DOMINANCE ORDER ON SIGNED INTEGER PARTITIONS

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Abstract. In 1973 Brylawski introduced and studied in detail the dominance partial order on the set $Par(m)$ of all the integer partitions of a fixed positive integer m. As it is well known, the dominance order is one of the most important partial orders on the finite set $Par(m)$. Therefore it is very natural to ask how it changes if we allow the summands of an integer partition to take also negative values. In such a case, m can be an arbitrary integer and $Par(m)$ becomes an infinite set. In this paper we extend the classical dominance order in this more general case. In particular, we consider the resulting lattice $Par(m)$ as an infinite increasing union on n of a sequence of finite lattices $O(m,n)$. The lattice $O(m,n)$ can be considered a generalization of the Brylawski lattice. We study in detail the lattice structure of $O(m,n)$.

1. Introduction

The dominance order for the integer partitions of a fixed positive integer m was introduced and studied by Brylawski in [11]. This partial order is very important in the symmetric functions theory (see [22]) and in symmetric group representation theory (see [26]). Up to now the dominance order has been studied only for integer partitions with positive summands. In this case, the set of all the integer partitions having fixed sum m is a finite set. What happens if we try to extend this order in the event that the summands of the partitions can also have negative value?

In [6] and [20], the authors have been recently studied some arithmetical properties of the signed integer partitions. If m is a fixed integer, a *signed partition* with sum m is a finite sequence of integers $a_k, \dots, a_1, a_{-1}, \dots, a_{-l}$ such that $a_k \geq \cdots \geq a_1 \geq 0 \geq a_{-1} \geq \cdots \geq a_{-l}$ and $a_k + \cdots + a_1 + a_{-1} + \cdots + a_{-l} = m$. In [11] Brylawski introduced and studied the lattice $L_B(m)$ of the classical integer partitions having sum $m \geq 0$ with respect to the dominance order. At present, however, an explicit study of the signed integer partitions from the order point of view has not been started yet. A poset $S(n,r)$, depending on two non negative integers $n \geq r \geq 0$, which can be considered a lattice of a particular type of signed partitions, has been introduced in [8] in order to study some extremal combinatorial sum problems. $S(n,r)$ is isomorphic to the direct product $M(r) \times M(n-r)^*$, where $M(n)$ is the lattice of all the integer partitions with distinct parts and maximum part not exceeding n, introduced by Stanley in [28]. In this paper we introduce the concept of dominance order $\trianglelefteq$ for signed integer partitions having sum m, where now m can be any integer. The main novelty in this new context is that the set of the signed partitions with sum m and order $\trianglelefteq$ is a infinite poset, that we denote by $Par(m)$. Therefore, in order to use finite methods, we introduce another integer $n \geq 0$ and we consider the sub-poset $O(m,n)$ of $Par(m)$ with exactly n non negative summands and exactly n non positive summands whose absolute value of the extremal summands is at most n. In this way we obtain an increasing chain $O(m,0) \subseteq O(m,1) \subseteq O(m,2) \subseteq \cdots$ of sub-posets of $Par(m)$ such that $Par(m) = \bigcup_{n\geq 0} O(m,n)$ and we study the poset $O(m,n)$ as a local model of $Par(m)$. At first we determine the not obvious covering relation in the poset $O(m,n)$. The better way to describe with precision the covering relation in $O(m,n)$ is to consider $O(m,n)$ as a discrete dynamical model with some evolution rules. The way to study a lattice of classical partitions as a discrete dynamic model having some particular evolution rules begins implicitly in [11], where Brylawski proposed a dynamical approach to study the lattice $L_B(m)$. However, the explicit theoretical association between integer partitions and discrete dynamical model begins in [15], where Goles and Kiwi introduced the Sand Piles Model $SPM(m)$. If m is a non negative integer, a configuration of $SPM(m)$ is represented by an ordered partition of m, i.e. a decreasing sequence $a = (a_1, ..., a_m)$ of non negative integers having sum m, and each positive entire unit is interpreted as a sand grain whose movement respects the following rule:

Rule 1 (vertical rule): *one grain can move from a column to the next column if the difference of height of these two columns is greater than or equal to 2.*

The discrete dynamical model $SPM(m)$ is a very interesting graded sub-lattice of $L_B(m)$ (see [15]) that has inspired a wide literature on topics related to it (see for example [12], [13], [14], [16], [17], [19], [21], [24]).
In the scope of the discrete dynamical systems (see [1], [2], [3], [4], [5], [7], [18], [23]), the Brylawski lattice can be interpreted then as the model $L_B(m)$, where the movement of a sand grain respects the previous Rule 1 and the following Rule 2:

Rule 2 (horizontal rule): *If a column containing $p+1$ grains, is followed by a sequence of columns containing p grains and next by one column containing $p-1$ grains, then one grain of the first column can slip to the last column.*

The previous rules in our model $O(m,n)$ become more complicated because we have the necessity to establish a balancing between positive and negative summands of the signed partitions.
We give now a brief description of the several sections of this paper. In section 2 we introduce the infinite set $P(m)$ of all the signed partitions of a fixed integer m. On $P(m)$ we define at first a quasi-order $\trianglelefteq$ (i.e. a binary reflexive and transitive relation) and next we define the signed dominance order poset $Par(m)$ as a quotient poset of the quasi-poset $(P(m), \trianglelefteq)$. We prove that $Par(m)$ has a lattice-structure by means of the increasing chain $\{O(m,n)\}_{n\geq 0}$ of its finite sub-posets whose union is $Par(m)$. In section 3 we study the lattice $O(m,n)$ as a discrete dynamical model by means of some evolution rules that generalize the classical evolution rules of the Brylawski model. We prove that these rules completely characterize the covering relations of $O(m,n)$. In section 4 we use the results established in section 3 in order to determine the maximum number of elements that an element of $O(m,n)$ can cover in $O(m,n)$. In section 5 we extend the classical duality concept in the case of signed partitions and we show that this duality is an anti-automorphism in the lattice $O(m,n)$. As a consequence of the duality we also deduce some structural properties of $O(m,n)$. In section 6 we describe in detail the structure of the local intervals in $O(m,n)$. This description is particularly useful if someone want to study the sequential and parallel dynamics of our model. For example, the characterization of the local intervals given in [11] is widely used in order to find non-trivial properties in all the discrete dynamical models that are extracted from the Brylawski lattice (see [17]). Finally, in section 7 we use all the results established in the previous sections in order to determine the Möbius function of the lattice $O(m,n)$.

2. The Signed Dominance Order Lattice

We begin with the definition of a signed partition, which was introduced in [6] and studied in [20] from an arithmetical point of view.

Definition 2.1. *Let p and q be two non-negative integers. A signed partition (briefly s-partition) with balance (p,q) is a finite sequence w of integers $w_1,\dots,w_p,w_{p+1},\dots,w_{p+q}$, called parts of w, such that $w_1 \geq \cdots \geq w_p \geq 0 \geq w_{p+1} \geq \cdots \geq w_{p+q}$. A s-partition is a s-partition with balance (p,q), for some non-negative integers p and q.*

In what follows we write w in the following form $w = (w_1,\dots,w_p|w_{p+1},\dots,w_{p+q})$, however in the numerical examples we write $w = w_1 \dots w_p\,|\,|w_{p+1}|\dots|w_{p+q}|$, where $|w_j|$ is the absolute value of w_j. If it is not necessary to distinguish which parts of a s-partition w are non-negative integers and which are non-positive integers, we simply write $w = l_1 \cdots l_n$, with l_i integer for $i = 1,\dots,n$. We shall denote by P the set of all the s-partitions. We call $w_1,\dots,w_p$ the *non-negative parts of* w and $w_{p+1},\dots,w_{p+q}$ the *non-positive parts of* w; also, we call *positive parts of* w the integers w_i with $w_i > 0$ and *negative parts of* w the integers w_j with $w_j < 0$. We denote by $(|)$ the empty s-partition, i.e. the s-partition without parts. If $w \in P$ and m is an integer such that $m = w_1 + \cdots + w_p + w_{p+1} + \cdots + w_{p+q}$, we say that w is a *s-partition of the integer* m and we shall write $w \vdash m$. We set then $P(m) := \{w \in P : w \vdash m\}$.

Remark on Notation 2.2. *If w is a s-partition, we denote by $\{w \geq 0\}$ [$\{w > 0\}$] the multi-set of all the non-negative [positive] parts of w and by $\{w \leq 0\}$ [$\{w < 0\}$] the multi-set of the absolute values of all the non-positive [negative] parts of w. We denote respectively with $|w|_{\geq}$, $|w|_{\leq}$, $|w|_{>}$, $|w|_{<}$ the cardinality of $\{w \geq 0\}$, $\{w \leq 0\}$, $\{w > 0\}$, $\{w < 0\}$. We call the ordered couple $(|w|_{>}, |w|_{<})$ the signature of w and we note that $(|w|_{\geq}, |w|_{\leq})$ is exactly the balance of w. Finally, we set $||w|| = |w|_{>} + |w|_{<}$.*

For example, if $w = 444221000|011333$, then $\{w \geq 0\} = \{4^3, 2^2, 1^1, 0^3\}$, $\{w \leq 0\} = \{0^1, 1^2, 3^3\}$, $\{w > 0\} = \{4^3, 2^2, 1^1\}$, $\{w < 0\} = \{1^2, 3^3\}$, w has balance $(|w|_{\geq}, |w|_{\leq}) = (9, 6)$ and signature $(|w|_{>}, |w|_{<}) = (6, 5)$.
If w is a s-partition having signature (t, s) and balance (p, q), then w has the form

$$w = (w_1, \ldots, w_t, w_{t+1}, \ldots, w_p | w_{p+1}, \ldots, w_{p+q-s}, w_{p+q-s+1}, \ldots, w_{p+q}) \tag{1}$$

where $w_1 \geq \cdots \geq w_t > 0 > w_{p+q-s+1} \geq \cdots \geq w_{p+q}$, $w_t = \ldots = w_p = 0$ and $w_{p+1} = \ldots = w_{p+q-s} = 0$. We also write the s-partition in (1) in the following form:

$$w = (w_1, \ldots, w_t, 0_{p-t} | 0_{q-s}, w_{p+q-s+1}, \ldots, w_{p+q}) \tag{2}$$

If w is a s-partition as in (2), we call *reduced s-partition of* w the following s-partition:

$$w^r = (w_1, \ldots, w_t | w_{p+q-s+1}, \ldots, w_{p+q}) \tag{3}$$

Let now m be a fixed integer. If $v = l_1 \cdots l_n$ and $w = l'_1 \cdots l'_{n'}$ are two s-partitions in $P(m)$ with the same balance we define:

$$w \trianglelefteq w' \quad \text{if and only if} \quad l_1 + l_2 + \cdots + l_i \leq l'_1 + l'_2 + \cdots + l'_i \tag{4}$$

for all $i \in \{1, \ldots, n\}$.
If w is a s-partition having signature (s, t) and N is an integer satisfying $N \geq \max\{s, t\}$ then we define a s-partition $\overline{w}^N = (W_1, \ldots, W_{2N})$, having balance (N, N), as follows:

$$W_i := \begin{cases} w_i^r & \text{if } 1 \leq i \leq s \\ 0 & \text{if } s < i \leq 2N - t \\ w_{i-2N+t+s}^r & \text{if } 2N - t < i \leq 2N \end{cases}$$

where w^r is the reduced s-partition of w.
For example if $w = (32100|014)$ and $N = 5$, then $w^r = (321|14)$ and $\overline{w}^N = (32100|00014)$.
We define then on $P(m)$ the following binary relation: if $v, w \in P(m)$, with signatures (s, t) and (i, j) respectively, we set

$$v \trianglelefteq w \quad \text{if and only if} \quad \overline{v}^N \trianglelefteq \overline{w}^N, \tag{5}$$

where the integer N needs to satisfy $N \geq \max\{s, t, i, j\}$. The following proposition is directly derived.

Proposition 2.3. *$\trianglelefteq$ is a quasi-order on the set $P(m)$; moreover, if v, w are two s-partition in $P(m)$ with signatures (s, t) and (i, j) respectively, then the following conditions are equivalent:*
(i) $v \trianglelefteq w$;
(ii) $\overline{v}^N \trianglelefteq \overline{w}^N$ for any $N \geq \max\{s, t, i, j\}$.

If $v, w \in P(m)$, we set

$$v \sim w \quad \text{if and only if} \quad \{v > 0\} = \{w > 0\} \text{ and } \{v < 0\} = \{w < 0\} \tag{6}$$

Then $\sim$ is an equivalence relation on $P(m)$ and it results that

$$v \sim w \quad \text{if and only if} \quad v \trianglelefteq w \text{ and } w \trianglelefteq v \tag{7}$$

Notice that if v and w are two s-partitions in $P(m)$, then $v = w$ iff v and w are uniform and $v \sim w$.
By (7) it follows that $\sim$ is exactly the equivalence relation on $P(m)$ induced from the quasi-order $\trianglelefteq$, therefore if $\mathcal{F}$ is any subset of $P(m)$, we can consider on the quotient set $\mathcal{F}/\sim$ the usual

partial order induced by $\trianglelefteq$, that we here denote $\trianglelefteq'$. We recall that $\trianglelefteq'$ is defined as follows: if $[v], [w] \in \mathcal{F}/\sim$ then

$$[v] \trianglelefteq' [w] \quad \text{if and only if} \quad v \trianglelefteq w \tag{8}$$

for any/all $v, w \in \mathcal{F}$. In the next definition we introduce our principal object of study.

Definition 2.4. *We call signed dominance order poset of the integer m the partially ordered set $Par(m) := (P(m)/\sim, \trianglelefteq')$.*

If $w \in \mathcal{F}$, in some case we set $[w]_\sim^{\mathcal{F}} = \{v \in \mathcal{F} : v \sim w\}$, that is the equivalence class of w in $\mathcal{F}/\sim$.

Remark 2.5. *If $\mathcal{F} \subseteq \mathcal{H} \subseteq P$ we can consider $\mathcal{F}/\sim$ as a subset of $\mathcal{H}/\sim$ through the identification of $[v]_\sim^{\mathcal{F}}$ with $[v]_\sim^{\mathcal{H}}$, for each $v \in \mathcal{F}$. Therefore, if $\mathcal{F} \subseteq \mathcal{H} \subseteq P$ we can always assume that $(\mathcal{F}/\sim \trianglelefteq')$ is a sub-poset of $(\mathcal{H}/\sim, \trianglelefteq')$.*

Let now n be a fixed non-negative integer and let $O(m, n)$ be the subset of $Par(m)$ with at most n positive summands, at most n negative summands, and no summands a_i with $|a_i| > n$. In what follows, by the previous discussion, we shall use w rather than $\overline{W}^n$ whenever working in $O(m, n)$, and thus it results:

$$O(m, n) := \{(w_1, \ldots, w_n | w_{n+1}, \ldots, w_{2n}) \in P(m) : \; n \geq w_1, \; w_{2n} \geq -n\}$$

Moreover, we continue to denote the partial order $\trianglelefteq'$ with the symbol $\trianglelefteq$. With the notation $v \triangleleft w$ we mean $v \trianglelefteq w$ with $v \neq w$. We also write $w \lessdot w'$ if w' covers w with respect to the partial order $\trianglelefteq$.
Let us note that $O(m, n)$ is non empty if and only if $-n^2 \leq m \leq n^2$, therefore we shall assume $-n^2 \leq m \leq n^2$. We set now

$$\hat{1}^{m,n} := \begin{cases} (n, \ldots, n, r, 0, \ldots, 0 | -n, \ldots, -n) & \text{if} \quad m < 0 \\ (n, \ldots, n | 0, \ldots, 0, -r, -n, \ldots, -n) & \text{if} \quad m > 0 \\ (n, \ldots, n | -n, \ldots, -n) & \text{if} \quad m = 0, \end{cases}$$

with n and $-n$ repeated exactly k times respectively when $m < 0$ and $m > 0$, where k and r are the unique non-negative integers such that $n^2 - |m| = kn + r$, with $r < n$. We also set

$$\hat{0}^{m,n} := \begin{cases} (0, \ldots, 0 | -h, \ldots, -h, -(h+1), \ldots, -(h+1)) & \text{if} \quad m < 0 \\ (h+1, \ldots, h+1, h, \ldots, h | 0, \ldots, 0) & \text{if} \quad m > 0 \\ (0, \ldots, 0 | 0, \ldots, 0) & \text{if} \quad m = 0, \end{cases}$$

with $-(h+1)$ and $h+1$ repeated exactly s times respectively when $m < 0$ and $m > 0$, where h and s are the unique non-negative integers such that $|m| = hn + s$, with $s < n$. Let us note that $s > 0$ implies $h + 1 \leq n$ because $|m| \leq n^2$.

