ATTRACTIVE AND QUASI-ATTRACTIVE PAIRINGS AND THEIR MODELS FROM GRAPH THEORY

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Abstract. In the present paper, given an arbitrary fixed non-empty set Ω and a triple (called *pairing*) $\mathfrak{P} := (U, F, \Lambda)$, where U and Λ are two sets and $F : U \times \Omega \longrightarrow \Lambda$ is a map, we consider a preorder $\leftarrow_{\mathfrak{P}}$ on the power set of Ω having the further property that $Y \leftarrow_{\mathfrak{P}} X$ if and only if $\{y\} \leftarrow_{\mathfrak{P}} X$ for any $y \in Y$. When we take the symmetrization of the above preorder, we obtain an equivalence relation $\approx_{\mathfrak{P}}$ on $\mathcal{P}(\Omega)$ which induces a Moore system $\mathcal{M}_{\mathfrak{P}}$ and an abstract simplicial complex $\mathcal{N}_{\mathfrak{P}}$ on Ω. They are defined by taking respectively the maximum and the minimal elements of any equivalence class with respect to $\approx_{\mathfrak{P}}$. We find a sufficient condition on $\mathfrak{P}$ ensuring that the family $\mathcal{N}_{\mathfrak{P}}$ is a matroid when Ω is a finite set. We call the resulting pairings *attractive*. The aforementioned condition can be generalized even for a non-finite ground set Ω. In this case, jointly with a finitess condition, it turns out that for each $X \in \mathcal{M}_{\mathfrak{P}}$ the minimal members of the corresponding equivalence class $[X]_{\approx_{\mathfrak{P}}}$ all have the same cardinality. Nevertheless, the converse does not hold. As a counterexample, we will consider a second kind of pairing, which we call *quasi-attractive* and whose main properties, above all in relation to exchange properties of specific set systems, have been largely investigated. Finally, we will consider the adjacency matrix of a graph G as a concrete model of pairing (still denoted by G) and interpret the corresponding relation $\leftarrow_G$ as a measure of how a kind of *local symmetry* between vertices and subsets of the graph itself is transmitted when we vary the vertex subsets. We will demonstrate that the pairing induced by the adjacency matrix of the Petersen graph is attractive, while that of the so-called *Erdös' friendship graph* induces a quasi-attractive but not attractive pairing on the vertex set.

1. Introduction

1.1. **General Premise.** In many mathematical scopes, given an arbitrary set Ω, a growing attention is going to be placed on the interactions between the properties of specific kinds of binary relations, set operators, families of subsets of Ω and set-theoretic operations between subsets or other algebraic and combinatorial structures. The development of the aforementioned interactions admits various interrelation with numerous branches of mathematics where it is applied, such as design theory [8, 29], phylogenetic analysis [7, 21, 22], discrete dynamical systems [2, 3], algebraic structures theory [1, 27, 35, 36, 37, 33] functional analysis [28, 38], combinatorial topology [25, 26], complex analysis [4, 5, 6], graph theory [20, 30, 31, 39] and granular computing [40].

Based on such a perspective, in [15] some kinds of subdomains of an integral domain U have been classified through a binary relation $\leftarrow_{mod}$ defined on U, involving modules and equivalently expressible by means of the vanishing of a specific subset of the polynomial ring in several variables with coefficients in suitable subdomains of U. In such a case, the aforementioned relation is *extended reflexive*, i.e. $X \subseteq Y \implies X \leftarrow_{mod} Y$ and also *union-additive*, i.e. $\bigcup\{Z \mid Z \in \mathcal{F}\} \leftarrow_{mod} X$ for any family $\mathcal{F}$ of subsets of Ω such that $Z \leftarrow_{mod} X$ for each $Z \in \mathcal{F}$.

On the other hand, in [16] monoid actions have been studied in order to investigate a decomposition of *Alexandroff topological spaces* in terms of a combinatorial property of *dependence on unions*. Through the use of specific set systems on Ω induced by monoid actions $S \times \Omega \longrightarrow \Omega$, new links between Alexandroff topologies on Ω and some kinds of congruence relations on Ω induced by the monoid action of S have been established.

However, indeed, one may generalize the monoid action by a map F from the Cartesian product of any two sets U and Ω to a set Λ. In general, the sets U and Ω differ and are not necessarily endowed with any algebraic or topological structures [13]; nevertheless, they may agree and could be chosen as the vertex set of a simple undirected graph [10, 14], as a metric space [11], in order to yield a more interesting study. The arising structure, which has been called a *pairing* on Ω, and its main properties have been deeply investigated in [11, 12].

The introduction of a pairing structure on Ω induces an abstract geometry (that has been called *relation geometry* in [15]) which depends on the properties satisfied by a particular binary relation relatively to

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set-theoretic operations between subsets of Ω or to pre-existing algebraic structures.
Based on the above remark, in the present paper we will introduce two subclasses of pairings, analyze how they affect the behaviour of some specific set systems definable in pairing theory and, finally, we find concrete models of these classes of pairings from graph theory. In view of the results obtained in [16], fixed a subset X of Ω, we can consider the following binary relation $\equiv_X$ on U defined as follows: for each $u, u' \in U$, we set $u \equiv_X u' :\Longleftrightarrow F(u,x) = F(u',x)$ for all $x \in X$. It may be easily shown that $\equiv_X$ turns out to be an equivalence relation on U, so we will denote by $[u]_X$ the equivalence class of $u \in U$ and, furthermore, we will set $\pi_{\mathfrak{P}}(X) := \{[u]_X \mid u \in U\}$ to denote the set partition induced by $\equiv_X$ on U. In our context the terminology X-*symmetry relation* in order to describe the above equivalence relation [14]. The nomenclature symmetry relation stems by the fact that when one interpretes the adjacency matrix of a simple undirected graph as a Boolean pairing, the previous equivalence relation takes actually account of a local symmetry, in the sense of Erdös [23]. In fact, given a vertex subset X and two non-adjacent vertices v, v' of G such that $v \equiv_X v'$, it may be easily verified that the automorphism group of the subgraph induced by $X \cup \{v, v'\}$ is non-trivial (for further details and interpretations from the viewpoint of local symmetry, we refer the reader to [10]).
On the other hand, the interpretation of the X-indiscernibility relations as a local symmetry and further developments in such a direction allow to *extend* a graph to another graph and to define on the vertex set of such an extension an algebraic operation whose properties are connected to the geometric properties of the starting graph (see [14] for further details).
Now, the X-symmetry relation induces the binary relation $\leftarrow_{\mathfrak{P}}$ on the power set $\mathcal{P}(\Omega)$ of Ω defined as follows: $Y \leftarrow_{\mathfrak{P}} X$ if and only if for each $u, u' \in U$ such that $u \equiv_X u'$ it also results that $u \equiv_Y u'$. The aforementioned relation takes into account how the local symmetries induced by any two subsets of Ω relate. This binary relation induces the relation geometry we want to investigate in the whole paper. It may be easily see that $\leftarrow_{\mathfrak{P}}$ satisfies extended reflexivity, union-additivity and transitivity. More in general, in [12] it has been proved that any binary relation $\mathcal{P}(\Omega)$ satisfying the three above properties agrees with the relation $\leftarrow_{\mathfrak{P}}$ induced by some pairing $\mathfrak{P}$ on Ω and, furthermore, in [13] an algorithm to construct such a pairing has been provided in the case where Ω is finite.

1.2. **Content of the Paper.** Starting from a given pairing $\mathfrak{P}$ on Ω, we may associate with $\mathfrak{P}$ a closure operator $M_{\mathfrak{P}} : \mathcal{P}(\Omega) \longrightarrow \mathcal{P}(\Omega)$, where $M_{\mathfrak{P}}(X) := \{z \in \Omega \mid \{z\} \leftarrow_{\mathfrak{P}} X\}$, and the corresponding Moore system $\mathcal{M}_{\mathfrak{P}}$ of all its fixed point [18]. Moreover, taking the symmetrization of the preorder $\leftarrow_{\mathfrak{P}}$, the resulting equivalence relation $\approx_{\mathfrak{P}}$ on $\mathcal{P}(\Omega)$ identifies any two subsets sharing the same image with respect to $M_{\mathfrak{P}}$. In this case, any equivalence class admits a maximum element, namely the image under $M_{\mathfrak{P}}$, which we call *maximum partitioner*. The aforementioned equivalence relation allows us to take into account the entire evolution of the set partitions of U induced by subsets of Ω when we undertake some deletion or additions of elements starting from some given subset (see [11]). In this context, the collection $\mathcal{N}_{\mathfrak{P}}$ of all the minimal members (called *minimal partitioners*) of each equivalence class with respect to $\approx_{\mathfrak{P}}$ assumes a relevant role. In fact, it turns out that $\mathcal{N}_{\mathfrak{P}}$ is an abstract simplicial complex and, when Ω is a finite set, it is possible to find examples of pairings for which $\mathcal{N}_{\mathfrak{P}}$ is a matroid. Therefore, in a more general perspective, it is interesting to find appropriate conditions on the pairing $\mathfrak{P}$ to ensure that $\mathcal{N}_{\mathfrak{P}}$ is a matroid.
In this paper we find the following sufficient condition for $\mathcal{N}_{\mathfrak{P}}$ to be a matroid when Ω is a finite set: for each $X \in \mathcal{P}(\Omega)$ and any $y \in \Omega$ and $x \in X$ such that $\{y\} \not\leftarrow_{\mathfrak{P}} X$ and $\{x\} \not\leftarrow_{\mathfrak{P}} X \smallsetminus \{x\}$, it results that $\{x\} \not\leftarrow_{\mathfrak{P}} X \bigtriangleup \{x, y\}$. We call a pairing $\mathfrak{P}$ satisfying the aforementioned property *attractive*.
By virtue of the link between attractiveness and the matroidality of $\mathcal{N}_{\mathfrak{P}}$, such a notion will be the starting point on which the present paper relies. In particular, for any arbitrary set Ω, assuming attractiveness jointly with a finiteness condition (which we call *locally finiteness*), it may be shown that all the minimal partitioners of a given $X \in \mathcal{M}_{\mathfrak{P}}$ have the same cardinality (see [12]). In [13] the aforementioned minimal partitioners have been called $\mathfrak{P}$-*reducts* of X and their computation has been brought back to the well-known problem of determining the minimal transversal set system of a specific family of subsets of Ω [32]; this problem has been already investigated for specific graph structures, such as the Petersen graph and a geometric characterization has been provided in [10].
In this paper, we will see that attractiveness implies the equality between the family of the $\mathfrak{P}$-reducts of X and the collection of the maximal members of $\mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(X)$, for any $X \in \mathcal{P}(\Omega)$; moreover, when Ω is finite, the previous set-theoretic equality and attractiveness become equivalent. On the other hand, attractiveness is also a sufficient condition for the collection of the maximal members of $\mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(X)$, for any $X \in \mathcal{P}(\Omega)$, to satisfy an exchange property generalizing the *exchange axiom* of the bases of a matroid [42]. So that, we deduce that if Ω is a finite set and $\mathfrak{P}$ is an attractive pairing on Ω, then $\mathcal{N}_{\mathfrak{P}}$ is the family of the independent sets of a matroid on Ω.

Nevertheless, the condition of uniform cardinality of the $\mathfrak{P}$-reducts of X, for each $X \in \mathcal{M}_{\mathfrak{P}}$ is only necessary for attractiveness when Ω is finite. Indeed, examples of non-attractive pairings for which the $\mathfrak{P}$-reducts of any maximum partitioner have uniform cardinality may be found in the context of graph. In the present work we weaken the condition of being attractive, requiring that for all subsets $X, Y \in \mathcal{P}(\Omega)$ such that $X \approx_{\mathfrak{P}} Y$ and any $x \in X$ the existence of $y_x \in Y$ for which $\{x\} \leftarrow_{\mathfrak{P}} (X \setminus \{x\}) \cup \{y_x\}$.
We call a pairing satisfying the above property *quasi-attractive* and show that quasi-attractiveness is actually a weaker version of attractiveness. Given a quasi-attractive pairing $\mathfrak{P}$ on an arbitrary set Ω (even infinite) satisfying locally finiteness, it may be proved that for any $X \in \mathcal{M}_{\mathfrak{P}}$ the $\mathfrak{P}$-reducts of X have the same cardinality. Furthermore, when Ω is a finite set, the collection of all the $\mathfrak{P}$-reducts of any maximum partitioner X satisfies the exchange property characterizing the bases of a matroid on Ω, though, $\mathcal{N}_{\mathfrak{P}}$ need not to be a matroid. However, quasi-attractiveness is also related to the matroidality of a specific sub-set system of $\mathcal{N}_{\mathfrak{P}}$.
In general, it is not easy an easy task to find non-trivial families of pairings which are either attractive or quasi-attractive; however, in this paper we will exhibit some specific models of attractive and quasi-attractive pairings from graph theory. To this regard, we interpret the adjacency matrix of a simple undirected graph G as a pairing on its vertex set, which we still denote by G. In this case, the corresponding relation $\leftarrow_G$ may be explicitly rewritten as follows: $Y \leftarrow_G X$ if and only if $((v \sim x \iff v' \sim x \;\; \forall x \in X) \implies (v \sim y \iff v' \sim y \;\; \forall b \in Y))$, for each $v, v' \in V(G)$ and where $\sim$ denotes the adjacency relation between vertices of G. From an intuitive standpoint, pairings taking account on how the local symmetry transmits when we vary the vertex subsets inducing it. In other terms, whenever two vertices are symmetric with respect to the vertex subset X, then they must be symmetric also with respecto to the vertex subset Y. In particular, we are interested in how the addition or the deletion of vertices from a given vertex subset causes changes in the induced local symmetries. In this sense, the behaviour of the minimal vertex subset describing the same information about local symmetry as X becomes fundamental. As we said above, these subsets correspond to $\mathcal{N}_G$. So, we deduce that when the graph induces an attractive pairing, then $\mathcal{N}_G$ forms a matroid on Ω. Thus, for attractive graphs, we have found another way to associate matroids with graphs, different from the matroid arising when one considers the circuits of the graph as the minimal dependent subsets [41, 42].
In graph context, moreover, the G-reducts of any $X \in \mathcal{M}_G$ have been also named *symmetry bases* of X [10] and, when G is attractive, they behave as the bases of the matroid $\mathcal{N}_G$. On the other hand, when we consider the pairing $\mathfrak{P}[G,d]$ induced by the distance matrix of a graph, it may be easily verified that the notion of $\mathfrak{P}[G,d]$-reduct of $V(G)$ translates into that of *resolvent subset*, which is a fundamental tool in order to compute the metric dimension of a graph [9, 34].
Now, one non-trivial example of graph whose adjacency matrix induces an attractive pairing is Petersen graph. As it is a strongly regular graph one may think the existence of a correlation between attractiveness and strongly regularity. The question arises spontaneously since it is placed within the problem of how to recognize through the structural properties (and therefore to characterize) the graphs inducing an attractive pairing. Nevertheless, such a correlation is disregarded as we will see in Section 5 and in Example 5.11.
On the other hand, as claimed before, quasi-attractiveness is weaker than attractiveness. To provide an example from graph theory, we will exhibit a family of graphs inducing a quasi-attractive but not attractive pairing, namely the *Erdös' friendship graph* F_n defined and studied in Section 6. These graphs stem from a well-known paper by Erdös et al. [24], where the authors investigated further extremal properties in graphs: in particular, Erdös' friendship graphs arise a solution to a problem which is classically exposed in an informative manner as follows: *if a group of people has the property that every pair of people has exactly one friend in common, there exists a person who is friend to all the other?*
In such a case, we still ask for the existence of a possible correlation between quasi-attractiveness and strong regularity but, again, we are able to prove that such a correlation is disregarded also in this case since, in view of the definition of F_n, it is immediate to verify that quasi-attractiveness of the induced pairing does not imply strong regularity, while, on the other hand, the 5-cycle provides an example of strongly regular graph which is not quasi-attractive.
We will now describe briefly the content of the sections. In Section 2 we provide the notations we will use within the whole paper, with a particular attention to main properties of pairings and pairings. In Section 3 we introduce the notion of attractive pairing and study various properties of such a kind of relation, above all in relation to the collection of the maximal members of $\mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(X)$, for each $X \in \mathcal{P}(\Omega)$. In Section 4 we introduce the notion of quasi-attractive pairing and, next, we will analyze further properties of quasi-attractive pairings relatively to the validity of the exchange property for the set system of all the $\mathfrak{P}$-reducts of any subset $X \in \mathcal{P}(\Omega)$. In Section 5 we will demonstrate the pairing induced by the

Petersen graph is attractive and next provide a negative answer to the existence of some interrelations between strongly regularity and attractiveness. Finally, in Section 6, we will prove that the Erdös' friendship graphs are an example of quasi-attractive but not attractive pairings and, next, provide a negative answer to the existence of some interrelations between strongly regularity and quasi-attractiveness.

2. Reviews, Notations and Basic Results

Notations. In what follows, we denote by Ω a given arbitrary (even infinite) set, by $\mathcal{P}(\Omega)$ its power set and by $\mathcal{P}_{fin}(\Omega)$ the family of the finite subsets of Ω. Let $X \in \mathcal{P}(\Omega)$. If $X \in \mathcal{P}_f(\Omega)$, we denote by $|X|$ the number of elements of X and, if $|X| = k$, we say that X is a k-*subset* of Ω. In general, for any $k \geq 1$ we denote the family of all k-subsets of Ω by $\mathcal{P}_k(\Omega)$. If $Y \in \mathcal{P}(\Omega)$, we denote by $X \setminus Y$ the difference between X and Y and by $X \vartriangle Y$ their symmetric difference. We use the symbol $BREL(\Omega)$ to denote the collection of all binary relations on $\mathcal{P}(\Omega)$. We call the elements of $\mathcal{P}(\mathcal{P}(\Omega))$ *set systems* on Ω. Let $\mathcal{F} \in \mathcal{P}(\mathcal{P}(\Omega))$. We denote by $Max(\mathcal{F})$ and $Min(\mathcal{F})$ the sub-set systems of all maximal and minimal members of $\mathcal{F}$ with respect to set-theoretical inclusion. If $X \in \mathcal{P}(\Omega)$, we call any member of the set system $Max(\mathcal{F} \cap \mathcal{P}(X))$ a X-*basis* of $\mathcal{F}$. In particular, we call an Ω-basis of $\mathcal{F}$ a *basis* of $\mathcal{F}$, therefore in our terminology the bases of $\mathcal{F}$ agree with the maximal elements of $\mathcal{F}$.
We say that a non-empty set system $\mathcal{F}$:

- has *uniform cardinality* if all members of $\mathcal{F}$ have the same cardinality;
- is a *Moore system* if $\Omega \in \mathcal{F}$ and $\forall \mathcal{F}' \subseteq \mathcal{F}\ [\bigcap \mathcal{F}' \in \mathcal{F}]$. We denote by $MSY(\Omega)$ the family of all Moore systems on Ω;
- is an *abstract simplicial complex* if $X \in \mathcal{F}$ and $Y \subseteq X \implies Y \in \mathcal{F}$;
- is *exchangeable* if $\forall X, Y \in \mathcal{F}$, $\forall x \in X \setminus Y\ [\exists y \in Y \setminus X\ (X \vartriangle \{x, y\} \in \mathcal{F})]$.

When Ω is a finite set, we say that $\mathcal{F}$ is a *matroid* on Ω if:

$(M1)$ $\mathcal{F}$ is an abstract simplicial complex on Ω;
$(M2)$ for any $X, Y \in \mathcal{F}$ such that $|X| = |Y| + 1$, there exists $x \in X \setminus Y$ such that $Y \cup \{x\} \in \mathcal{F}$.

