Monomial Togliatti Systems of Cubics

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CURVES OF MAXIMUM GENUS IN RANGE A AND STICK-FIGURES

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Abstract. In this paper we show the existence of smooth connected space curves not contained in a surface of degree less than \( m \), with genus maximal with respect to the degree, in half of the so-called range A. The main tool is a technique of deformation of stick-figures due to G. Fløystad.

0. Introduction

A classical problem, which goes back to Halphen [H], is to determine, for given integers \( d \) and \( m \), the maximal genus \( G(d, m) \) of a smooth projective curve of degree \( d \) not contained in a surface of degree \( < m \). This problem is actually very natural, and it is in fact one of the cornerstones of the numerical classification of space curves.

The problem of determining \( G(d, m) \) depends on the size of \( d \) with respect to \( m \). Following a long tradition, we distinguish four ranges:

Range \( \emptyset \). If \( d < \frac{m^2+4m+6}{6} \), then every curve \( X \) of degree \( d \) satisfies \( h^0(\mathbb{P}^3, \mathcal{I}_X(m-1)) \neq 0 \).

Range A. If \( \frac{m^2+4m+6}{6} \leq d < \frac{m^2+4m+6}{3} \), then \( G(d, m) \leq G_A(d, m) \), where

\[ G_A(d, m) = 1 + \frac{d(m-1)}{m+2} - \frac{m+2}{3} \]

Range B. 1. If \( \frac{m^2+4m+6}{3} \leq d < m^2 - 2m + 2 \), then \( G(d, m) \geq G_B(d, m) \) (see [HH2] for a complete definition of \( G_B(d, m) \)). It is known that \( G(d, m) \) is equal to \( G_B(d, m) \) in several cases (see [Ha2], [GP2], [E], [ES], [S1], [S2]).

2. If \( d = m^2 - 2m + 2 \), then \( G(d, m) = 1 + \frac{(m-3)d}{3} \).

Range C. ([GP1] and [GP2]). If \( m^2 - 2m + 3 \leq d \), then \( G(d, m) \) is exactly known.

In range A, the upper bound for the genus is obtained by using Clifford's theorem (see [Ha1]). A long-standing conjecture is that \( G(d, m) = G_A(d, m) \); i.e. this genus is actually obtained by some smooth curve.

Two of the present authors announced [BE] an asymptotic solution for it; later on Ch. Walter announced a complete result about curves with seminatural cohomology which implied a positive answer to the conjecture. No written proof has appeared since then, due mainly to the technical nature of both proofs, which heavily used the so-called "m´ethode d’Horace", which leads to heavy, intricate, and sometimes cumbersome constructions.

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Joint work with E. Mezzetti, G. Ottaviani
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GOAL: To establish a close relationship between a priori two unrelated problems:

(1) The existence of homogeneous artinian ideals $I \subset k[x_0, x_1, \ldots, x_n]$ which fail the Weak Lefschetz Property;

(2) The existence of (smooth) projective varieties $X \subset \mathbb{P}^n$ satisfying at least one Laplace equation of order $s \geq 2$. 
An $n$-dimensional projective variety $X \subset \mathbb{P}^N$ satisfies $\delta$ independent Laplace equations of order $s$ if its $s$-osculating space at a general point $p \in X$ has dimension $\left(\begin{array}{c} n+s \\ s \end{array}\right) - 1 - \delta$.

A homogeneous artinian ideal $I \subset k[x_0, x_1, \cdots, x_n]$ has the Weak Lefschetz Property (WLP) if $\exists L \in k[x_0, x_1, \cdots, x_n]$ such that, for all integers $j$, the multiplication map

$$\times L : (k[x_0, x_1, \cdots, x_n]/I)_j \to (k[x_0, x_1, \cdots, x_n]/I)_{j+1}$$

has maximal rank, i.e. it is injective or surjective.
Brenner-Kaid (2009): \((x^3, y^3, z^3, f(x, y, z)) \subseteq k[x, y, z]\) with \(\deg(f) = 3\) fails WLP iff \(f \in (x^3, y^3, z^3, xyz)\). In addition, \((x^3, y^3, z^3, xyz)\) is the only monomial artinian ideal generated by 4 cubics that fails WLP.

Togliatti (1929): The only non-trivial smooth surface \(X \subseteq \mathbb{P}^5\) obtained by projecting the Veronese surface \(V(2, 3) \subseteq \mathbb{P}^9\) and satisfying a Laplace equation of order 2 is the image of \(\mathbb{P}^2\) via the linear system

\[
\langle x^2 y, xy^2, x^2 z, xz^2, y^2 z, yz^2 \rangle \subseteq |\mathcal{O}_{\mathbb{P}^2}(3)|.
\]
Remark:

The linear system of cubics given by Brenner-Kaid’s example $\langle x^3, y^3, z^3, xyz \rangle$ is apolar to the linear system of cubics $\langle x^2y, xy^2, x^2z, xz^2, y^2z, yz^2 \rangle$ given in Togliatti’s example.