Theorem 2.6. *$O(m, n)$ is a lattice with maximum $\hat{1}^{m,n}$ and minimum $\hat{0}^{m,n}$.*

Proof. We prove at first that $\hat{0}^{m,n} \trianglelefteq w \trianglelefteq \hat{1}^{m,n}$ for all $w \in O(m, n)$. We consider only the case $m > 0$, because with a similar argument we can also prove the other cases. Let $w = (w_1, \ldots, w_n | w_{n+1}, \ldots, w_{2n}) \in O(m, n)$. Since $n \geq w_1 \geq \cdots \geq w_n \geq 0$, we have $\sum_{i=1}^{j} w_i \leq jn$ for each $j \in \{1, \ldots, n\}$ and the same inequality obviously still holds for each $j \in \{n+1, \ldots, 2n-k-1\}$. Moreover, for each $j \in \{2n-k, \ldots, 2n\}$, we have $\sum_{i=j+1}^{2n} w_j \geq (2n-j)(-n)$, which is equal to the sum of the last $2n-j$ entries in $\hat{1}^{m,n}$. Hence $w \trianglelefteq \hat{1}^{m,n}$. On the other hand, since $\sum_{i=1}^{n} w_i \geq m$, we have $\sum_{i=1}^{j-1} w_i \geq j(h+1)$ for each j in $\{1, \ldots, s\}$ and $\sum_{i=1}^{j} w_i \geq s(h+1) + (j-s)h = s + jh$ for each j in $\{s+1, \ldots, n\}$. Note that the members on the right-hand side in the previous inequalities are equal to the sums of the first j integers in $\hat{0}^{m,n}$. Because $w_{n+1}, \ldots, w_{2n}$ are non-positive, the inequality $\sum_{i=1}^{j} w_i \geq m$ holds for each $j \in n+1, \cdots 2n$. Hence $\hat{0}^{m,n} \trianglelefteq w$.
We denote now by $H(m, n)$ the set of all the $(2n+1)$-ples of integers $T = (t_0, t_1, \ldots, t_n : t_{n+1}, \ldots, t_{2n})$ having the following properties:

(H1): $0 = t_0 \leq t_1 \leq \cdots \leq t_n \geq t_{n+1} \geq \cdots \geq t_{2n} = m$ (unimodality).

(H2): $2t_i \geq t_{i-1} + t_{i+1}$ for $i = 1, \dots, 2n-1$ (concavity).

(H3): $t_1 - t_0 \leq n$ and $t_{2n} - t_{2n-1} \geq -n$.

If $w = (w_1, \dots, w_n | w_{n+1}, \dots, w_{2n}) \in O(m,n)$, we set

$$\hat{w} = (s_0, s_1, \dots, s_n : s_{n+1}, \dots, s_{2n}),$$

where $s_0 := 0$, $s_k := \sum_{i=1}^{k} w_i$ for $k = 1, \dots, 2n$. It is immediate to verify that $\hat{w} \in H(m,n)$. Therefore we can consider the application $\Lambda : O(m,n) \to H(m,n)$ such that $\Lambda(w) = \hat{w}$. The map Λ is bijective. It follows immediately from the definition of $\hat{w}$ and simple arithmetic that Λ is injective, so it remains to show that Λ is surjective. If $T = (t_0, t_1, \dots, t_n : t_{n+1}, \dots, t_{2n}) \in H(m,n)$, we take $w_i := t_i - t_{i-1}$ for $i = 1, \dots, 2n$. By (H1), (H2) and (H3) we deduce then that $w = (w_1, \dots, w_n | w_{n+1}, \dots, w_{2n})$ is an element of $O(m,n)$ such that $\hat{w} = T$.
If $T' = (t'_0, t'_1, \dots, t'_n : t'_{n+1}, \dots, t'_{2n})$ and $T'' = (t''_0, t''_1, \dots, t''_n : t''_{n+1}, \dots, t''_{2n})$ are two elements of $H(m,n)$, we set $\omega(T', T'') := (t_0, t_1, \dots, t_n : t_{n+1}, \dots, t_{2n})$, where $t_i = \min\{t'_i, t''_i\}$ for $i = 0, 1, \dots, 2n$. It is easy then to verify that $\omega(T', T'') \in H(m,n)$. Let now $w', w'' \in O(m,n)$ and $T = \omega(\Lambda(w'), \Lambda(w''))$. Since $T \in H(m,n)$ and Λ is bijective, there exists a unique element $w \in O(m,n)$ such that $\hat{w} = T$. From the definitions of $\hat{w}$ and of dominance order it follows that w is exactly the maximal lower bound of the elements w' and w'' with respect to the partial order $\trianglerighteq$. Hence $w = w' \wedge w''$. Thus $O(m,n)$ has always a maximum element $\hat{1}^{m,n}$.
Finally, if $w, w' \in O(m,n)$ the set of the upper bounds of w and w' is not-empty because it always contains $\hat{1}^{m,n}$, therefore we take $u = \bigwedge\{z \in O(m,n) : z \sqsupseteq w, z \sqsupseteq w'\}$ in order to have $u = w' \vee w''$. ☐

Example 2.7. Let $m = -7$ and $n = 8$. If we take $w' = 75444220|44444555$ and $w'' = 85433220|33334468$ as elements of $O(-7,8)$, then

$$\Lambda(w') = (0, 7, 12, 16, 20, 24, 26, 28, 28 : 24, 20, 16, 12, 8, 3, -2, -7)$$

and

$$\Lambda(w'') = (0, 8, 13, 17, 20, 23, 25, 27, 27 : 24, 21, 18, 15, 11, 7, 1, -7).$$

So that, if $T = \omega(\Lambda(w'), \Lambda(w''))$, then

$$T = (0, 7, 12, 16, 20, 23, 25, 27, 27 : 24, 20, 16, 12, 8, 3, -2, -7).$$

The signed partition $w \in O(-7,8)$ such that $\hat{w} = T$ is $w = 75443220|34444555$. Hence $w = w' \wedge w''$.

Below we draw the Hasse diagram of the lattice $O(-4,3)$:

$O(-4,3)$:

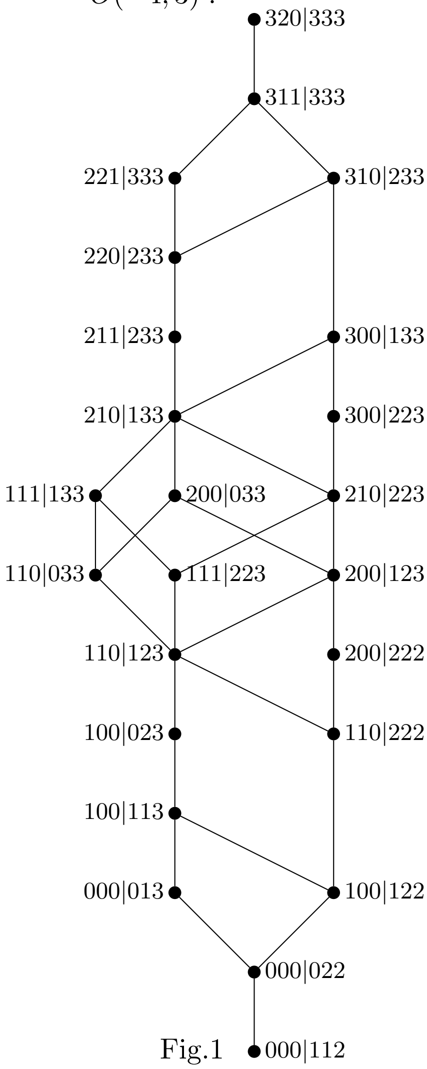


Fig.1

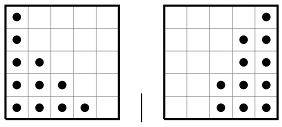
As a direct consequence of the previous theorem we obtain the following result.

Corollary 2.8. *$Par(m)$ is a lattice.*

Proof. It suffices to note that the poset $Par(m)$ can be identified with the increasing union of its sub-posets $O(m,n)$, for $n = 0,1,2,\ldots$. $\square$

3. $O(m,n)$ as a Discrete Dynamical Model

In order to describe better the covering rules of the lattice $O(m,n)$, in this section we consider $O(m,n)$ as a particular type of discrete dynamical model. A *discrete dynamical model* is a system whose elements (called *configurations*) evolve in discrete time under certain evolution rules (see [1], [2], [3], [4], [5], [7], [17], [18], [23] for several interesting studies concerning such models). In our case a configuration is an element of $O(m,n)$. We describe now the evolution rules of our model. In order to illustrate how such rules act in $O(m,n)$, we represent any signed partition $w \in O(m,n)$ by using two square-boxes, one for the non negative parts (on the left) and the other for the non positive parts (on the right). Each positive unit is represented as a black ball in the left square-box and each negative unit as a black ball in the right square-box. Therefore, in the following pictures, when we sum $+1$ $[-1]$ over a negative part, the corresponding negative column of black balls (in the right square-box) loses [gains] a ball. For example if $w = 53210|00245 \in O(0,5)$, then we represent w as in the next picture:



If $i \in \{1, 2, \ldots, 2n-1\}$ we define $c_i : O(m, n) \to O(m, n)$, where $c_i(w) = v$ $(w, v \in O(m, n))$, as follows:

- if $i \neq n$, then:

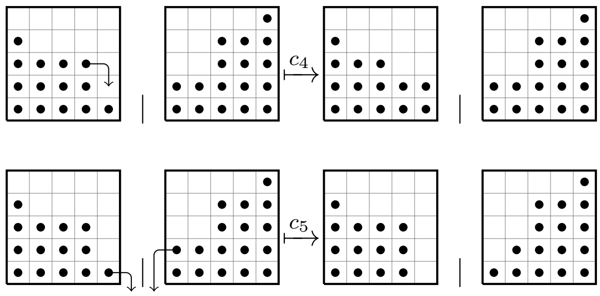
when it happens that $w_i - w_{i+1} \geq 2$, we put $v_i = w_i - 1$, $v_{i+1} = w_{i+1} + 1, v_k = w_k$ if $k \notin \{i, i+1\}$;

- if $i = n$, then:

when it happens that $w_n > 0 > w_{n+1}$, we put $v_n = w_n - 1$, $v_{n+1} = w_{n+1} + 1, v_k = w_k$ if $k \notin \{n, n+1\}$;

in all the other cases we put $v = w$.

Example 3.1. In $O(-3, 5)$, if $w = 43331|22445$ then $c_4(w) = 43322|22445$ and $c_5(w) = 43330|12445$. In pictures:



If $i \in \{1, 2, \ldots, 2n-2\}$ and $j \in \{i+2, \ldots, 2n\}$, we define $s_{ij} : O(m, n) \to O(m, n)$, where $s_{ij}(w) = v$ $(w, v \in O(m, n))$, as follows:

- if $i \in \{1, 2, \ldots, n-2\}$ and $j \in \{i+2, \ldots, n\}$, or if $i \in \{n+1, \ldots, 2n-2\}$ and $j \in \{i+2, \ldots, 2n\}$, then:

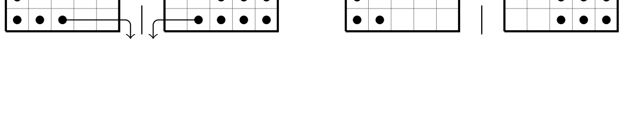
when it happens that $w_i - 1 = w_{i+1} = \cdots = w_{j-1} = w_j + 1$, we put $v_i = w_i - 1$, $v_j = w_j + 1$, $v_k = w_k$ if $k \notin \{i, j\}$;

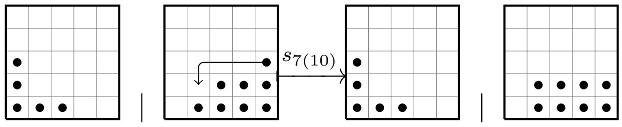
- if $i \in \{1, 2, \ldots, n\}$ and $j \in \{\max\{i+2, n+1\}, \ldots, 2n\}$, then:

when it happens that $w_i - 1 = w_{i+1} = \cdots = w_{j-1} = w_j + 1 = 0$, we put $v_i = v_j = 0$, $v_k = w_k$ if $k \notin \{i, j\}$.

in all the other cases we put $v = w$.

Example 3.2. In $O(-3, 5)$, if $w = 31100|01223$ then $s_{37}(w) = 31000|00223$ and $s_{7(10)}(w) = 31100|02222$. In pictures:

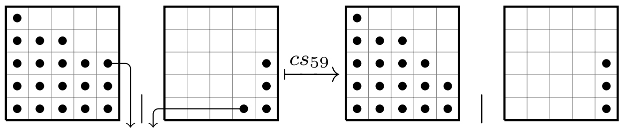




If $i = n$ and $j \in \{n+2, \ldots, 2n\}$, we define $cs_{ij} : O(m,n) \to O(m,n)$, where $cs_{ij}(w) = v$ $(w, v \in O(m,n))$, as follows:
when it happens that $w_n \geq 2$ and $w_{n+1} = \cdots = w_{j-1} = w_j + 1 = 0$, we put $v_n = w_n - 1$, $v_j = 0$, $v_k = w_k$ if $k \notin \{n, j\}$;

in all the other cases we put $v = w$.

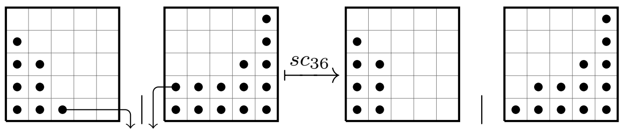
Example 3.3. In $O(15,5)$, if $w = 54433|00013$ then $cs_{59}(w) = 54432|00003$. In picture:



If $i \in \{1, 2, \ldots, n-1\}$ and $j = n+1$, we define $sc_{ij} : O(m,n) \to O(m,n)$, where $sc_{ij}(w) = v$ $(w, v \in O(m,n))$, as follows:
when it happens that $w_i - 1 = w_{i+1} = \cdots = w_n = 0$ and $w_{n+1} \leq -2$, we put $v_i = 0$, $v_{n+1} = w_{n+1} + 1$, $v_k = w_k$ if $k \notin \{i, n+1\}$;

in all the other cases we put $v = w$.