In this case, we call *independent set* any element of $\mathcal{F}$. If $\mathcal{F}$ is a matroid on Ω, it may be easily verified that the set system $Max(\mathcal{F} \cap \mathcal{P}(X))$ is exchangeable for any $X \in \mathcal{P}(\Omega)$. Conversely, if Ω is a finite set, any non-empty exchangeable set system $\mathcal{B}$ on Ω agrees with the family of all the bases of a matroid on Ω. For other general results on matroids, the reader can consult [42].
X *set operator* on Ω is a map $\sigma : \mathcal{P}(\Omega) \longrightarrow \mathcal{P}(\Omega)$, and we denote by $OP(\Omega)$ the set of all set operators on Ω. Let $\sigma \in OP(\Omega)$. We say that σ is:

- *extensive* if $X \in \mathcal{P}(\Omega) \implies X \subseteq \sigma(X)$;
- *monotone* if $X, B \in \mathcal{P}(\Omega), B \subseteq X \implies \sigma(B) \subseteq \sigma(X)$;
- *idempotent* if $X \in \mathcal{P}(\Omega) \implies \sigma(\sigma(X)) = \sigma(X)$;
- *a closure operator* if it is extensive, monotone and idempotent.

Posets and Lattices. A *partially ordered set* (abbreviated *poset*) is a pair $\mathbb{X} = (X, \leq)$, where X is a set and $\leq$ is a binary, reflexive, antisymmetric and transitive relation on X. Let X be a given poset and $x \in X$. We call *upset* of x the subset $(x)^{\uparrow}_X := \{y \in X \mid x \leq y\}$. Let $y \in X$. We write $x < y$ if $x \leq y$ and $x \neq y$. Moreover we use the symbol $x \| y$ to say that x and y are two non-comparable elements in $\mathbb{X}$. We say that y *covers* x (or that x is a *co-cover* of y), denoted by $x \lessdot y$, if $x < y$ and there exists no element $z \in X$ such that $x < z < y$. We denote by $(x)^{\downarrow, \lessdot}_X$ the family of all the co-covers of x in X.
We set

$$\mathcal{I}_X(x) := \{z \in (x)^{\downarrow, \lessdot}_X \mid z' \in X \text{ and } z' < x \implies z' \leq z\}$$

and

$$\mathcal{I}(X) := \{x \in X \mid \mathcal{I}_X(x) \neq \varnothing\}.$$

Clearly, if $\mathcal{I}_X(x) \neq \varnothing$, it contains only one element. If $Y \subseteq X$, we call an element $z \in X$ such that $y \leq z$ for any $y \in Y$ an *upper bound* of Y. We call the minimum (if it exists) of all upper bounds of Y the *least upper bound* of Y. The notions of *lower bound* and *greatest lower bound* are dual. $\mathbb{X}$ is said a *lattice* if any two elements of X have both a least upper bound and a greatest lower bound. Moreover, $\mathbb{X}$ is said a *complete lattice* if $\mathbb{X}$ is a lattice and if each of its subsets has both a least upper bound and greatest lower bound in the lattice.

Pairings. We call a triple $\mathfrak{P} = (U, F, \Lambda)$, where U, Λ are non-empty sets and $F : U \times \Omega \longrightarrow \Lambda$ is a map having domain $U \times \Omega$ and codomain Λ, a *pairing* on Ω. Let $PAIR(\Omega)$ denote the set of all pairings on Ω.

Fix an arbitrary $\mathfrak{P} = (U, F, \Lambda) \in PAIR(\Omega)$. For any $X \in \mathcal{P}(\Omega)$, we consider the equivalence relation $\equiv_X$ on the set U, that we call X-*symmetry relation*, defined by

$$\forall u, u' \in U \ [u \equiv_X u' :\Longleftrightarrow F(u, x) = F(u', x) \ \ \forall x \in X],$$

Let $[u]_X$ the equivalence class of u with respect to $\equiv_X$ and $\pi_{\mathfrak{P}}(X) := \{[u]_X \mid u \in U\}$.
Starting from the equivalence relation $\equiv_X$, we may define the binary relation $\leftarrow_{\mathfrak{P}}$ on $\mathcal{P}(\Omega)$ as follows: for each $X, Y \in \mathcal{P}(\Omega)$ we set

$$Y \leftarrow_{\mathfrak{P}} X :\Longleftrightarrow (\forall u, u' \in U \ [u \equiv_X u' \implies u \equiv_Y u']),$$

We call $\leftarrow_{\mathfrak{P}}$ a *pairing relation* on Ω. From now on we will always assume that $\mathfrak{P} \in PAIR(\Omega)$ is a given arbitrary pairing on Ω. With regard to pairing relations, it is immediate to verify the following properties:

- $(P1)$ $Y \subseteq X \implies Y \leftarrow_{\mathfrak{P}} X$;
- $(P2)$ $Z \leftarrow_{\mathfrak{P}} Y$ and $Y \leftarrow_{\mathfrak{P}} X \implies Z \leftarrow_{\mathfrak{P}} X$;
- $(P3)$ $Y \leftarrow_{\mathfrak{P}} X \iff \forall y \in Y \ [\{y\} \leftarrow_{\mathfrak{P}} X]$;

for any $X, Y, Z \in \mathcal{P}(\Omega)$.
For each $X, Y \in \mathcal{P}(\Omega)$ we set

$$X \approx_{\mathfrak{P}} Y :\Longleftrightarrow X \leftarrow_{\mathfrak{P}} Y \text{ and } Y \leftarrow_{\mathfrak{P}} X \iff \forall u, u' \in U \ (u \equiv_X u' \iff u \equiv_Y u')$$

The relation $\approx_{\mathfrak{P}}$ is an equivalence relation and we will denote by $[X]_{\approx_{\mathfrak{P}}}$ the equivalence class of any subset $X \in \mathcal{P}(\Omega)$.
At this point, let us provide the notion of $\mathfrak{P}$-reduct of a subset X.

Definition 2.1. *Let $X, Y \in \mathcal{P}(\Omega)$ be such that $Y \subseteq X$. We say that Y is a* $\mathfrak{P}$-reduct *of X if:*

- $(R1)$ *$X \leftarrow_{\mathfrak{P}} Y$;*
- $(R2)$ $\forall y \in Y \ [X \not\leftarrow_{\mathfrak{P}} Y \setminus \{y\}]$.

We denote by $\mathcal{R}_{\mathfrak{P}}(X)$ the set of all $\mathfrak{P}$-reducts of X. If $X = \Omega$, we will use the notation $\mathcal{R}_{\mathfrak{P}}$ instead of $\mathcal{R}_{\mathfrak{P}}(\Omega)$.

In the next result we recall some properties of the relation $\leftarrow_{\mathfrak{P}}$ and of two associated set systems and of the family of the X-reducts, for each subset $X \in \mathcal{P}(\Omega)$.

Theorem 2.2. *Let $X, Y, Z \in \mathcal{P}(\Omega)$. The following conditions hold:*

- (i) *If $X \leftarrow_{\mathfrak{P}} Y$, then $X \cup Z \leftarrow_{\mathfrak{P}} Y \cup Z$. In particular, if $X \approx_{\mathfrak{P}} Y$, then $X \cup Z \approx_{\mathfrak{P}} Y \cup Z$;*
- (ii) *the subset family $[X]_{\approx_{\mathfrak{P}}}$ has a maximum $M_{\mathfrak{P}}(X)$ that coincides with $\bigcup [X]_{\approx_{\mathfrak{P}}}$;*
- (iii) $M_{\mathfrak{P}}(X) = \{z \in \Omega \mid \{z\} \leftarrow_{\mathfrak{P}} X\} = \{z \in \Omega \mid X \cup \{z\} \approx_{\mathfrak{P}} X\} = \{z \in \Omega \mid (u, u' \in \Omega \wedge u \equiv_X u') \implies F(u, z) = F(u', z)\}$;
- (iv) *the set operator $M_{\mathfrak{P}} : W \in \mathcal{P}(\Omega) \mapsto M_{\mathfrak{P}}(W) \in \mathcal{P}(\Omega)$ is a closure operator on Ω and the set system $\mathcal{M}_{\mathfrak{P}} := \{M_{\mathfrak{P}}(W) \mid W \in \mathcal{P}(\Omega)\} = \{W \in \mathcal{P}(\Omega) \mid M_{\mathfrak{P}}(W) = W\} \in MSY(\Omega)$ and, hence $\mathbb{M}(\mathfrak{P}) := (\mathcal{M}_{\mathfrak{P}}, \subseteq)$ is a complete lattice;*
- (v) *the set system $\mathcal{N}_{\mathfrak{P}} := \bigcup \{Min([X]_{\approx_{\mathfrak{P}}}) \mid X \in \mathcal{M}_{\mathfrak{P}}\} = \{X \in \mathcal{P}(\Omega) \mid \forall x \in X \ [x \in \Omega \setminus M_{\mathfrak{P}}(X \setminus \{x\})]\}$ is an abstract simplicial complex on Ω;*
- (vi) $\mathcal{R}_{\mathfrak{P}}(X) = Min([X]_{\approx_{\mathfrak{P}}})$;
- (vii) $\mathcal{R}_{\mathfrak{P}}(X) \subseteq Max(\mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(X))$.

Proof. See [11]. □

The members of the set system $\mathcal{M}_{\mathfrak{P}}$ are usually called *maximum partitioners*, while those of $\mathcal{N}_{\mathfrak{P}}$ are said *minimal partitioners* [11].

Remark 2.3. *The reverse inclusion in part (vii) of Theorem 2.2 does not hold in general. We refer the reader to* [11] *for some counterexamples.*

2.1. **Graphs and Pairings.** We refer the reader to [19] for any general notion concerning graph theory. Let $G = (V(G), E(G))$ be a finite simple (i.e. no loops and no multiple edges are allowed) undirected graph, with vertex set $V(G) = \{v_1, \dots, v_n\}$ and edge set $E(G)$. If $v, v' \in V(G)$, we will write $v \sim v'$ if $\{v, v'\} \in E(G)$ and $v \nsim v'$ otherwise. We call the set $N_G(v) := \{w \in V(G) \mid v \sim w\}$ the *neighborhood* of v in G and we set $N_G^c(v) := V(G) \setminus N_G(v)$. Two vertices v and w are said *twin* if $N_G(v) = N_G(w)$. A graph is said *twin-free* if it has no twin vertices. We say that a graph G is *regular* if $|N_G(v)| = |N_G(w)|$ for each $v, w \in V(G)$ and, more specifically, we say that G is k-*regular* if $|N_G(v)| = k$ for each $v \in V(G)$. Moreover, we say that G is *strongly regular with parameters* (n, k, λ, ν) if it is a k-regular graph on n vertices, every two adjacent vertices have λ common neighbors and every two non-adjacent vertices have ν common neighbors.

Two graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ are said *isomorphic*, denoted by $G \cong H$, if there exists a bijection $\phi : V(G) \longrightarrow V(H)$ such that for all $v, v' \in V(G)$ it results that $\{v, v'\} \in E(G) \iff \{\phi(v), \phi(v')\} \in E(H)$. We say that $H = (V(H), E(H))$ is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $X \subseteq V(G)$, the *generated subgraph* by X in G, denoted by $G[X]$, is the graph having X as vertex set and such that if v and v' are two distinct vertices in X, then $\{v, v'\} \in E(G[X])$ if and only if $\{v, v'\} \in E(G)$.
If v and w are two distinct vertices of G and $k \geq 1$, a *k-path* (or sometimes simply a *path*) between v and w is a graph $P = (V(P), E(P))$, where $V(P) = \{v_0, \dots, v_k\}$, $E(P) = \{\{v_0, v_1\}, \dots, \{v_{k-1}, v_k\}\}$, $v_0 = v$ and $v_k = w$. In such a case, the number $k - 1$ is called the *length* of the path. In particular, any 1-path is a single vertex. We denote by $d(v, w)$ the distance between v and w, i.e. the length of any shortest path between v and w. By convention, we also assume that $d(u, u) = 0$ for any $u \in V(G)$. We say that a graph G is *connected* if for any two distinct vertices $v, w \in V(G)$ there exists a path between them. If v is a vertex, we call the maximal connected subgraph of G containing v the *connected component* of v in G. Each maximal connected subgraph of G is said *connected component* of G.
In this paper we denote by $P_{k_1,\dots,k_s}$ any disjoint union of s paths $P_{k_1}, \dots, P_{k_s}$, where P_{k_i} is a k_i-path, for $i = 1, \dots, s$.
Some classical examples of graphs with vertex set $\{v_1, \dots, v_n\}$ we will use within the paper are given below:

- *Complete graph on n vertices.* It is denoted by K_n and is the graph such that $\{v_i, v_j\}$ is an edge, for each pair of indexes $i \neq j$.
- $(r_1, \dots, r_s)$*–complete multipartite graph on n vertices.* It is denoted by $K_{r_1,\dots,r_s}$, where $r_1 + \dots + r_s = n$ and there exist s non-empty subsets $B_1, \dots, B_s$ of $V(G)$ such that $|B_i| = r_i$, $B_i \cap B_j = \varnothing$ if $i \neq j$, $\bigcup_{i=1}^{s} B_i = V(G)$ and $E(G) = \{\{x, y\} \mid x \in B_i, y \in B_j, i \neq j\}$. In this case, we also denote $K_{r_1,\dots,r_s}$ by the symbol $(B_1|\dots|B_s)$. Moreover, if $s = 2$, we say that K_{r_1,r_2} is a *complete bipartite graph.*
- *n-cycle.* It is denoted by C_n and has edge set $E(C_n) = \{\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\}$.

Let G be a n-graph. We now consider the pairing $\mathfrak{P}[G] := (V(G), F, \{0, 1\}) \in PAIR(V(G))$, where

$$F(u, v) := \begin{cases} 1 & \text{if } u \sim v \\ 0 & \text{otherwise} \end{cases}$$

We call $\mathfrak{P}[G]$ the *adjacency pairing* of G. From now on, we write G instead of $\mathfrak{P}[G]$. For any $X \in \mathcal{P}(V(G))$, the equivalence relation $\equiv_X$ can be translated as follows:

$$v \equiv_X v' :\iff N_G(v) \cap X = N_G(v') \cap X$$

3. Attractive Pairings

In this section we provide the notion of attractive pairing on Ω and next analyze some basic properties of such structures. More in detail, we will see that the family of all $\mathfrak{P}$-reducts of each subset $X \in \mathcal{P}(\Omega)$ agrees with the family of the maximal members of $\mathcal{N}_{\mathfrak{P}}$ contained in X and form an exchangeable set system. We finally observe that there exist non-attractive pairing for which the condition $Max(\mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(X))$ holds for any $X \in \mathcal{P}(\Omega)$; while, when Ω is a finite set, the coincidence between $\mathcal{R}_{\mathfrak{P}}(X)$ and $Max(\mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(X))$ becomes also a sufficient condition.
Let us provide the notion of *locally finite* and *attractive* pairings on Ω.

Definition 3.1. *We say that a pairing $\mathfrak{P} \in PAIR(\Omega)$ is:*

- locally finite *if*
$$\forall X, Y \in \mathcal{P}(\Omega), \forall y \in Y \; [Y \leftarrow_{\mathfrak{P}} X \implies \exists X_y \subseteq X \; (\{y\} \leftarrow_{\mathfrak{P}} X_y)$$
We denote by $PAIR_{lf}(\Omega)$ the set of all locally finite pairings on Ω;
- attractive *if*
$$\forall X \in \mathcal{P}(\Omega), \forall y \in \Omega, \forall x \in X \; [\{y\} \not\leftarrow_{\mathfrak{P}} X \text{ and } \{x\} \not\leftarrow_{\mathfrak{P}} X \smallsetminus \{x\} \implies \{x\} \not\leftarrow_{\mathfrak{P}} X \triangle \{x, y\}]$$
We denote by $PAIR_a(\Omega)$ the collection of all attractive pairings on Ω. Moreover, we use the symbol $PAIR_{lfa}(\Omega) := PAIR_a(\Omega) \cap PAIR_{lf}(\Omega)$ to denote the collection of all locally finite, attractive pairings on Ω.

Let us characterize in another way the condition for a pairing of being attractive.

Proposition 3.2. *The following two conditions are equivalent:*

(*i*) $\mathfrak{P} \in PAIR_a(\Omega)$;
(*ii*) $\forall y \in \Omega, \forall x \in X \; [\{y\} \leftarrow_{\mathfrak{P}} X$ *and* $\{y\} \not\leftarrow_{\mathfrak{P}} X \smallsetminus \{x\} \implies \{x\} \leftarrow_{\mathfrak{P}} (X \smallsetminus \{x\}) \cup \{y\}]$.

Proof. See [12]. □

In the next result we provide a property satisfied by any attractive pairing.

Proposition 3.3. *Let* $\mathfrak{P} \in PAIR_a(\Omega)$, $X \in \mathcal{P}(\Omega)$ *and* $x, y \in \Omega$ *be such that*

$$M_{\mathfrak{P}}(\{x\}) \cap M_{\mathfrak{P}}(X) = \varnothing \text{ and } M_{\mathfrak{P}}(\{x\}) \cap M_{\mathfrak{P}}(X \cup \{y\}) \neq \varnothing \tag{1}$$

Then $M_{\mathfrak{P}}(\{y\}) \cap M_{\mathfrak{P}}(X \cup \{x\}) \neq \varnothing$.

Proof. Let $X \in \mathcal{P}(\Omega)$ and $x, y \in \Omega$ be such that $M_{\mathfrak{P}}(\{x\}) \cap M_{\mathfrak{P}}(X) = \varnothing$ and $M_{\mathfrak{P}}(\{x\}) \cap M_{\mathfrak{P}}(X \cup \{y\}) \neq \varnothing$. Hence, by part (*iii*) of Theorem 2.2 there exists $u \in \Omega$ such that

$$\{u\} \leftarrow_{\mathfrak{P}} \{x\} \text{ and } \{u\} \leftarrow_{\mathfrak{P}} X \cup \{y\} \tag{2}$$

As $\{u\} \leftarrow_{\mathfrak{P}} \{x\}$, the first condition in (1) implies that

$$\{u\} \not\leftarrow_{\mathfrak{P}} X \tag{3}$$

Assume by contradiction that

$$M_{\mathfrak{P}}(\{y\}) \cap M_{\mathfrak{P}}(X \cup \{x\}) = \varnothing \tag{4}$$

Hence, by part (*iii*) of Theorem 2.2 and by $(P1)$ and $(P2)$, we easily deduce that $\{y\} \not\leftarrow_{\mathfrak{P}} X$. Now, by (2) and (3), using Proposition 3.2 we get $\{y\} \leftarrow_{\mathfrak{P}} X \cup \{u\}$, whence $X \cup \{y\} \leftarrow_{\mathfrak{P}} X \cup \{u\}$. At this point, from the second condition of (2) we easily deduce that

$$X \cup \{y\} \approx_{\mathfrak{P}} X \cup \{u\}$$

So, using the first condition of (2), we readily verify that $X \cup \{y\} \leftarrow_{\mathfrak{P}} X \cup \{x\}$. Thus, by $(P1)$, it results that $\{y\} \leftarrow_{\mathfrak{P}} X \cup \{x\}$, whence $y \in M_{\mathfrak{P}}(\{y\}) \cap M_{\mathfrak{P}}(X \cup \{x\})$, in contrast with (4). This shows that $M_{\mathfrak{P}}(\{y\}) \cap M_{\mathfrak{P}}(X \cup \{x\}) \neq \varnothing$. □

In [12] it has been proved that the collection of all the $\mathfrak{P}$-reduct of a maximum partitioner X is always non-empty set when the pairing is both locally finite and attractive. Let us recall such an important result.

