that $w \mapsto w^c$. Then it is immediate to verify that

Proposition 4.1. *$(P(n, r), \sqsubseteq, c)$ is an involution poset.*

If $w = a_r \dots a_1|b_1 \dots b_{n-r} \in P(n, r)$ we also set $w^t = (-b_{n-r})\dots(-b_1)|(-a_1)\dots(-a_r)$ and we call w^t the *transposed* of w. Let us note that $w^t \in P(n, n - r)$.

Proposition 4.2.

(i) *$P(n, d, r)$ is a sub-lattice of $P(n, r)$.*
(ii) *$P(n, r) = \dot{\bigcup}_{n \geq d \geq 0} P(n, d, r)$.*
(iii) *$|P(n, r)| = \binom{2r}{r}\binom{2(n-r)}{n-r}$.*
(iv) *$|P(n, d, r)| = \sum_{\substack{0 \leq k \leq \min\{r,d\} \\ 0 \leq d-k \leq n-r}} \binom{r+k-1}{k}\binom{n-r+d-k-1}{d-k}$.*
(v) *$P(n, r) \cong P(n, n - r)$.*
(vi) *$P(n, r)$ is peck.*

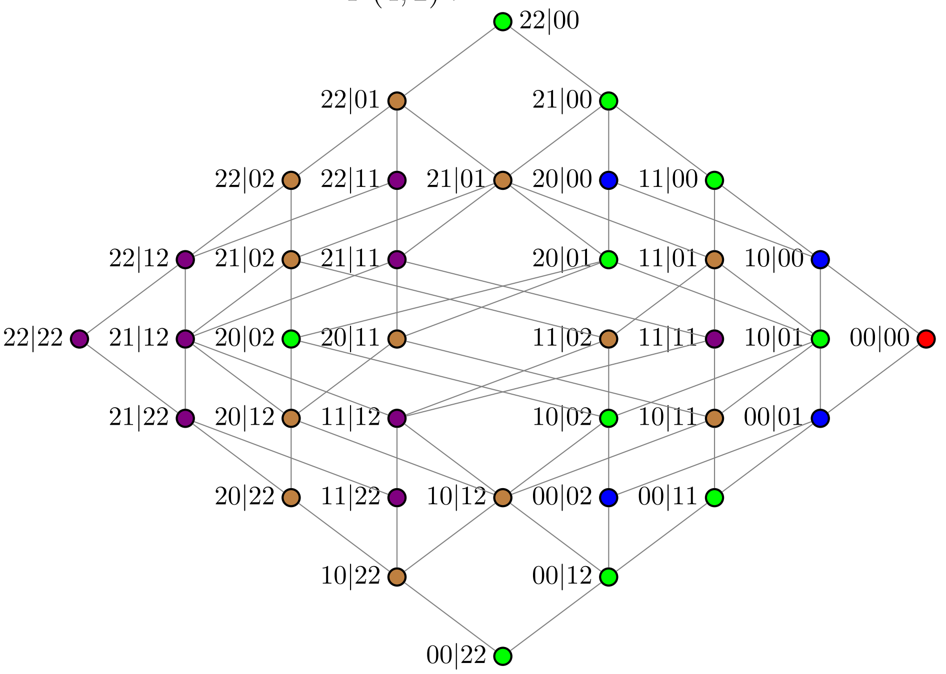
Proof. (i) and (ii) are obvious.
(iii) It follows at once since it is well known that $|L(m, n)| = \binom{m+n}{n}$, however we give a rapid proof. It is sufficient to observe that $P(n, r)$ can be identified with the set of all the ordered pairs (z_1, z_2), where z_1 is a decreasing string of length r on the ordered alphabet $r > \dots > 1 > 0$ and z_2 is a decreasing string of length $n - r$ on the ordered alphabet $0 > -1 > \dots > -(n - r)$, therefore $|P(n, r)| = \binom{(r+1)+r-1}{r}\binom{(n-r+1)+(n-r)-1}{n-r} = \binom{2r}{r}\binom{2(n-r)}{n-r}$.
(iv) A generic g-partition w of $P(n, d, r)$ has the form $w = a_r \dots a_1|b_1 \dots b_n - r$, where $r \geq a_r$, $b_{n-r} \geq -(n - r)$ and such that the non-zero parts of w are exactly d. If we take an integer k such that $0 \leq k \leq \min\{r, d\}$ and $0 \leq d - k \leq n - r$, we can count at first the g-partitions of $P(n, d, r)$ that have exactly k positive parts and $d - k$ negative parts. This is equivalent to count the ordered pairs (z_1, z_2), where z_1 is a decreasing string of length k on the ordered alphabet $r > \dots > 1$ and z_2 is a decreasing string of length $d - k$ on the ordered alphabet $-1 > \dots > -(n - r)$. The number of these ordered pairs is $\binom{r+k-1}{k}\binom{(n-r)+(d-k)-1}{d-k}$. The

number of all the g-partitions in $P(n,d,r)$ is then obtained taking the sum on k of all the numbers $\binom{r+k-1}{k}\binom{(n-r)+(d-k)-1}{d-k}$.
(v) We define the map $\phi : P(n,r) \to P(n,n-r)$ such that $\phi(w) = (w^t)^c$, for all $w \in P(n,r)$. We prove then that ϕ is an isomorphism of posets. By (3) it follows that $|P(n,r)| = |P(n,n-r)|$, therefore ϕ is bijective since it is easy to see that it is injective. If $w, w' \in P(n,r)$, the condition $w \sqsubseteq w' \iff \phi(w) \sqsubseteq \phi(w')$ follows at once from the conditions $I1$), $I2$) and because the transposed map $w \mapsto w^t$ is order-reversing and such that $(w^t)^t = w$.
(vi) $L(r,r)$ and $L(n-r,n-r)^*$ are both peck lattices and since the peck-property is preserved from the direct product operation (see [7]), the result follows by (12).