Question:

Is there a relationship between artinian ideals $I \subset k[x_0, \cdots, x_n]$ generated by $r$ forms of degree $d$ that fails WLP and projections of the Veronese variety $V(n, d)$ satisfying at least a Laplace equation of order $d – 1$?
$X \subset \mathbb{P}^N$ projective variety of dim $n$, $p \in X$ a smooth point. Choose affine coordinates around $p$ and a local parametrization of $X$ of the form $\phi(t_1, \ldots, t_n)$ where $p = \phi(0, \ldots, 0)$. The tangent space to $X$ at $p$ is the $k$-vector space generated by the $n$ partial derivatives of $\phi$ at $p$.

$$\dim T_p X = n.$$  

We define the $s$-th osculating space $T_p^{(s)} X$ to be the span of all partial derivatives of $\phi$ of order $\leq s$.

$$\dim T_p^{(s)} X \leq \binom{n + s}{s} - 1.$$

We say that $X$ satisfies $\delta > 0$ Laplace equations of order $s$ if strict inequality holds for all smooth points $p$ of $X$, and

$$\dim T_p^{(s)} X = (n+s) - 1 - \delta$$

for general $p$. We will also consider the projective $s$th osculating space $\mathbb{T}_p^{(s)} X$, embedded in $\mathbb{P}^N$. 

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Remarks.

(1) A non-degenerate curve $X \subset \mathbb{P}^N$ does not satisfy any Laplace equation.

(2) If $N < \binom{n+s}{s} - 1$, then $X$ satisfies at least one Laplace equation of order $s$.

(3) If $X \subset \mathbb{P}^N$ is a rational variety of dimension $n$, $\exists \mathbb{P}^n \dashrightarrow X$ a birational map given by $F_0, \ldots, F_N \in k[x_0, \ldots, x_n]_d$ and for a general point $p \in X$, the projective $s$-th osculating space $T_p^{(s)} X$ of $X$ at $p$ is generated by the $s$-th partial derivates of $F_0, F_1, \ldots, F_N$ at $p$.

Problems:

(1) To classify all rational surfaces $X \subset \mathbb{P}^N, N \geq 5$, which satisfy at least a Laplace equation of order 2.

(2) To classify all rational surfaces $X \subset \mathbb{P}^N, N \geq \binom{2+s}{s} - 1$, which satisfy at least a Laplace equation of order $s$.

(3) To classify all $n$-dimensional rational varieties $X \subset \mathbb{P}^N, N \geq \binom{n+s}{s} - 1$, which satisfy a Laplace equation of order $s$. 

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Macaulay-Matlis duality

- \( V = k^{n+1} \), \( R = \bigoplus_{i \geq 0} \text{Sym}^i V^* \cong k[x_0, x_1, \cdots, x_n] \), \( \mathcal{D} = \bigoplus_{i \geq 0} \text{Sym}^i V \cong k[y_0, y_1, \cdots, y_n] \).
- There are products

\[
\text{Sym}^i V^* \otimes \text{Sym}^j V \longrightarrow \text{Sym}^{i-j} V
\]

\[
F \otimes D \quad \leftrightarrow \quad F \cdot D
\]

making \( \mathcal{D} \) into a graded \( R \)-module. If \( F(x_0, x_1, \cdots, x_n) \in R \) and \( D(y_0, y_1, \cdots, y_n) \in \mathcal{D} \), then

\[
F \cdot D = F(\partial/\partial y_0, \partial/\partial y_1, \cdots, \partial/\partial y_n)D.
\]
• If $I \subset R$ is a homogeneous ideal, we define the Macaulay’s inverse system $I^{-1}$ for $I$ as

$$I^{-1} := \{ D \in \mathcal{D}, F \cdot D = 0 \text{ for all } F \in I \} \subset \mathcal{D}.$$ 

• If $M \subset \mathcal{D}$ is a graded $R$-submodule, then

$$\text{Ann}(M) := \{ F \in R, F \cdot D = 0 \text{ for all } D \in M \} \subset R.$$ 

• The pairing $R_i \times \mathcal{D}_i \longrightarrow k$ \quad $(F, D) \mapsto F \cdot D$ is exact; it is called the apolarity or Macaulay-Matlis duality action of $R$ on $\mathcal{D}$. If $F \cdot D = 0$ and $\text{deg}(F) = \text{deg}(D)$, then $F$ and $D$ are said to be apolar to each other.