Theorem 3.4. *Let* $\mathfrak{P} \in PAIR_{lfa}(\Omega)$ *and* $X \in \mathcal{M}_{\mathfrak{P}}$. *Then* $\mathcal{R}_{\mathfrak{P}}(X) \neq \varnothing$ *and* $\mathcal{R}_{\mathfrak{P}}(X)$ *has uniform cardinality.*

Proof. See [12]. □

At this point, we will establish two specific properties of attractive pairings, namely the fact that the X-reducts agree with the maximal members of $\mathcal{N}_{\mathfrak{P}}$ contained in X and form an exchangeable set system. In particular, when Ω is a finite set and $\mathfrak{P}$ is an attractive pairing, we will demonstrate that the family of all the minimal partitioners is a matroid.

Theorem 3.5. *Let* $\mathfrak{P} \in PAIR_a(\Omega)$. *Then:*

(*i*) $\mathcal{R}_{\mathfrak{P}}(X) = Max(\mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(X))$;
(*ii*) $Max(\mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(X))$.

for any $X \in \mathcal{P}(\Omega)$. *Moreover, if* Ω *is a finite set, it also results that:*

(*iii*) $\mathcal{N}_{\mathfrak{P}}$ *is a matroid and* $\mathcal{R}_{\mathfrak{P}}(X)$ *agrees with the family of all its* X*-bases, for each* $X \in \mathcal{P}(\Omega)$.

Proof. (*i*): Let $X \in \mathcal{P}(\Omega)$. We claim that $\mathcal{R}_{\mathfrak{P}}(X) = Max(\mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(X))$. To this regard, it suffices to show the inclusion $Max(\mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(X)) \subseteq \mathcal{R}_{\mathfrak{P}}(X)$ since, by part (*vii*) of Theorem 2.2, the reverse inclusion is always true. Take therefore $Y \in Max(\mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(X))$. We will first demonstrate that

$$X \leftarrow_{\mathfrak{P}} Y \tag{5}$$

Assume by contradiction the existence of $x \in X$ such that

$$\{x\} \not\leftarrow_{\mathfrak{P}} Y \tag{6}$$

Hence $x \in \Omega \smallsetminus M_{\mathfrak{P}}(Y)$ by part (*iii*) of Theorem 2.2. Moreover, as $Y \in \mathcal{N}_{\mathfrak{P}}$, it results that $\{y\} \not\leftarrow_{\mathfrak{P}} Y \smallsetminus \{y\}$ for any $y \in Y$. Using (6) and the fact that $\mathfrak{P} \in PAIR_a(\Omega)$, it follows that the condition

$$\{y\} \not\leftarrow_{\mathfrak{P}} Y \vartriangle \{x, y\} \tag{7}$$

holds for any $y \in Y$. Consider now the subset $Y \cup \{x\}$. In view of (6) and of (7), it easily follows that $Y \cup \{x\} \in \mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(X)$, contradicting the fact that $Y \in Max(\mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(X))$. Thus (5) holds.
Now, fix some $y \in Y$. We have

$$X \not\leftarrow_{\mathfrak{P}} Y \smallsetminus \{y\}$$

since, otherwise, we would get

$$\{y\} \leftarrow_{\mathfrak{P}} Y \leftarrow_{\mathfrak{P}} X \leftarrow_{\mathfrak{P}} Y \smallsetminus \{y\},$$

whence $\{y\} \leftarrow_{\mathfrak{P}} Y \smallsetminus \{y\}$, which contradicts the assumption that $Y \in \mathcal{N}_{\mathfrak{P}}$. Therefore, Y also satisfies $(R2)$ of Definition 2.1. This proves that $Y \in \mathcal{R}_{\mathfrak{P}}(X)$.
(ii): Let $B, C \in Max(\mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(X))$ and $x \in B \smallsetminus C$. In view of the previous part (i), we have that $\mathcal{R}_{\mathfrak{P}}(X) = Max(\mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(X))$. Thus

$$B \approx_{\mathfrak{P}} C. \tag{8}$$

Now, as $B \in \mathcal{N}_{\mathfrak{P}}$, it results that $\{x\} \not\leftarrow_{\mathfrak{P}} B \smallsetminus \{x\}$. Therefore $B \not\leftarrow_{\mathfrak{P}} B \smallsetminus \{x\}$ and, by (8), we deduce that $C \not\leftarrow_{\mathfrak{P}} B \smallsetminus \{x\}$. In particular, there exists $y \in C$ such that

$$\{y\} \not\leftarrow_{\mathfrak{P}} B \smallsetminus \{x\}. \tag{9}$$

By (9), we must clearly have $y \in C \smallsetminus B$. At this point, by (8) and (9), we have that $\{y\} \leftarrow_{\mathfrak{P}} B$ and $\{y\} \not\leftarrow_{\mathfrak{P}} B \smallsetminus \{x\}$. Therefore, in view of part (ii) of Proposition 3.2, we get

$$\{x\} \leftarrow_{\mathfrak{P}} B \vartriangle \{x, y\}. \tag{10}$$

Hence

$$B \vartriangle \{x, y\} \approx_{\mathfrak{P}} B \cup \{y\} \approx_{\mathfrak{P}} B. \tag{11}$$

Set $B' := B \vartriangle \{x, y\}$. Clearly, $B' \in \mathcal{P}(X)$. We claim that $B' \in \mathcal{N}_{\mathfrak{P}}$ or, equivalently, that

$$\{b'\} \not\leftarrow_{\mathfrak{P}} B' \smallsetminus \{b'\} \tag{12}$$

for each $b' \in B'$. In view of (9), the claim holds when we take $b' := y$. So, take $b' \in B \smallsetminus \{x\}$ and assume by contradiction that (12) does not hold. Thus

$$\{b'\} \leftarrow_{\mathfrak{P}} B' \smallsetminus \{b'\} = (B \smallsetminus \{x\}) \vartriangle \{y, b'\}. \tag{13}$$

Notice that

$$\{b'\} \not\leftarrow_{\mathfrak{P}} B \smallsetminus \{x, b'\}, \tag{14}$$

otherwise $\{b'\} \leftarrow_{\mathfrak{P}} B \smallsetminus \{b'\}$, contradicting the fact that $B \in \mathcal{N}_{\mathfrak{P}}$. Now, using part (ii) of Proposition 3.2 on the conditions (13) and (14), we get

$$\{y\} \leftarrow_{\mathfrak{P}} (B \smallsetminus \{x, b'\}) \cup \{b'\} = B \smallsetminus \{x\},$$

which contradicts (9). Therefore, we conclude that (12) holds for each $b' \in B'$. This shows that $B' \in \mathcal{N}_{\mathfrak{P}}$. Finally, let $B'' \in Max(\mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(X))$ be such that $B' \subsetneq B''$. As $Max(\mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(X)) = \mathcal{R}_{\mathfrak{P}}(X)$, we get $B'' \in \mathcal{R}_{\mathfrak{P}}(X)$, which is in contrast with (11). So, $B' \in Max(\mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(X))$ and this proves that $Max(\mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(X))$ is exchangeable.
(iii): In view of the previous two parts, it follows that the collection $Max(\mathcal{N}_{\mathfrak{P}}) = \mathcal{R}_{\mathfrak{P}}$ agrees with the family of the bases of a matroid. Such a matroid is exactly $\mathcal{N}_{\mathfrak{P}}$. □

Remark 3.6. *In general, the converse of part* (ii) *of Theorem 3.5 does not hold. See* [11] *for some counterexamples.*

In the following example, we will see that if the set system $Max(\mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(X))$ is exchangeable, then the corresponding pairing should not be attractive.

Example 3.7. Let us consider the pairing $\mathfrak{P}$ induced by the adjacency matrix of the graph P_5. Let $V := V(P_5)$. In such a case, it may be easily verified that

$$\begin{aligned}
Max(\mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(V)) = \{&\{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_2, v_5\}, \{v_1, v_3, v_4\}, \\
&\{v_1, v_4, v_5\}, \{v_2, v_3, v_5\}, \{v_2, v_4, v_5\}, \{v_3, v_4, v_5\}\},
\end{aligned}$$

$$\begin{aligned}
Max(\mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(\{v_1, v_2, v_3, v_4\})) &= \{\{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}\}, \\
Max(\mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(\{v_1, v_2, v_3, v_5\})) &= \{\{v_1, v_2, v_3\}, \{v_1, v_2, v_5\}, \{v_2, v_3, v_5\}\}, \\
Max(\mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(\{v_1, v_2, v_4, v_5\})) &= \{\{v_1, v_2, v_4\}, \{v_1, v_2, v_5\}, \{v_1, v_4, v_5\}, \{v_2, v_4, v_5\}\}, \\
Max(\mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(\{v_1, v_3, v_4, v_5\})) &= \{\{v_1, v_3, v_4\}, \{v_1, v_4, v_5\}, \{v_3, v_4, v_5\}\}, \\
Max(\mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(\{v_2, v_3, v_4, v_5\})) &= \{\{v_2, v_3, v_5\}, \{v_2, v_4, v_5\}, \{v_3, v_4, v_5\}\}, \\
Max(\mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(\{v_1, v_3, v_5\})) &= \{\{v_1, v_3\}, \{v_1, v_5\}, \{v_3, v_5\}\},
\end{aligned}$$

$$Max(\mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(\{v_2, v_3, v_4\})) = \{\{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}\},$$

and $Max(\mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(X)) = \{X\}$ for the remaining vertex subsets X.
On the other hand, let us observe that P_5 is not an attractive graph. In fact, just take $X = \{v_2, v_3, v_4\}$, $a = v_3$ and $b = v_1$. Then $\{v_3\} \leftarrow_{\mathfrak{P}} X \triangle \{v_1, v_3\} = \{v_1, v_2, v_4\}$ since $\{v_1, v_2, v_4\} \approx_{\mathfrak{P}} V$.

When Ω is a finite set, it results that the condition $Max(\mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(X)) = \mathcal{R}_{\mathfrak{P}}(X)$ for any $X \in \mathcal{P}(\Omega)$ becomes sufficient for attractiveness. This we now present.

Theorem 3.8. *Let Ω be a finite set. The following conditions are equivalent:*

(*i*) $\mathfrak{P} \in PAIR_a(\Omega)$;
(*ii*) $\forall X \in \mathcal{P}(\Omega)\ [Max(\mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(X)) = \mathcal{R}_{\mathfrak{P}}(X)]$.

Proof. $(i) \implies (ii)$: It has been already shown in part (i) of Theorem 3.5.
$(ii) \implies (i)$: Fix $X \in \mathcal{P}(\Omega)$ and consider first the monotone map $\rho_{\mathfrak{P}} : \mathcal{P}(\Omega) \to \mathbb{N}$ defined as follows:

$$\rho_{\mathfrak{P}}(X) := \max\{|X'| \mid X' \in Max(\mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(X))\}. \tag{15}$$

We claim that

$$M_{\mathfrak{P}}(X) = \{w \in \Omega \mid \rho_{\mathfrak{P}}(X) = \rho_{\mathfrak{P}}(X \cup \{w\})\}. \tag{16}$$

To this regard, let $z \in M_{\mathfrak{P}}(X)$. Hence $X \cup \{z\} \approx_{\mathfrak{P}} X$. Let $Y \in \mathcal{R}_{\mathfrak{P}}(X \cup \{z\}) = Max(\mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(X \cup \{z\}))$ be such that $|Y| = \rho_{\mathfrak{P}}(X)$. If $z \in \Omega \setminus Y$, then we clearly have $Y \in \mathcal{R}_{\mathfrak{P}}(X)$. Thus, we get $|Y| \leq \rho_{\mathfrak{P}}(X) \leq \rho_{\mathfrak{P}}(X \cup \{z\}) = |Y|$, i.e. $\rho_{\mathfrak{P}}(X) = \rho_{\mathfrak{P}}(X \cup \{z\})$.
On the other hand, if $z \in Y$, we have $Y = C \cup \{z\}$ for some $C \in \mathcal{P}(X)$. Notice that $C \notin \mathcal{R}_{\mathfrak{P}}(X)$, otherwise we would have $\pi_{\mathfrak{P}}(Y \setminus \{z\}) = \pi_{\mathfrak{P}}(C) = \pi_{\mathfrak{P}}(X) = \pi_{\mathfrak{P}}(X \cup \{z\})$, contradicting the fact that $Y \in \mathcal{R}_{\mathfrak{P}}(X \cup \{z\})$. Nevertheless, since $Y \in \mathcal{N}_{\mathfrak{P}}$, we must necessarily have $C \in \mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(X)$ and, thus, $C \subsetneq D$ for some $D \in Max(\mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(X)) = \mathcal{R}_{\mathfrak{P}}(X)$. This means that $|Y'| < \rho_{\mathfrak{P}}(X)$ and, hence, we deduce that

$$\rho_{\mathfrak{P}}(X \cup \{z\}) = |Y| = |Y'| + 1 \leq \rho_{\mathfrak{P}}(X)$$

Since $\rho_{\mathfrak{P}}$ is monotone, we also have $\rho_{\mathfrak{P}}(X) \leq \rho_{\mathfrak{P}}(X \cup \{z\})$. This proves that $M_{\mathfrak{P}}(X) \subseteq \{w \in \Omega \mid \rho_{\mathfrak{P}}(X) = \rho_{\mathfrak{P}}(X \cup \{w\})\}$.
Conversely, to prove the reverse inclusion, let $z \in \Omega$ be such that $\rho_{\mathfrak{P}}(X) = \rho_{\mathfrak{P}}(X \cup \{z\})$ and assume by contradiction that $z \in \Omega \setminus M_{\mathfrak{P}}(X)$, i.e. $X \not\approx_{\mathfrak{P}} X \cup \{z\}$. Let moreover $Y \in Max(\mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(X)) = \mathcal{R}_{\mathfrak{P}}(X)$ be such that $|Y| = \rho_{\mathfrak{P}}(X)$. Clearly, it follows that $Y \in \mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(X \cup \{z\})$ and, by our assumption, that $|Y| = \rho_{\mathfrak{P}}(X) = \rho_{\mathfrak{P}}(X \cup \{z\})$. This implies that $Y \in Max(\mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(X \cup \{z\}) = \mathcal{R}_{\mathfrak{P}}(X \cup \{z\})$ and, hence,

$$\pi_{\mathfrak{P}}(X) = \pi_{\mathfrak{P}}(Y) = \pi_{\mathfrak{P}}(X \cup \{z\})$$

in contrast with the fact that $z \in \Omega \setminus M_{\mathfrak{P}}(X)$. This concludes the proof of (16).
Take now $\{y\} \not\leftarrow_{\mathfrak{P}} X$ and $\{x\} \not\leftarrow_{\mathfrak{P}} X \setminus \{x\}$. Assume by contradiction that

$$\{x\} \leftarrow_{\mathfrak{P}} X \triangle \{x, y\} \tag{17}$$

Then, in view of part (iii) of Theorem 2.2, (15) and (16), our choices of the elements x and y may be expressed in terms of the map $\rho_{\mathfrak{P}}$ as follows:

$$\rho_{\mathfrak{P}}(X \cup \{y\}) = \rho_{\mathfrak{P}}(X) + 1, \quad \rho_{\mathfrak{P}}(X) = \rho_{\mathfrak{P}}(X \setminus \{x\}) + 1, \quad \rho_{\mathfrak{P}}(X \cup \{y\}) = \rho_{\mathfrak{P}}(X \triangle \{x, y\}). \tag{18}$$

Let $X' \in Max(\mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(X \triangle \{x, y\}))$ be such that $\rho_{\mathfrak{P}}(X \triangle \{x, y\}) = |X'|$. As $X' \in Max(\mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(X \triangle \{x, y\})) \subseteq \mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(X \cup \{y\})$, by (17) and by the condition $\rho_{\mathfrak{P}}(X \cup \{y\}) = \rho_{\mathfrak{P}}(X \triangle \{x, y\})$ given in (18), we easily deduce that

$$X' \in Max(\mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(X \cup \{y\})) = \mathcal{R}_{\mathfrak{P}}(X \cup \{y\}),$$

where the last equality holds in view of the hypothesis. Now, since $\{y\} \not\leftarrow_{\mathfrak{P}} X$ and $X' \in \mathcal{R}_{\mathfrak{P}}(X \cup \{y\})$, it must necessarily be $X' = X'' \cup \{y\}$, where $X'' \in \mathcal{P}(X \setminus \{x\})$. By part (v) of Theorem 2.2 it results that $X'' \in \mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(X)$ and thus we get

$$|X''| = |X'| - 1 = \rho_{\mathfrak{P}}(X \cup \{y\}) - 1 = \rho_{\mathfrak{P}}(X) > \rho_{\mathfrak{P}}(X \setminus \{x\}). \tag{19}$$

Nevertheless, as $X'' \in \mathcal{P}(X \setminus \{x\})$, we also have $\rho_{\mathfrak{P}}(X \setminus \{x\}) \geq |X''| = \rho_{\mathfrak{P}}(X)$, which contradicts (19). Therefore, (17) cannot hold and $\mathfrak{P}$ must be an attractive pairing. $\square$

4. Quasi-Attractive Pairings

In this section we introduce the notion of *quasi-attractive pairing.* It originates by assuming a specific property which is indeed weaker than attractiveness, as we will see in Proposition 4.1. Furthermore, we will firstly provide a characterization of quasi-attractiveness in terms of order-theoretical properties of $\mathcal{M}_{\mathfrak{P}}$. Next, we investigate the main properties of quasi-attractive pairings, above all in relation to the behaviour of the set system $\mathcal{R}_{\mathfrak{P}}(X)$, for any $X \in \mathcal{P}(\Omega)$. In particular, we will demonstrate that when Ω is a finite set, quasi-attractiveness becomes equivalent to require the exchangeability of $\mathcal{R}_{\mathfrak{P}}(X)$ for any $X \in \mathcal{P}(\Omega)$. In this way, we relate quasi-attractiveness to the matroidality of a sub-set system of $\mathcal{N}_{\mathfrak{P}}$.
Let us now provide a necessary condition for attractive pairings, allowing us to introduce a new subclass of pairings.

Proposition 4.1. *Let $\mathfrak{P} \in PAIR_a(\Omega)$ and $X, Y \in \mathcal{P}(\Omega)$ be such that $X \approx_{\mathfrak{P}} Y$. Then*

$$\forall x \in X \; [\exists y_x \in Y \; (\{x\} \leftarrow_{\mathfrak{P}} (X \smallsetminus \{x\}) \cup \{y_x\})]$$

Proof. Let $X, Y \in \mathcal{P}(\Omega)$ be such that $X \approx_{\mathfrak{P}} Y$ and fix $x \in X$. There is nothing to prove if $x \in Y$. So assume that $x \in \Omega \smallsetminus Y$. First note that if $\{x\} \leftarrow_{\mathfrak{P}} X \smallsetminus \{x\}$, then the condition $\{x\} \leftarrow_{\mathfrak{P}} (X \smallsetminus \{x\}) \cup \{y\}$ holds for each $y \in Y$. Therefore, we may also suppose that

$$\{x\} \not\leftarrow_{\mathfrak{P}} X \smallsetminus \{x\}$$

Then, using the assumption that $X \approx_{\mathfrak{P}} Y$, we may find an element $y_x \in Y$ such that $\{y_x\} \not\leftarrow_{\mathfrak{P}} X \smallsetminus \{x\}$. Hence, by $(P1)$, $(P2)$ and the fact that $Y \approx_{\mathfrak{P}} X$, it follows that $\{y_x\} \leftarrow_{\mathfrak{P}} X$. Now, since $\mathfrak{P} \in PAIR_a(\Omega)$, by part (ii) of Proposition 3.2 we get

$$\{x\} \leftarrow_{\mathfrak{P}} (X \smallsetminus \{x\}) \cup \{y_x\}$$

□

In view of Proposition 4.1, we may now provide the fundamental notion of quasi-attractive pairing and the corresponding notion of quasi-attractive pairing.