□

Below we draw the Hasse diagram of $P(4,2)$ using different colors for the nodes of its sublattices $P(4,d,2)$, where $d = 0,1,2,3,4$. Specifically, we draw red the unique node of $P(4,0,2)$ (i.e. 00|00), blue the nodes of $P(4,1,2)$, green the nodes of $P(4,2,2)$, brown the nodes of $P(4,3,2)$ and violet the nodes of $P(4,4,2)$.

$P(4,2)$:



If $n' \geq n$, $r' \geq r$ and $n'-r' \geq n-r$, then we can consider $P(n,r)$ as a sub-lattice of $P(n',r')$ in the following way: we identify a g-partition

$$a_r \dots a_1 | b_1 \dots b_{n-r} \in P(n,r)$$

with the g-partition

$$a_r \dots a_1 0_{r'-r} | 0_{(n'-r')-(n-r)} b_1 \dots b_{n'-r'} \in P(n',r').$$

Sometimes we need to specify that $(P(n,r), \sqsubseteq)$ is considered a sub-lattice of $(P(n',r'), \sqsubseteq)$, in such case we will write $(P(n,r)_{(n',r')}, \sqsubseteq)$. Let us note, in particular, that

$$P(0,0) \subset P(2,1) \subset P(4,2) \subset \cdots \subset P(2n,n) \subset \dots \tag{13}$$

and

$$P_* = \bigcup_{n\geq 0} P(2n,n) \tag{14}$$

Hence $\{P(2n,n)\}_{n\geq 0}$ is a filling chain of P_* which we call *square chain* of P_*. The reason for such a denomination is clear if we observe the Hasse diagrams respectively of $P(0,0)$, $P(2,1)$ and $P(4,2)$. Let us note that in all these diagrams the position of $(|)$ is always in the est corner. The next result concerns the way to generate the elements that cover a fixed element of the lattice $P(n,r)$.

Proposition 4.3.
(i) $P(n,r)$ is 1-covering.
(ii) The rank function of $P(n,r)$ is $\rho : P(n,r) \to \mathbb{N}$ such that

$$\rho(a_r \dots a_1 | b_1 \dots b_{n-r}) = a_r + \cdots a_1 + b_1 \cdots + b_{n-r} + (n-r)^2. \tag{15}$$

(iii) The rank of $P(n,r)$ is $r^2 + (n-r)^2$.
(iv) $N_k(P(n,r)) = \{w \in P(n,r) : w \vdash k - (n-r)^2\}$.

Proof. (i) Let $w = l_1 \dots l_n$ and $w' = l'_1 \dots l'_n$ be two g-partitions in $P(n,r)$ such that w' covers w. We must show that there exists exactly a place k where w and w' are different and that in this place, it hlds $l'_k = l_k + 1$. By contradiction, we must distinguish three cases:

1) there exists exactly one place k where w and w' are different, but $l'_k \neq l_k + 1$. Since by hypothesis w' covers w, we have $w \sqsubset w'$; therefore it must be $l_k < l'_k$. This implies that $l_k < l_k + 1 < l'_k$. We consider now the g-partition

$$u = l_1 \cdots l_{k-1}(l_k + 1)l_{k+1} \cdots l_n$$

Let us observe then that $w \sqsubset u \sqsubset w'$ and $u \in P(n,r)$ (since $r \geq l_1 \geq \cdots \geq l_{k-1} = l'_{k-1} \geq l'_k > l_k + 1 > l_k \geq l_{k+1} \geq \cdots \geq l_n \geq -(n-r)$), and this is a contradiction because w' covers w.

2) There exist at least two places k and s, with $s > k$, where w and w' differ, with $l'_k > l_k$ and $l'_s > l_s$. We consider the g-partition

$$u = l'_1 \dots l'_{k-1} l'_k l_{k+1} \dots l_{s-1} l_s l_{s+1} \dots l_n$$

Then $r \geq l'_1 \geq \cdots \geq l'_{k-1} \geq l'_k > l_k \geq l_{k+1} \geq \cdots \geq l_{s-1} \geq l_s \geq l_{s+1} \geq \cdots \geq l_n \geq -(n-r)$, therefore $u \in P(n,r)$, and since $w \sqsubset u \sqsubset w'$ this is against the hypothesis that w' covers w.

3) w and w' are equal in all their parts. In this case $w = w'$, against the hypothesis.

We assume now that w and w' are different only in a place k and that $l'_k = l_k + 1$. It is obvious then that $w \sqsubset w'$. If w' doesn't cover w then there exists a $w'' \in P(n,r)$ such that $w \sqsubset w'' \sqsubset w'$. Let $w'' = l''_1 \cdots l''_n$. By hypothesis it follows that $l_i = l''_i = l'_i$ if $i \neq k$, and $l_k < l''_k < l'_k$, against the hypothesis that $l'_k = l_k + 1$.

(ii) Since $\hat{0} = 0 \cdots 0| - (n-r) \cdots - (n-r)$ is the minimum of $P(n,r)$ and $\hat{0} \vdash -(n-r)^2$, the assertion (ii) follows at once by Proposition 3.2.

(iii) The maximum element of $P(n,r)$ is $r \dots r|0 \dots 0$, therefore the rank of $P(n,r)$ is $\rho(r \dots r|0 \dots 0) - \rho(0 \dots 0| - (n-r) \cdots - (n-r)) = r^2 + (n-r)^2$.

(iv) Since $\hat{0} \vdash -(n-r)^2$ it follows at once by Proposition 3.2.