Example 3.4. In $O(-6,5)$, if $w = 43100|22235$ then $sc_{36}(w) = 43000|12235$. In picture:



If $w \neq v \in O(m,n)$ we write $w \downarrow v$ if v can be obtained from w by applying one of the previous evolution rules.
In the next result we prove that the previous evolution rules are exactly the covering relations in $O(m,n)$ with respect to the dominance partial order.

Theorem 3.5. *If $v, w \in O(m,n)$ then*

$$w \gtrdot v \quad \textit{if and only if} \quad w \downarrow v$$

Proof. Take $w = (w_1, \ldots, w_n | w_{n+1}, \ldots, w_{2n})$ and $v = (v_1, \ldots, v_n | v_{n+1}, \ldots, v_{2n})$. We assume at first that $w \downarrow v$. In this case w and v differ between them in exactly two places h and k, with $h < k$, where $v_h = w_h - 1$ and $v_k = w_k + 1$. Hence $w \rhd v$. Let $u \in O(m,n)$ such that $w \unrhd u \unrhd v$. We must show that $u = v$ or $u = w$. For all $t \in \{1, 2, \ldots, 2n\}$ with $t < h$, from the conditions $w \unrhd u \unrhd v$ and $w_t = v_t$ we obtain $u_t = w_t = v_t$. In a similar way it also results $u_t = w_t = v_t$ for all $t \in \{1, 2, \ldots, 2n\}$ with $t > k$ because $v_1 + \cdots + v_h + \cdots + v_k = w_1 + \cdots + (w_h - 1) + \cdots + (w_k + 1) = w_1 + \cdots + w_h + \cdots + w_k$. Let us observe that $u_h = v_h$ or $u_h = w_h$ because $v_h = w_h - 1$ and $w \unrhd u \unrhd v$. We distinguish now several cases. If $k = h+1$, from the conditions $w_h + w_k = u_h + u_k = v_h + v_k$ we deduce respectively that $u = v$ or $u = w$. Let $k > h+1$ with $h \neq n$ and $k \neq n+1$. If $u_h = v_h$, then $v_{h+1} = v_h = u_h \geq u_{h+1} \geq v_{h+1}$, therefore $u_{h+1} = v_{h+1}$; iterating this procedure we find $u_t = v_t$ for all $0 \leq t \leq 2n-1$, with $t \neq k$. Hence also $u_k = v_k$ because $\sum_{s=1}^{2n} u_s = \sum_{s=1}^{2n} v_s$. This proves that $u = v$. We assume now $u_h = w_h$. Since $v_h + v_{h+1} \leq u_h + u_{h+1} \leq w_h + w_{h+1}$, it follows that $w_{h+1} - 1 \leq u_{h+1} \leq w_{h+1}$. If $u_{h+1} = w_{h+1} - 1$ take $t \in \{h+2, \ldots, k\}$ and we obtain $u_t \leq u_{t-1} \leq \cdots \leq u_{h+1} = w_{h+1} - 1 = w_t - 1 \leq w_{t+1}$. Therefore $w_h + w_{h+1} + w_{h+2} + \cdots + w_k = u_h + u_{h+1} + u_{h+2} \cdots + u_k \leq w_h + (w_{h+1} - 1) + w_{h+2} + \cdots + w_k < w_h + w_{h+1} + w_{h+2} + \cdots + w_k$, that is absurd. Hence it must

be necessarily $u_{h+1} = w_{h+1}$. Iterating this procedure we deduce that $u_t = w_t$ for $1 \le t \le 2n$ with $t \ne k$. Hence, as before, $u_k = w_k$. This proves that $u = w$.
We consider now $k > h+1$ and $h = n$. As before let $u \in O(m,n)$ such that $w \trianglerighteq u \trianglerighteq v$ and so $u_h = v_h$ or $u_h = w_h$. If $u_h = v_h$, since $w \downarrow v \trianglelefteq u$ and $u_t \le 0$ for all $h+1 \le t \le k-1$, we have $u_{h+1} = \cdots = u_{k-1} = 0 = v_{h+1} = \cdots = u_{k-1}$. Hence $u_k = v_k$ and this prove that $u = v$. We assume now $u_h = w_h$. Since $v_h + v_{h+1} \le u_h + u_{h+1} \le w_h + w_{h+1}$, it follows that $-1 \le u_{h+1} \le 0$. If $u_{h+1} = -1$ take $t \in \{h+2, \ldots, k\}$ and we obtain $u_t \le u_{t-1} \le \cdots \le u_{h+1} = -1 \le w_t$. Therefore $w_h + w_{h+1} + w_{h+2} + \cdots + w_k = u_h + u_{h+1} + u_{h+2} \cdots + u_k \le w_h + (w_{h+1} - 1) + w_{h+2} + \cdots + w_k < w_h + w_{h+1} + w_{h+2} + \cdots + w_k$, that is absurd. Hence it must be necessarily $u_{h+1} = w_{h+1} = 0$. Iterating this procedure we deduce that $u_t = 0$ for $n \le t \le k-1$ with $t \ne k$. Hence, as before, $u_k = w_k$. This proves that $u = w$.
Finally we assume $k > h+1$ and $k = n+1$. If $u_h = v_h = 0$ then obviously $u_{h+1} = \cdots = u_n = 0$ and so $u_t = v_t$ for all $1 \le t \le 2n$, with $t \ne k$. Hence also $u_k = v_k$ and finally $u = v$. If $u_h = w_h = 1$ then $0 = w_{h+1} \ge u_{h+1} = \cdots = u_n \ge 0$. It follows $u_t = w_{t+1}$ for all $t \in \{1, \cdots, 2n\} \setminus \{k\}$. Therefore also $u_k = w_k$. This proves that $u = w$.
We must prove now that

$$w \gtrdot v \Longrightarrow w \downarrow v.$$

More generally, we prove that if $w \triangleright v$ then there exists $u \in O(m,n)$ such that $w \downarrow u \trianglerighteq v$. In fact, if this holds, then $w \gtrdot v$ implies $w \triangleright v$ and so there exists u with $w \downarrow u \trianglerighteq v$. Hence $w \ne u$ and, since we assume $w \gtrdot v$, we obtain $u = v$ and $w \downarrow v$. We divide the proof in four cases.
Case 1. Assume that there exists $i \in \{1, \cdots, 2n\}$ such that $w_i - w_{i+1} \ge 2$, $v_1 + \cdots + v_i < w_1 + \cdots + w_i$. Assume also that either $i \ne n$ or $w_{i+1} \ne 0$. Then $u := c_i(w)$ satisfies $w \downarrow u \trianglerighteq v$. In fact all partial sums of the u_t are trivially equal to the correspondent partial sums of the w_t, except for $t = i$. But in this case $v_1 + \cdots + v_i \le w_1 + \cdots + w_i - 1 = u_1 + \cdots + u_i$ and so $u \trianglerighteq v$.
Case 2. Alternatively, assume that $w_n \ge 2$, $w_{n+1} = 0$ and $v_1 + \cdots + v_n < w_1 + \cdots + w_n$. Let us note that in such a case there exists $j > n+1$ such that $w_j < 0$ since necessarily there exists $j > n+1$ such that $v_j > w_j$ and this is impossible if each w_j with $j > n+1$ is equal to 0. Let w_j the first negative part of w. Then we set

$$u := \begin{cases} c_{j-1}(w) & \text{if } w_j \le -2, \\ cs_{ij}(w) & \text{if } w_j = -1. \end{cases}$$

In both cases u satisfies $w \downarrow u \trianglerighteq v$.
Case 3. Assume that there exists $i \in \{1, \cdots, 2n\}$ such that $w_i - w_{i+1} = 1$, $v_1 + \cdots + v_i < w_1 + \cdots + w_i$ and $w_i \ne 1$. Let $j \in \{i+2,\, i+3,\, \cdots,\, 2n\}$ such that $w_{j-1} > w_j$. Then we set

$$u := \begin{cases} c_{j-1}(w) & \text{if } w_{j-1} - w_j \ge 2, \\ s_{ij}(w) & \text{if } w_{j-1} - w_j = 1. \end{cases}$$

We have that u satisfies $w \downarrow u \trianglerighteq v$.
Case 4. Assume that there exists $i \in \{1, \cdots, 2n\}$ such that $w_i - w_{i+1} = 1$, $v_1 + \cdots + v_i < w_1 + \cdots + w_i$ and $w_i = 1$. Let $j \in \{i+2,\, i+3,\, \cdots,\, 2n\}$ such that $w_{j-1} > w_j$. Then we set

$$u := \begin{cases} c_{j-1}(w) & \text{if } w_{j-1} - w_j \ge 2 \text{ and } j > n, \\ sc_{j-1}(w) & \text{if } w_{j-1} - w_j \ge 2 \text{ and } j = n, \\ s_{ij}(w) & \text{if } w_{j-1} - w_j = 1. \end{cases}$$

In each case $w \downarrow u \trianglerighteq v$. This completes the proof. □

The result of the previous theorem tell us that $O(m,n)$ is also a discrete dynamical model whose local evolution rules are described by means of the operators c_i, for $i = 1, 2, \ldots, 2n-1$ and s_{ij}, cs_{ij} and sc_{ij} for $i = 1, 2, \ldots, 2n-2$ and $j = i+2, i+3, \ldots, 2n-2$. Since $O(m,n)$ is a finite lattice, when we see it as a discrete dynamical model, its starting configuration is $\hat{1}^{m,n}$ and its unique fixed point is $\hat{0}^{m,n}$.
If $w = (w_1, \ldots, w_n | w_{n+1}, \ldots, w_{2n}) \in O(m,n)$, we set $\overline{w} := (-w_{2n}, \ldots, -w_{n+1} | -w_n, \ldots, -w_0)$. Then $\overline{w}$ is an element of $O(-m,n)$ which we will call *symmetric* of w.

Remark 3.6. *It is easy to verify that the map*

$$\psi : w \in O(m,n) \mapsto \overline{w} \in O(-m,n)$$

is a lattice isomorphism which transforms each covering rule in $O(m,n)$ into a corresponding "symmetric" rule in $O(-m,n)$. For example, if $v = cs_{ij}(w)$ in $O(m,n)$ then $\overline{v} = sc_{ji}(\overline{w})$ in $O(-m,n)$. Analogously for all the other covering rules. Therefore, we restrict ourselves to the case $0 \leq m \leq n^2$.

Now let:

$$Q(n) := \{(w_1, \ldots, w_n | w_{n+1}, \ldots, w_{2n}) \in P : \ n \geq w_1, \ w_{2n} \geq -n\}.$$

On the subset $Q(n)$ we consider the following partial order $\sqsubseteq$:

$$(w_1, \ldots, w_n | w_{n+1}, \ldots, w_{2n}) \sqsubseteq (w'_1, \ldots, w'_n | w'_{n+1}, \ldots, w'_{2n})$$

if and only if

$$w_k \leq w'_k, \ \text{for } k = 1, \ldots, 2n.$$

In the following result we give the basic properties of the poset $(Q(n), \sqsubseteq)$. We recall at first the definition of the classical lattice $L(l,n)$. If l is a non-negative integer, the set $L(l,n)$ is the set of all the usual partitions with at most l parts and with largest part at most n. Such lattice was introduced by Stanley in [28], who showed that $L(l,n)$ is Peck (a graded poset is called *Peck poset* if it is rank-symmetric, rank-unimodal and strongly Sperner (see [29])).

Proposition 3.7. *(i) $(Q(n), \sqsubseteq)$ is a finite distributive (hence also graded) lattice.*
(ii) The rank function of $Q(n)$ is $\rho : Q(n) \to \mathbb{N}$ such that

$$\rho((w_1, \ldots, w_n | w_{n+1}, \ldots, w_{2n})) = w_1 + \cdots w_n + w_{n+1} \cdots + w_{2n} + n^2. \tag{9}$$

(iii) The rank of $Q(n)$ is $2n^2$.
(iv) If $-n^2 \leq m \leq n^2$, then $O(m,n)$ is exactly the $(m+n^2)$-th rank level (i.e. the subset of the elements having rank $m+n^2$) of $Q(n)$.
(v) $Q(n)$ is isomorphic to $L(n,n) \times L(n,n)^*$.
(vi) $Q(n)$ is Peck.
(vii) $|Q(n)| = \binom{2n}{n}^2$.

Proof. (i) Immediate.
(ii) Let $w = l_1 \ldots l_n$ and $w' = l'_1 \ldots l'_n$ be two elements in $Q(n)$ such that w' covers w. Then there exists exactly a place k where w and w' are different and that in this place results $l'_k = l_k + 1$. In fact, by contradiction, we must distinguish three cases:
1) there exists exactly one place k where w and w' are different, but $l'_k \neq l_k + 1$. Since by hypothesis w' covers w, we have $w \sqsubset w'$; therefore it must be $l_k < l'_k$. This implies that $l_k < l_k + 1 < l'_k$. But if we take $u = l_1 \cdots l_{k-1}(l_k+1)l_{k+1} \cdots l_n$, then $w \sqsubset u \sqsubset w'$ and $u \in Q(n)$. This contradicts our assumption because w' covers w.
2) There exist at least two places k and s, with $s > k$, where w and w' differ, with $l'_k > l_k$ and $l'_s > l_s$. We take then $u = l'_1 \ldots l'_{k-1} l'_k l_{k+1} \ldots l_{s-1} l_s l_{s+1} \ldots l_n$ and we obtain $r \geq l'_1 \geq \cdots \geq l'_{k-1} \geq l'_k > l_k \geq l_{k+1} \geq \cdots \geq l_{s-1} \geq l_s \geq l_{s+1} \geq \cdots \geq l_n \geq -(n-r)$, therefore $u \in Q(n)$, and since $w \sqsubset u \sqsubset w'$ contradicting the hypothesis that w' covers w.
3) w and w' are equal in all their parts. In this case $w = w'$, against the hypothesis.
We assume now that w and w' are different only in a place k and that $l'_k = l_k + 1$. It is obvious then that w' covers w in $Q(n)$.
Hence: w' covers w in $Q(n)$ iff there exists exactly a place k where w and w' are different and that in this place results $l'_k = l_k + 1$. Finally, since $\hat{0} = 0 \cdots 0 | -n \cdots -n$ is the minimum of $Q(n)$ and $\hat{0} \vdash -n^2$, the (ii) follows.
(iii) and (iv) are both direct consequences of (ii).
(v) It follows at once from the definitions of $Q(n)$ and $L(l,n)$.
(vi) $L(l,n)$ and $L(l,n)^*$ are in general both Peck lattices and the Peck-property is preserved

from the direct product operation, hence the result follows by (v).
(vii) Since $|L(l,n)| = \binom{l+n}{n}$, (vii) follows from (v). $\square$

As a direct consequence of the previous proposition we obtain the following properties of the lattices $O(m,n)$ when n is fixed:

Corollary 3.8. *If $n \geq 0$ is fixed, the sequence $\{|O(m,n)|\}_{-n^2 \leq m \leq n^2}$ is symmetric, unimodal and moreover*
$\sum_{m=-n^2}^{n^2} |O(m,n)| = \binom{2n}{n}^2$.

4. Covering Properties in $O(m,n)$

We denote by $c(m,n)$ the maximum number of elements that a configuration of $O(m,n)$ can cover in $O(m,n)$. In this section we use the results of the previous section in order to compute explicitly the number $c(m,n)$. Let p be an integer such that $0 \leq p \leq n^2$. If we start from the maximum $\hat{1}^{p-n^2,n}$ of $O(p-n^2,n)$ and we apply only c_i and s_{ij} where $0 \leq i < j \leq n-1$, we obtain a sub-poset of $O(p-n^2,n)$ that we denote by $B^+(p,n)$. Analogously, if we start from the maximum $\hat{1}^{n^2-p,n}$ of $O(n^2-p,n)$ and we apply only c_i and s_{ij} where $n \leq i < j \leq 2n-1$, we obtain a sub-poset of $O(n^2-p,n)$ that we denote by $B^-(p,n)$.