Definition 4.2. *We say that a pairing $\mathfrak{P} \in PAIR(\Omega)$ is* quasi-attractive *if for each $X, Y \in \mathcal{P}(\Omega)$ such that $X \approx_{\mathfrak{P}} Y$ and any $x \in X$*

$$\exists y_x \in Y \; [\{x\} \leftarrow_{\mathfrak{P}} (X \smallsetminus \{x\} \cup \{y_x\}]$$

We denote by $PAIR_{qa}(\Omega)$ the collection of all quasi-attractive pairings on Ω. We set moreover $PAIR_{lfqa}(\Omega) := PAIR_{lf}(\Omega) \cap PAIR_{qa}(\Omega)$.

Let us characterize the condition for a pairing of being quasi-attractive in terms of a specific property of the complete lattice $\mathbb{M}(\mathfrak{P})$. More in detail we will demonstrate that a pairing is quasi-attractive if and only if for any $X \in \mathcal{P}(\Omega)$ the elements of $\mathcal{I}(\mathcal{F})$, where $\mathcal{F}$ is the upset of $M_{\mathfrak{P}}(X)$ in $\mathbb{M}(\mathfrak{P})$, are exactly those of the form $M_{\mathfrak{P}}(X \cup \{z\})$, where $\{z\} \not\leftarrow_{\mathfrak{P}} X$.

Proposition 4.3. *The following conditions are equivalent:*

- (i) $\mathfrak{P} \in PAIR_{qa}(\Omega)$;
- (ii) $\forall Z \in \mathcal{M}_{\mathfrak{P}}, \forall Y \in \mathcal{P}(\Omega), \forall x \in \Omega$ $[\{x\} \not\leftarrow_{\mathfrak{P}} Z$ *and* $Y \smallsetminus Z \neq \varnothing$ *and* $Z \cup Y \approx_{\mathfrak{P}} Z \cup \{x\} \implies \exists y \in Y \; (\{x\} \leftarrow_{\mathfrak{P}} Z \cup \{y\})]$;
- (iii) *for any $X \in \mathcal{P}(\Omega)$ the elements of $\mathcal{I}(\mathcal{F})$, where $\mathcal{F} = (M_{\mathfrak{P}}(X))^{\uparrow}_{\mathcal{M}_{\mathfrak{P}}}$, are exactly those of the form $M_{\mathfrak{P}}(X \cup \{z\})$, where $\{z\} \not\leftarrow_{\mathfrak{P}} X$.*

Proof. $(i) \implies (ii)$: Let $Z \in \mathcal{M}_{\mathfrak{P}}$ and $Y \in \mathcal{P}(\Omega)$ be such that $Y \smallsetminus Z \neq \varnothing$. Let moreover $x \in \Omega$ be such that $Z \cup Y \approx_{\mathfrak{P}} Z \cup \{x\}$. If $\{x\} \leftarrow_{\mathfrak{P}} Z$, there is nothing to prove. So, assume that $\{x\} \not\leftarrow_{\mathfrak{P}} Z$. As $\mathfrak{P} \in PAIR_{qa}(\Omega)$, there must be some $y \in Y \smallsetminus Z$ such that $\{x\} \leftarrow_{\mathfrak{P}} Z \cup \{y\}$.
$(ii) \implies (i)$: Let $X, Y \in \mathcal{P}(\Omega)$ be such that $X \approx_{\mathfrak{P}} Y$ and $x \in X$. We claim that $\mathfrak{P} \in PAIR(\Omega)$. If $\{x\} \leftarrow_{\mathfrak{P}} X \smallsetminus \{x\}$, there is nothing to prove. Analogously, if $x \in Y \cap X$, the claim is obvious. So, take $x \in X \smallsetminus Y$ and assume that $\{x\} \not\leftarrow_{\mathfrak{P}} X \smallsetminus \{x\}$. Set $Z := M_{\mathfrak{P}}(X \smallsetminus \{x\})$. Hence $\{x\} \not\leftarrow_{\mathfrak{P}} Z$. It is immediate to verify that

$$X \approx_{\mathfrak{P}} Z \cup \{x\}. \tag{20}$$

Furthermore, since $X \approx_{\mathfrak{P}} Y$, there exists $y \in Y$ such that

$$\{y\} \not\leftarrow_{\mathfrak{P}} Z. \tag{21}$$

In view of (20), we also get $\{y\} \not\leftarrow_{\mathfrak{P}} Z \cup \{x\}$. Notice also that $Y \cup Z \approx_{\mathfrak{P}} Y$. Thus, by (20) and of our choice of Y, it follows that $Z \cup \{x\} \approx_{\mathfrak{P}} Z \cup Y$. In view of (21), we also have $y \in Y \smallsetminus Z$. So, we can use our assumption to find an element $y' \in Y$ such that

$$\{x\} \leftarrow_{\mathfrak{P}} Z \cup \{y'\}. \tag{22}$$

As $Z \approx_{\mathfrak{P}} X \smallsetminus \{x\}$, we conclude that $\{x\} \leftarrow_{\mathfrak{P}} (X \smallsetminus \{x\}) \cup \{y'\}$.
$(i) \implies (iii)$: Let $\mathfrak{P} \in PAIR_{qa}(\Omega)$. We claim that for any $X \in \mathcal{P}(\Omega)$ the elements of $\mathcal{I}(\mathcal{F})$, where $\mathcal{F} = (M_{\mathfrak{P}}(X))^{\uparrow}_{\mathcal{M}_{\mathfrak{P}}}$, are exactly those of the form $M_{\mathfrak{P}}(X \cup \{z\})$, where $\{z\} \not\leftarrow_{\mathfrak{P}} X$. To this aim, fix $X \in \mathcal{P}(\Omega)$ and an element $z \in \Omega$ such that $\{z\} \not\leftarrow_{\mathfrak{P}} X$. Consider the subsets

$$Y = \{z \in \Omega \mid M_{\mathfrak{P}}(X \cup \{z\}) = M_{\mathfrak{P}}(X \cup \{z\})\}$$

and

$$X' := M_{\mathfrak{P}}(X \cup \{z\}) \smallsetminus Y.$$

It clearly follows that $M_{\mathfrak{P}}(X) \subseteq X' \subsetneq M_{\mathfrak{P}}(X \cup \{z\})$. We now claim that $X' \in \mathcal{M}_{\mathfrak{P}}$. Let $z \in M_{\mathfrak{P}}(X') \smallsetminus X'$. Clearly, it results that $z \in Y$. Moreover, it follows that $X' \approx_{\mathfrak{P}} X \cup \{z\}$. As $\mathfrak{P} \in PAIR_{qa}(\Omega)$, there exists $x' \in X'$ such that

$$\{z\} \leftarrow_{\mathfrak{P}} [(X \cup \{z\}) \smallsetminus \{z\}] \cup \{x'\} = X \cup \{x'\}.$$

Thus, we deduce that $X \cup \{z\} \leftarrow_{\mathfrak{P}} X \cup \{x'\}$. It is also evident that $X \cup \{x'\} \leftarrow_{\mathfrak{P}} X \cup \{z\}$ since $x' \in X'$. In this way, we are saying that $X \cup \{x'\} \approx_{\mathfrak{P}} X \cup \{z\}$, whence $x' \in Y$, contradicting our choice of x'. This shows that $X' \in \mathcal{M}_{\mathfrak{P}}$.
Let now $X'' \in \mathcal{M}_{\mathfrak{P}}$ be such that $M_{\mathfrak{P}}(X) \subseteq X'' \subsetneq M_{\mathfrak{P}}(X \cup \{z\})$. We clearly have $X'' \subseteq X'$. Therefore, X' is the only co-cover of $M_{\mathfrak{P}}(X \cup \{z\})$ in the upset of $M_{\mathfrak{P}}(X)$ in the lattice $\mathbb{M}(\mathfrak{P})$.
Conversely, let $W \in \mathcal{I}(\mathcal{F})$, where $\mathcal{F} = (M_{\mathfrak{P}}(X))^{\uparrow}_{\mathcal{M}_{\mathfrak{P}}}$. Denote by W' the corresponding co-cover. Then $M_{\mathfrak{P}}(X) \subseteq W' \subsetneq W$. Suppose by contradiction that no element of the form $M_{\mathfrak{P}}(X \cup \{z\})$, with $\{z\} \not\leftarrow_{\mathfrak{P}} X$, agrees with W. Since $W' \in \mathcal{I}_{\mathcal{F}}(W)$, it follows that $M_{\mathfrak{P}}(X \cup \{z\}) \subseteq W'$ for any $x \in \Omega$ such that $\{z\} \not\leftarrow_{\mathfrak{P}} X$. Thus, taking all the elements $w \in W$ such that $\{w\} \not\leftarrow_{\mathfrak{P}} X$, we get $W \subseteq W'$, which is an absurd. Therefore, there exists $z \in \Omega$ such that $\{z\} \not\leftarrow_{\mathfrak{P}} X$ and $W = M_{\mathfrak{P}}(X \cup \{z\})$.
$(iii) \implies (i)$: Take $X \approx_{\mathfrak{P}} Y$ and fix $x \in X$. Set $X' := X \smallsetminus \{x\}$. There is nothing to prove if $\{x\} \leftarrow_{\mathfrak{P}} X'$. Therefore, assume that $\{x\} \not\leftarrow_{\mathfrak{P}} X'$. Set $\mathcal{G} := (M_{\mathfrak{P}}(X'))^{\uparrow}_{\mathcal{M}_{\mathfrak{P}}}$. In view of our assumption, it results that $\mathcal{I}_{\mathcal{G}}(M_{\mathfrak{P}}(X' \cup \{x\})) \neq \varnothing$. Let B denote the corresponding co-cover. Clearly, B cannot contain elements $z \in \Omega$ such that $X' \cup \{x\} \approx_{\mathfrak{P}} X' \cup \{z\}$. This implies that $B = M_{\mathfrak{P}}(X' \cup \{x\}) \smallsetminus Z$, where $Z = \{z \in \Omega \mid X' \cup \{x\} \approx_{\mathfrak{P}} X' \cup \{z\}\}$. At this point, notice that $Y \cap Z \neq \varnothing$, otherwise we would have $M_{\mathfrak{P}}(Y) = M_{\mathfrak{P}}(X) = M_{\mathfrak{P}}(X' \cup \{x\}) \subseteq M_{\mathfrak{P}}(X' \cup \{x\}) \smallsetminus Z \subsetneq M_{\mathfrak{P}}(X' \cup \{x\})$, which is impossible. Thus, let $y \in Y \cap Z$. We get $\{x\} \leftarrow_{\mathfrak{P}} X' \cup \{y\} = (X \smallsetminus \{x\}) \cup \{y\}$, i.e. $\mathfrak{P} \in PAIR_{qa}(\Omega)$. □

In this section we will demonstrate that when one has a locally finite quasi-attractive pairing, then all the $\mathfrak{P}$-reducts of a maximum partitioner A have the same cardinality. Moreover, we will also deduce that quasi-attractiveness on a finite set Ω may be characterized by the fact that the $\mathfrak{P}$-reducts of any subset $A \in \mathcal{P}(\Omega)$ form an exchangeable set system, property which characterizes the A-bases of a matroid on Ω, though in general $\mathcal{N}_{\mathfrak{P}}$ is not a matroid.

In the next result we will prove that all the reducts of a maximum partitioner of a locally finite quasi-attractive pairing have the same cardinality.

Theorem 4.4. *Let $\mathfrak{P} \in PAIR_{lfqa}(\Omega)$ and $X \in \mathcal{M}_{\mathfrak{P}}$ be such that $\mathcal{R}_{\mathfrak{P}}(X) \neq \varnothing$. Then $\mathcal{R}_{\mathfrak{P}}(X)$ has uniform cardinality.*

Proof. Let $W = \{w_1, \dots, w_n\}, V = \{v_1, \dots, v_m\} \in \mathcal{R}_{\mathfrak{P}}(X)$ be such that $m < n$. Take $v_1 \in V$. As $\mathfrak{P} \in PAIR_{qa}(\Omega)$, there exists an element of W, say w_1, such that $\{v_1\} \leftarrow_{\mathfrak{P}} (V \smallsetminus \{v_1\}) \cup \{w_1\}$. Set $V^{(1)} := V$ and

$$V^{(2)} := (V \smallsetminus \{v_1\}) \cup \{w_1\} = \{w_1, v_2, \dots, v_m\}.$$

It is immediate to verify that $V^{(2)} \approx_{\mathfrak{P}} V^{(1)} \approx_{\mathfrak{P}} V \approx_{\mathfrak{P}} W$. Therefore, we may use again the assumption that $\mathfrak{P} \in PAIR_{qa}(\Omega)$ on the subsets $V^{(2)}$ and W and to the element v_2 in order to find $w_2 \in W$ such that $\{v_2\} \leftarrow_{\mathfrak{P}} (V^{(2)} \smallsetminus \{v_2\}) \cup \{w_2\}$. Set

$$V^{(3)} := (V^{(2)} \smallsetminus \{v_2\}) \cup \{w_2\} = \{w_1, w_2, v_3, \dots, v_m\}.$$

Clearly we get $V^{(3)} \approx_{\mathfrak{P}} W$ and, thus, we may iterate the above procedure until we reach a subset $V^{(m)} := \{w_1, \dots, w_m\} \subsetneq W$ such that $V^{(m)} \approx_{\mathfrak{P}} W$. Thus, we conclude that $W \notin \mathcal{R}_{\mathfrak{P}}(X)$, contradicting our choice of W. This proves the claim when Ω is finite.

Let now Ω be an infinite set and also assume $W \in \mathcal{R}_{\mathfrak{P}}(X)$ to be infinite. Also the subset V must be infinite. In fact, note that for any $v \in V$ there must correspond some finite subset $F_v \in \mathcal{P}_{fin}(W)$ such that $\{v\} \leftarrow_{\mathfrak{P}} F_v$. Let now

$$Y := \bigcup \{F_v \mid v \in V\}$$

Clearly, it results $V \leftarrow_{\mathfrak{P}} Y$ and $Y \subseteq W$, whence $Y \approx_{\mathfrak{P}} V \approx_{\mathfrak{P}} W \approx_{\mathfrak{P}} X$. Since $W \in \mathcal{R}_{\mathfrak{P}}(X)$, we must necessarily have $Y = W$. If V were finite, even the subset Y (and so W) would be finite, which is impossible by our assumption. Thus, V is an infinite subset.

At this point, notice that $|W| \leq \sum_{v \in V} |F_v| \leq \aleph_0 |V| = |V|$. The thesis holds exchanging the role of W and V in the above argument. □

Corollary 4.5. *If Ω is a finite set, $\mathfrak{P} \in PAIR_{qa}(\Omega)$ and $X \in \mathcal{P}(\Omega)$, then $\mathcal{R}_{\mathfrak{P}}(X)$ has uniform cardinality.*

Proof. It is an immediate consequence of Theorem 4.4. □

In the next result we will characterize quasi-attractiveness on finite sets through the fact that the set system of the $\mathfrak{P}$-reducts of any subset X is exchangeable.

Theorem 4.6. *Let Ω be a finite set. Then the following conditions are equivalent:*

- *(i) $\mathfrak{P} \in PAIR_{qa}(\Omega)$;*
- *(ii) for any $X \in \mathcal{P}(\Omega)$ the set system $\mathcal{R}_{\mathfrak{P}}(X)$ is exchangeable.*

Proof. $(i) \implies (ii)$: Let $X \in \mathcal{P}(\Omega)$, $Y, Z \in \mathcal{R}_{\mathfrak{P}}(X)$ and $y \in Y \setminus Z$. As $Y \approx_{\mathfrak{P}} Z$ and $\mathfrak{P} \in PAIR_{qa}(\Omega)$, there exists $z \in Z$ such that

$$\{y\} \leftarrow_{\mathfrak{P}} (Y \setminus \{y\}) \cup \{z\}. \tag{23}$$

Set $W := (Y \setminus \{y\}) \cup \{z\}$. In order to show that $\mathcal{R}_{\mathfrak{P}}(X)$ is exchangeable, we will demonstrate that $z \in Z \setminus Y$ and $W \in \mathcal{R}_{\mathfrak{P}}(X)$. First of all, notice that

$$z \in Z \setminus Y, \tag{24}$$

otherwise, by (23), we would get $z \neq y$ and $\{y\} \leftarrow_{\mathfrak{P}} W = Y \setminus \{y\}$, contradicting the assumption $Y \in \mathcal{R}_{\mathfrak{P}}(X)$. Therefore, since $y \in Y \setminus Z$, by (24) it follows that $|W| = |Y|$. Moreover, again by (23), we also deduce that

$$W \approx_{\mathfrak{P}} Y \cup \{z\} \approx_{\mathfrak{P}} Y, \tag{25}$$

where the second equivalence follows by the fact that $Y \approx_{\mathfrak{P}} Z$ and $z \in Z$.

Assume now by contradiction that $W \notin \mathcal{R}_{\mathfrak{P}}(X)$. As $Y \in \mathcal{R}_{\mathfrak{P}}(X)$, we have that $X \leftarrow_{\mathfrak{P}} Y$, therefore, by (25), we also have that $X \leftarrow_{\mathfrak{P}} W$. Hence the fact that $W \notin \mathcal{R}_{\mathfrak{P}}(X)$ implies the negation of condition $(R2)$ of Definition 2.1, so that we can find some $w \in W$ such that $X \leftarrow_{\mathfrak{P}} W \setminus \{w\}$. Consider the set system

$$\mathcal{G}_w := \{W' \in \mathcal{P}(W \setminus \{w\}) \mid X \leftarrow_{\mathfrak{P}} W'\}.$$

Then $\mathcal{G}_w \neq \emptyset$ because $X \leftarrow_{\mathfrak{P}} W \setminus \{w\}$ and, since Ω is finite, we have $Min(\mathcal{G}_w) \neq \emptyset$. Let then $W^* \in Min(\mathcal{G}_w)$. As $X \leftarrow_{\mathfrak{P}} W^*$, it follows that W^* satisfies $(R1)$ of Definition 2.1 and, by the minimality of W^* in $\mathcal{G}_w$, we deduce that W^* also satisfies $(R2)$ of Definition 2.1. Thus

$$W^* \in \mathcal{R}_{\mathfrak{P}}(X) \text{ and } |W^*| < |W| = |Y| \tag{26}$$

As $Y \in \mathcal{R}_{\mathfrak{P}}(X)$, the conditions in (26) are in contrast with the statement of Corollary 4.5. So, we conclude that $W \in \mathcal{R}_{\mathfrak{P}}(X)$.

$(ii) \implies (i)$: Let $X, Y \in \mathcal{P}(\Omega)$ be such that $X \approx_{\mathfrak{P}} Y$ and let $x \in X$. We claim the existence of $y \in Y$ such that $\{x\} \leftarrow_{\mathfrak{P}} (X \setminus \{x\}) \cup \{y\}$. If $\{x\} \leftarrow_{\mathfrak{P}} X \setminus \{x\}$ we clearly have $\{x\} \leftarrow_{\mathfrak{P}} (X \setminus \{x\}) \cup \{y\}$ for any $y \in Y$, so there is nothing to prove. Similarly, if $x \in X \cap Y$, just choose $y = x$ to get the claim.