□

The following result is the analogue of the Sperner theorem for the Boolean lattice in the case of $P(n,r)$:

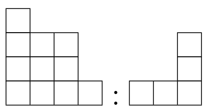
Proposition 4.4.
(i) If n is even, then $P(n,r)\bigcap Par(\frac{n(2r-n)}{2})$ is a maximal antichain in $P(n,r)$.
(ii) If n is odd, then $P(n,r)\bigcap Par(\frac{n(2r-n)-1}{2})$ and $P(n,r)\bigcap Par(\frac{n(2r-n)+1}{2})$ are two maximal antichains in $P(n,r)$.
(iii) $P(2n,n)\bigcap Par(0)$ is a maximal antichain in $P(2n,n)$.

Proof. By Proposition 4.3 (iii), the rank of $P(n,r)$ is $r^2+(n-r)^2$. We set $h=r^2+(n-r)^2$. Since $h\equiv n \mod 2$, if n is even also h is even, therefore, by Proposition 4.2(vi) it follows that the $\lfloor\frac{h}{2}\rfloor$-th rank level of $P(n,r)$ is a maximal chain. Moreover, by Proposition 4.3 (iv) it results that such rank level is the set $\{w\in P(n,r) : w\vdash\lfloor\frac{h}{2}\rfloor-(n-r)^2\}=\{w\in P(n,r) : w\vdash\frac{r^2+(n-r)^2}{2}-(n-r)^2\}=\{w\in P(n,r) : w\vdash\frac{n(2r-n)}{2}\}$.
If n is odd, also h is odd and by Proposition 4.2(vi) it follows that the $\lfloor\frac{h}{2}\rfloor$-th rank level and the $\lceil\frac{h}{2}\rceil$-th rank level of $P(n,r)$ are both maximal chains. By Proposition 4.3 (iv) it results that such rank levels are respectively the sets $\{w\in P(n,r) : w\vdash\lfloor\frac{h}{2}\rfloor-(n-r)^2\}=\{w\in P(n,r) : w\vdash\frac{r^2+(n-r)^2-1}{2}-(n-r)^2\}=\{w\in P(n,r) : w\vdash\frac{n(2r-n)-1}{2}\}$ and $\{w\in P(n,r) : w\vdash\lceil\frac{h}{2}\rceil-(n-r)^2\}=\{w\in P(n,r) : w\vdash\frac{r^2+(n-r)^2+1}{2}-(n-r)^2\}=\{w\in P(n,r) : w\vdash\frac{n(2r-n)+1}{2}\}$.
This proves (i) and (ii).
(iii) follows by (i). □

Remark 4.5. *$P(n,d,r)$ is not 1-covering. For example, in $P(4,2,2)$ the g-partition* 10|02 *covers the g-partition* 00|12 *in $P(4,2,2)$, but they differ in two places. Obviously also the lattice $P(n,d,r)$ is graded because it is finite and distributive, however we cannot use the Proposition 3.2 to compute its rank functions.*

5. $P(n,d,r)$ as a Sand Piles Model with three Evolution Rules

In this section we describe the covering relation in the lattice $P(n,d,r)$, which is not 1-covering (a similar description for a particular sub-lattice of $P(n,d,r)$ is given in [10]). We describe the covering relation in $P(n,d,r)$ using a discrete dynamical model with three evolution rules. In this context we also call *configurations* the elements of $P(n,d,r)$. In the sequel, to comply with the terminology concerning the Sand Piles Models, if $w\in P(n,d,r)$, we represent the sequence of the positive parts of w as a not-increasing sequence of columns of stacked squares and the sequence of the negative parts of w as a not-decreasing sequence of columns of stacked squares. We call *pile* a column of stacked squares and *grain* each square of a pile. For example, if $n=10, r=6, d=7$, the configuration



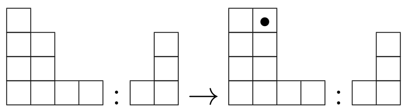
is identified with the partitions $(4,3,3,1,0,0|0,-1,-1,-3)=433100|0113\in P(10,7,6)$. In this section we denote by $D(w):=D^+(w):D^-(w)$ the configuration associated to w, where $D^+(w)$ is the Young diagram (represented with not-increasing columns) of the partition $(a_r,\ldots,a_1)$ and $D^-(w)$ is the Young diagram (represented with not-decreasing columns) of

the partition $(-b_1, \ldots, -b_{n-r})$. Our goal is to define some rules of evolution that starting from the minimum of $P(n, d, r)$ allow us to reconstruct the Hasse diagram of $P(n, d, r)$ (and therefore to determine the covering relations in $P(n, d, r)$).
Let $w = a_r \ldots a_1 | b_1 \ldots b_{n-r} \in P(n, d, r)$. We formally set $a_0 = 0$, $a_{r+1} = r$ and $b_0 = 0$. If $0 \leq i \leq r+1$ we call a_i the i^{th}-*plus pile* of w, and if $0 \leq j \leq n-r$ we call b_j the j^{th}-*minus pile* of w. We call a_i *plus singleton pile* if $a_i = 1$ and b_j *minus singleton pile* if $b_j = -1$. If $1 \leq i \leq r+1$ we set $\Delta_i^+(w) = a_i - a_{i-1}$ and we call $\Delta_i^+(w)$ the *plus height difference* of w in i. If $1 \leq j \leq n-r$ we set $\Delta_j^-(w) = |b_j| - |b_{j-1}|$ and we call $\Delta_j^-(w)$ the *minus height difference* of w in j. If $1 \leq i \leq r+1$, we say that w has a *plus step* at i if $\Delta_i^+(w) \geq 1$. If $1 \leq j \leq n-r$, we say that w has a *minus step* at j if $\Delta_j^-(w) \geq 1$.