Proposition 4.1. *$B^+(n,n)$ is a lattice isomorphic to the classical Brylawski lattice $L_B(n)$.*

Proof. The starting configuration of $B^+(n,n)$ is $(n,0,\ldots,0|-n,\ldots,-n)$. Each configuration $w \in B(n,n)$ has the form $w = (w_1,\ldots,w_n|-n,\ldots,-n)$, where $w_1 \geq w_1 \geq \cdots\ w_n \geq 0$ is a usual partition of the non-negative integer n obtained from the starting configuration $(n,0,\ldots,0)$ with the two Brylawski's evolution rules described in Proposition 2.3 of [11]. It is clear then that $B^+(n,n)$ is a sub-poset of $O(n-n^2,n)$ isomorphic to $L_B(n)$, hence it is also a lattice. $\square$

In order to establish the main result of this section, we define at first the two following parameters:
- if $0 \leq p \leq n^2$ we set

$$\alpha(p,n) := \begin{cases} 0 & \text{if } p \in \{0,1,n^2-1,n^2\} \\ 1 & \text{if } p=2 \text{ or } p=n^2-2 \\ 2 & \text{if } n=4 \text{ and } p=8 \\ \lfloor \frac{1}{2}(-3+\sqrt{8p+1}) \rfloor & \text{if } 1 \leq p \leq \frac{n^2-n+2}{2} \\ n-3 & \text{if } \frac{n^2-n+2}{2} < p < \frac{n^2+n-2}{2} \text{ and } n \neq 4 \\ \lfloor \frac{1}{2}(-3+\sqrt{8(n^2-p)+1}) \rfloor & \text{if } \frac{n^2+n-2}{2} \leq p \leq n^2; \end{cases}$$

- if $0 \leq m \leq n^2$, we set

$$\delta(m,n) := \begin{cases} 0 & \text{if } m=n^2 \text{ or } m=n^2-4 \text{ or } m=n-2 \\ 1 & \text{otherwise .} \end{cases}$$

Then our main result of this section is the following:

Theorem 4.2. *If $0 \leq m \leq n^2$ then*

$$c(m,n) = \begin{cases} 3 & \text{if } n=3 \text{ and } m=4 \\ 2n-4+\delta(m,n) & \text{if } 0 \leq m \leq 1 \text{ or } n-2 \leq m \leq n \\ 2n-5+\delta(m,n) & \text{if } \begin{pmatrix} 2 \leq m \leq n-3 \\ \text{or} \\ n+1 \leq m \leq 2n-1 \end{pmatrix} \text{ and } \begin{pmatrix} n \neq 3 \\ \text{or} \\ m \neq 4 \end{pmatrix} \\ \alpha(m_1,n)+\alpha(m_2,n)+\delta(m,n) & \text{otherwise,} \end{cases}$$

where $m_1 = n^2 - \frac{\left(\alpha(\lceil \frac{n^2+m}{2} \rceil,n)+1\right)\left(\alpha(\lceil \frac{n^2+m}{2} \rceil,n)+2\right)}{2}$ *and* $m_2 = m_1 - m$.

In order to prove the previous theorem we introduce some new lattices and we divide the proof in several steps.

Lemma 4.3. *Let $0 \leq p \leq n^2$. The map $\phi : B^+(p,n) \to B^+(n^2-p,n)$ given by*

$$w = (w_1, \dots, w_n| -n, \dots, -n) \mapsto \widetilde{w} = (n - w_n, \dots, n - w_1| -n, \dots, -n)$$

is such that:

i *If $v = c_i(w)$ in $B^+(p,n)$, then $\widetilde{v} = c_{n-i}(\widetilde{w})$;*
ii *if $v = s_{ij}(w)$ in $B^+(p,n)$, then $\widetilde{v} = s_{(n+1-j)(n+1-i)}(\widetilde{w})$.*

In particular, ϕ is a lattice-isomorphism.

Proof. Let us note that $B^+(p,n) \subset O(p-n^2,n)$ and $B^+(n^2-p,n) \subset O(-p,n)$. Let us suppose that $v = c_i(w)$ in $B^+(p,n)$. In this case we have $w_i - w_{i+1} \geq 2$, $v_i = w_i - 1$, $v_{i+1} = w_{i+1} + 1$ and $v_k = w_k$ if $k \notin \{i, i+1\}$ and so $\widetilde{w}_{n-2-i} - \widetilde{w}_{n-1-i} = n - w_{i+1} - n + w_i \geq 2$, $\widetilde{v}_{n-2-i} = n - v_{i+1} = n - w_{i+1} - 1 = \widetilde{w}_{n-2-i} - 1$, $\widetilde{v}_{n-1-i} = n - v_i = n - w_i + 1 = \widetilde{w}_{n-1-i} + 1$ and obviously $\widetilde{v}_k = \widetilde{w}_k$ if $k \notin \{n-2-i, n-1-i\}$. Hence $\widetilde{v} = c_{n-2-i}(\widetilde{w})$. The inverse implication is exactly identical. In a similar way we can prove **ii.** $\square$

If $0 \leq p \leq n^2$, we denote now by $c^+(p,n)$ $[c^-(p,n)]$ the maximum number of elements that an element of $B^+(p,n)$ $[B^-(p,n)]$ can cover in $B^+(p,n)$ $[B^-(p,n)]$.

Lemma 4.4. *If $0 \leq p \leq n^2$ then $c^+(p,n) = c^-(p,n) = \alpha(p,n)$.*

Proof. The number of elements that a given $w \in B^+(p,n)$ can cover is less than the number of times $w_i \geq w_{i+1}$ and they are equal if and only if $w_i - w_{i+1} \geq 2$ for all $w_i \neq w_{i+1}$.
If $n = 4$ then as noted before $c^+(p,n) \leq 2$ $(c^-(p,n) \leq 2)$. The element $w = (4220|4444) \in B^+(8,4)$ $(w = (4444|0224) \in B^-(8,4))$ can cover 2 elements, so in this case $c^+(8,4) = c^-(8,4) = \alpha(8,4)$. In what follows we consider thus $n \neq 4$ or $p \neq 8$.
If $p = 0$ or $p = n^2$ the result is obvious. Let us suppose now $1 \leq p \leq \frac{n^2-n+2}{2}$. As noted by Brylawski in [11], if $\frac{(m+1)(m+2)}{2} \leq p \leq \frac{(m+1)(m+2)}{2} + m + 1 = \frac{(m+1)(m+4)}{2}$, with m positive integer, then $c^+(p,n) = m$. The first inequality gives $0 \leq m \leq \frac{1}{2}(-3+\sqrt{8p+1})$ and since $m \in \mathbb{Z}$, $m \leq \lfloor \frac{1}{2}(-3+\sqrt{8p+1}) \rfloor$. An element in $B^+(p,n)$ which covers such a number of elements is given by $(m+1+\varepsilon(p,m,0), m+\varepsilon(p,m,m+1), m-1+\varepsilon(p,m,m), \dots, 1+\varepsilon(p,m,2), \varepsilon(p,m,1), 0, \dots, 0|-n, \dots, -n)$, where $\varepsilon(p,m,k)$ is defined by

$$\varepsilon(p,m,k) = \begin{cases} 1 & \text{if } k+1 \leq p - \frac{(m+1)(m+2)}{2} \\ 0 & \text{elsewhere .} \end{cases}$$

If $\frac{n^2-n+2}{2} < p \leq \frac{n^2}{2}$, then it is easy to see that the maximal number of elements that an element $w \in B^+(p,n)$ can cover is equal to $n-3$. The other claims follow directly by Lemma 4.3. $\square$

Given an element $w = (w_1, \dots, w_n|w_{n+1}, \dots, w_{2n}) \in O(m,n)$, let $p_1 := w_1 + \cdots + w_n$, $p_2 := -w_{n+1} - \cdots - w_{2n}$ and let us consider the elements $w^+ = (w_1, \dots, w_n|-n, \dots, -n) \in B^+(p_1,n)$ and $w^- = (n, \dots, n|w_{n+1}, \dots, w_{2n}) \in B^-(p_2,n)$.

Lemma 4.5. *Let $w \in O(m,n)$, w^+, and w^- as before such that the numbers of elements that $w^+ \in B^+(p_1,n)$ and $w^- \in B^-(p_2,n)$ can cover are exactly $c^+(p_1,n)$ and $c^-(p_2,n)$. Then $c_n(w) = s_{ij}(w) = cs_{nj}(w) = sc_{i(n+1)}(w) = w$ for each $i \in \{1, \dots, n-1\}$ and $j \in \{n+2, \dots, 2n\}$ if and only if $p_1, p_2 \in \{0, 2, \frac{n^2+n-2}{2}\}$.*

Proof. If $p_1 = 0$ $(p_2 = 0)$, $c^+(p_1,n) = 0$ $(c^-(p_2,n) = 0)$ and obviously the claim holds in this case. If $p_1 = 2$ $(p_2 = 2)$, $c^+(p_1,n) = 1$ $(c^-(p_2,n) = 1)$ and $w^+ = (2,0,\dots,0|-n,\dots,-n)$ $(w^- = (n,\dots,n|0,\dots,0,-2))$, so the claim holds again. If $p_1 = \frac{n^2+n-2}{2}$ $(p_2 = \frac{n^2+n-2}{2})$ then $c^+(p_1,n) = n-2$ $(c^-(p_2,n) = n-2)$ and the element in $B^+(p_1,n)$ $(B^-(p_2,n))$ which covers such a number of elements is $w^+ = (n, n-1, \dots, 2, 0|-n, \dots, -n)$ $(w^- = (n, \dots, n|0, -2, \cdots -n))$ and thus the first implication is proved.
One between $c_n(w)$, $s_{ij}(w)$, $cs_{nj}(w)$ and $sc_{i(n+1)}(w)$ is different from w if and only if $w_n \neq 0$ or $w_n = 0$ and the last entry greater than zero is equal to 1 and at the same time $w_{n+1} \neq 0$ or $w_{n+1} = 0$ and the first entry lesser than zero is equal to -1. If $p_1 = 1$ $(p_2 = 1)$ then obviously the assertion holds. If $3 \leq p_1 \leq \frac{n^2-n+2}{2}$ $(3 \leq p_2 \leq \frac{n^2-n+2}{2})$ then by taking $w \in O(m,n)$ such

that w^+ $\left(\overline{w^-}\right)$ is equal to $(c^++1+\varepsilon(p_1,c^+,0),c^++\varepsilon(p_1,c^+,1),\ldots,1+\varepsilon(p_1,c^+,m),\varepsilon(p_1,c^+,c^++1),0,\ldots,0|-n,\ldots,-n)$, where $c^+=c^+(p_1,n)$ and ε is defined as in the Lemma 4.4, we have that w^+ (w^-) covers exactly c^+ (c^-) elements, $w_n=0$ $(w_{n+1}=0)$ and the last (first) entry greater (lesser) than zero in w is equal to 1 (-1). If $\frac{n^2+n-2}{2}\le p_1\le n^2$ $(\frac{n^2+n-2}{2}<p_2\le n^2)$ then the element $v=\widetilde{w}$, where w is defined as before, is such that v^+ (v^-) covers $c^+(p_1,n)$ $(c^-(p_2,n))$ elements and $v_{n-1}\neq 0$ $(v_n\neq 0)$. Finally if $\frac{n^2-n+2}{2}<p_1<\frac{n^2+n-2}{2}$ then if w^+ $\left(\overline{w^-}\right)$ is equal to $(n,n-2+\xi(p_1,n,1),\ldots,1+\xi(p_1,n,n-2),\xi(p_1,n,n-1)|-n,\ldots,-n)$, where ξ is defined by

$$\xi(p,n,k)=\begin{cases}1 & \text{if } p-\frac{n^2-n+2}{2}\ge k\\ 0 & \text{elsewhere ,}\end{cases}$$

clearly satisfies all our requests and the lemma is proved. $\square$

Lemma 4.6. *If* $0\le m\le n^2$ *then*

$$c(m,n)=\max\{c^+(m_1,n)+c^-(m_2,n)+\delta(m,n):0\le m_2\le m_1\le n^2,m_1-m_2=m\}.$$

Proof. Let $w\in O(m,n)$, $w^+\in B^+(p_1,n)$ and $w^-\in B^-(p_2,n)$ as before. Obviously the number of elements that w covers is less than or equal to $c^+(p_1,n)+c^-(p_2,n)+1$. Thus

$$c(m,n)\le\max\{c^+(m_1,n)+c^-(m_2,n)+1:0\le m_2\le m_1\le n^2,m_1-m_2=m\}.$$

By Lemma 4.5 the equality holds if and only if there exists p_1 and p_2, with $0\le p_2\le p_1\le n^2$, $p_1-p_2=m$ such that $c^+(p_1,n)+c^-(p_2,n)=\max\{c^+(m_1,n)+c^-(m_2,n):0\le m_2\le m_1\le n^2,m_1-m_2=m\}$ and p_1,p_2 are different from 0, 2 and $\frac{n^2+n-2}{2}$. This is the case when $m\in\{0,\ldots n^2\}\setminus\{n^2,n^2-4,n-2\}$. $\square$

Proof of the Theorem 4.2
By Lemma 4.6 we have to find for each $m\in\{0,\ldots,n^2\}$ the values of m_1 and m_2, with $m_1-m_2=m$ that maximize the quantity $\{c^+(m_1,n)+c^-(m_2,n)+\delta(m,n)\}$.
If $n=3$ and $m=4$ then by Lemma 4.4 the value of $c^+(7,3)$ and $c^-(3,3)$ is maximum and it holds that $c^+(7,3)=c^-(3,3)=1$ and moreover $\delta(4,3)=1$. Thus by Lemma 4.6 $c(m,n)=3$. In what follows we can consider thus $n\neq 3$ or $m\neq 4$.
If $m\in\{0,1,n-2,n-1,n\}$ then we can choose m_1 and m_2 such that $m_1-m_2=m$ and $c^+(m_1,n)$ and $c^-(m_2,n)$ are maximum. By Lemma 4.4 they are equal to $n-2$ and so $c(m,n)=2n-4+\delta(m,n)$.
If $2\le m\le n-3$ or $n+1\le m\le 2n-1$ then we can choose m_1 and m_2 such that $c^+(m_1,n)$ and $c^-(m_2,n)$ are equal, one to $n-2$, the other to $n-3$ and so $c(m,n)=2n-5+\delta(m,n)$.
If $2n\le m\le n^2$ then a value close to the searched maximum can be found by taking values of m_1 and m_2 *symmetric* by respect to $\frac{n^2}{2}$. Therefore we consider

$$\overline{m_1}=\left\lceil\frac{n^2+m}{2}\right\rceil \quad\text{and}\quad \overline{m_2}=\overline{m_1}-m.$$

We set now

$$m_1=n^2-\frac{(c_1+1)(c_1+2)}{2}\quad\text{and}\quad m_2=m_1-m,$$

where $c_1:=c^+(\overline{m_1})=\left\lfloor\frac{1}{2}\left(-3+\sqrt{8(n^2-\overline{m_1})+1}\right)\right\rfloor$.
Let's note that