Therefore, we may assume that X are incomparable Y with respect to set-theoretical inclusion and take $x \in X \setminus Y$ such that

$$\{x\} \not\leftarrow_{\mathfrak{P}} X \setminus \{x\} \tag{27}$$

Consider at this point the set system

$$\mathcal{H}_X := \{Y \in [X]_{\approx_{\mathfrak{P}}} \mid x \in Y\}$$

As $X \in \mathcal{H}_X$ and Ω is finite, we have that $Min(\mathcal{H}_X) \neq \emptyset$. Let $\hat{X} \in Min(\mathcal{H}_X)$. We claim that $\hat{X} \in \mathcal{R}_{\mathfrak{P}}(M_{\mathfrak{P}}(X))$. As $\hat{X} \in [X]_{\approx_{\mathfrak{P}}}$, we clearly have $M_{\mathfrak{P}}(X) \leftarrow_{\mathfrak{P}} \hat{X}$, so that Property $(R1)$ of Definition 2.1 holds. Let now $X' \subsetneq \hat{X}$ and assume that $X' \in [X]_{\approx_{\mathfrak{P}}}$, i.e.

$$X' \approx_{\mathfrak{P}} \hat{X} \tag{28}$$

Then $x \in \Omega \smallsetminus X'$. By (28) and by $(P1)$,$(P2)$,$(P3)$, we easily deduce that $\{x\} \leftarrow_{\mathfrak{P}} X \smallsetminus \{x\}$, in contrast with (27). So $\hat{X}$ satisfies also Property $(R2)$ of Definition 2.1 and, thus, $\hat{X} \in \mathcal{R}_{\mathfrak{P}}(M_{\mathfrak{P}}(X))$.
Denote now by Y' some $\mathfrak{P}$-reduct of Y. In view of our assumptions, we easily deduce that $\hat{X}$ and Y' are incomparable with respect to set-theoretical inclusion. So, as $\hat{X}, Y' \in \mathcal{R}_{\mathfrak{P}}(M_{\mathfrak{P}}(X))$, by the exchangeability of $\mathcal{R}_{\mathfrak{P}}(M_{\mathfrak{P}}(X))$ there exists $y_x \in Y' \smallsetminus \hat{X}$ such that $\{x\} \leftarrow_{\mathfrak{P}} (\hat{X} \smallsetminus \{x\}) \cup \{y_x\}$. As $\hat{X} \in \mathcal{P}(X)$, it follows that $(\hat{X} \smallsetminus \{x\}) \cup \{y_x\} \leftarrow_{\mathfrak{P}} (X \smallsetminus \{x\}) \cup \{y_x\}$, so that we conclude that

$$\{x\} \leftarrow_{\mathfrak{P}} (X \smallsetminus \{x\}) \cup \{y_x\}$$

This shows that $\mathfrak{P} \in PAIR_{qa}(\Omega)$. □

Remark 4.7. *When Ω is a finite set and $\mathfrak{P} \in PAIR_{qa}(\Omega)$, the equality $\mathcal{R}_{\mathfrak{P}}(X) = Max(\mathcal{N}_{\mathfrak{P}} \cap \mathcal{P}(X))$ does not hold, as we will see in the last section. Nevertheless, notice that in general $\mathcal{R}_{\mathfrak{P}}$ agrees with the family of the bases of a matroid on Ω which is a sub-set system of $\mathcal{N}_{\mathfrak{P}}$.*

5. The Petersen Graph is Attractive

Non-trivial examples of attractive pairings from metric space theory have been found in [12]. Concerning graphs, using the results of Section 8 in [11], it may be easily shown that the complete graph K_n and the complete multipartite graph $K_{r_1,\dots,r_s}$ are attractive. Furthermore, the n-cycle is not in general attractive since if we take $G = C_5$, $X = \{v_1, v_2\}$, $x = v_2$ and $y = v_3$ it results that $\{y\} \not\leftarrow_G X$, $\{x\} \not\leftarrow_G X \smallsetminus \{x\}$ but $\{x\} \leftarrow_G X \vartriangle \{x, y\}$.
On the other hand, a second tipology of pairing can be introduced on a graph G by taking the triple $(V(G), d, \mathbb{N})$, where d denotes the distance between the vertices of the graph G. Such a pairing has been called the *distance pairing* of G. It may be easily shown that the distance pairing of the n-cycle C_n is attractive for each $n \geq 3$. Moreover, when we consider the distance pairing of a graph, the notion of reduct becomes the well-known notion of *resolvent subset* [9]. In particular, the minimum cardinality of the family of all the reducts of G agrees with the *metric dimension* of the graph G. Hence, the problem of the determination of the metric dimension of a graph has been translated as the problem of finding the minimum cardinality of the reducts of the distance pairing of a graph. In view of Theorem 4.4, we deduce that if the distance pairing associated with a graph G is quasi-attractive, then the resolvent subsets of G have uniform cardinality, which agrees with the metric dimension of G itself. In particular, an interesting task consists of the geometric characterization of the resolvent subsets and the computation of the corresponding metric dimension through the reducts of the distance pairing.
In this section we provide an example of non-trivial simple undirected graph whose adjacency pairing is attractive. This is the so-called Petersen graph, here denoted by Pet. It is the graph whose vertices can be identified with the 2-subsets of $\hat{5} := \{1, 2, 3, 4, 5\}$, such that two vertices X and Y are adjacent if and only if their corresponding 2-subsets are disjoint. Therefore, we write v_{ij} to denote the vertex identified with $\{i, j\}$. In what follows the letters h, i, j, k, l will denote all elements in $\hat{5}$ in an arbitrary order.
In what follows, we will provide a characterization of the Moore system $\mathcal{M}_{Pet}$ and, next, demonstrate that Pet is an attractive graph.
We recall now, without giving any proofs, some important well known properties of the Petersen graph.

Proposition 5.1. *We have that:*
1. *Pet is a strongly regular graph with parameters* $(10, 3, 0, 1)$.
2. *The girth of Pet is equal to* 5.

We want to provide a characterization for the members of $\mathcal{M}_{Pet}$. To this regard, we divide the proof in some propositions, where we analyze the various cases occurring. First of all notice that the emptyset and the singletons are all maximum partitioners since the Petersen graph is connected and twin-free. Furthermore, all pairs of vertices are maximum partitioners. This we now present.

Proposition 5.2. *We have that $\mathcal{P}_2(V(Pet)) \subseteq \mathcal{M}_{Pet}$.*

Proof. Set $G := Pet$. Let us now prove that any 2-subset A belongs to $\mathcal{M}_G$. To this regard, it suffices to show that whenever we take $u \in V(G) \smallsetminus A$, then $\pi_G(A) \neq \pi_G(A \cup \{u\})$, i.e. there exists $z, z' \in V(G)$ such that $z \equiv_A z'$ but $z \not\equiv_{A \cup \{u\}} z'$. Take $A = \{v, w\}$, where $v \sim w$ and let $u \notin A$. Two cases may occur: either $u \nsim v$ and $u \nsim w$ or $u \sim w$ and $u \nsim v$. In the first case, as $u \nsim v$, there exists a vertex $z \in V(G)$ such that $N_G(u) \cap N_G(v) = \{z\}$. Let moreover $z' \in N_G(v) \smallsetminus \{z\}$. It may be easily verified that $z \equiv_A z'$ but $z \equiv_{A \cup \{u\}} z'$. In the second case, v, w and u form a 3-path. Assume that $u \sim v$. Then there exist $h, i, j, k \in \hat{5}$ such that $w = v_{ij}$, $v = v_{hk}$ and $u = v_{il}$. Using the strong regularity of G, it may be easily verified that $N_G(A) = \{v_{ij}, v_{hk}, v_{il}, v_{kl}, v_{hl}, v_{jl}, v_{jk}, v_{hj}\} \neq V(G)$. Therefore there exists

$z \in V(G) \smallsetminus [N_G(w) \cup N_G(v) \cup N_G(u)]$. Moreover, let us note that there exists $z' \in V(G)$ such that $z' \nsim v$, $z' \nsim w$ and $z' \sim u$. In fact, as $N_G(u) = \{v, u', u''\}$, using the strong regularity it is straightforward to see that $u', u'' \notin N_G(v) \cup N_G(w)$. Let $z' := u'$ (the case $z' := u''$ is the same). Then $z \equiv_A z'$ but $z \not\equiv_{A \cup \{u\}} z'$.
Take now $A = \{v, w\}$, where $v \nsim w$ and let $u \in A$. Three cases may occur: or v, w and u form a 3-path, or $u \sim v$ and $u \nsim w$ or u, v and w are three non-adjacent vertices. In the first situation, the vertex u is the only element of $N_G(v) \cap N_G(w)$. Let $v = v_{ij}$ and $w = v_{ih}$ for some $h, i, j \in \hat{5}$. We may assume that $u = v_{kl}$. Take $z = v_{hj}$ and $z' = v_{ik}$. It is straightforward to see that $z \equiv_A z'$ and $z \not\equiv_{A \cup \{u\}} z'$.
In the second situation, let $v = v_{ij}$ and $w = v_{ih}$ for some $h, i, j \in \hat{5}$. We may take $u = v_{hk}$. Notice that there exist $z, t \in V(G)$ such that $\{z\} = N_G(u) \cap N_G(w)$ and $\{t\} = N_G(w) \cap N_G(v)$. Denote by z' the third vertex which is adjacent to w. It may be easily seen that $z \equiv_A z'$ and $z \not\equiv_{A \cup \{u\}} z'$.
In the third case, we have that u, v and w. If $v = v_{ij}$ and $w = v_{ih}$ for some $h, i, j \in \hat{5}$, then there are exactly three vertices non-adjacent to both v and w, namely $u = v_{ik}$, $z = v_{jh}$ and $z' = v_{il}$. Hence, we get $z \sim u$ and $z' \nsim u$. Therefore $z \equiv_A z'$ and $z \not\equiv_{A \cup \{u\}} z'$. Thus $A \in \mathcal{M}_G$ when A is a 2-subset. □

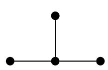
Let us now characterize the maximum partitioners with three elements.

Proposition 5.3. *Let $A \in \mathcal{P}_3(V(Pet))$. Then $A \in \mathcal{M}_{Pet}$ if and only if $G[A] \cong P_{2,1}$.*

Proof. Set $G := Pet$ and let $A = \{u, v, w\}$ be 3-subset. First of all, notice that A may assume three possible configurations, namely $P_{1,1,1}$, $P_{2,1}$ and P_3. Let $G[A] \cong P_{1,1,1}$. Then there exists $z \in V(G)$ such that either $A = N_G(z)$ or z is not adjacent to any of the three vertices of A. In the first case, the addition of the vertex z to A does not affect the induced symmetry partition, so $\pi_G(A) = \pi_G(A \cup \{z\})$. In the second case, let us note that $\pi_G(A) = A \cup \{z\} | \{t\}_{t \in (A \cup \{z\})^c}$. In fact, there is no pair of adjacent vertices in $A \cup \{z\}$ and, because of the strong regularity, the vertices of $(A \cup \{z\})^c$ are the common neighbors of pairs of vertices of $A \cup \{z\}$. Nevertheless, by the same reason, it results that $\pi_G(A \cup \{z\}) = \pi_G(A)$. Therefore, when $G[A] \cong P_{1,1,1}$, we conclude that $A \notin \mathcal{M}_G$.
Let $G[A] \cong P_3$. Without loss of generality, suppose that $u \sim v$ and $v \sim w$. Let $z \in V(G)$ be such that $v \sim z$, $z \nsim u$ and $z \nsim w$. Then we have that $\pi_G(A) = \pi_G(A \cup \{z\})$. In fact, the other two vertices $t, t' \in N_G(z) \smallsetminus \{v\}$ are not adjacent to the vertices of A, so $t \equiv_A t'$ and $t \equiv_{A \cup \{z\}} t'$. The other relations A-symmetry classes are the same since the involved vertices are not adjacent to z. Thus $A \notin \mathcal{M}_G$.
Finally, let us consider the case where $G[A] \cong P_{2,1}$. Clearly, there exist $h, i, j, k \in \hat{5}$ such that $v = v_{ij}$, $w = v_{hk}$ and $u = v_{ih}$. Using the strong regularity, it may be easily verified that $|N_G(A)| = 7$. Thus, the three vertices in $V(G) \smallsetminus N_G(A)$ form a single A-symmetry block. Set $Y := A \cup \{t\}$, where $t \in V(G) \smallsetminus A$. We will demonstrate that $\varnothing \neq V(G) \smallsetminus N_G(Y) \subsetneq V(G) \smallsetminus N_G(A)$. This clearly implies the existence of $z, z' \in V(G)$ such that $z \equiv_A z'$ and $z \not\equiv_Y z'$, ensuring that $A \in \mathcal{M}_G$. Without loss of generality we may set $v = v_{ij}$, $w = v_{hk}$ and $u = v_{ih}$. The addition of the vertex t involves some possibilities. We may have $t \sim w$, $t \nsim v$ and $t \nsim u$. Then, using the strong regularity, we may easily verify that $N_G(Y) = \{v_{il}, v_{hk}, v_{ij}, v_{jl}, v_{kl}, v_{hl}, v_{ik}, v_{jk}\}$, whence $\varnothing \neq V(G) \smallsetminus N_G(Y) \subsetneq V(G) \smallsetminus N_G(A)$.
Secondly, we may have $t \sim w$, $t \sim u$ and $t \nsim v$. In such a case, let $t = v_{lj}$. Then, using again the strong regularity, we may easily verify that $N_G(Y) = \{v_{hl}, v_{ij}, v_{hk}, v_{lj}, v_{ih}, v_{li}, v_{ik}, v_{jk}, v_{kl}\}$, whence $\varnothing \neq V(G) \smallsetminus N_G(Y) \subsetneq V(G) \smallsetminus N_G(A)$.
Moreover, it may happen that $t \sim u$, $t \nsim v$ and $t \nsim w$. Then, through the strong regularity, it may be easily shown that $N_G(Y) = \{v_{ij}, v_{hk}, v_{kl}, v_{jl}, v_{hl}, v_{il}, v_{hi}, v_{jk}\}$, whence again $\varnothing \neq V(G) \smallsetminus N_G(Y) \subsetneq V(G) \smallsetminus N_G(A)$.
Finally, we may have $t \nsim u$, $t \nsim v$ and $t \nsim v$. Again by the strong regularity, it may be easily shown that $N_G(Y) = \{v_{ij}, v_{hk}, v_{kl}, v_{hl}, v_{jl}, v_{il}, v_{ik}, v_{jk}\}$, whence again $\varnothing \neq V(G) \smallsetminus N_G(Y) \subsetneq V(G) \smallsetminus N_G(A)$. □

In the following result we will exhibit all the 4-subsets of Pet which are also maximum partitioners.

Proposition 5.4. *Let $A \in \mathcal{P}_4(V(Pet))$. Then $A \in \mathcal{M}_{Pet}$ if and only if $G[A] \cong P_{2,2}$ or $G[A] \cong P_{1,1,1,1}$ or $G[A] \cong H_1$, where H_1 has the following form:*



Proof. Set $G := Pet$ and let $A = \{u, v, w, t\}$. Assume that $G[A] \cong P_4$, with the adjacencies given by $u \sim v$, $v \sim w$ and $w \sim t$. Without loss of generality, we may suppose that $v = v_{ij}$, $u = v_{hl}$, $w = v_{hk}$ and $t = v_{il}$. Therefore, it may be easily verified through the strong regularity that $N_G(A) = \{v_{ij}, v_{hl}, v_{hk}, v_{il}, v_{hj}, v_{jk}, v_{jl}, v_{kl}, v_{ik}\}$. Thus $|N_G^c(A)| = 1$. Let now $z, z' \in V(G)$ be two distinct vertices such that $z \equiv_A z'$. Then there exists $a \in A$ such that $z \sim a$ and $z' \sim a$. Such an element a must be unique. Moreover, $z \nsim z'$. We cannot have neither $a = u$ nor $a = t$. In fact, if $z, z' \in N_G(u)$ (the case of t is similar), since $u \nsim t$, only one between z and z' must be adjacent to t and this implies $z \not\equiv_A z'$, contrarily to our assumption. So, either $a = v$ or $a = w$. This means that $z \equiv_A z'$ if and only if either $z \in N_G(v)$ and $z' = u$

or $z \in N_G(w)$ and $z' = t$. Let $B := A \cup \{z\}$. It is now immediate that $\pi_G(A) = \pi_G(A \cup \{z\})$. So, $A \notin \mathcal{M}_G$. Assume now that $G[A] \cong P_{3,1}$, with he adjacencies given by $u \sim v$ and $v \sim w$. Without loss of generality, we may suppose that $v = v_{ij}$, $u = v_{hl}$, $w = v_{hk}$ and $t = v_{hi}$. Then, using the strong regularity, we have $N_G(A) = \{v_{hl}, v_{ij}, v_{hk}, v_{ik}, v_{il}, v_{kl}, v_{jk}, v_{jl}\}$. Thus $|N_G^c(A)| = 2$ and $t \in N_G^c(A)$. Take two distinct vertices $z, z' \in V(G)$ such that $z \equiv_A z'$. This happens if and only if either $z, z' \in N_G^c(A)$ or $z = u$ and $z = w$. In fact, let $a \in A$ be the only vertex which is both adjacent to z and to z'. It cannot be $a = t$ since the strong regularity of G implies that every neighbor of t is also a neighbor of one between u, v and w; moreover it cannot be $a = u$ (the case $a = w$ is similar) since there exists a single common neighbor between u and t by the strong regularity. Thus $a = v$ and then $z = u$ and $z' = w$. Denote by x the vertex in $V(G) \setminus (N_G(A) \cup \{t\})$ and set $B := A \cup \{x\}$. Then we clearly have $\pi_G(A) = \pi_G(B)$.

Let now assume $G[A] \cong P_{2,1,1}$, with $v \sim w$. Without loss of generality, we may suppose that $v = v_{ij}$, $w = v_{hl}$, $u = v_{jl}$ and $t = v_{il}$. Using the strong regularity, we have $N_G(A) = \{v_{jl}, v_{kl}, v_{ij}, v_{hk}, v_{hj}, v_{hl}, v_{ik}, v_{jk}\}$. Thus $N_G^c(A) = \{v_{jl}, v_{il}\} = \{u, t\}$. Furthermore, in view of the previous computation, we observe that $N_G(v) \cap N_G(u) \cap N_G(t) = \varnothing$ and $N_G(w) \cap N_G(u) \cap N_G(t) = \{x\}$. Take two distinct vertices $z, z' \in V(G)$ such that $z \equiv_A z'$. Then either $z, z' \in N_G^c(A)$ or $z, z' \in N_G(A)$. In the latter situation, there exists only a vertex $a \in A$ such that $z \sim a$ and $z' \sim a$. The previous computations force a to be w. Therefore we get $z = y$ and $z' = v$, where y denotes the third vertex in $N_G(w)$. In other terms, $z \equiv_A z'$ if and only if either $z = u$ and $z' = t$ or $z = y$ and $z' = v$. Set now $B := A \cup \{x\}$. In view of our choice of x, we clearly have $\pi_G(A) = \pi_G(B)$.