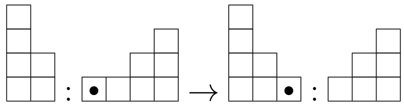
Remark 5.1. *The choice to set $a_0 = 0$, $a_{r+1} = r$ and $b_0 = 0$ is a formal trick for decreasing the number of rules necessary for our model. This means that when we apply the next rules to one element $w \in P(n, d, r)$ we think that there is an "invisible" extra pile in the imaginary place $r+1$ having exactly r grains, an "invisible" extra pile with 0 grains in the imaginary place to the right of a_1 and to the left of $|$ and another "invisible" extra pile with 0 grains in the imaginary place to the left of b_1 and to the right of $|$. However the piles corresponding respectively to $a_0 = 0$, $a_{r+1} = r$ and $b_0 = 0$ must be not considered as parts of w.*

Evolution rules:

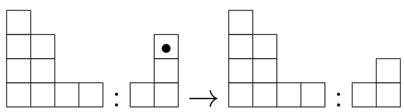
Rule 1: If the i^{th}-plus pile has at least one grain and if w has a plus step at $i+1$ then one grain must be added on the i^{th}-plus pile.



Rule 2: If there are some minus singleton piles, then the first of them from the left must be shifted to the side of the lowest not empty plus pile.



Rule 3: One grain must be deleted from the j^{th}-minus pile if w has a minus step at j and $|b_j| > 1$.



Remark 5.2. *(i) In the Rule 2 the lowest not empty plus pile can also be the invisible column in the place $r+1$. In this case all the plus piles are empty and an eventual minus singleton pile must be shifted in the place r.*
(ii) We take implicitly intended for that the shift of one minus singleton pile into a plus singleton pile can be made if the number of plus piles (excluding $a_{r+1} = r$) with at least a grain is less than r (otherwise we obtain a configuration that does not belong to $P(n, d, r)$).

We write $w \to^k w'$ (or $w' = w \to^k$) to denote that w' is a n-tuple of integers obtained from w applying the Rule k, for $k = 1, 2, 3$. We also set

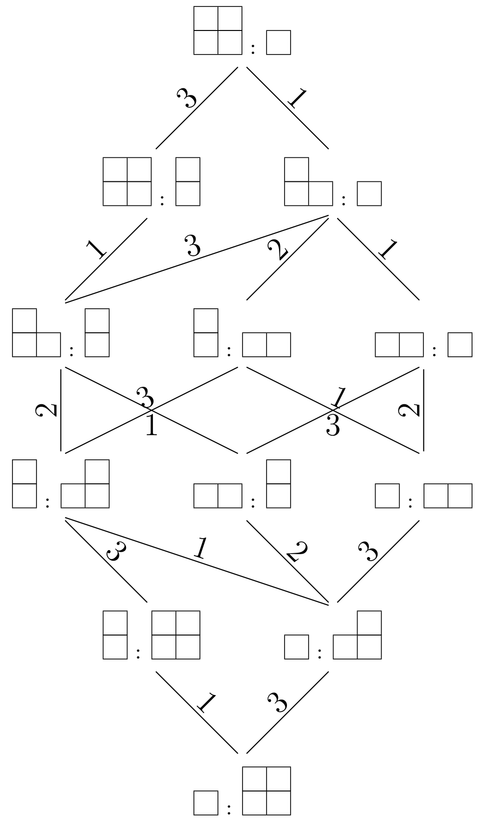
$$\nabla(w) = \{w' : w \to^k w', k = 1, 2, 3\}$$

Theorem 5.3. *If $w \in P(n, d, r)$ then $\nabla(w) = \{w' \in P(n, d, r) : w' \gtrdot w\}$.*

Proof. We start to show that $\nabla(w) \subseteq \{w' \in P(n, d, r) : w' \gtrdot w\}$. Let $w = a_r \dots a_1 | b_1 \dots b_{n-r} \in P(n, d, r)$ and $a_{r+1} = r$ the invisible pile in place $r + 1$. We distinguish the three possible cases related to the previous rules.
Case 1: Let us assume that $r \geq i \geq 1$, $a_i \neq 0$ and that w has a plus step at $i + 1$. If $w' = w \to^1$, then $w' = a_r \dots a_{i+1}(a_i+1)a_{i-1} \dots a_1 | b_1 \dots b_{n-r}$. It is clear that $||w'|| = d$ because $a_i \neq 0$. Since there is a plus step at $i+1$ we have $a_{i+1} - a_i \geq 1$, hence $a_{i+1} \geq a_i + 1 > a_i \geq a_{i-1}$, and this implies that $w' \in P(n, d, r)$. We must show now that w' covers w in $P(n, d, r)$. Since w and w' differ between them only in the place i for a_i and $a_i + 1$ respectively, it is clear that there does not exist an element $z \in P(n, d, r)$ such that $w \sqsubset z \sqsubset w'$. Hence $w' \gtrdot w$.
Case 2: Let us assume that in w there is a minus singleton pile b_j, for some $1 \leq j \leq n - r$. Since $a_{r+1} = r > 0$, we can assume that $a_{i+1} > 0$, $a_i = 0$, for some $1 \leq i \leq r$. This means that w has the following form: $w = a_r \dots a_{i+1}00 \dots 0|0 \dots 0(-1)b_{j+1} \dots b_{n-r}$. Applying the Rule 2 to w we obtain $w' = w \to^2$, where $w' = a_r \dots a_{i+1}10 \dots 0|0 \dots 00b_{j+1} \dots b_{n-r}$. It is clear then that $w' \in P(n, r)$ and $||w'|| = d$ since w' is obtained from w with only a shift of the pile -1 to the left in the place i. Let us note that the only elements $z_1, z_2 \in P(n, r)$ such that $w \sqsubset z_1 \sqsubset w'$ and $w \sqsubset z_2 \sqsubset w'$ are $z_1 = a_r \dots a_{i+1}10 \dots 0|0 \dots 0(-1)b_{j+1} \dots b_{n-r}$ and $z_2 = a_r \dots a_{i+1}00 \dots 0|0 \dots 00b_{j+1} \dots b_{n-r}$, but $||z_1|| = d + 1$ and $||z_2|| = d - 1$, hence z_1, z_2 are not elements of $P(n, d, r)$. This implies that w' covers w in $P(n, d, r)$.
Case 3: If $1 \leq j \leq n - r$ and w has a minus step at j, we apply the Rule 3 to w on the minus pile b_j and we obtain $w' = w \to^3$, where $w' = a_r \dots a_1 | b_1 \dots b_{j-1}(b_j + 1)b_{j+1} \dots b_{n-r}$. Since w has a minus step at j, we have $-b_j + b_{j-1} = |b_j| - |b_{j-1}| \geq 1$, therefore $w' \in P(n, r)$ because $0 \geq b_{j-1} \geq b_j + 1 > b_j \geq b_{j+1}$ and $||w'|| = d$ since $b_j \leq -2$ implies $b_j + 1 < 0$. As in the case 1, we note that w' covers w in $P(n, d, r)$ because they differ between them only for a grain in the place j.
We now must show that $\{w' \in P(n, d, r) : w' \gtrdot w\} \subseteq \nabla(w)$. Let $w'' = a''_r \dots a''_1 | b''_1 \dots b''_{n-r}$ a generic element of $P(n, d, r)$ such that $w'' \sqsupset w$. If we show that there exists an element $w' = a'_r \dots a'_1 | b'_1 \dots b'_{n-r}$ of $P(n, d, r)$ such that $w' \in \nabla(w)$ and $w'' \sqsupseteq w'$ we complete the proof of the theorem. Since $w'' \sqsupset w$, there is a place where the corresponding component of w'' is an integer strictly bigger than the integer component of w corresponding to the same place. We distinguish several cases.
Case A: We assume that $a''_i > a_i$ and $a_{i+1} \geq a_i + 1$ for some $i \in \{r - 1, \dots, 1\}$. In this case we can apply the Rule 1 in place i to obtain $w' = w \to^1$ such that $w'' \sqsupseteq w'$.
Case B: We assume that $a''_i > a_i$ and $a_{i+1} = a_i + 1$ for some $i \in \{r - 1, \dots, 1\}$. In this case