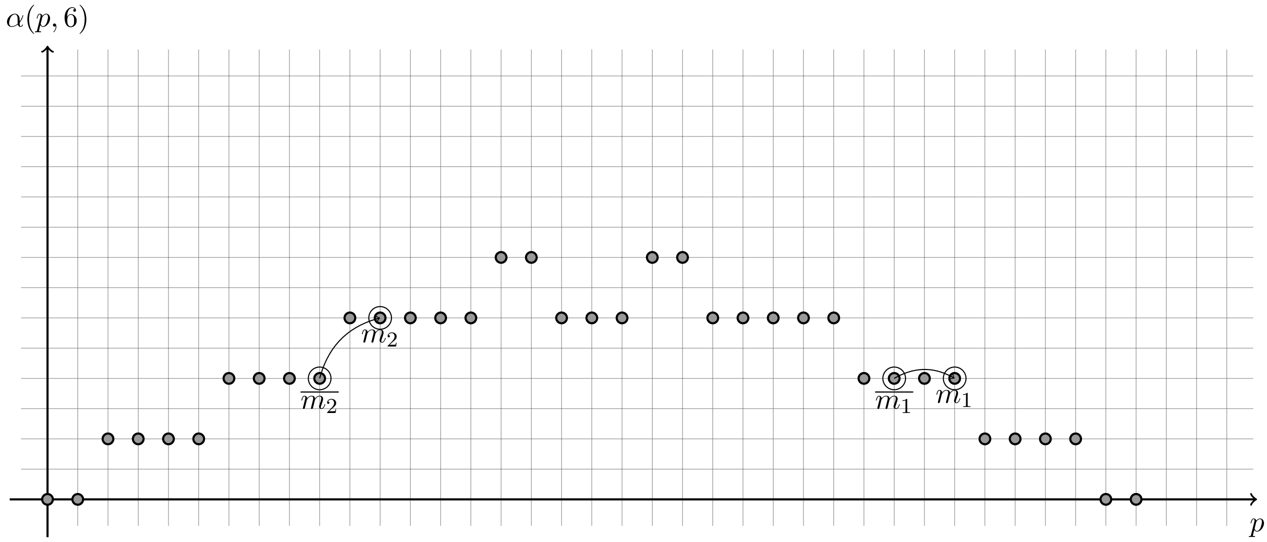
$$\begin{aligned}c^+(m_1,n)&=c^+\left(\frac{(c_1+1)(c_1+2)}{2},n\right)=\\&=\left\lfloor\frac{1}{2}\left(-3+\sqrt{4(c_1+1)(c_1+2)+1}\right)\right\rfloor=\\&=\left\lfloor\frac{1}{2}\left(-3+\sqrt{4c_1^2+12c_1+9}\right)\right\rfloor=\\&=\left\lfloor\frac{1}{2}\left(-3+2c_1+3\right)\right\rfloor=c_1.\end{aligned}$$

while since $m_2 = m_1 - m \geq \overline{m_2}$ and $m_2 \leq \frac{n^2-n+2}{2}$ we have

$$c^-(m_2, n) \geq c^-(\overline{m_2}, n).$$

This completes the proof of Theorem 4.2.

Example 4.7. Let us consider for example $n = 6$ and $m = 19$. In such case $\overline{m_1} = \lceil \frac{36+19}{2} \rceil = 28$, $\overline{m_2} = 28 - 19 = 9$. It follows $c_1 = 2$, $m_1 = 30$ and $m_2 = 11$.
Thus $c(m, n) = 2 + 3 + 1 = 6$.



5. Duality in $O(m, n)$

In this section we extend the concept of Brylawski duality to the lattice $O(m, n)$. Also in our case we show that the duality is an anti-automorphism of $O(m, n)$ and we deduce some important structural properties of the lattice $O(m, n)$.

Definition 5.1. *Let $w \in O(m, n)$. For each $1 \leq k \leq n$ let $d^-(w, k)$ be the number of w_i such that $w_i + k \leq 0$ and let $d^+(w, k)$ be the number of w_i such that $w_i - k \geq 0$. The* dual *or* conjugate *of w, that we denote by w^*, is the signed partition $(-d^-(w, n) + n, \ldots, -d^-(w, 1) + n | d^+(w, 1) - n, \ldots, d^+(w, n) - n)$. If $W \subseteq O(m, n)$ we set $W^* := \{w^* : w \in W\}$.*

If $w \in O(m, n)$, let $M^+(w)$ be the $n \times n$ 0-1 matrix with rows sum equal to the vector of the nonnegative part of w, $(w_1, \ldots, w_n)$ and no 0 lying left to a 1 and let $M^-(w)$ be the $n \times n$ 0-1 matrix with rows sum equal to the opposite of the vector of the non-positive part of w, $-(w_{n+1}, \ldots, w_{2n})$ and no 0 lying left to a 1. Note that $M^+(w^*) = {}^tM^-(w)$ and $M^-(w^*) = {}^tM^+(w)$. Hence $M^+(w^{**}) = {}^tM^-(w^*) = {}^t\big({}^tM^+(w)\big) = M^+(w)$, $M^-(w^{**}) = {}^tM^+(w^*) = {}^t\big({}^tM^-(w)\big) = M^-(w)$ and thus $w^{**} = w$.

Proposition 5.2. *Let $v, w \in O(m, n)$. Then w covers v if and only if v^* covers w^*. Hence duality is an anti-automorphism of $O(m, n)$.*

Proof. Let $u \in O(m, n)$ such that $u \trianglelefteq v^*$ and u covers w^*. By the claim and the transitive property it follows $w = w^{**} \triangleleft u^* \trianglelefteq v^{**} = v$ and since v covers w, we have $u^* = v$ and thus $u = u^{**} = v^*$, so v^* covers u^*. The reverse implication follows directly from the identity $w^{**} = w$.
Let v, w in $O(m, n)$ such that w covers v. In each case there exist two indices i and j such that $i < j$, $w_i = v_i + 1$ and $w_j = v_j - 1$.
Let us suppose $0 \leq i < j \leq n - 1$. Since $w_i = v_i + 1$, $w_j = v_j - 1$ and $w_k = v_k$ for each $k \in \{0, \ldots, 2n - 1\} \setminus \{i, j\}$ we have $d^+(w, w_i) > d^+(v, w_i)$, $d^+(w, w_j) < d^+(v, w_j)$, $d^+(w, w_k) = d^+(v, w_k)$ for each $k \in \{0, \ldots, n-1\} \setminus \{i, j\}$ and $d^-(w, w_k) = d^-(v, w_k)$ for each $k \in \{n, \ldots, 2n - 1\}$. Thus $w^*_{n-1+w_i} = d^+(w, w_i) - n > d^+(v, w_i) - n = v_{n-1+w_i}$, $w_{n-1+w_j} = d^+(w, w_j) - n <$

$d^+(v, w_j) - n = v_{n-1+w_j}$, and $w_k^* = v_k^*$ for each $k \in \{0, \dots, 2n-1\} \setminus \{n-1+w_i, n-1+w_j\}$. Moreover $i < j$ implies $w_i > w_j$. Hence $w \lhd v$. In a similar way we prove the same result when $0 \le i \le n-1 < j \le 2n-1$ or $n \le i < j \le 2n-1$. The proposition is proved. □

Corollary 5.3. *$v \vee w = (v^* \wedge w^*)^*$.*

In [11] (Proposition 2.11) a class of particular types of sublattices is examined by virtue of their specific symmetry properties. In our context we can describe a similar situation as follows. If $0 \le k \le n$ we set

$$^kO(m,n) := \{w \in O(m,n) : w_1 = k\},$$

$$^k\hat{1}^{(m,n)} := \begin{cases} (k, \dots, k, r_1, 0, \dots, 0| -n, \dots, -n) & \text{if} \quad k - n^2 \le m \le kn - n^2 \\ (k, \dots, k|0, \dots, 0, -r_2, -n, \dots, -n) & \text{if} \quad kn - n^2 \le m \le kn \end{cases},$$

where $r_1 = m + n^2 - \left\lfloor \frac{m+n^2}{k} \right\rfloor k$, $r_2 = -(kn-m) + \left\lfloor \frac{kn-m}{n} \right\rfloor n$,

$$^k\hat{0}^{(m,n)} := \begin{cases} (k, 0, \dots, 0| -h_1, \dots, -h_1, -(h_1+1), \dots, -(h_1+1)) & \text{if} \quad k - n^2 \le m \le k \\ (k, h_2+1, \dots, h_2+1, h_2, \dots, h_2|0, \dots, 0) & \text{if} \quad k \le m \le kn \end{cases}$$

where $h_1 = \left\lfloor \frac{k-m}{n} \right\rfloor$, $h_2 = \left\lfloor \frac{m-k}{n-1} \right\rfloor$, and also

$$_kO(m,n) := \{w \in O(m,n) : w_{2n} = -k\},$$

$$_k\hat{1}^{(m,n)} := \begin{cases} (n, \dots, n, R_1, 0, \dots, 0| -k, \dots, -k) & \text{if} \quad -kn \le m \le n^2 - kn \\ (n, \dots, n|0, \dots, 0, -R_2, -k, \dots, -k) & \text{if} \quad n^2 - kn \le m \le n^2 - k \end{cases},$$

where $R_1 = m + kn - \left\lfloor \frac{m+kn}{n} \right\rfloor n$, $R_2 = n^2 - m + \left\lfloor \frac{n^2-m}{k} \right\rfloor k$,

$$_k\hat{0}^{(m,n)} := \begin{cases} (0, \dots, 0| -H_1, \dots, -H_1, -(H_1+1), \dots, -(H_1+1), -k) & \text{if} \quad -kn \le m \le -k \\ (H_2+1, \dots, H_2+1, H_2, \dots, H_2|0, \dots, 0, -k) & \text{if} \quad -k \le m \le n^2 - k \end{cases}$$

where $H_1 = \left\lfloor \frac{-k-m}{n-1} \right\rfloor$, $H_2 = \left\lfloor \frac{m+k}{n} \right\rfloor$.

Proposition 5.4. *(i) $^kO(m,n)$ is a sublattice of $O(m,n)$ having maximum $^k\hat{1}^{(m,n)}$ and minimum $^k\hat{0}^{(m,n)}$.*
(ii) $_kO(m,n)$ is a sublattice of $O(m,n)$ having maximum $_k\hat{1}^{(m,n)}$ and minimum $_k\hat{0}^{(m,n)}$.

Proof. It is easy to see that if $w \in {}^kO(m,n)$ ($w \in {}_kO(m,n)$) then $^k\hat{0}^{(m,n)} \unlhd w \unlhd {}^k\hat{1}^{(m,n)}$ ($_k\hat{0}^{(m,n)} \unlhd w \unlhd {}_k\hat{1}^{(m,n)}$). Now by definition of infimum and supremum, if $w, w' \in {}^kO(m,n)$ ($w, w' \in {}_kO(m,n)$), then $^k\hat{0}^{(m,n)} \unlhd w \wedge w' \unlhd w$ ($_k\hat{0}^{(m,n)} \unlhd w \wedge w' \unlhd w$) and $w \unlhd w \vee w' \unlhd {}^k\hat{1}^{(m,n)}$ ($w \unlhd w \vee w' \unlhd {}_k\hat{1}^{(m,n)}$). Thus both $w \wedge w'$ and $w \vee w'$ are in $^kO(m,n)$ ($_kO(m,n)$). □

In order to determine the dual of the previous sublattices, we set

$$O^k(m,n) := \{w \in O(m,n) : w_{n+k-1} > w_{n+k} = -n\},$$

$$\hat{1}^{k,(m,n)} := \begin{cases} (n, \dots, n, s_1, 0, \dots, 0|1-n, \dots, 1-n, -n, \dots, -n) & \text{if} \quad k - n^2 \le m \le k \\ (n, \dots, n|0, \dots, 0, -s_2, 1-n, \dots, 1-n, -n, \dots, -n) & \text{if} \quad k \le m \le kn \end{cases},$$

where s_1 is the remainder of the division between $m - k + n^2$ and n, while s_2 is the remainder of the division between $kn - m$ and $n - 1$,

$$\hat{0}^{k,(m,n)} := \begin{cases} (0, \dots, 0| -q_1, \dots, -q_1, -q_1 - 1, \dots, -q_1 - 1, -n, \dots, -n) & \text{if} \quad k - n^2 \le m \le kn - n^2 \\ (q_2+1, \dots, q_2+1, q_2, q_2|k-n, \dots, k-n) & \text{if} \quad kn - n^2 \le m \le kn \end{cases},$$

where $q_1 = \left\lfloor \frac{kn-n^2-m}{n-k} \right\rfloor$ and $q_2 = \left\lfloor \frac{m-kn+n^2}{n} \right\rfloor$, and also

$$O_k(m,n) := \{w \in O(m,n) : n = w_{n-k-1} > w_{n-k}\},$$

$$\hat{1}_k^{(m,n)} := \begin{cases} (n, \dots, n, n-1, \dots, n-1, S_1, 0, \dots, 0| -n, \dots, -n) & \text{if} \quad -kn \le m \le -k \\ (n, \dots, n, n-1, \dots, n-1|0, \dots, 0, -S_2, -n, \dots, -n) & \text{if} \quad -k \le m \le n^2 - k \end{cases},$$

where S_1 is the remainder of the division between $m+kn$ and $n-1$ and S_2 is the remainder of the division between n^2-k+m and n,

$$\hat{0}_k^{(m,n)} := \begin{cases} (n-k,\dots,n-k|Q_1,\dots,Q_1,Q_1+1,\dots,Q_1+1) & \text{if} \quad -kn \le m \le n^2-kn \\ (n,\dots,n,Q_2+1,\dots,Q_2+1,Q_2,\dots,Q_2|0,\dots,0) & \text{if} \quad n^2-kn \le m \le n^2-k \end{cases},$$

where $Q_1 = \left\lfloor \frac{n^2-kn-m}{n} \right\rfloor$ and $Q_2 = \left\lfloor \frac{m-n^2+kn}{n-k} \right\rfloor$.

Let us note that if $-n^2 \le m < k$ or $kn < m \le n^2$ the subsets ${}^kO(m,n)$ and $O^k(m,n)$ are empty, while if $-n^2 \le m < -kn$ or $n^2-k < m \le n^2$ the subsets ${}_kO(m,n)$ and $O_k(m,n)$ are empty.

Proposition 5.5. *(i) $O^k(m,n)$ is a sublattice of $O(m,n)$ having maximum $\hat{1}^{k,(m,n)}$ and minimum $\hat{0}^{k,(m,n)}$.*
(ii)$O_k(m,n)$ is a sublattice of $O(m,n)$ having maximum $\hat{1}_k^{(m,n)}$ and minimum $\hat{0}_k^{(m,n)}$.
(iii) $O^k(m,n) = ({}^kO(m,n))^*$ and $O_k(m,n) = ({}_kO(m,n))^*$.
(iv) The following identities hold: ${}^k_pO(m,n) = {}^kO(m,n) \cap {}_pO(m,n)$, ${}^kO_p(m,n) = {}^kO(m,n) \cap O_p(m,n)$, ${}^kO^p(m,n) = {}^kO(m,n) \cap O^p(m,n)$, ${}_kO^p(m,n) = {}_kO(m,n) \cap O^p(m,n)$, ${}_kO_p(m,n) = {}_kO(m,n) \cap O_p(m,n)$ and $O^k_p(m,n) = O^k(m,n) \cap O_p(m,n)$.