Let $G[A] \cong P_{1,1,1,1}$. Hence, the vertices of A are all isolated and, by the strong regularity, any pair of vertices of A admits one common neighbor. This suffices to show that $\pi_G(A) = A|\{v\}_{v \in N_G(A)}$. Take $x \in V(G) \setminus A$. Hence there are two vertices of A, say v and w, such that $x \in N_G(v) \cap N_G(w)$. Hence $v \equiv_A u$ but $v \not\equiv_{A \cup \{x\}} u$ and this proves that $A \in \mathcal{M}_G$.

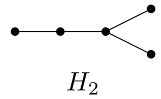
Let now assume that $G[A] \cong H_1$, with the adjacencies $u \sim v$, $v \sim w$ and $v \sim t$. Then there exists a vertex v such that $A = \{v\} \cup N_G(v)$. As v is the common neighbor of each pair of non-adjacent vertices in A, we deduce that any vertex of $A \setminus \{v\}$ admits two other distinct neighbors, so $N_G(A) = V(G)$. Now, let $z, z' \in V(G)$ be such that $z \equiv_A z'$. Then there exists only a vertex $a \in A$ such that $z \sim a$ and $z' \sim a$. In view of the above observation, it may be easily verified that $\pi_G(A) = v|N_G(v)|(N_G(a) \setminus \{v\})_{a \in N_G(v)}$. Take now $x \in V(G) \setminus A$. Denote by z, z' the elements of $N_G(t) \setminus \{v\}$. If $x \in N_G(z)$ (the case $x \in N_G(z')$ is similar), then $z \not\equiv_{A \cup \{x\}} z'$. Moreover, if $x = z$ (the case $x = z'$ is similar), then $t \not\equiv_{A \cup \{z\}} u$. Thus $A \in \mathcal{M}_G$.

Finally, assume that $G[A] \cong P_{2,2}$, with adjacencies $u \sim v$ and $w \sim t$. We firstly find the symmetry partition induced by A. To this regard, let us denote by u, v, w, t the vertices in A, with $u \sim v$ and $w \sim t$. Since $u \sim v$, there exist four distinct indexes h, j, k, $l \in \hat{5}$ such that $u = v_{hj}$ and $v = v_{kl}$. Furthermore, as $w \sim t$ without being adjacent to u and v, we may have $\{w, t\} = \{v_{ik}, v_{jl}\}$ or $\{w, t\} = \{v_{il}, v_{jk}\}$. Assume that $\{w, t\} = \{v_{ik}, v_{jl}\}$ (the proof in the other case is similar). Since $v_{ij} \nsim v_{jl}$, there exists only a vertex, namely v_{hk}, which is adjacent to both v_{ij} and v_{jl}. Similarly, let v_{hl} be the only vertex adjacent to both v_{ij} and v_{ik}, v_{hi} be the only vertex adjacent to both v_{kl} and v_{jl}, and v_{hj} be the only vertex adjacent to both v_{kl} and v_{ik}. In this way, we get $N_G(A) = \{v_{ij}, v_{kl}, v_{ik}, v_{jl}, v_{hk}, v_{hl}, v_{hi}, v_{hj}\} = V(G) \setminus \{v_{il}, v_{jk}\}$. Thus, it results that $v_{il} \equiv_A v_{jk}$ since they are not adjacent to the vertices in A. At this point, let $z, z' \in N_G(A)$ be such that $z \equiv_A z'$. Hence, using the above argument and the strong regularity of G, there exists only a vertex $a \in A$ such that $z \sim a$ and $z' \sim a$. Without loss of generality, let $a = v_{ij}$. Thus $\{z, z'\} \subseteq \{v_{kl}, v_{hl}.v_{hk}\}$. Nevertheless, $v_{hl} \sim v_{ik}$ but $v_{kl} \nsim v_{ik}$; similarly, $v_{hk} \sim v_{jl}$ but $v_{kl} \nsim v_{jl}$ and, finally, $v_{hk} \nsim v_{ik}$. This forces $z = z'$ and, therefore, we proved that $\pi_G(A) = \{v\}_{v \in N_G(A)}|N_G^c(A)$. We claim that $A \in \mathcal{M}_G$. It suffices to prove that any vertex $x \notin A$ is always adjacent to only one among the vertices of $N_G^c(A)$. To this aim, just notice that $x \in \{v_{hl}, v_{hj}, v_{hk}, v_{hi}, v_{jk}, v_{il}\}$, that $N_G(v_{il}) = \{v_{hj}, v_{jk}, v_{hk}\}$ and $N_G(v_{jk}) = \{v_{hl}, v_{hi}, v_{il}\}$. Hence, there always exists an edge between the vertices in $V(G) \setminus A$ and those of $N_G^c(A)$. However, by the strong regularity no pair of vertices may have two common neighbors. This proves the claim and concludes the proof. □

In the next proposition we will describe the behaviour of the 5-subsets of Pet.

Proposition 5.5. *Let $A \in \mathcal{P}_5(V(Pet))$. Then either $A \in \mathcal{R}_{Pet}$ or there exists a 6-subset $B \in \mathcal{M}_{Pet}$ such that $A \subseteq B$. Moreover, in the second case it results that $\pi_{Pet}(A) = \pi_{Pet}(B)$.*

Proof. Let $A \in \mathcal{P}(V(Pet))$ be a 5-subset. In view of the proof of Theorem 7.10 of [10], it may be verified that $A \in \mathcal{R}_{Pet}$ or there exists a 6-subset $B \in \mathcal{M}_{Pet}$ such that $A \subseteq B$. In the latter situation, it results that either $G[A] \cong P_{3,1,1}$ or $G[A] \cong H_2$, where H_2 has the following form:

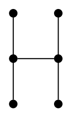


H_2

Assume firstly that $G[A] \cong P_{3,1,1}$. Let $A = \{u, v, w, z, t\}$, where z and t are the isolated vertices of $G[A]$, $u \sim v$ and $v \sim w$. Notice that two distinct vertices $u', u'' \in V(G)$ are such that $u' \equiv_A u''$ if and only if either $u' = u$ and $u'' = w$ or $u' = z$ and $u'' = t$. At this point, taking the vertex subset B and denoted by x the vertex missing from A, we get $N_G(x) = \{z, t, v\}$. Hence, again, $u' \equiv_B u''$ if and only if either $u' = u$ and $u'' = w$ or $u' = z$ and $u'' = t$, i.e. $\pi_G(A) = \pi_G(B)$.
Assume now that $G[A] \cong H_2$. As before, denote by x the vertex in $B \smallsetminus A$. Let moreover $u \sim v$, $v \sim w$, $v \sim z$ and $z \sim t$. Notice that two distinct vertices $u', u'' \in V(G)$ are such that $u' \equiv_A u''$ if and only if either $u' = u$ and $u'' = w$ or $u' = x$ and $u'' = t$. At this point, taking the vertex subset B, we clearly have that $u \equiv_B w$ and $z \equiv_B t$, whence $\pi_G(A) = \pi_G(B)$. □

In the following result we will describe the behaviour of the 6-subsets of Pet.

Proposition 5.6. *Let $A \in \mathcal{P}_6(V(Pet))$. Then the following conditions are equivalent:*
- (i) $A \not\approx_{Pet} V(Pet)$;
- (ii) $A = \Delta_{Pet}(v, w)$ *where* $v \sim w$;
- (iii) $Pet[A] \cong H_3$, *where H_3 has the following form:*



H_3

Moreover, if $Pet[A] \cong H_3$ then $A \in \mathcal{M}_{Pet}$.

Proof. Set $G := Pet$. The equivalence between (ii) and (iii) has been provided in [10]. Let us prove the equivalence between (i) and (ii). To this regard, first notice that since G is a twin-free graph, Then $\pi_G(V(G)) = v_1|\ldots|v_{10}$. Now, assume that $A = \Delta_G(v, w)$ for some two adjacent vertices v and w. Let $N_G(v) := \{w, v', v''\}$ and $N_G(w) := \{v, w', w''\}$, with v', v'', w', w'' distinct vertices. Hence $v' \equiv_A v''$ (and similarly $w' \equiv_A w''$). In fact, if $v' \sim w'$ (or $v' \sim w''$) we would obtain $N_G(v') \cap N_G(w) = \{v, w'\}$ (or $N_G(v') \cap N_G(w) = \{v, w''\}$), contradicting the fact that non-adjacent vertices in G must have only one common neighbor. Thus $A \not\approx_G V(G)$.
Conversely, let $A \not\approx_G V(G)$. We claim that $A = \Delta_G(v, w)$, for some two adjacent vertices v and w. First of all, as A is a 6-subset, let us note that $N_G(A) = V(G)$, otherwise we would find a vertex $u \in V(G)$ which is not adjacent to all the vertices of A and, thus, $|N_G(u)| \le 2$, contradicting the 3-regularity of G. Moreover, as $A \not\approx_G V(G)$, there exist $v', v'' \in V(G)$ such that $v' \equiv_A v''$. In particular, in view of the strongly regularity, it results that $v' \nsim v''$ and, thus, there exists exactly one vertex $v \in A$ such that $v' \sim v$ and $v'' \sim v$. The other vertices of $A \smallsetminus \{v\}$ are not adjacent neither to v' nor to v''. So, setting $N_G(v') = \{v, v_1', v_2'\}$ and $N_G(v'') = \{v, v_1'', v_2''\}$, we clearly have $A^c = \{v_1', v_2', v_1'', v_2''\}$. At this point, notice that $v', v'' \in A$. In fact, if one of them, say v', does not belong to A, we conclude that A^c contains five points.
As $|N_G(v)| = 3$ and v cannot be adjacent to the vertices of A^c in view of the strong regularity, there exists $w \in A$ such that $v \sim w$. Let $N_G(w) = \{v, w', w''\}$. The vertices w' and w'' must belong to A. In fact, assume $w' \notin A$. Without loss of generality, we may suppose that $w' = v_1'$ (the other cases are similar). Then w and v' are not adjacent but have two common neighbors, namely v and w', contradicting the strong regularity. So, we must have $w' \in A$. The same argument holds for w''. This proves that $A = \Delta_G(v, w)$, with $v \sim w$. In this way, we showed the equivalence between $(i), (ii)$ and (iii).
At this point, we claim that $A \in \mathcal{M}_G$. To this aim, let $u \in V(G) \smallsetminus A$. We have to find two vertices $z, z' \in V(G)$ such that $z \equiv_A z'$ but $z \not\equiv_{A \cup \{u\}} z'$. In view of the previous argument, without loss of generality we may assume $u \in N_G(v_1')$. Hence set $z := v'$ and $z' := v''$. It follows that $z \equiv_A z'$. Moreover, $z \sim u$ and $z' \nsim u$ in view of the strong regularity of G. So $z \not\equiv_{A \cup \{u\}} z'$ and $A \in \mathcal{M}_G$. □

Corollary 5.7. *We have that $\mathcal{P}_7(V(Pet)) \subseteq [V(Pet)]_{\approx_{Pet}}$.*

Let us provide the following properties of specific 4-subsets which we will be useful when showing the attractiveness of the relation $\leftharpoonup_{Pet}$.

Lemma 5.8. *Let $A = \{u, v, w, t\} \in \mathcal{P}_4(V(Pet))$. The following conditions hold:*
- (i) *if $Pet[A] \cong P_{3,1}$, then there exists $B \in \mathcal{P}(V(Pet))$ such that $Pet[B] \cong H_3$, $A \subseteq B$ and $\pi_{Pet}(A) = \pi_{Pet}(B)$;*
- (ii) *if $Pet[A] \cong P_{2,1,1}$, then there exists $B \in \mathcal{P}(V(Pet))$ such that $Pet[B] \cong H_3$, $A \subseteq B$ and $\pi_{Pet}(A) = \pi_{Pet}(B)$;*

(*iii*) *if* $Pet[A] \cong P_4$*, then there exists* $B \in \mathcal{P}(V(Pet))$ *such that* $Pet[B] \cong H_3$*,* $A \subseteq B$ *and* $\pi_{Pet}(A) = \pi_{Pet}(B)$*.*

Proof. (*i*): Let $Pet[A] \cong P_{3,1}$, where $u \sim v$ and $v \sim w$. In view of the proof of Proposition 5.4, there exists only a vertex $x \in V(G) \smallsetminus (N_G(A) \cup \{t\})$ and $\pi_{Pet}(A) = \pi_{Pet}(A \cup \{x\})$. As $Pet[A \cup \{x\}] \cong P_{3,1,1}$, by Proposition 5.6 there exists $B \in \mathcal{P}(V(Pet))$ such that $Pet[B] \cong H_3$ and $A \subseteq B$. In particular $\pi_{Pet}(A \cup \{x\}) = \pi_{Pet}(B)$.
(*ii*): Let $Pet[A] \cong P_{2,1,1}$, where $v \sim w$. In view of the proof of Proposition 5.4, there exists only a vertex $x \in N_G(w) \cap N_G(u) \cap N_G(t)$ and $\pi_{Pet}(A) = \pi_{Pet}(A \cup \{x\})$. As $Pet[A \cup \{x\}] \cong H_2$, by Proposition 5.6 there exists $B \in \mathcal{P}(V(Pet))$ such that $Pet[B] \cong H_3$ and $A \subseteq B$. In particular $\pi_{Pet}(A \cup \{x\}) = \pi_{Pet}(B)$.
(*iii*): Let $Pet[A] \cong P_4$, where $u \sim v$, $v \sim w$ and $w \sim t$. In view of the proof of Proposition 5.4, just add the third vertex of $N_G(v)$ and the third vertex of $N_G(w)$ in order to obtain the wanted 6-subset B. □

Let us recollect the above results in the following theorem.

Theorem 5.9. *Let* $A \in \mathcal{P}(V(Pet))$*. Then* $A \in \mathcal{M}_{Pet}$ *if and only if one of the following conditions hold:*

- $|A| \leq 2$*;*
- $A = V(Pet)$*;*
- $|A| = 3$ *and* $G[A] \cong P_{2,1}$*;*
- $|A| = 4$ *and* $G[A] \cong P_{2,2}$*, or* $G[A] \cong P_{1,1,1,1}$ *or* $G[A] \cong H_1$*;*
- $|A| = 6$ *and* $A = \Delta_{Pet}(v,w)$*, for some* $v \sim w$*.*

In the next result we will demonstrate that the Petersen graph is attractive.

Theorem 5.10. *Let* $V := V(Pet)$*. We have that* $Pet \in PAIR_a(V)$*.*

Proof. Set $G := Pet$. In view of Theorem 5.9, we may assume that $1 \leq |A| \leq 6$. Furthermore, in view of Propositions 5.5 and 5.6, notice that when A is a 6-subset it is never possible to choose at the same time two vertices $b \in \Omega$ and $a \in A$ such that $\{b\} \not\to_G A$ and $\{a\} \not\to_G A \smallsetminus \{a\}$.
Let now A be a singleton or a 2-subset. Hence, in view of Theorem 5.9 for any choice of $b \in \Omega$ and $a \in A$ such that $\{b\} \not\to_G A$ and $\{a\} \not\to_G A \smallsetminus \{a\}$ it easily follows that $\{a\} \not\to_G A \triangle \{a,b\}$.
Let A be an arbitrary 3-subset. Clearly, $\{a\} \not\to_G A \smallsetminus \{a\}$ for any $a \in A$. Let $\{b\} \not\to_G A$ and set $X := A \smallsetminus \{a\} \cup \{b\}$. Assume that $G[A] \cong P_{1,1,1}$. As we argued in Proposition 5.3, there exists $z \in V(G)$ such that either $A = N_G(z)$ or z is not adjacent to any of the three vertices of A. In the first case, we get $b \in V(G) \smallsetminus (A \cup \{z\})$. Hence either $G[X] \cong P_{2,1}$ or $G[X] \cong P_{1,1,1}$ and there exists $w \in V(G)$ which is not adjacent to the vertices of X. If $G[X] \cong P_{2,1}$, in view of Proposition 5.3 we clearly have $\{a\} \not\to_G X$. Otherwise $G[X] \cong P_{1,1,1}$ and there exists $w \in V(G)$ which is not adjacent to the vertices of X. Again by the proof of Proposition 5.3, it results that $X \approx_G X \cup \{z\}$ and by Proposition 5.4, it results that $X \cup \{z\} \in \mathcal{M}_G$. So, $\{a\} \not\to_G X$. In the second case, $X \approx_G X \cup \{z\}$ by the proof Proposition 5.3 and $X \cup \{z\} \in \mathcal{M}_G$. So, we get $b \neq z$. On the one hand, if b is adjacent to two vertices of A, when a is one of them and replace it with b, we obtain $G[X] \cong P_{2,1}$ and, thus $\{a\} \not\to_G X$; while when a is the remaining vertex and replace it with b, we obtain $G[X] \cong P_3$. By Proposition 5.3, it results that $M_G(X) = X \cup \{y\}$, where $X \cup \{y\} \cong H_1$ and, again, $\{a\} \not\to_G X$.
On the other hand, if b is adjacent to only one vertex of A, when a is one of the two remaining vertices and replace it with b, we obtain $G[X] \cong P_{2,1}$ and, thus $\{a\} \not\to_G X$; while when $a \sim b$, we get $G[X] \cong P_{1,1,1}$. By our choice of b, there must necessarily exists $z \in V(G)$ such that $N_G(z) = X$. Therefore $M_G(X) = X \cup \{z\}$ and, thus, $\{a\} \not\to_G X$.
Assume that $G[A] \cong P_3$. Let z be the third vertex adjacent to the central vertex of the 3-path A. Then $b \neq z$ in view of Proposition 5.4. Thus, either b is adjacent to only one extreme of the 3-path A or it is not adjacent to any vertex of A. In the first case, when we choose a to be the second extreme of the 3-path A, we obtain $G[X] \cong P_3$, so taking the vertex y which is adjacent to the central vertex of the 3-path X, we conclude that $y \neq a$ and $M_G(X) = X \cup \{y\}$, whence $\{a\} \not\to_G X$; otherwise when we choose a to be one of the other vertices, it results that $G[X] \cong P_{2,1}$ and, hence $\{a\} \not\to_G X$. In the second case, whenever we replace a vertex a with b we get $G[X] \cong P_{2,1}$ and, hence $\{a\} \not\to_G X$.
Assume that $G[A] \cong P_{2,1}$. Fix $a \in A$ and replace it with $b \in V(G) \smallsetminus A$. Set $X := A \triangle \{a,b\}$. If a is the isolated vertex, b can be adjacent to at least one of the remaining vertices, so that $G[X] \cong P_3$. In such a situation, the central vertex of the resulting 3-path is one among the previous two vertices, say v. As $M_G(X) = X \cup \{y\}$ where y is the third vertex of $N_G(v)$, and $y \neq a$, we conclude that $\{a\} \not\to_G X$. Otherwise, b can be an isolated vertex, i.e. $G[X] \cong P_{2,1}$, and then $\{a\} \not\to_G X$ as $X \in \mathcal{M}_G$.
Let now $A = \{u,v,w,t\}$ be a 4-subset. In view of Propositions 5.3 and 5.4 it may be easily verified that $A \smallsetminus \{a\} \approx_G A$ for any $a \in A$ if and only if either $G[A] \cong P_{1,1,1,1}$ or $G[A] \cong H_1$.