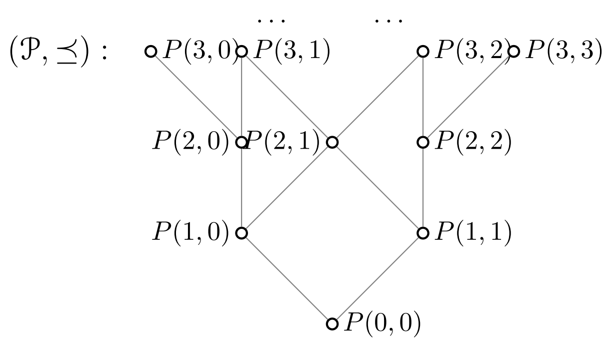
we have $a''_{i+1} \geq a''1 > a_i = a_{i+1}$, i.e. $a''_{i+1} \geq a_{i+1} + 1$, therefore, if $a_{i+2} \geq a_{i+1} + 1$ we can apply the Rule 1 in place $i + 1$ to obtain w', otherwise we have $a_{i+2} = a_{i+1} = a_i$. Iterating this procedure, to each step $k \geq 1$ we can apply the Rule 1 in place $i + k$ to obtain w' or it necessarily results that $a_{i+k} = \cdots = a_{i+1} = a_i$. Hence, if for no one k we can apply the Rule 1 in place $i+k$ we necessarily arrive to the condition $a_r = \cdots = a_{i+1} = a_i$. Since $r \geq a''_i > a_i$, it must be $a_r < r$, therefore we can apply the Rule 1 in place r to obtain $a'_r = a_r + 1$, with $a''_r \geq a'_r$ because $a''_r \geq \cdots \geq a''_i > a_i = \cdots = a_r$.
Case C: We assume that $a''_r > a_r$. In this case we can apply the Rule 1 in place r.
Case D: We assume that $0 > b''_j > b_j$, for some $j \in \{1, \ldots, n-r\}$. In this case we can apply the Rule 3 in place j.
Case E: We assume that $0 = b''_j > b_j$ and $b_j \leq -2$, for some $j \in \{1, \ldots, n-r\}$. Also in this case we can apply the Rule 3 in the place j.
Case F: We assume that $0 > b''_j > b_j = -1$, for some $j \in \{1, \ldots, n - r\}$. In this case the number of negative parts of w'' is strictly lower than the number of negative parts of w, and since $||w|| = ||w''|| = d$, it follows that there exists at least one index $i \in \{1, \ldots, r\}$ such that $a''_i > 0$ and $a_i = 0$. We choose then such index i maximal, so that we have $i = r$ or $i \leq r - 1$ and $a_{i+1} > 0$.
We suppose at first that $i = r$. In this case we have $a''_r \geq 1$ and $0 = a_r = \cdots = a_1$, therefore we can apply the Rule 2 and to move the "negative" grain from the place j into the place r, so that the (n, r)-partition $w' = 100 \ldots 0|0 \ldots 0b_{j+1} \cdots_{n-r}$ is such that $||w'|| = d$ and $w'' \sqsupseteq w'$.
If $i \leq r - 1$ and $a_{i+1} > 0$ we apply again the Rule 2 to move the "negative" grain from the place j into the place i, so that the (n, r)-partition $w' = a_r \ldots a_{i+1}1 \ldots 0|0 \ldots 0b_{j+1} \cdots_{n-r}$ is such that $||w'|| = d$ and $w'' \sqsupseteq w'$. □

Below we draw the Hasse diagram of the lattice $P(4, 3, 2)$ by using the evolution rules 1,2,3 starting from the minimum element of this lattice, which is $10|22$. We label a generic edge of the next diagram with the symbol k if it leads to a production that uses the Rule k, for $k \in \{1, 2, 3\}$.