Proof. Let us note that $\hat{1}^{k,(m,n)} = \left({}^k\hat{0}^{(m,n)}\right)^*$, $\hat{0}^{k,(m,n)} = \left({}^k\hat{1}^{(m,n)}\right)^*$, $\hat{1}_k^{(m,n)} = \left({}_k\hat{0}^{(m,n)}\right)^*$ and $\hat{0}_k(m,n) = \left({}_k\hat{1}^{(m,n)}\right)^*$. Now let us observe that $w \in ({}^kO(m,n))^*$ if and only if $w_1 = k$, that is $d^+(w,j) = 0\, \forall j > k$ and $d^+(w,k) > 0$. This is equivalent to $w^*_{n+k-1} > w^*_{n+k} = -n$, and thus to $w^* \in O^k(m,n)$. This proves (iii).
Parts (i) and (ii) follow directly by part (iii) and Proposition 5.2. The identities in part (iv) are a direct consequence of the definitions. □

The following lemma is necessary in order to describe a fundamental structural property of the lattice $O(m,n)$.

Lemma 5.6. *Let $A = \{w_1,\dots,w_r\} \subset O(m,n)$ be a set of signed partitions. Then the following are equivalent:*

(i) *A has a common pairwise infimum;*
(ii) *for each $l \in \{1,\dots,2n\}$ there exists $s \in \{1,\dots r\}$ such that*

$$\sum_{k=0}^{l} w_{i,k} = \sum_{k=1}^{l} w_{j,k} \le \sum_{k=1}^{l} w_{s,k} \text{ for each } i,j \in \{1,\dots r\} \setminus \{s\}; \tag{10}$$

(iii) *for each $l \in \{1,\dots,2n\}$ there exists $s \in \{1,\dots r\}$ such that*

$$\sum_{k=l}^{2n-1} w_{i,k} = \sum_{k=l}^{2n} w_{j,k} \ge \sum_{k=l}^{2n} w_{s,k} \text{ for each } i,j \in \{1,\dots r\} \setminus \{s\}. \tag{11}$$

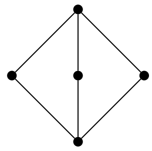
Proof. The equivalence of (ii) and (iii) is obvious since $\sum_{k=1}^{2n} w_{i,k} = m$, for each $i \in \{1,\dots,r\}$. Let us prove now (i)$\Longrightarrow$(ii). Let us assume (ii) false. Let l be the minimum number in $\{1,\dots,2n\}$ such that (10) is false. So there exist s_0, s_1 and s_2 in $\{1,\dots,r\}$ such that

$$\sum_{k=1}^{l} w_{s_0,k} < \sum_{k=1}^{l} w_{s_1,k} \le \sum_{k=1}^{l} w_{s_2,k}.$$

It is easy to see that $\beta := w_{s_1} \wedge w_{s_2} \not\trianglelefteq w_{s_0}$, so that A has not a common pairwise infimum. Finally we prove (ii) $\Longrightarrow$(i). Let i,j two different integers in $\{1,\dots,r\}$. Condition (ii) implies $\omega\left(\Lambda(w_i),\Lambda(w_j)\right) := \omega\left(\Lambda(w_1),\Lambda(w_2)\right)$, so $w_i \wedge w_j = w_1 \wedge w_2$ and thus A has a common pairwise infimum. □

The following result generalizes in our case the classical Proposition 2.13 of [11].

Proposition 5.7. *In $O(m,n)$ there are not sublattices of the form:*



Proof. Let w_1, w_2 and w_3 be distinct signed partitions with a common pairwise infimum and let $l \in \{0, \ldots, 2n-1\}$ be the largest integer such that $\sum_{k=l}^{2n-1} w_{i,k}$ are not all equal. Hence by the previous lemma we may assume that

$$\sum_{k=l}^{2n-1} w_{1,k} < \sum_{k=l}^{2n-1} w_{2,k} \leq \sum_{k=l}^{2n-1} w_{3,k}$$

and

$$\sum_{k=j}^{2n-1} w_{1,k} < \sum_{k=j}^{2n-1} w_{2,k} \leq \sum_{k=j}^{2n-1} w_{3,k} \text{ for all } j > l.$$

Thus $w_{1,l} < w_{2,l} \leq w_{3,l}$, while $w_{1,j} = w_{2,j} = w_{3,j}$ for all $j > l$.
If $l \leq n-1$, then we have

$$d^-(w_1, k) = d^-(w_2, k) = d^-(w_3, k) \qquad \forall k \in \{1, \ldots, n\},$$
$$d^+(w_1, k) = d^+(w_2, k) = d^+(w_3, k) \qquad \forall k < w_{1,l},$$

and if $w_{1,l} \geq 1$,

$$d^+(w_1, w_{1,l}) < d^+(w_2, w_{1,l}) = d^+(w_3, w_{1,l}) = l + 1.$$

So $w^*_{1,j} = w^*_{2,j} = w^*_{3,j}$ $\forall j \in \{0, \ldots, h\}$, where $h := \max\{n-1, n + w_{1,l} - 2\}$, while $w^*_{1,h+1} < w^*_{2,h+1} = w^*_{3,h+1} = l + 1 - n$. Hence by lemma 5.6 w^*_1, w^*_2, w^*_3 have not a common infimum and so w_1, w_2, w_3 have not a common supremum.
If $n \leq l \leq 2n-1$, then we have

$$d^-(w_1, k) = d^-(w_2, k) = d^-(w_3, k) \qquad \forall k > -w_{1,l},$$
$$d^-(w_1, -w_{1,l}) > d^-(w_2, -w_{1,l}) \geq d^-(w_3, -w_{1,l}) = 2n - l - 1,$$

so

$$n - d^-(w_1, -w_{1,l}) < n - d^-(w_2, -w_{1,l}) \leq n - d^-(w_3, -w_{1,l}).$$

Thus $w^*_{1,j} = w^*_{2,j} = w^*_{3,j}$ $\forall j \in \{0, \ldots, h\}$, where $h := n + w_{1,l} - 1$, while $w^*_{1,h+1} < w^*_{2,h+1} = w^*_{3,h+1} = n + l + 1$. Hence by lemma 5.6 w^*_1, w^*_2, w^*_3 have not a common infimum and so w_1, w_2, w_3 have not a common supremum. This completes the proof. $\square$

6. Local Structure of $O(m, n)$

Let w, w' and w'' be three signed partitions in $O(m, n)$ such that $w' \lessdot w$, $w'' \lessdot w$ and $w' \neq w''$. Let $v = w' \wedge w''$. We call $[v, w]$ a *local interval* in $O(m, n)$. In this section we completely characterize the structure of the local intervals in $O(m, n)$. Moreover, for each given structure of a local interval, we also describe all the possible evolution rules that determine such a structure. We call *form* of the local interval $[v, w]$, the complete description of its structure and of the evolution rules which determine such a structure.
If I is any local interval in $O(m, n)$, we will prove that the possible structures for I are of four types (as in the Brylawski case), but the different possible forms for I are 61 (whereas in the Brylawski case are 9).
By virtue of Theorem 3.5, when $w, w' \in O(m, n)$ and w covers w' we can put $\Delta(w', w) = (i, j)$, where i and j are exactly the two indexes in $\{0, 1, \ldots, 2n-1\}$ such that either $j = i + 1$ and $w' = c_i(w)$ or $j - i \geq 2$ and it holds exactly one of the following: $w' = s_{ij}(w)$, $w' = cs_{ij}(w)$, $w' = sc_{ij}(w)$. When $w' \lessdot w$ we say that w' is a *cocover* of w.
In what follows we take w' and w'' two distinct cocovers of w in $O(m, n)$, where $\Delta(w', w) = (i, j)$ and $\Delta(w'', w) = (k, l)$. Since w' and w'' are two different cocovers of w we have $i \neq k$, so we can suppose $i < k$. There are three possible cases:

$j < k$: w' and w'' are called *nonoverlapping* cocovers of w;

$j > k$: w' and w'' are called *fully overlapping* cocovers of w;
$j = k$: w' and w'' are called *partially overlapping* cocovers of w.
We will study now the interval $I = [v, w]$ in $O(m, n)$ where $v = w' \wedge w''$.

Proposition 6.1. *If w' and w'' are nonoverlapping cocovers of w then I has the following form:*

$$\begin{tikzcd} & w \arrow[dl, dash, "f"'] \arrow[dr, dash, "g"] & \\ w' \arrow[dr, dash, "g"'] & & w'' \arrow[dl, dash, "f"] \\ & v & \end{tikzcd}$$

where f (g) is one evolution rule between c_i, s_{ij}, cs_{ij}, sc_{ij} (c_k, s_{kl}, cs_{kl}, sc_{kl}). Thus there are 16 *possibles cases.*

Proof. Since $j < k$ we can write:

$$\begin{array}{ccccccccccc} w & = & \dots, & w_i, & \dots, & w_j, & \dots, & w_k, & \dots, & w_l, & \dots \\ w' & = & \dots, & w_i - 1, & \dots, & w_j + 1, & \dots, & w_k, & \dots, & w_l, & \dots \\ w'' & = & \dots, & w_i, & \dots, & w_j, & \dots, & w_k - 1, & \dots, & w_l + 1, & \dots \end{array}$$

and by Theorem 2.6, it is easy to see that

$$v = \dots, w_i - 1, \dots, w_j + 1, \dots, w_k - 1, \dots, w_l + 1, \dots.$$

We note first that w' and w'' cover v and second that the evolution rule between w and w'' is the same of that between w' and v. We have a similar situation for the other pairs (w, w') and (w'', v). □

Lemma 6.2. *If w' and w'' are fully overlapping cocovers of w, i.e. $j > k$, then necessarily $l > j = k + 1$ and $j \neq n$.*

Proof. Since $i < k < j$ then $j - i \geq 2$. So the case $w' = c_i(w)$ cannot happen. Therefore we have the following situation:

$$w_i - 1 \geq w_{i+1} = \dots = w_k = w_{k+1} = \dots = w_{j-1} \geq w_j + 1. \tag{12}$$

Since $\Delta(w'', w) = (k, l)$ it holds $w_k > w_{k+1}$. Thus $k = j - 1$. For the second part we note that if $j = n$ we have $w_k = 0$ and this is in contradiction with the assumption that $\Delta(w'', w) = (k, l)$. □

Proposition 6.3. *If w' and w'' are fully overlapping cocovers of w then the form of the interval I is exactly one of the types listed in the table (f.o. table).*

Proof. By lemma 6.2, we can write:

$$\begin{array}{cccccccccc} w & = & \dots, & w_i, & \dots, & w_k, & w_j, & \dots, & w_l, & \dots \\ w' & = & \dots, & w_i - 1, & \dots, & w_k, & w_j + 1, & \dots, & w_l, & \dots \\ w'' & = & \dots, & w_i, & \dots, & w_k - 1, & w_j, & \dots, & w_l + 1, & \dots \end{array}$$

and it holds that $j - i \geq 2$, $l - k \geq 2$. Thus we have

$$v \;=\; \dots, \;\; w_i - 1, \;\; \dots, \;\; w_k, \;\; w_j, \;\; \dots, \;\; w_l + 1, \;\; \dots.$$

By assumption that w covers both w' and w'' and since $j \neq n$ (equivalently $k \neq n - 1$), we have:

$$w_i - 1 \geq w_{i+1} = \dots = w_{j-1} = w_j + 1$$

and

$$w_k - 1 = w_{k+1} = \dots = w_{l-1} \geq w_l + 1.$$

If $w_i - 1 = w_{i+1}$ then $w' = s_{ij}(w)$, while if $w_i - 1 > w_{i+1}$ then $i = n - 1$ and $w' = cs_{ij}(w)$. Similarly if $w_{l-1} = w_l + 1$ then $w'' = s_{kl}(w)$, while if $w_{l-1} > w_l + 1$ then $l = n$ and $w'' = sc_{kl}(w)$. It is obvious that $w' = cs_{ij}(w)$ and $w'' = sc_{kl}(w)$ cannot happen at the same time.
Let us compare now w' and w'' with v. First note that w' and v are different only in the positions j and l, while w'' and v in the positions i and k. It holds that:

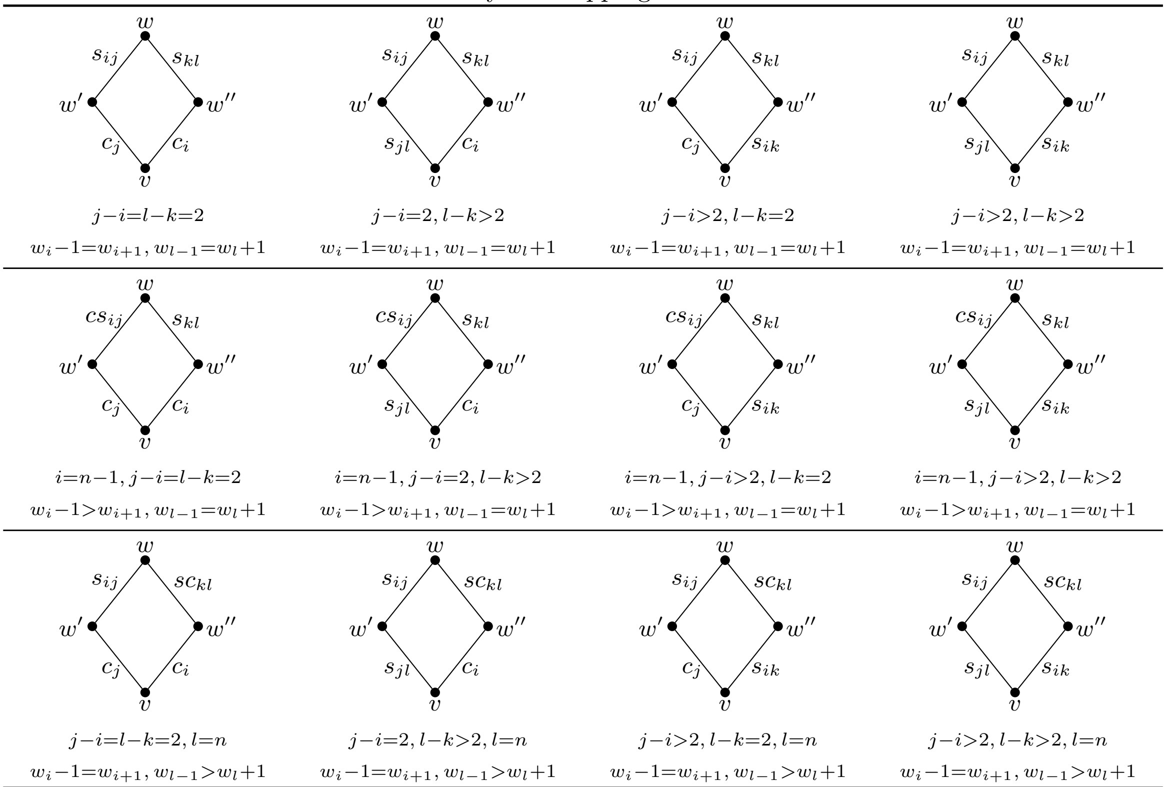
$$w'_j - 1 = w_j = w'_{j+1} = \dots = w'_{l-1} \geq w'_l + 1$$

and

$$w''_i - 1 = w_i - 1 \geq w''_{i+1} = \cdots = w''_{k-1} = w''_k + 1.$$

Note now that $v_j = w'_j - 1$, $v_l = w'_l + 1$, $v_i = w''_i - 1$ and $v_k = w''_k + 1$, so w' and w'' cover v. All the possible forms are listed in the Fully Overlapping Table. □

Fully Overlapping Table



Proposition 6.4. *If w' and w'' are partially overlapping cocovers of w then the form of the interval I is exactly one of the types listed in the table (Partially Overlapping Table).*