Assume then $G[A] \cong P_{2,2}$, with adjacencies $u \sim v$ and $w \sim t$. We may choose $a \in A$ and $b \in V(G) \setminus A$. Set $X := A \triangle \{a,b\}$. We may have either $b \in V(G) \setminus N_G(A)$ or b is adjacent to only one vertex of any of the two 2-paths, for instance $b \in N_G(u) \cap N_G(w)$. On the one hand, if $b \notin N_G(A)$, then $G[X] \cong P_{2,1,1}$. Moreover, in view of Lemma 5.8 and of Proposition 5.4, we get $M_G(X) = X \cup \{x,y\}$ where x is adjacent to b, to the other isolated vertex of X and to one of the vertices of the 2-path of X, while y is the third neighbor of the previous vertex of the 2-path of X. As $a \notin \{x,y\}$, we conclude that $\{a\} \not\leftarrow_G X$.
On the other hand, if $b \in N_G(u) \cap N_G(w)$ we have either $G[X] \cong P_{3,1}$ or $G[X] \cong P_4$. When $G[X] \cong P_{3,1}$, by the proof of Proposition 5.4 we get $M_G(X) = X \cup \{x,y\}$, where $x \in V(G) \setminus (N_G(A) \cup \{v\})$ and $y \in N_G(x) \cap N_G(v)$; while when $G[X] \cong P_4$, we get $M_G(X) = X \cup \{x,y\}$, where x is the third vertex of $N_G(b)$ and y is the third vertex of $N_G(u)$. Clearly, in both situations, we get $a \notin \{x,y\}$ and, thus, $\{a\} \not\leftarrow_G X$.
Assume now that $G[A] \cong P_4$, with adjacencies $u \sim v$, $v \sim w$ and $w \sim t$. Let $\{b\} \not\leftarrow_G A$. Hence, in view of Proposition 5.4, it follows that $b \notin N_G(v) \cup N_G(w)$. Now, we may remove an extreme a of the 4-path, say u, or an inner vertex, say v. Set $X := A \triangle \{a,b\}$. Let $a = u$. On the one hand, we may have $b \sim t$, whence $G[X] \cong P_4$. Then $M_G(X) = X \cup \{x,y\}$ where $x \in N_G(w)$ and $y \in N_G(t)$. Clearly, $a \notin \{x,y\}$ and hence $\{a\} \not\leftarrow_G X$. On the other hand, we may have $b \sim u$ or $b \notin N_G(A)$. We get $G[X] \cong P_{3,1}$ and reasoning as in the proof of Proposition 5.4, it follows that $M_G(X) = X \cup \{x,y\}$ where $x \in N_G(w) \cap N_G(b)$ and y is the third vertex of $N_G(x)$. Thus again $a \notin \{x,y\}$ and, in particular, $\{a\} \not\leftarrow_G X$.
Furthermore, if $a = v$, we may have $b \sim u$ and $b \nsim t$, whence $G[X] \cong P_{2,2}$. By Proposition 5.4 it readily follows that $\{a\} \not\leftarrow_G X$. We may also have $b \in N_G(u) \cap N_G(t)$. Then $G[X] \cong P_4$ and, by the proof of Proposition 5.8, $M_G(X) = X \cup \{x,y\}$ where $x \in N_G(t)$ and $y \in N_G(b)$. Again $\{a\} \not\leftarrow_G X$. We may have $b \sim t$ and $b \nsim u$. In such a case, we get $G[X] \cong P_{3,1}$. By Lemma 5.8, we get $M_G(X) = X \cup \{x,y\}$, where $x \in N_G(t) \cap N_G(u)$, $y \in N_G(x)$ and $y \neq u$. Clearly $a \notin \{x,y\}$ and therefore $\{a\} \not\leftarrow_G X$. Finally, we may have $b \notin N_G(A)$, whence $G[X] \cong P_{2,1,1}$. In this situation, by Lemma 5.8 notice that $M_G(X) = X \cup \{x,y\}$ where $x \in N_G(t) \cap N_G(u)$ and $y \in N_G(t)$. Thus, once again $\{a\} \notin \{x,y\}$, so $\{a\} \not\leftarrow_G X$.
Assume now that $G[A] \cong P_{3,1}$, with adjacencies $u \sim v$ and $v \sim w$. In view of Lemma 5.8, it follows that b cannot be the common neighbor of v and t and, moreover, called such a common neighbor z, we cannot have $b \sim z$. Therefore, we may choose b to be adjacent to only one extreme of the 3-path or to be adjacent to one extreme of the 3-path and to the isolated vertex. Let $a \in A$ and set $X := A \triangle \{a,b\}$. In the first case, if $a = v$ we get $G[X] \cong P_{2,1,1}$ and thus, by Lemma 5.8, we get $M_G(X) = X \cup \{x,y\}$, where $x \in N_G(w) \cap N_G(b)$ and $y \sim b$. Clearly $v \notin \{x,y\}$, so $\{a\} \not\leftarrow_G X$. If $a = w$ we get $G[X] \cong P_{3,1}$ and thus, by Lemma 5.8, we get $M_G(X) = X \cup \{x,y\}$, where $x \in N_G(u) \cap N_G(t)$ and $y \sim x$. Clearly $w \notin \{x,y\}$, whence $\{a\} \not\leftarrow_G X$. If $a = t$ we get $G[X] \cong P_4$ and thus, by Lemma 5.8, we get $M_G(X) = X \cup \{x,y\}$, where $x \in N_G(u)$ and $y \in N_G(v)$. Clearly, $t \notin \{x,y\}$ and hence $\{a\} \not\leftarrow_G X$.
In the second case, if $a = w$ we get $G[X] \cong P_4$ and thus, by Lemma 5.8, we get $M_G(X) = X \cup \{x,y\}$, where $x \in N_G(b)$ and $y \in N_G(u)$. Clearly $w \notin \{x,y\}$ and hence $\{a\} \not\leftarrow_G X$. If $a = v$ we get $G[X] \cong P_{3,1}$ and thus, by Lemma 5.8, we get $M_G(X) = X \cup \{x,y\}$, where $x \in N_G(b) \cap N_G(w)$ and $y \in N_G(x)$. Clearly $v \notin \{x,y\}$ and hence $\{a\} \not\leftarrow_G X$. If $a = t$ we get $G[X] \cong P_4$ and thus, by Lemma 5.8, we get $M_G(X) = X \cup \{x,y\}$, where $x \sim v$ and $y \sim u$. Clearly $t \notin \{x,y\}$ and hence $\{a\} \not\leftarrow_G X$. Finally, if $a = u$, we get $G[X] \cong P_{2,2}$ which belongs to $\mathcal{M}_G$ by Theorem 5.9. So again $\{a\} \not\leftarrow_G X$.
Assume now that $G[A] \cong P_{2,1,1}$, with $u \sim v$. Let $z \in N_G(w) \cap N_G(t)$. In view of the proof of Proposition 5.4, it follows that only one of the vertices u and v is adjacent to z. Suppose that u is such a vertex. Then the condition $\{b\} \not\leftarrow_G A$ becomes $b \notin N_G(u) \cup A$ by Lemma 5.8. Therefore, it may happen that either b is adjacent to only one isolated vertex of A or b is adjacent to v and to one isolated vertex of A. Let $a \in A$ and set $X := A \triangle \{a,b\}$. In the first situation, if $a = t$, we get either $G[X] \cong P_{2,2}$. By Theorem 5.9 we deduce that $\{a\} \not\leftarrow_G X$. If $a = w$, we get $G[X] \cong P_{2,1,1}$. Our choice of b and Lemma 5.8 imply that $M_G(X) = X \cup \{x,y\}$, where $x \in N_G(t) \cap N_G(v)$ and $y \in N_G(v)$. Clearly, $w \notin \{x,y\}$ and hence $\{a\} \not\leftarrow_G X$. If $a = u$, we again get $G[X] \cong P_{2,1,1}$. We supposed that $N_G(w) \cap N_G(v) \cap N_G(t) = \varnothing$. So, by Lemma 5.8 we deduce that $M_G(X) = X \cup \{x,y\}$, where $x \in N_G(v) \cap N_G(b)$ and $y \in N_G(b)$. Clearly, $u \notin \{x,y\}$ and hence $\{a\} \not\leftarrow_G X$. If $a = v$, once again we get $G[X] \cong P_{2,1,1}$. We supposed that $N_G(w) \cap N_G(u) \cap N_G(t) \neq \varnothing$. So, by Lemma 5.8 we deduce that $M_G(X) = X \cup \{x,y\}$, where $x \in N_G(u) \cap N_G(w)$ and $y \in N_G(w)$. Clearly, $v \notin \{x,y\}$ and hence $\{a\} \not\leftarrow_G X$.
In the second situation, let $b \sim v$ and assume, without loss of generality, $b \sim w$. Fix $a \in A$ and set $X := A \triangle \{a,b\}$. If $a = t$, we get $G[X] \cong P_4$. By Lemma 5.8 we deduce that $M_G(X) = X \cup \{x,y\}$, where $x \in N_G(b)$ and $y \in N_G(v)$ and, as $t \notin \{x,y\}$, we conclude that $\{a\} \not\leftarrow_G X$. If $a = w$, we get $G[X] \cong P_{3,1}$. By Lemma 5.8 we deduce that $M_G(X) = X \cup \{x,y\}$, where $x \in N_G(v) \cap N_G(t)$ and $y \in N_G(x)$. Clearly $w \notin \{x,y\}$ and thus $\{a\} \not\leftarrow_G X$. If $a = u$, we again get $G[X] \cong P_{3,1}$. By Lemma 5.8 we deduce that $M_G(X) = X \cup \{x,y\}$, where $x \in N_G(b) \cap N_G(t)$ and $y \in N_G(x)$. Clearly, $u \notin \{x,y\}$ and hence $\{a\} \not\leftarrow_G X$.

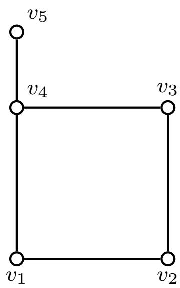


FIGURE 1. The Graph of Example 5.11.

	v_1	v_2	v_3	v_4	v_5
v_1	0	1	0	1	0
v_2	1	0	1	0	0
v_3	0	1	0	1	0
v_4	1	0	1	0	1
v_4	0	0	0	1	0

FIGURE 2. The Adjacency Matrix of the Graph of Example 5.11

If $a = v$, we get $G[X] \cong P_{2,1,1}$. In view of Lemma 5.8, there exists a 6-subset B containing X and inducing the same symmetry partition as X. Let $x \in N_G(u) \cap N_G(t)$. If $x \in N_G(b)$, we would have $v \in B$ and, in particular, $\{b\} \leftarrow_G A$, contradicting our choice of b. Therefore, $B = M_G(X)) = X \cup \{x, y\}$, where $x \in N_G(w) \cap N_G(t)$ and $y \in N_G(w)$. Clearly $u \notin \{x, y\}$ and hence $\{a\} \not\leftarrow_G X$.
Finally, assume that $A = \{u, v, w, t, z\}$ is a 5-subset. By Proposition 5.5 and Theorem 5.9, we may only suppose either $G[A] \cong P_{3,1,1}$ or $G[A] \cong H_2$. In the first case, let $u \sim v$ and $v \sim w$. Let $\{b\} \not\leftarrow_G A$. In view of Proposition 5.5, we can choose b to be adjacent to one extreme of the 3-path and to one isolated vertex. Without loss of generality, take $b \in N_G(u) \cap N_G(t)$. Let $a \in A$ be such that $\{a\} \not\leftarrow_G A \setminus \{a\}$. In view of Lemma 5.8, the only choice of a is v. Set $X := A \triangle \{a, b\}$. Then $G[X] \cong P_{3,1,1}$. By Propositions 5.5 and 5.6 it follows that $X \subseteq B$, where $B \in \mathcal{M}_G$ is a 6-subset such that $G[B] \cong H_3$. The only way to obtain such a subset B consists of adding to X the vertex $y \in N_G(b) \cap N_G(z) \cap N_G(w)$. Clearly, $y \neq a$ and thus $\{a\} \not\leftarrow_G X$.
In the second case, let $v \sim w$, $w \sim u$, $u \sim t$ and $u \sim z$. Let $\{b\} \not\leftarrow_G A$. Then $b \notin N_G(w)$. Without loss of generality, we may choose have either $b \in N_G(v) \cap N_G(t)$ or $b \in N_G(t) \setminus N_G(v)$. In both situations, it follows that $X \cong H_2$ and adding the remaining vertex y of $N_G(t)$, we get $X \subseteq X \cup \{y\}$. As $y \neq a$, we conclude $\{a\} \not\leftarrow_G X$. This proves the thesis. □

The example of the Petersen graph may induce to think the existence of a relation between strongly regularity and attractiveness. Nevertheless, there is no link between the above properties. In fact, in reference to the discussion of the 5-cycle at the beginning of Section 5, notice that when a graph is strongly regular, it does not necessarily turn out to be an attractive graph.
The converse does not even hold as we will see in the following example.

Example 5.11. Consider the graph G in Figure 1 and whose adjacency matrix is given in Figure 2. The aforementioned graph is not regular (and hence it is not strongly regular). It may be easily shown that

$$\mathcal{M}_G = \{\varnothing, \{v_1, v_3, v_4\}, \{v_2\}, \{v_5\}, \{v_1, v_2, v_3, v_4\}, \{v_1, v_3, v_4, v_5\}, V(G)\}$$

and that

$$[\varnothing]_{\approx_G} = \{\varnothing\},\ [\{v_1, v_2, v_3\}]_{\approx_G} = \{\{v_1\}, \{v_3\}, \{v_4\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_3, v_4\}, \{v_1, v_3, v_4\}\},\ [\{v_2\}]_{\approx_G} = \{\{v_2\}\},$$

$$[\{v_5\}]_{\approx_G} = \{\{v_5\}\},\quad [\{v_2, v_5\}]_{\approx_G} = \{\{v_2, v_5\}\},$$

$$[\{v_1, v_2, v_3, v_4\}]_{\approx_G} = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_2, v_3, v_4\}, \{v_1, v_2, v_3, v_4\}\},$$

$$[\{v_1, v_3, v_4, v_5\}]_{\approx_G} = \{\{v_1, v_5\}, \{v_3, v_5\}, \{v_4, v_5\}, \{v_1, v_3, v_5\}, \{v_1, v_4, v_5\}, \{v_3, v_4, v_5\}, \{v_1, v_3, v_4, v_5\}\},$$

$$[V(G)]_{\approx_G} = \{\{v_1, v_2, v_5\}, \{v_2, v_3, v_5\}, \{v_2, v_4, v_5\}, \{v_1, v_2, v_3, v_5\}, \{v_1, v_2, v_4, v_5\}, \{v_2, v_3, v_4, v_5\}, V(G)\}.$$

The reader can now easily check that G is an attractive graph.

6. Friendship Graphs are Quasi-Attractive but not Attractive

We now find a model of quasi-attractive pairing induced by the adjacency matrix of a specific graph family. To be more detailed, in this section we deal with a family of graphs inducing a quasi-attractive relation. This is the so-called *Erdös' friendship graph* F_n. We will characterize the members of the Moore system $\mathcal{M}_{F_n}$ and, next, we will prove that the pairing induced by F_n satisfies quasi-attractive but not attractiveness.

Let us consider the set $V := \{v_1, v_2, \dots, v_{2n+1}\}$ and take the partition $\mathcal{T} := \{S_1, \dots, S_n, S_{n+1}\}$, where $S_i := \{v_{2i-1}, v_{2i}\}$ for each $i = 1, \dots, n$, and $S_{n+1} := \{v_{2n+1}\}$.

We will consider the graph F_n whose vertex set is $V(F_n) := V$ and edge set $E(G) := \{S_1, \dots, S_n, \{v_j, v_{2n+1}\}_{j=1}^{2n}\}$.

In Figure 3, we represent F_4.

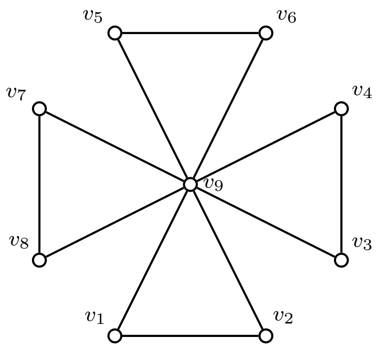


Figure 3. The Graph F_4.

Set

$$\mathcal{G}_X := \{S \in \mathcal{T} \smallsetminus \{S_{n+1}\} \mid S \subseteq X\}, \quad W_X := \bigcup \mathcal{G}_X$$

$$\mathcal{F}_X := \{S \in \mathcal{T} \mid |S \cap X| = 1\}, \quad Z_X := \bigcup \{S \cap X \mid S \in \mathcal{F}_X\} \text{ and } Y_X := \bigcup \{S \smallsetminus X \mid S \in \mathcal{F}_X\}.$$

We now exhibit the X-symmetry partition of F_n for each non-empty vertex subset $X \in \mathcal{P}(V(F_n))^*$.

Proposition 6.1. *For each non-empty vertex subset $X \in \mathcal{P}(V(F_n))^*$, we have that*

$$\pi_{F_n}(X) = \begin{cases} Y_X \cup \{v_{2n+1}\} | [Y_X \cup \{v_{2n+1}\}]^c & \text{if } X = \{v_i\} \text{ for some } i = 1, \dots, 2n \\ \{v\}_{v \in Y_X} | \{w\}_{w \in W_X} | v_{2n+1} | (Y_X \cup W_X \cup \{v_{2n+1}\})^c & \text{otherwise} \end{cases} \tag{29}$$

Proof. Set $G := F_n$. Let $w \in X \smallsetminus \{v_{2n+1}\}$. Then $w \in S_i$ for some $i = 1, \dots, n$. In particular, we have either $w = v_{2i-1}$ or $w = v_{2i}$. The only vertices adjacent to w are exactly the remaining vertex of S_i and v_{2n+1}. Therefore, if $X = \{v_{2i-1}\}$, we get $\pi_G(X) = v_{2i}v_{2n+1} | \{v_{2i}v_{2n+1}\}^c$; while if $X = \{v_{2i}\}$, we get $\pi_G(X) = v_{2i-1}v_{2n+1} | \{v_{2i-1}v_{2n+1}\}^c$.

Furthermore, if $X = \{v_{2n+1}\}$, we have $X = Z_X$, $W_X = Y_X = \varnothing$ and any vertex is adjacent to v_{2n+1} except for v_{2n+1} itself. Thus, in this case, $\pi_G(X) = v_{2n+1} | (v_{2n+1})^c$.

Assume that $|X| \geq 2$. Take $v \in W_X$. Hence there exists $S \in \mathcal{T} \smallsetminus \{S_{n+1}\}$ such that $v \in S$. Let moreover v' be the remaining vertex of S. Then, v is the only vertex which is adjacent to v' but not v. Thus, the vertices of W_X form single blocks. Let now $v \in Y_X$. Then $v \in S \smallsetminus X$ for some $S \in \mathcal{T} \smallsetminus \{S_{n+1}\}$. Denote by v' the remaining vertex of S. Clearly, $v' \in X$. The only vertices which are adjacent to v' are v and v_{2n+1} but, since $|X| \geq 2$, there exists another vertex $u \in X$ and $v_{2n+1} \sim u$ while $v \nsim u$. This proves that the vertices of Y_X form single blocks.

Furthermore, the vertex v_{2n+1} constitutes a single symmetry block, since it is the only vertex of the graph which is adjacent to all the vertices of $X \smallsetminus \{v_{2n+1}\}$ and not adjacent to v_{2n+1} itself.

Finally, the remaining vertices can be adjacent to at most one vertex of X, i.e. the vertex v_{2n+1} (if it belongs to X). Thus they form a single symmetry block and this concludes the proof of (29). ☐

In the next result we will characterize the maximum partitioners of the graph F_n.