6. Signed Young Diagrams

To study the order properties in P_* we need now to compare between them two generic elements in P_*; to such aim we use the "minimal" lattice $P(n,r)$ that contains them. At first we set $\mathcal{P} = \{P(n,r) : n \geq r \geq 0\}$ and let us note that we can see the couple $(\mathcal{P}, \preceq)$ as an infinite lattice, where the relation $P(n,r) \preceq P(n',r')$ means that $P(n,r)$ is a sub-lattice of $P(n',r')$. A part of the Hasse diagram of the lattice $(\mathcal{P}, \preceq)$ is the following:



Let $w = a_t \ldots a_1 | b_1 \ldots b_s \in P_*$ have signature (t,s). Then we set $w^+ = a_t - t$, $w^- = |b_s| - s$ and also

$$w^* = \begin{cases} a_t \dots a_1 | b_1 \dots b_s & \text{if } w^+ \leq 0 \text{ and } w^- \leq 0 \\ a_t \dots a_1 0_{w^+} | b_1 \dots b_s & \text{if } w^+ > 0 \text{ and } w^- \leq 0 \\ a_t \dots a_1 | 0_{w^-} b_1 \dots b_s & \text{if } w^+ \leq 0 \text{ and } w^- > 0 \\ a_t \dots a_1 0_{w^+} | 0_{w^-} b_1 \dots b_s & \text{if } w^+ > 0 \text{ and } w^- > 0 \end{cases} \tag{16}$$

$$r = \begin{cases} t & \text{if } w^+ \leq 0 \\ t + w^+ & \text{if } w^+ > 0 \end{cases} \tag{17}$$

$$n = \begin{cases} r + s & \text{if } w^- \leq 0 \\ r + s + w^- & \text{if } w^- > 0 \end{cases} \tag{18}$$

It is immediate to verify then that $w^* \in P(n,r)$ and also:

Proposition 6.1. *With the previous notation, it results that $w^* \in P(n',r')$, for some non-negative integers n', r', if and only if $P(n,r) \preceq P(n',r')$. Equivalently, $\{P(n',r') \in \mathcal{P} : w^* \in P(n',r')\}$ is the principal up-set in $\mathcal{P}$ generated by $P(n,r)$.*

The result of the previous proposition justifies the following definition. If $w \in P_*$ has signature (t,s) and w^*, r, n are respectively as in (16), (17), (18), we call the integers n and r the *minimal parameters* of w and the g-partition $w^* \in P(n,r)$ the *minimal regularized* of w. Let $w_1, w_2 \in P_*$ and let (n_i, r_i) be the minimal parameters of w_i for $i = 1,2$, so that $w_i^* \in P(n_i, r_i)$ for $i = 1,2$. Then we consider w_1^* and w_2^* as elements of $P(n',r')$, where $n' = max\{r_1, r_2\} + max\{n_1 - r_1, n_2 - r_2\}$ and $r' = max\{r_1, r_2\}$, and we can compare them with respect to the partial order $\sqsubseteq$ of $P(n',r')$. It is easy to observe that the previous $P(n',r')$ is exactly the sup between the elements $P(n_1,r_1)$ and $P(n_2,r_2)$ in the lattice $(\mathcal{P}, \preceq)$; we call it the *minimal square lattice* of w_1 and w_2 and we denote it by the symbol $ML(w_1,w_2)$. Let us note that if $w_1, w_2 \in P_*$ then

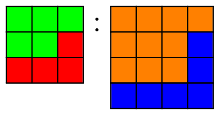
$$w_1 \sqsubseteq w_2 \Longleftrightarrow w_1^* \sqsubseteq w_2^* \text{ in } ML(w_1,w_2) \tag{19}$$

Therefore, by (19) we can identify w_i with w_i^*, for $i = 1,2$. For example, if we take $w_1 = 331|45 \in P_*$ and $w_2 = 621|3 \in P_*$, we have (with the previous notations) $w_1^* = 331|00045$, $w_2^* = 621000|003$, therefore $n_1 = 8$ and $r_1 = 3$, $n_2 = 9$ and $r_2 = 6$. Then we have $ML(w_1,w_2) = P(11,6)$ and the elements w_1^* and w_2^* will be identified respectively with the following two elements of $P(11,6)$: $w_1 = 331000|00045$ and $w_2 = 621000|00003$. Since in $P(11,6)$ we have $w_1 \wedge w_2 = 321000|00045$, it follows that w_1 and w_2 are not comparable in $P(11,6)$, therefore by (19) this means that w_1 and w_2 are not comparable in P_*. The following proposition shows that locally there is no loss of information if we work in the minimal square lattice of two signed partitions:

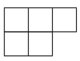
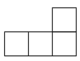
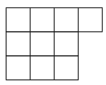
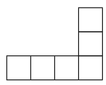
Proposition 6.2. *If $w_1, w_2 \in P_*$ and $X = ML(w_1,w_2)$, then $[w_1^*, w_2^*]_X = [w_1, w_2]_{P_*}$. In particular, w_2 covers w_1 in P_* if and only if w_2 covers w_1 in X.*

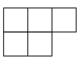
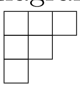
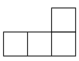
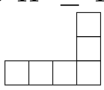
Proof. Immediate by (19). $\square$

If $n \geq r \geq 0$, we call (n,r)-*signed Young diagram* (briefly (n,r)-SYD) an ordered couple $T = T_1 : T_2$, where T_1 is a $r \times r$ colored table whose green squares form a classical Young diagram and the remaining squares are colored red and T_2 is a $(n-r) \times (n-r)$ colored table whose orange squares form a classical Young diagram and the remaining squares are colored blue. For example, a $(7,3)$-SYD is the following:

(20) 