• For any integer $i$, we have $h_{R/I}(i) = \dim_k (R/I)_i = \dim_k (I^{-1})_i.$
Theorem

We have a bijective correspondence

\[
\{\text{Homogeneous ideals } I \subset R\} \iff \{\text{Graded } R - \text{submodules of } D\}
\]

\[
I \quad \rightarrow \quad I^{-1}
\]

\[
\text{Ann}(M) \quad \leftarrow \quad M
\]

Furthermore, \(I^{-1}\) is a finitely generated \(R\)-module if and only if \(R/I\) is an artinian ring.

When considering only monomial ideals, we can simplify by regarding the inverse system in the same polynomial ring \(R\), and in any degree, \(d\), the inverse system \(I_d^{-1}\) is spanned by the monomials in \(R_d\) not in \(I_d\).

Example

If \(I = (x^4, y^4, z^4, x^3y, x^3z, xy^3, xz^3, y^3z, yz^3) \subset k[x, y, z]\), then \(I^{-1} = (x^2y^2, x^2yz, x^2z^2, xy^2z, xyz^2, y^2z^2)\).
• Let \( I \) be an artinian ideal generated by \( r \) forms of degree \( d \): \( F_1, \cdots, F_r \in R = k[x_0, x_1, \cdots, x_n] \). Let \( I^{-1} \subset \mathcal{D} \) be its Macaulay inverse system. Associated to \((I^{-1})_d\) there is a rational map

\[
\varphi(I^{-1})_d : \mathbb{P}^n \dashrightarrow \mathbb{P}^{\binom{n+d}{d} - r - 1}.
\]

\( \text{Im}(\varphi(I^{-1})_d) \subset \mathbb{P}^{\binom{n+d}{d} - r - 1} \) is the projection of \( V(n, d) \) from the linear system \( \langle F_1, \cdots, F_r \rangle \subset |\mathcal{O}_{\mathbb{P}^n}(d)| \). Let us call it \( X_{n,(I^{-1})_d} \).

• Associated to \( I_d \) there is a morphism

\[
\varphi_{I_d} : \mathbb{P}^n \longrightarrow \mathbb{P}^{r-1}.
\]

\( \varphi_{I_d} \) is regular because \( I \) is artinian. Its image \( \text{Im}(\varphi_{I_d}) \subset \mathbb{P}^{r-1} \) is the projection of the \( n \)-dimensional Veronese variety \( V(n, d) \) from the linear system \( \langle (I^{-1})_d \rangle \subset |\mathcal{O}_{\mathbb{P}^n}(d)| \). Let us call it \( X_{n,I_d} \).

• The varieties \( X_{n,I_d} \) and \( X_{n,(I^{-1})_d} \) are called apolar.
Thm (M-MR-O) Let $I \subset R$ be an artinian ideal generated by $r$ forms $F_1, \ldots, F_r$ of degree $d$, $r \leq \binom{n+d-1}{n-1}$. Then TFAE:

1. The ideal $I$ fails the WLP in degree $d - 1$, 
2. The homogeneous forms $F_1, \ldots, F_r$ become $k$-linearly dependent on a general hyperplane $H$ of $\mathbb{P}^n$, 
3. The $n$-dimensional variety $X_{n,(I^{-1})_d}$ satisfies at least one Laplace equation of order $d - 1$.

Remarks:

- The assumption $r \leq \binom{n+d-1}{n-1}$ ensures that the Laplace equations obtained in (3) are not trivial. In the particular case $n = 2$, this assumption gives $r \leq d + 1$.
- For $n = 2$, $d = 3$ and $I = (x_0^3, x_1^3, x_2^3, x_0 x_1 x_2) \subset k[x_0, x_1, x_2]$, we recover Togliatti’s example.

Definition: With notation as above, we will say that $I^{-1}$ (or $I$) defines a **Togliatti system** if it satisfies the three equivalent conditions in the above Theorem.
EXAMPLE.

Let $d = 2q + 1$ be an odd number and $n = 2$. Let $l_1, \ldots, l_d$ be general linear forms in $k[x, y, z]$. The ideal $(l_1^d, \ldots, l_d^d, l_1 l_2 \cdots l_d)$ is generated by $d + 1$ forms of degree $d$ and it fails the WLP in degree $d - 1$ because $l_1^d, \ldots, l_d^d, l_1 l_2 \cdots l_d$ become dependent on a general line $L \subset \mathbb{P}^2$.