Proposition 6.2. *Let $X \in \mathcal{P}(V(F_n))$. Then $X \in \mathcal{M}_{F_n}$ if and only if it results that*

(a) $|X| \leq 1$ *or*
(b) $X = V(F_n)$ *or*
(c) $X = Z_X \cup W_X \cup \{v_{2n+1}\}$, *where Z_X and W_X satisfy one of the following conditions:*
  - $|Z_X \smallsetminus \{v_{2n+1}\}| \leq 1$ *and* $|Z_X \smallsetminus \{v_{2n+1}\}| + \frac{|W_X|}{2} \leq n - 1$;
  - $|Z_X \smallsetminus \{v_{2n+1}\}| \geq 2$.

Proof. Set $G := F_n$. There is nothing to prove when $X = V(G)$. Moreover, in view of (29), it follows that $\pi_G(\{v\}) \neq \pi_G(\varnothing)$ for each $v \in V(G)$, thus $\varnothing \in \mathcal{M}_G$. It may be also easily verified that all singletons are maximum partitioners of G. In fact, if $X = \{v, w\}$, then one of the vertices, say v, must belong to some S_i, where $i = 1, \dots, n$. Let $v' \in S_i \setminus \{v\}$. We may have either $v' \in Y_X$ or $v' \in W_X$. However, in both cases, we get $v' \equiv_{\{v\}} v_{2n+1}$ while $v' \not\equiv_X v_{2n+1}$ by (29). Thus all singletons are maximum partitoners of G.
Consider a vertex subset $X \neq V(G)$ such that $|X| \geq 2$ and $X = Z_X \cup W_X \cup \{v_{2n+1}\}$. We will demonstrate the existence of two vertices u, u' such that $u \equiv_X u'$ and $u \not\equiv_{X\cup\{w\}} u'$ when either $|Z_X \setminus \{v_{2n+1}\}| \leq 1$ and $|Z_X \setminus \{v_{2n+1}\}| + \frac{|W_X|}{2} \leq n-1$ or $|Z_X \setminus \{v_{2n+1}\}| \geq 2$. In other terms, we want to show that $X \in \mathcal{M}_G$ when either $|Z_X \setminus \{v_{2n+1}\}| \leq 1$ and $|Z_X \setminus \{v_{2n+1}\}| + \frac{|W_X|}{2} \leq n-1$ or $|Z_X \setminus \{v_{2n+1}\}| \geq 2$.
First of all, take $w \notin X$. Notice that there exists an index $i \in \{1, \dots, n\}$ such that $w \in S_i$. Let $w' \in S_i \setminus \{w\}$ and assume that $w' \notin X$. Hence $w, w' \in [Y_X \cup W_X \cup \{v_{2n+1}\}]^c$ and, thus, $w \equiv_X w'$. Furthermore, as $w' \in Y_{X\cup\{w\}}$ and $w \in Z_{X\cup\{w\}} \setminus \{v_{2n+1}\}$, we get $w \not\equiv_{X\cup\{w\}} w'$.
On the one hand, assume that $|Z_X \setminus \{v_{2n+1}\}| \leq 1$ and $|Z_X \setminus \{v_{2n+1}\}| + \frac{|W_X|}{2} \leq n-1$. Take $w \notin X$. Clearly, $w \in S_i$ for some $i = 1, \dots, n$. Let us denote by w' the vertex in $S_i \setminus \{w\}$. Assume firstly that $w' \in X$. Hence $w' \in Z_X \setminus \{v_{2n+1}\}$ and, in particular, $\frac{|W_X|}{2} \leq n-2$. Thus, there exists S_j, wher $j \neq i$, such that $S_j \cap X = \varnothing$. Let $u \in S_j$ and set $u' := w$. Hence, by (29), we easily get $u \equiv_X u'$. Nevertheless, we get $w' \in W_{X\cup\{w\}}$ and $S_j \cap (X \cup \{w\}) = \varnothing$. Therefore, again by (29), we conclude that $u \not\equiv_{X\cup\{w\}} u'$, as wanted. On the other hand, assume now that $|Z_X \setminus \{v_{2n+1}\}| \geq 2$. Take $w \notin X$. Hence there exists some $i \in \{1, \dots, n\}$ such that $w \in S_i$. Denote by w' the vertex of $S_i \setminus \{w\}$. As $|Z_X \setminus \{v_{2n+1}\}| \geq 2$ there must be a vertex $u \in Z_X \setminus \{v_{2n+1}\}$ which is different from w' and which clearly belongs to S_j, for some $j \neq i$. Then, after setting $u' := w'$, by (29) we get $u \equiv_X u'$. Nevertheless, it results that $u \in W_{X\cup\{w\}}$ and $u' \in Z_{X\cup\{w\}}$, thus $u \not\equiv_{X\cup\{w\}} u'$.
This shows the right implication. Conversely, take $X \in \mathcal{M}_G$ such that $|X| \geq 2$ and $X \neq V(G)$. In view of (29), if $v_{2n+1} \notin X$ and $|X| \geq 2$, then $\pi_G(X) = \pi_G(X \cup \{v_{2n+1}\})$. Moreover, in view of the definition of Z_X and of W_X and by the above remark, we get

$$X = Z_X \cup W_X \cup \{v_{2n+1}\}.$$

We will demonstrate that either $|Z_X \setminus \{v_{2n+1}\}| \leq 1$ and $|Z_X \setminus \{v_{2n+1}\}| + \frac{|W_X|}{2} \leq n-1$ or $|Z_X \setminus \{v_{2n+1}\}| \geq 2$. To this regard, let $w \notin X$. Hence there exists an index $i \in \{1, \dots, n\}$ such that $w \in S_i$. Denote by w' the remaining vertex of $S_i \setminus \{w\}$. As $X \in \mathcal{M}_G$, we have that $\pi_G(X) \neq \pi_G(X \cup \{w\})$. Assume firstly that $w' \notin X$. Thus $S_i \cap X = \varnothing$ and, in particular, we deduce that $\frac{|W_X|}{2} \leq n-1$. In particular, if $|Z_X \setminus \{v_{2n+1}\}| \leq 1$, it must necessarily be $|Z_X \setminus \{v_{2n+1}\}| + \frac{|W_X|}{2} \leq n-1$.
Let us now assume that $w' \in X$. Hence we get $w \in Z_X \setminus \{v_{2n+1}\}$ and $w \in W_{X\cup\{w\}}$. As $\pi_G(X) \neq \pi_G(X \cup \{w\})$, there must exist $u \in [Y_X \cup W_X \cup \{v_{2n+1}\}]^c$ such that $w \equiv_X u$ and $w \not\equiv_{X\cup\{w\}} u$. Two cases may occur: either $u \in Z_X \setminus \{v_{2n+1}\}$ and, hence $|Z_X \setminus \{v_{2n+1}\}| \geq 2$ or $|Z_X \setminus \{v_{2n+1}\}| = 1$. In the latter case, $u \in S_j$ for some $j \neq i$. In particular, it results that $S_j \cap X = \varnothing$ and, thus, $\frac{|W_X|}{2} \leq n-2$. This concludes the proof. □

We are now able to characterize the minimal partitioners of F_n.

Proposition 6.3. *Let $X \in \mathcal{P}(V(F_n))$. Then $X \in \mathcal{N}_{F_n}$ if and only if*

- (a) $|X| \leq 1$ *or*
- (b) $|X| = 2$ *and* $v_{2n+1} \in X$ *or*
- (c) $|X| = 2n-1$ *and* $v_{2n+1} \notin X$ *or*
- (d) $X = Z_X \cup W_X$, $v_{2n+1} \notin X$ *and* Z_X *and* W_X *satisfy one of the following conditions:*
  - $|Z_X| \leq 1$ *and* $|Z_X| + \frac{|W_X|}{2} \leq n-1$;
  - $|Z_X| \geq 2$.

Proof. Set $G := F_n$. There is nothing to prove if $|X| \leq 1$. Assume therefore that $|X| \geq 2$. We firstly claim that $X \approx_G V(G)$ if and only if either $|X| \geq 2n$ or $|X| = 2n-1$ and $v_{2n+1} \notin X$. To this regard, in view of Proposition 6.2 it results that $X \approx_G V(G)$ if and only if the two following cases occur: either $|Z_X \setminus \{v_{2n+1}\}| = 1$ and $\frac{|W_X|}{2} = n-1$ or $Z_X \setminus \{v_{2n+1}\} = \varnothing$ and $\frac{|W_X|}{2} = n$. The first case implies that either $|X| = 2n$ or $|X| = 2n-1$, depending on whether v_{2n+1} belongs or not to X; while the second implies that $|X| = 2n$.
Suppose now $X \not\approx_G V(G)$. Let us prove that $[X]_{\approx_G}$ contains two elements if and only if $X \neq \{v_i, v_{2n+1}\}$. To this aim, assume that $[X]_{\approx_G}$ contains two elements. Take $X := \{v_i, v_{2n+1}\}$ for some $i = 1, \dots, 2n$. In view of Proposition 6.2, it results that $X \in \mathcal{M}_G$ and, moreover, by the same result we have that each singleton belongs to $\mathcal{M}_G$. So it results that $[X]_{\approx_G} = \{X\}$, contradicting our assumption.
Conversely, let us assume that $X \neq \{v_i, v_{2n+1}\}$. Our assumptions on X imply that $X = Z_X \cup W_X$, where

either $|Z_X \smallsetminus \{v_{2n+1}\}| \geq 2$ or $|Z_X \smallsetminus \{v_{2n+1}\}| \leq 1$ and $2 \leq |Z_X \smallsetminus \{v_{2n+1}\}| + \frac{|W_X|}{2} \leq n-1$. In view of Proposition 6.2 the previous configurations belong to $\mathcal{M}_G$ when $v_{2n+1} \in X$. Nevertheless, by (29) just notice that if $v_{2n+1} \notin X$, then $X \approx_G X \cup \{v\}$ and this implies that $[X]_{\approx_G}$ contains two elements. □

Remark 6.4. *Corollary 6.3 and part (vii) of Theorem 2.2 ensure that $\mathcal{R}_{\mathfrak{P}} \subsetneq Max(\mathcal{N}_{\mathfrak{P}})$ since the last family contains the vertex subsets of the form $\{v_i, v_{2n+1}\}$, with $i = 1, \dots, 2n$, while the former does not.*

Remark 6.5. *We clearly have $F_1 = K_3$, i.e. it is complete graph on 3 vertices. We have that K_3 is attractive and, hence, also quasi-attractive.*

In the next theorem we will demonstrate that friendship graphs are quasi-attractive but not attractive.

Theorem 6.6. *Let $V := V(F_n)$. Then $F_n \in PAIR_{qa}(V) \smallsetminus PAIR_a(V)$ for each $n \geq 2$.*

Proof. In view of Proposition 6.3, it is immediate to check that $RED_{F_n}(X)$ forms an exchangeable set system for any $X \in \mathcal{P}(\Omega)$. Thus, by Theorem 4.6 we conclude that F_n is quasi-attractive for each $n \geq 2$. To show that F_n is not attractive, in view of part (iii) of Theorem 3.5 it suffices to prove that $\mathcal{N}_{F_n}$ is not a matroid. To this regard, take the vertex subsets $X = \{v_{2n}, v_{2n+1}\}$ and $Y = \{v_1, v_2, v_{2n}\}$. In view of Corollary 6.3, it results that $X, Y \in \mathcal{N}_{F_n}$. Nevertheless, notice that neither $X \cup \{v_1\}$ nor $X \cup \{v_2\}$ belong to $\mathcal{N}_{F_n}$. Thus, $\mathcal{N}_{F_n}$ is not a matroid and the graph F_n cannot be attractive. □

Remark 6.7. *The previous result gives an example of quasi-attractive graph which is not regular and, a fortiori, which is not strongly regular. The converse does not even hold, as one may see taking again the 5-cycle C_5. Set $G := C_5$ and let $X = \{v_1, v_2, v_4\}$, $Y = \{v_1, v_3, v_5\}$ and $x = v_4$. It may be easily verified that $X \approx_G Y$ and that neither $\{x\} \leftarrow_G X \vartriangle \{x, v_3\}$ nor $\{x\} \leftarrow_G X \vartriangle \{x, v_5\}$.*

References

[1] S. Akbari, R. Nikandish, M. J. Nikmehr, Some results on the intersection graphs of ideals of rings, *Journal of Algebra and its Applications*, Vol. 12, No. 4 (2013) 1250200 (13 pages).

[2] J.A. Aledo, S. Martínez, J. C. Valverde, Parallel dynamical systems over directed dependency graphs, *Applied Mathematics and Computation*, 219 no.3, (2012), 1114–1119.

[3] J.A. Aledo, L.G. Diaz, S. Martinez, J.C. Valverde, Solution to the predecessors and Gardens-of-Eden problems for synchronous systems over directed graphs, *Applied Mathematics and Computation*, 347 (2019), 22–28.

[4] C. Bisi, On commuting polynomial automorphisms of $\mathbb{C}^k$, $k \geq 3$, *Mathematische Zeitschrift*, 258(4) (2008), 875–891.

[5] C. Bisi, On Closed Invariant Sets in Local Dynamics, *Journal of Mathematical Analysis and Applications*, 350(1) (2009), 327–332.

[6] C. Bisi, A Landau's theorem in several complex variables. *Annali di Matematica Pura ed Applicata*, Vol. 196, Issue 2, (2017), 737-742.

[7] S. Bocker, A. Dress, Patchworks, *Advances in Mathematics*, 157 (2001), 1–21.

[8] P. Bonacini, M. Gionfriddo, L. Marino, Nesting House-designs, *Discrete Mathematcis* 339 (4) (2016), 1291–1299.

[9] G. Chartrand, L. Eroh, M.A. Johnson, O.R.Oellermann, Resolvability in graphs and the metric dimension of a graph. *Discrete Applied Mathematics* 2000;105: 99–113.

[10] G. Chiaselotti, T. Gentile, F. Infusino, P. A. Oliverio, The Adjacency Matrix of a Graph as a Data Table. A Geometric Perspective. *Annali di Matematica Pura e Applicata*, Vol. 196, No. 3, (2017), 1073–1112.

[11] G. Chiaselotti, T. Gentile, F. Infusino, Simplicial Complexes and Closure Systems induced by Indistinguishability Relations. *C. R. Acad. Sci. Paris, Ser. I*, 355 (2017), 991–1021.

[12] G. Chiaselotti, T. Gentile, F. Infusino. New Perspectives of Granular Computing in Relation Geometry induced by Pairings. *Annali dell'Universitá di Ferrara* Vol.65, Issue 1, May 2019, 57–94.

[13] G. Chiaselotti, T. Gentile, F. Infusino, Symmetry Geometry by Pairings, *Journal of the Australian Mathematical Society* Vol. 106, Issue 3, 2019, 342–360.

[14] G. Chiaselotti, T. Gentile, F. Infusino, Local Dissymmetry on Graphs and Related Algebraic Structures. *International Journal of Algebra and Computation*, Vol. 29, No. 8, 2019, 1499–1526.

[15] G. Chiaselotti, F. Infusino, P. A. Oliverio, Set Relations and Set Systems induced by some Families of Integral Domains. *Advances in Mathematics*, 363 (2020) 106999, https://doi.org/10.1016/j.aim.2020.106999.

[16] G. Chiaselotti, F. Infusino, Alexandroff Topologies and Monoid Actions, *Forum Mathematicum*, Volume 32, Issue 3 (2020), 795–826.

[17] G. Chiaselotti, F. Infusino, Some Classes of Abstract Simplicial Complexes motivated by Module Theory, *Journal of Pure and Applied Algebra* (2020), 106471, doi: https://doi.org/10.1016/j.jpaa.2020.106471.

[18] B. A. Davey, H. A. Priestley, *Introduction to lattices and order*. Second edition. Cambridge University Press, New York, (2002).

[19] R. Diestel, Graph Theory (4th edition), Graduate Text in Mathematics, Springer 2010.

[20] P. Dorbec, M. Montassier, P. Ochem, Vertex Partitions of Graphs into Cographs and Stars; *Journal of Graph Theory*, 75 (1), pp.75–90, 2014.

[21] A. Dress, Even Set Systems, *Annals of Combinatorics*, 13 (2009), 193–198.

[22] A. Dress, M. Steel, Phylogenetic Diversity Over an Abelian Group, *Annals of Combinatorics*, 11 (2007), 143–160.

[23] P. Erdös, A. Rényi; Asymmetric Graphs, *Acta Mathematica Hungarica*, Vol.14, Issue 3–4, pp. 295–315, 1963.

[24] P. Erdös, A. Rényi, V.T. Sos, On a problem of graph theory, *Studia Sci. Math. Hungar.* (1966), 1: 215–235.

[25] S. Faridi, The Facet Ideal of a Simplicial Complex, *Manuscripta Mathematica*, 109 (2002), 159–174.

[26] S. Faridi, Simplicial Trees Are Sequentially Cohen-Macaulay, *Journal of Pure and Applied Algebra*, Volume 190, Issues 1-3 (2004), 121–136.
[27] M. Farshi, B. Davvaz, S. Mirvakili, Hypergraphs and Hypergroups Based on a Special Relation, *Communications in Algebra*, 42 (2014), 3395–3406.
[28] A. R. Gairola, K. K. Singh, V. N. Mishra, Rate of approximation by q-Durrmeyer operators in $L_p([0,1])$, $1 \le p \le \infty$, *Annals of Functional Analysis*, 8 (3), 303–313 (2017).
[29] M. Gionfriddo, E. Guardo, L. Milazzo, Extending bicolorings for Steiner triple systems, *Applicable Analysis and Discrete Mathematics* (2013), 225–234.
[30] M. Groshaus, J. L. Szwarcfiter, Biclique graphs and biclique matrices, Journal of Graph Theory 63 (1), 2010, 1–16.
[31] M. Groshaus, L. P. Montero, On the Iterated Biclique Operator, Journal of Graph Theory 73 (2), 2013, 181–190.
[32] M. Hagen; Lower bounds for three algorithms for transversal hypergraph generation; *Discrete Applied Mathematics*, Vol.157, pp. 1460–1469, 2009.
[33] G. Hahn and G. Sabidussi (Editors); Graph Symmetry. Algebraic Methods and Applications; *NATO ASI Series*, Vol.497, 1997, Springer.
[34] F. Harary, R.A. Melter, On the metric dimension of a graph, *Ars Combinatoria*, (1976) 2: 191–195.
[35] E. Katsoulis, D. Kribs, Isomorphisms of algebras associated with directed graphs, *Mathematische Annalen*, 330 (2004), 709–728.
[36] A. V. Kelarev, S. J. Quinn, Directed Graphs and Combinatorial Properties of Semigroups. *Journal of Algebra*, 251 (2002), 16–26.
[37] M. Mahmoudi, A. Mousivand, M. Crupi, G. Rinaldo, N. Terai, Vertex decomposability and regularity of very well-covered graphs, *Journal of Pure and Applied Algebra*, Volume 215, Issue 3 (2011), 2473–2480.
[38] V.N. Mishra, R.N. Mohapatra, P. Sharma, On approximation properties of Baskakov-Szász-Stancu operators using hypergeometric representation, *Applied Mathematics and Computation*, 294, 77–86 (2017).
[39] D.B. Parker, R.F. Westhoff, M.J. Wolf, On two-path convexity in multipartite tournaments, *European Journal of Combinatorics*, 29 (2008), 641–651.
[40] S. M. Sanahuja, New rough approximations for n-cycles and n-paths, *Applied Mathematics and Computation*, (2016) 276, 96–108.
[41] W.T. Tutte, An algorithm for determining whether a given binary matroid is graphic, *Proceedings of the American Mathematical Society*, 11 (1960), 905–917.
[42] D. J. A. Welsh; Matroid Theory. Academic Press, 1976.

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