Proof. In this case $j = k$ and

$$\begin{array}{cclcccccccc} w & = & \ldots, & w_i, & \ldots, & w_{j-1}, & w_j, & w_{j+1} & \ldots, & w_l, & \ldots \\ w' & = & \ldots, & w_i - 1, & \ldots, & w_{j-1}, & w_j + 1, & w_{j+1} & \ldots, & w_l, & \ldots \\ w'' & = & \ldots, & w_i, & \ldots, & w_{j-1}, & w_j - 1, & w_{j+1} & \ldots, & w_l + 1, & \ldots \end{array}$$

Thus we have

$$v \;=\; \ldots,\; w_i - 1,\; \ldots,\; w_{j-1},\; w_j,\; w_{j+1}\; \ldots,\; w_l + 1,\; \ldots$$

By assumption that w covers both w' and w'' it holds:

$$w_i - 1 \geq w_{i+1} = \cdots = w_{j-1} \geq w_j + 1 \tag{13}$$

and

$$w_j - 1 \geq w_{j+1} = \cdots = w_{l-1} \geq w_l + 1. \tag{14}$$

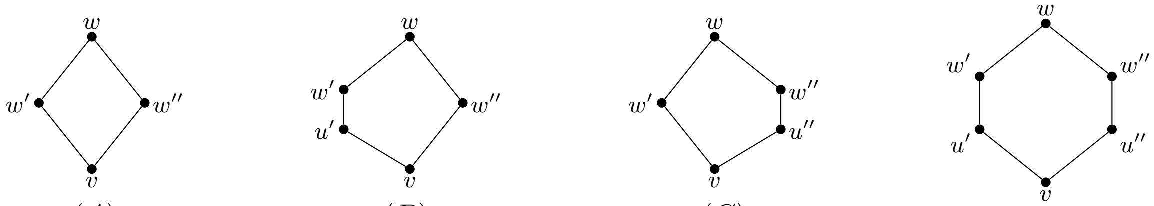
Let us note that, if $j = i + 1$ then (13) becomes $w_i - w_j \geq 2$ and $w' = c_i(w)$. Similarly if $l = j + 1$ then (14) becomes $w_l - w_j \geq 2$ and $w'' = c_j(w)$. Moreover we have $w' = s_{ij}(w)$, $w' = cs_{ij}(w)$ or $w' = sc_{ij}(w)$ when $j - i \geq 2$ and in (13) the inequalities are respectively $(=, =)$, $(>, =)$ or $(=, >)$ and likewise for the covering $w'' \lessdot w$. It is obvious that the cases $w' = cs_{ij}(w)$ and $w' = sc_{ij}(w)$ are incompatible with cases $w'' = cs_{jl}(w)$ and $w'' = sc_{jl}(w)$.
Let us compare now w' and w'' with v. First note that w' and v are different only in the positions j and l, while w'' and v in the positions i and j. It holds that:

$$w'_j - 1 = w_j > w'_{j+1} = \cdots = w'_{l-1} \geq w_l + 1 = w'_l + 1 \tag{15}$$

and

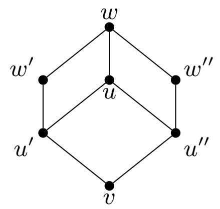
$$w_i'' - 1 = w_i - 1 \geq w_{i+1}'' = \cdots = w_{j-1}'' \geq w_j + 1 > w_j'' + 1. \tag{16}$$

The presence of strict inequalities in (15) and (16) implies that it is possible that w' and w'' do not cover v. For example if $w_j' - 1 = w_j > w_{j+1}' = \cdots = w_{l-1}' = w_l + 1 = w_l' + 1$, with $j \neq n-1$, then $s_{jl}(w) = w''$, $w' \gtrdot u' := c_j(w') = \ldots, w_i - 1, \ldots, w_{j-1}, w_j, w_{j+1} + 1, \ldots, w_l, \ldots$. Therefore $u' \rhd v$. Note that u' and v are different only in the positions $j+1$ and l and we have w' covers v, in particular if $l - j = 2$ then $v = c_{j+1}(u')$, while if $l - j > 2$ then $v = s_{(j+1)l}(u')$. In a similar way we have that either w' covers v or there exists u'' in $O(m,n)$ such that $v \lessdot u'' \lessdot w''$, with $u'' = c_{j-1}(w'')$. So there are 4 possible structures of I:



(A) (B) (C) (D)

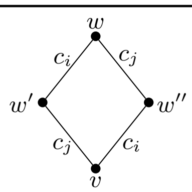
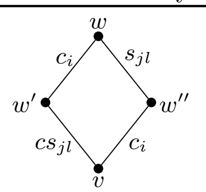
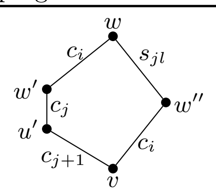
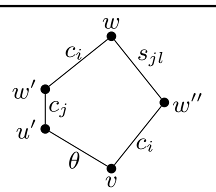
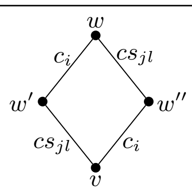
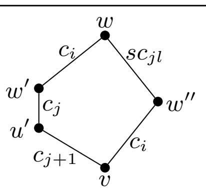
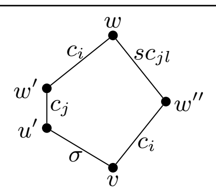
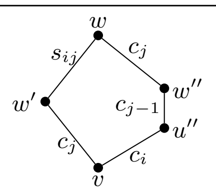
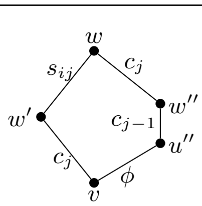
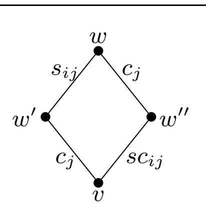
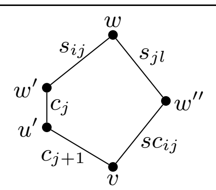
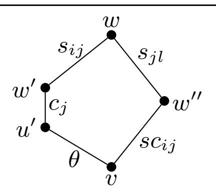
In case (D) by taking $u = s_{(j-1)(j+1)}(w)$ it is easy to see that u covers both u' and u'', so (D) becomes

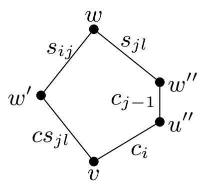


(D')

All the possible forms are listed in the Partially Overlapping Table, where $\varphi = s_{(j-1)(j+1)}$, $\phi = s_{i(j-1)}$, $\psi = cs_{i(j-1)}$, $\theta = s_{(j+1)l}$ and $\sigma = sc_{(j+1)l}$. $\square$

Partially Overlapping Table

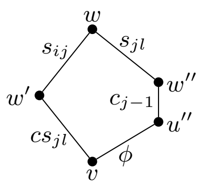
 $j+1=j=l-1$	 $i+1=j=n-1, l-j\geq 2$ $w_j-1=w_{j+1}=w_{l-1}=w_l+1$	 $i+1=j\neq n-1, l-j=2$ $w_j-1=w_{j+1}=w_{l-1}=w_l+1$	 $i+1=j\neq n-1, l-j>2$ $w_j-1=w_{j+1}=w_{l-1}=w_l+1$
 $i+1=j=n-1, l-j\geq 2$ $w_j-1>w_{j+1}=w_{l-1}=w_l+1$	 $i+1=j, l=n, l-j=2$ $w_j-1=w_{j+1}=w_{l-1}>w_l+1$	 $i+1=j, l=n, l-j>2$ $w_j-1=w_{j+1}=w_{l-1}>w_l+1$	 $j-i=2, j\neq n, j+1=l$ $w_i-1=w_{i+1}=w_{j-1}=w_j+1$
 $j-i>2, j\neq n, j+1=l$ $w_i-1=w_{i+1}=w_{j-1}=w_j+1$	 $j-i\geq 2, j=n, j+1=l$ $w_i-1=w_{i+1}=w_{j-1}=w_j+1$	 $j-i\geq 2, j=n, l-j=2$ $w_i-1=w_{i+1}=w_{j-1}=w_j+1$ $w_j-1=w_{j+1}=w_{l-1}=w_l+1$	 $j-i\geq 2, j=n, l-j>2$ $w_i-1=w_{i+1}=w_{j-1}=w_j+1$ $w_j-1=w_{j+1}=w_{l-1}=w_l+1$



$j-i=2, j=n-1, l-j\geq 2$

$w_i-1=w_{i+1}=w_{j-1}=w_j+1$

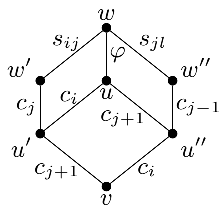
$w_j-1=w_{j+1}=w_{l-1}=w_l+1$



$j-i>2, j=n-1, l-j\geq 2$

$w_i-1=w_{i+1}=w_{j-1}=w_j+1$

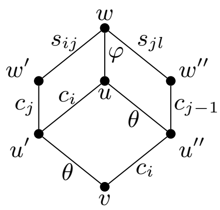
$w_j-1=w_{j+1}=w_{l-1}=w_l+1$



$j-i=l-j=2, j\neq n-1, j\neq n$

$w_i-1=w_{i+1}=w_{j-1}=w_j+1$

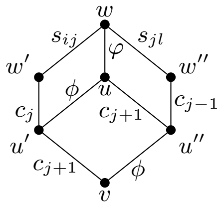
$w_j-1=w_{j+1}=w_{l-1}=w_l+1$



$j-i=2, l-j>2, j\neq n-1, j\neq n$

$w_i-1=w_{i+1}=w_{j-1}=w_j+1$

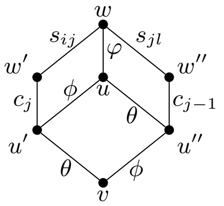
$w_j-1=w_{j+1}=w_{l-1}=w_l+1$



$j-i>2, l-j=2, j\neq n-1, j\neq n$

$w_i-1=w_{i+1}=w_{j-1}=w_j+1$

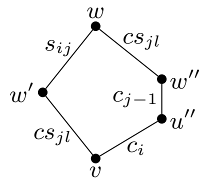
$w_j-1=w_{j+1}=w_{l-1}=w_l+1$



$j-i>2, l-j>2, j\neq n-1, j\neq n$

$w_i-1=w_{i+1}=w_{j-1}=w_j+1$

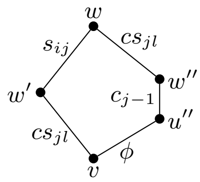
$w_j-1=w_{j+1}=w_{l-1}=w_l+1$



$j-i=2, j=n-1, l-j\leq l$

$w_i-1=w_{i+1}=w_{j-1}=w_j+1$

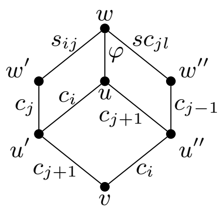
$w_j-1>w_{j+1}=w_{l-1}=w_l+1$



$j-i>2, j=n-1, l-j\leq l$

$w_i-1=w_{i+1}=w_{j-1}=w_j+1$

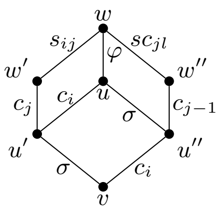
$w_j-1>w_{j+1}=w_{l-1}=w_l+1$



$j-i=l-j=2, j\neq n-1, j\neq n$

$w_i-1=w_{i+1}=w_{j-1}=w_j+1$

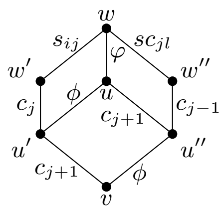
$w_j-1=w_{j+1}=w_{l-1}>w_l+1$



$j-i=2, l-j>2, j\neq n-1, j\neq n$

$w_i-1=w_{i+1}=w_{j-1}=w_j+1$

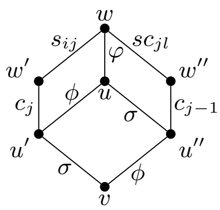
$w_j-1=w_{j+1}=w_{l-1}>w_l+1$



$j-i>2, l-j=2, j\neq n-1, j\neq n$

$w_i-1=w_{i+1}=w_{j-1}=w_j+1$

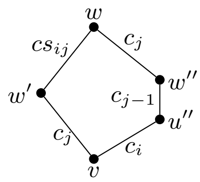
$w_j-1=w_{j+1}=w_{l-1}>w_l+1$



$j-i>2, l-j>2, j\neq n-1, j\neq n$

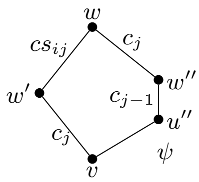
$w_i-1=w_{i+1}=w_{j-1}=w_j+1$

$w_j-1=w_{j+1}=w_{l-1}>w_l+1$



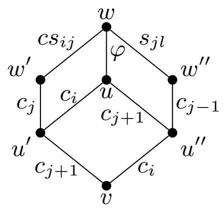
$j-i=2, j\neq n, j+1=l$

$w_i-1>w_{i+1}=w_{j-1}=w_j+1$



$j-i>2, j\neq n, j+1=l$

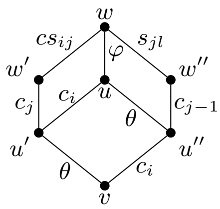
$w_i-1>w_{i+1}=w_{j-1}=w_j+1$



$j-i=l-j=2, j\neq n-1, j\neq n$

$w_i-1>w_{i+1}=w_{j-1}=w_j+1$

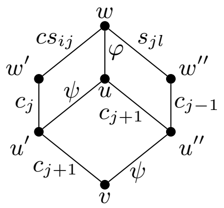
$w_j-1=w_{j+1}=w_{l-1}=w_l+1$



$j-i=2, l-j>2, j\neq n-1, j\neq n$

$w_i-1>w_{i+1}=w_{j-1}=w_j+1$

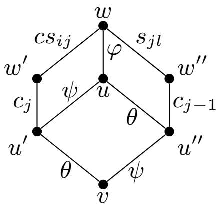
$w_j-1=w_{j+1}=w_{l-1}=w_l+1$



$j-i>2, l-j=2, j\neq n-1, j\neq n$

$w_i-1>w_{i+1}=w_{j-1}=w_j+1$

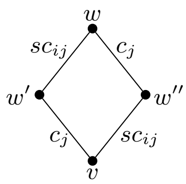
$w_j-1=w_{j+1}=w_{l-1}=w_l+1$



$j-i>2, l-j>2, j\neq n-1, j\neq n$

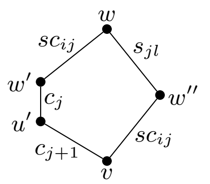
$w_i-1>w_{i+1}=w_{j-1}=w_j+1$

$w_j-1=w_{j+1}=w_{l-1}=w_l+1$



$j-i\geq 2, j=n, j+1=l$

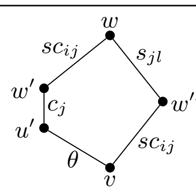
$w_i-1=w_{i+1}=w_{j-1}>w_j+1$



$j-i\geq 2, j=n, l-j=2$

$w_i-1=w_{i+1}=w_{j-1}>w_j+1$

$w_j-1=w_{j+1}=w_{l-1}=w_l+1$



$j-i\geq 2, j=n, l-j>2$

$w_i-1=w_{i+1}=w_{j-1}>w_j+1$

$w_j-1=w_{j+1}=w_{l-1}=w_l+1$

Corollary 6.5. *Let $u \lhd w$ be two signed partitions in $O(m,n)$ and let p, q respectively the minimal and the maximal length of a saturated chain from u to w in $O(m,n)$. Then for each $i \in \{p, p+1 \ldots, q\}$ then there exists a saturated chain from u to w of length i.*

Proof. We start from w and we take two any cocovers w' and w'' of w. Let $v = w' \wedge w''$ and $I = [v, w]$. From the previous three propositions, we know that the possible structures for I are (A), (B), (C) or (D'). The proof follows then exactly as in [11]. □

7. Overlapping Paths in $O(m,n)$ and Möbius Function

In this section we determine the Möbius Function of our lattice $O(m,n)$. Let w be a fixed signed partition of $O(m,n)$. At first we introduce in $O(m,n)$ particular types of subsequences which are associated with given sequences of "overlapping" cocovers of w. In the case of the usual integer partitions, these subsequences are called *overlapping paths* of w and they were introduced and studied in [11]. We recall at first the definition of overlapping path as given in [11].