If we think each green row with k boxes as the positive integer k and each blue row with l boxes as the negative integer $-l$, we can identify each (n,r)-SYD with an unique (n,r)-partition in $P(n,r)$ and viceversa. For example, if T is the $(7,3)$-SYD in (20), we obviously can identify it with the $(7,3)$-partition $320|0114$. When a (n,r)-partition $w \in P(n,r)$ is identified with a (n,r)-SYD $T = T_1 : T_2$, we write $T^{n,r}(w) := T$, $T_+^{n,r}(w) := T_1$, $T_-^{n,r}(w) := T_2$ and $p^{n,r}(T) := w$, moreover, we denote respectively by $G^{n,r}(w)$ and $R^{n,r}(w)$ the sub-tables of $T_+^{n,r}(w)$ with green and red squares and respectively by $O^{n,r}(w)$ and $B^{n,r}(w)$ the sub-tables of $T_-^{n,r}(w)$ with orange and blue squares. Sometimes we call $G^{n,r}(w)$, $R^{n,r}(w)$, $O^{n,r}(w)$, $B^{n,r}(w)$ respectively the *green, red, orange, blue diagrams* of w. From our definitions, if $w = a_r \dots a_1|b_1 \dots b_{n-r} \in P(n,r)$, it follows that $G^{n,r}(w)$ is the Young diagram of the partition $a_r, \dots, a_1$ and $O^{n,r}(w)$ is the Young diagram of the partition $(n-r)+b_1, \dots, (n-r)+b_{n-r}$. By (15) it results then that the sum of the number of boxes in $G^{n,r}(w)$ and of the number of the boxes in $O^{n,r}(w)$ is $a_r + \cdots + a_1 + b_1 + \cdots + b_{n-r} + (n-r)^2 = \rho(w)$, where ρ is the rank function of $P(n,r)$. In the sequel, when we have to represent someone of the green, red, orange, or blue diagrams of w outside of its corresponding square table, we draw the diagram without colors. For example, if T is the $(7,3)$-SYD in (20) and $w = p^{7,3}(T) = 320|0114$, then

$G^{7,3}(w) =$  $R^{7,3}(w) =$  $O^{7,3}(w) =$  $B^{7,3}(w) =$ 

If H and H' are sub-tables of some rectangular table (in particular, they can be usual Young diagrams), we write $H \subseteq H'$ to denote that H is a sub-table of H'. For example,  $\subseteq$  and  $\subseteq$ 

Proposition 6.3. *If $w_1, w_2 \in P(n,r)$, then:*
(i) $w_1 \sqsubseteq w_2 \iff G^{n,r}(w_1) \subseteq G^{n,r}(w_2)$ and $O^{n,r}(w_1) \subseteq O^{n,r}(w_2)$;
(ii) w_2 covers w_1 if and only if exactly one of the following two conditions is verified: $G^{n,r}(w_2)$ has only one square more w.r.t. $G^{n,r}(w_1)$ or $O^{n,r}(w_2)$ has only one square more w.r.t. $O^{n,r}(w_1)$.
If $w_1, w_2 \in P_*$ and $ML(w_1, w_2) = P(n,r)$, then:
(iii) $w_1 \sqsubseteq w_2$ iff $G^{n,r}(w_1^*) \subseteq G^{n,r}(w_2^*)$ and $O^{n,r}(w_1^*) \subseteq O^{n,r}(w_2^*)$.

Proof. (i) Straightforward from the definitions.
(ii) It is equivalent to say that $P(n,r)$ is 1-covering.
(iii) It follows by (19) and by (i). □

If $w \in P_*$ has minimal parameters n, r, we set $T(w) := T^{n,r}(w^*)$, $T_+(w) := T_+^{n,r}(w^*)$, $T_-(w) := T_-^{n,r}(w^*)$, $G(w) := G^{n,r}(w^*)$, $R(w) := R^{n,r}(w^*)$, $O(w) := O^{n,r}(w^*)$, $B(w) := B^{n,r}(w^*)$, and we call $T(w)$ the *signed Young diagram* (briefly SYD) of *shape* w. If $T = T_1 : T_2$

is a (n,r)-SYD, we denote by $T^t = T_1^t : T_2^t$ the (n,r)-SYD such that T_i^t is obtained from a reflection of T_i around the principal diagonal, for $i = 1, 2$. We call T^t the *transposed* of T. If $w \in P(n,r)$ and $T = T^{n,r}(w)$, we call (n,r)-*conjugate* of w the (n,r)-partition $p^{n,r}(T^t)$, and we denote it by $\mathrm{conj}^{n,r}(w)$. For example, if $w = 7222100|01225 \in P(12,7)$, then $\mathrm{conj}^{12,7}(w) = 5411111|11134$. If $w \in P_*$ has minimal parameters n, r and $v = \mathrm{conj}^{n,r}(w)$, we set $\mathrm{conj}(w) := v_*$ and we call $\mathrm{conj}(w)$ the *conjugate* of w.
If $w = a_q \dots a_1 | c_1 \dots c_p \in P$ and i is a non-negative integer we set $m_i^+(w) = |\{j \in \{1, \dots, r\} : a_j = i\}|$ and $m_i^-(w) = |\{j \in \{1, \dots, n-r\} : c_j = i\}|$. In particular, if w is a usual partition $a_q, \dots, a_1$, then the multiplicity of the integer i in w is exactly $m_i^+(w)$.

Proposition 6.4. *Let $n \geq r$ be fixed, $T = T_1 : T_2$ a (n,r)-SYD and $w = p^{n,r}(T)$. If R_1 and C_1 are respectively the numbers of rows and of columns of $G^{n,r}(w)$ and R_2 and C_2 are respectively the numbers of rows and of columns of $O^{n,r}(w)$, then*

$$m_0^+(w) = \max\{0, r - R_1\} \text{ and } m_0^-(w) = \begin{cases} m_{n-r}^+(w_2) & \text{if } C_2 \geq R_2 \\ 0 & \text{otherwise} \end{cases},$$

where $w_2 = p^{n,r}(D_2 : \varnothing)$.