Definition 7.1. *Let $S = (w^1, w^2, \ldots, w^r)$ be a sequence of cocovers of w, with $\Delta(w^l, w) = (i_l, j_l)$ for $l = 1, 2, \ldots, r$. Assume further that for all $l \in \{1, 2, \ldots, r-1\}$, w^l and w^{l+1} are (partially or fully) overlapping cocovers of w. Then we associate to S the subsequence of parts of w $P = (w_{i_1}, w_{i_1+1}, \ldots, w_{j_r})$. We call P the* overlapping path *of w associated to S.*

In our model the characterization of the overlapping paths is more complicated with respect to the classical Brylawski case. This greater difficulty is due to the presence of the negative summands in w, which produce the new evolution rules cs_{ij} and sc_{ij}, not present in the Brylawski model. In order to explain as these rules act on the overlapping paths, we establish the next two results. In the following propositions, we use the same notations as in Definition 7.1.

Proposition 7.2. *Let P be the overlapping path of w associated to S. If $k \in \{i_1, i_1+1, \ldots, j_r - 1\}$ and $w_k = w_{k+1}$ then:*

a) *$i_1 < k < j_r - 1$;*
b) *either $w_{k-1} - w_k \leq 1$ or $w_{k-1} - w_k \geq 2$, $w_k = 0$, $k = n+1$;*
c) *either $w_{k+1} - w_{k+2} \leq 1$ or $w_{k+1} - w_{k+2} \geq 2$, $w_k = 0$, $k = n-1$.*

Proof. a) Since $\Delta(w^1, w) = (i_1, j_1)$ and $\Delta(w^r, w) = (i_r, j_r)$, we have $w_{i_1} > w_{i_1+1}$ and $w_{j_r-1} > w_{j_r}$. Therefore $k \neq i_1$ and $k \neq j_r - 1$ because $w_k = w_{k+1}$.

b) We assume $w_{k-1} - w_k \geq 2$. Then $k - 1 = i_l$ for some $l \in \{1, \ldots, r\}$ and either $k = j_l$ (in this case $w_l = c_{k-1}(w)$) or $w_k = 0$ and $k = n+1$. Since P is an overlapping path, if $k = j_l$, then $l \neq r$, $k = i_{l+1}$ and $w_k > w_{k+1}$, which is a contradiction.

c) We assume $w_{k+1} - w_{k+2} \geq 2$. Then $k+2 = j_l$ for some $l \in \{1, \ldots, r\}$ and either $k+1 = i_l$ (in this case $w_l = c_{k+1}(w)$) or $w_k = 0$ and $k = n-1$. Since P is an overlapping path, if $k+1 = i_l$, then $l \neq 1$, $k+1 = j_{l-1}$ and $w_k > w_{k+1}$, which is a contradiction. □

Proposition 7.3. *Let P be the overlapping path of w associated to S. If $k \in \{i_1, i_1+1, \ldots, j_r - 1\}$ and $w_k = w_{k+1} + 1$ then exactly one of the following conditions holds:*

a) *$k > i_1$ and $w_{k-1} - w_k \leq 1$ or $k < j_r - 1$ and $w_{k+1} - w_{k+2} \leq 1$;*
b) *$i_1 < k = n-1 < j_r - 1$, $w_{k-1} \geq 3$, $w_k = 1$ and $w_{k+2} \leq -2$;*
c) *$i_1 < k = n+1 < j_r - 1$, $w_{k-1} \geq 2$, $w_k = 0$ and $w_{k+2} \leq -3$;*
d) *$k = i_1 = n-1$, $w_k = 1$ and $w_{k+2} \leq -2$;*
e) *$k = j_r - 1 = n+1$, $w_{k-1} \geq 2$ and $w_k = 0$.*

Proof. Let us suppose that case *a)* does not hold. If $k = i_1$ and $w_{k+1} - w_{k+2} \geq 2$ then either $k+1 = i_2$ or $k = n-2$, $j_1 = n+1$ and $w_k = 1$ (case *d)*). The first case can not occur since $k = i_1$ and $w_k - w_{k+1} = 1$. If $k = j_r - 1$ and $w_{k-1} - w_k \geq 2$ then necessarily $k - 1 = i_r$, $k = n+1$

and $w_k = 0$ (case *e)*). If $i_1 < kj_r - 1$, $w_{k-1} - w_k \geq 2$ and $w_{k+1} - w_{k+2} \geq 2$ then it holds one of the following:

i either $k+1 = i_l$ for some $l \in \{2, \ldots, r\}$ or $k = n-1$, $j_l = n$ and $w_k = 1$ (case *b)*);
ii either $k-1 = i_h$ for some $h \in \{1, \ldots, r-1\}$ or $k-1 = i_h$, $k = n+1$ and $w_k = 0$ (case *c)*).

Note now that since P is an overlapping path and $w_k - w_{k+1} = 1$, if $k-1 = i_h$ and $k+1 = i_l$, it must be $l = h+2$, $k = i_{l+1}$ and $w_k - w_{k+1} \geq 2$, which is a contradiction. $\square$

Here we give six examples which illustrate the difference of our case with respect to the Brylawski case.

$$\begin{tikzcd} & 4432|0012 \arrow[dl, dash, "s_{24}"'] \arrow[d, dash, "cs_{47}"] \arrow[dr, dash, "s_{68}"] & \\ 4333|0012 & 4431|0002 & 4432|0111 \end{tikzcd}$$

Proposition 7.2 *b)*

$$\begin{tikzcd} & 2100|2344 \arrow[dl, dash, "s_{13}"'] \arrow[d, dash, "sc_{25}"] \arrow[dr, dash, "s_{57}"] & \\ 1110|2344 & 2000|1344 & 2100|3334 \end{tikzcd}$$

Proposition 7.2 *c)*

$$\begin{tikzcd} & 4310|2444 \arrow[dl, dash, "c_{2}"'] \arrow[d, dash, "sc_{35}"] \arrow[dr, dash, "c_{5}"] & \\ 4220|2444 & 4300|1444 & 4310|3344 \end{tikzcd}$$

Proposition 7.3 *b)*

$$\begin{tikzcd} & 4432|0144 \arrow[dl, dash, "s_{24}"'] \arrow[d, dash, "cs_{46}"] \arrow[dr, dash, "c_{6}"] & \\ 4333|0144 & 4431|0044 & 4432|0234 \end{tikzcd}$$

Proposition 7.3 *c)*

$$\begin{tikzcd} & 33110|23455 \arrow[dl, dash, "sc_{46}"'] \arrow[d, dash, "s_{68}"] \arrow[dr, dash, "s_{79}"] & \\ 33100|13455 & 33110|33355 & 33110|24445 \end{tikzcd}$$

Proposition 7.3 *d)*

$$\begin{tikzcd} & 54432|01115 \arrow[dl, dash, "s_{14}"'] \arrow[d, dash, "s_{35}"] \arrow[dr, dash, "cs_{57}"] & \\ 44442|01115 & 54333|01115 & 54431|00115 \end{tikzcd}$$

Proposition 7.3 *e)*

Given $w \in O(m,n)$ let us give now an important characterization of the signed partitions which are meet of cocovers of w.

Proposition 7.4. *Let $w \in O(m,n)$ and let $S = (w^1, w^2, \ldots, w^r)$ be an overlapping sequence of cocovers of w, associated to the overlapping path $P = (w_{i_1}, w_{i_1+1}, \cdots, w_{j_r})$. An element $v \in O(m,n)$ is the infimum of the elements in S if and only if the vector $\delta := w - v$ is equal to $(0, \cdots, 0, 1, 0 \cdots, 0, -1, 0 \cdots, 0)$ where 1 and -1 are in positions i_1 and j_r respectively.*

Proof. Let $v = \bigwedge_i w^i$ and let $k \in \{0, 1, \cdots, 2n-1\}$. If $0 \leq k < i_1$ then since for all $j \in \{0, 1, \cdots, k\}$ and for all $j \in \{1, 2, \cdots, r\}$, $w^i_j = w_j$, we have $v_k = w_k$ and so $\delta_k = 0$. When $k = i_1$ then $w^1_k + 1 = w^2_k = \cdots = w^r_k = w_k$, $v_k = \min_i\{\sum_{j=0}^k w^i_j\} - \sum_{j=0}^{k-1} v_j = w_k - 1$ and so $\delta_k = 1$. If $i_1 < k < j_r$ it is easy to see that $\min_i\{\sum_{j=0}^k w^i_j\} = \sum_{j=0}^k w_j - 1$ and so $v_k = w_k$. When $k = j_r$ then, for each i $\sum_{j=0}^k w^i_j = \sum_{j=0}^k w_j$ and $\sum_{j=0}^{k-1} v_j = \sum_{j=0}^{k-1} w_j - 1$. Thus $v_k = w_k + 1$ and $\delta_k = -1$. Finally since for all $j_r \leq l \leq 2n-1$, $\sum_{j=0}^l w^i_j = \sum_{j=0}^l w_j \sum_{j=0}^l v_j$, it holds $v_k = w_k$ and thus $\delta_k = 0$ for all $k \in \{j_r+1, j_r+2, \cdots, 2n-1\}$. $\square$

Corollary 7.5. *Let $v, w \in O(m,n)$ with $v \lhd w$ and let $\{w^i\}_{i \in I}$ be a set of cocovers of w. Then $v = \bigwedge_{i \in I} w^i$ if and only if $\delta := w - v = (d_0, d_1, \cdots, d_{2n-1})$ is a vector of alternating 1s and -1s, eventually spaced by 0s such that if $d_i - 1 = d_{i+1} = \cdots = d_{i+j-1} = d_{i+j} + 1 = 0$, then $(w_i, \cdots, w_{i+j})$ is an overlapping path of w.*

Proof. It follows by applying the Proposition 7.4 for each overlapping sequence in $\{w^i\}_{i \in I}$. $\square$

We can compute now the Möbius function μ of $O(m,n)$ in some particular cases.

Corollary 7.6. *Let $v, w \in O(m,n)$ with $v \lhd w$. If $\delta = w - v$ is not a vector of alternanting 1 and -1 eventually spaced by 0s or if it is but the parts in w between the position of any 1 and that of the succeeding -1 does not form an overlapping path, then $\mu(v, w) = 0$.*

Proof. The thesis follows at once by Corollary 7.5 and Corollary 3.9.5. of [27]. □

Let us consider now a particular class of cocovers of a fixed s-partition w.

Proposition 7.7. *Let $w \in O(m,n)$, $\{w^i\}_{i\in I}$ be a set of cocovers of w and $v = \bigwedge_{i\in I} w^i$. Then for $j \in I$, $v = \bigwedge_{i\in I,\ i\neq j} w^i$ if and only if there exists $p \in \{2, 3, \cdots, 2n-3\}$ such that $w_j = s_{(p-1)(p+1)}(w)$ and w_j fully overlap with w_k and w_l for some $k, l \in I$. In this case w_k and w_l partially overlap and we call w_j an* acritical cocover*.*

Proof. By Corollary 7.5 $v = \bigwedge_{i\in I,\ i\neq j} w^i$ if and only if the overlapping paths of w associated to the set of cocovers $\{w^i\}_{i\in I}$ are the same of those associated to $\{w^i\}_{i\in I,\ i\neq j}$. This is possible only if w_j fully overlaps with w_k and w_l for some $k, l \in I$ and at the same time w_k and w_l overlap. So the only possibility is that w_k and w_l partially overlap and there exists $p \in \{2, 3, \cdots, 2n-3\}$ such that $w_j = s_{(p-1)(p+1)}(w)$. In this case the interval $[w_k \wedge w_l, w]$ has the structure (D') (see Proposition 6.4) where $w_j = u$. The reverse implication follows from Corollary 7.5. □

An *acritical chain* $C = \{v_1, v_2, \ldots, v_r\}$ is a fully overlapping sequence of acritical cocovers of $w \in O(m,n)$. The *length* of an acritical chain C is the number of cocovers in it and it is denoted by $l(C)$. A *critical cocover* of w is a cocover of w which is not acritical. The *length* of an overlapping path P is the number of critical cocovers in it and it is denoted by $l(P)$.
The calculus of the Möbius function in the other cases makes use of the following combinatorial lemma.

Lemma 7.8. *The number of ways to extract a subsequence of m elements from a sequence of k elements such that no two consecutive elements are both left out is given by*

$$f(k,m) = \binom{m+1}{k-m}.$$

Further, by setting $F(k) := \sum_{m=0}^{k} (-1)^m f(k,m)$, it results that $F(k)$ is equal to 0, 1 or -1 and $F(k) \equiv 1-k \pmod 3$.

Proof. See [11] p. 213. □

Proposition 7.9. *Let $v, w \in O(m,n)$ such that v is a meet of cocovers of w. Let $n(v,w)$ be the number of critical cocovers of w in the interval $[v,w]$, $C_1, \ldots C_r$ be the acritical chains of cocovers of w in $[v,w]$ and $m(v,w) := \prod_i (1 - l(C_i))$. Then $\mu(v,w)$ is equal to 0, 1 or -1 and $\mu(v,w) \equiv (-1)^{n(v,w)} m(v,w) \pmod 3$.*

Proof. Rota's Cross-cut Theorem (see [25] for the classical reference or [27], Corollary 3.9.4.) applied to the interval $[v,w]$ implies that $\mu(v,w) = \sum_k (-1)^k N_k$ where N_k is the number of k-subsets of cocovers of w whose infimum is v. But any such subset must contain the set K of all critical cocovers of w in $[v,w]$. So we can write

$$\mu(v,w) = (-1)^{n(v,w)} \sum_k (-1)^k\ N'_k,$$

where N'_k is the number of k-subsets of acritical cocovers of w whose infimum with K is v. Now note that

$$\sum_k (-1)^k\ N'_k = \prod_{i=1}^{r} \left(\sum_k (-1)^k\ N'_{k_i} \right),$$

where N'_{k_i} is the number of k-subsets of C_i whose infimum with K is $K \wedge C_i$. But any such number N_{k_i} is equal to the number of k-subsets of the C_i such that no two fully overlapping cocovers in C_i are both left out. The proposition then follows easily by the Lemma 7.8. □

We note that by using Corollary 7.6 and Proposition 7.9 we can compute the Möbius function $\mu(v,w)$ for all $v \trianglelefteq w$ in $O(m,n)$. In fact, if v is not a meet of cocovers of w then, by Corollary 7.5, v and w satisfy the hypotheses of the Corollary 7.6 and thus $\mu(v,w) = 0$, while if v is a meet of cocovers of w then Proposition 7.9 gives us the value of $\mu(v,w)$.

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