Proof. It is a straightforward consequence of the definition of (n,r)-SYD and of the definition of $p^{n,r}(T)$. $\square$

Proposition 6.5. *Let $w = a_r \dots a_1 | c_1 \dots c_{n-r} \in P(n,r)$ and $w' = \mathrm{conj}^{n,r}(w)$, then:*
i) $w' \in P(n,r)$;
ii) if w has rank k in $P(n,r)$, also w' has rank k in $P(n,r)$;
iii) if $w' = a'_r \dots a'_1 | c'_1 \dots c'_{n-r}$, then
$m_0^+(w) = r - a'_r$, $m_i^+(w) = a'_{r-i+1} - a'_{r-i}$ for $i = 1, \dots, r-1$, $m_r^+(w) = a'_1 - 0$ and
$m_0^-(w) = n - r - c'_{n-r}$, $m_j^-(w) = c'_{n-r-j+1} - c'_{n-r-j}$, for $j = 1, \dots, n-r-1$, $m_{n-r}^-(w) = c'_1 - 0$.

Proof. i) If $w \in P(n,r)$, then $a_r \leq r$, $a_r \geq a_{r-1} \geq \dots \geq a_1$, $c_1 \leq c_2 \dots \leq c_{n-r}$ and $c_{n-r} \leq (n-r)$. From the definition of $\mathrm{conj}^{n,r}$, it follows that $a'_i =$ number of elements of w greater or equal to $(r+1-i)$ (analogously for c'_j). Hence the a'_is of w' appear obviously in a decreasing order and $a'_r \leq r$ (similarly for c'_j), therefore $w' \in P(n,r)$.
ii) The rank of w is the number of squares of w_* in its Young Diagramm, hence it stays constant under transposition of the Young Diagram.
iii) Without loss of generality, we prove that $m_i^+(w) = a'_{r-i+1} - a'_{r-i}$. Since $a'_i =$ number of elements of w greater or equal to $(r+1-i)$, it follows that $a'_{r-i+1} - a'_{r-i}$ is the number of elements of w greater or equal than i, but not greater or equal to $i+1$, which means that it is exactly the number of elements equal to i and the statement follows. $\square$

In the sequel, any way of putting a positive integer in each box of some sub-table H will be called a *filling* of H. If H and H' are two sub-tables (not necessarily of the same table), we call *filling* of $H : H'$ a filling of H and a filling of H'. If $w \in P_*$, we call *signed Young tableau* (briefly SYT) of *shape* w a filling of $G(w) : O(w)$ that is weakly increasing across each row and strictly increasing down each column. For example, if $w = 32|114 \in P_*$, then w has minimal parameters 7, 4, $w^* = 320|0114$ and

$$\begin{array}{|c|c|c|} \hline 2 & 3 & 3 \\ \hline 3 & 4 \\ \cline{1-2} \end{array} : \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline 5 & 5 & 6 \\ \cline{1-3} 6 & 7 & 7 \\ \cline{1-3} \end{array}$$

is a $(7,3)$-SYT of shape w.
If $w \in P_*$ has minimal parameters n, r, a *standard signed Young tableau* (briefly SSYT) of *shape* w is a SYT of shape w whose entries are filled with all the integers from 1 to $\rho(w^*)$, each occurring once (where $\rho(w^*)$ is the rank of the minimal regularized w^* in the lattice $P(n,r)$). Since the total number of boxes in both $G(w)$ and $O(w)$ is exactly $\rho(w^*)$, when we have a SSYT of shape w, the integers across each row of $G(w)$ and of $O(w)$ must appear in strictly increasing order. For example, if $w = 32|114$ then

$$\begin{array}{ccc|c|c|c|c|}
 & & & & \cline{2-5} & & & \multicolumn{1}{c|}{} & 1 & 3 & 12 & 15 \\
\hline
\multicolumn{1}{|c|}{5} & 7 & 9 & & 2 & 4 & 13 & \multicolumn{1}{c}{} \\
\hline
\multicolumn{1}{|c|}{6} & 8 & \multicolumn{1}{c}{} & : & 10 & 11 & 14 & \multicolumn{1}{c}{} \\
\cline{1-2}\cline{5-7}
\end{array}$$

is a $(7,3)$-SSYT of shape w.
If D is a classical Young diagram, the square of D located on the i-th row and on the j-th column is called *extremal* if it is the last square of the row i and also the last square of the column j.

Proposition 6.6. *Let $w \in P_*$ be with minimal parameters n, r and minimal regularized w^*. Then there is a natural bijection between saturated chains in $P(n,r)$ starting in $\hat{0}$ and ending in w^* and SSYT's of shape w.*

Proof. We apply the same technique described in [42] for the case of the Young lattice. Let $w_0 = \hat{0}, w_1, \ldots, w_{h-1}, w_h = w^*$ be a saturated chain starting in $\hat{0}$ and ending in w^*, where $h = \rho(w^*)$. When we consider w_i and w_{i-1}, by proposition 6.3 ii) it results that one and only one of the two following conditions is realized : $G^{n,r}(w_i) \setminus G^{n,r}(w_{i-1})$ is a unique square or $O^{n,r}(w_i) \setminus O^{n,r}(w_{i-1})$ is a unique square. Then, in such square we place the integer i. Now, also if the placement of the integers $h, \ldots, 1$ can occur in an alternating way between $G^{n,r}(w^*)$ and $O^{n,r}(w^*)$, however, this leads always to an increasing placement of the previous integers on the rows and on the columns of $G^{n,r}(w^*)$ and of $O^{n,r}(w^*)$. $\square$

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