- For $d = 3$ we recover Togliatti example.
- A similar construction in even degree produces ideals which do satisfy the WLP.
**Proposition**

Let $I \subset R := k[x_0, x_1, \cdots, x_n]$ be an artinian monomial ideal. Then $R/I$ has the WLP if and only if $x_0 + x_1 + \cdots + x_n$ is a Lefschetz element for $R/I$.

$$\mathcal{L}_{n,d} := |\mathcal{O}_{\mathbb{P}^n}(d)| \text{ and } n_d := \dim(\mathcal{L}_{n,d}) = \binom{n+d}{n} - 1.$$ 

**Definition**

- A linear subspace $\mathcal{L} \subset \mathcal{L}_{n,d}$ is called a **monomial linear system** if it can be generated by monomials.

- $\mathcal{L} \subset \mathcal{L}_{n,d}$ is called a **monomial Togliatti’s system** if, in addition, its apolar system $\mathcal{L}'$ generates an artinian ideal which fails WLP in degree $d - 1$, or equivalently, $X = \overline{\text{Im}(\varphi_{\mathcal{L}})}$ ($\varphi_{\mathcal{L}} : \mathbb{P}^n \dashrightarrow \mathbb{P}^\dim \mathcal{L}$ the rational map associated to $\mathcal{L}$) satisfies a Laplace equation of order $d - 1$. 
A linear subspace $\mathcal{L} \subset \mathcal{L}_{n,d}$ is called a monomial linear system if it can be generated by monomials.

$\mathcal{L} \subset \mathcal{L}_{n,d}$ is called a monomial Togliatti’s system if, in addition, its apolar system $\mathcal{L}'$ generates an artinian ideal which fails WLP in degree $d - 1$, or equivalently, $X = \overline{\text{Im}(\varphi_{\mathcal{L}})}$ ($\varphi_{\mathcal{L}} : \mathbb{P}^n \dashrightarrow \mathbb{P}^{\dim \mathcal{L}}$ the rational map associated to $\mathcal{L}$) satisfies a Laplace equation of order $d - 1$.

A monomial Togliatti’s system $\mathcal{L} \subset \mathcal{L}_{n,d}$ is said to be smooth if, in addition, $X$ is a smooth variety.

A monomial Togliatti’s system $\mathcal{L} \subset \mathcal{L}_{n,d}$ is said to be minimal if its apolar system $\mathcal{L}'$ is generated by monomials $m_1, \ldots, m_r$ and there is no a proper subset $m_{i_1}, \ldots, m_{i_{r-1}}$ defining a monomial Togliatti system.
Togliatti: For \( n = 2 \), the only smooth minimal monomial Togliatti system of cubics is \( I = (x_0^3, x_1^3, x_2^3, x_0 x_1 x_2) \).

**Theorem (Michałek - MR)**

Let \( I \subset R := k[x_0, x_1, \ldots, x_n] \) be a minimal smooth monomial Togliatti system of quadrics and \( n \geq 3 \). Then, there is a bipartition of \( n + 1 \): \( n + 1 = a_1 + a_2 \) with \( n - 1 \geq a_1 \geq a_2 \geq 2 \), such that, up to permutation of the coordinates

\[ I = (x_0, \ldots, x_{a_1 - 1})^2 + (x_{a_1}, \ldots, x_n)^2. \]
GOAL:

To classify ALL smooth minimal monomial Togliatti systems of cubics \( I \subset R := k[x_0, x_1, \cdots, x_n], n \geq 2. \)
Conjecture

The only smooth minimal monomial Togliatti system $\mathcal{L} \subset \mathcal{L}_{n,3}$ of cubics of dimension $n(n+1) - 1$ is

$$\mathcal{L} = |\{x_i^2 x_j\}_{0 \leq i \neq j \leq n}| \subset \mathcal{L}_{n,3}.$$

(1) $\mathcal{L} = |\{x_i^2 x_j\}_{0 \leq i \neq j \leq n}| \subset \mathcal{L}_{n,3}$ is a smooth minimal monomial Togliatti system of cubics of dimension $n(n+1) - 1$.

(2) $\mathcal{M} := |\{x_0^2 x_2, x_0 x_2^2, x_0^2 x_3, x_0 x_3^2, x_1^2 x_2, x_1 x_2^2, x_1^2 x_3, x_1 x_3^2, x_2^2 x_3, x_2 x_3^2, x_0 x_1 x_2, x_0 x_1 x_3\}| \subset \mathcal{L}_{3,3}$ is a smooth minimal monomial Togliatti system of cubics of dimension 11 (n=3).
Problem:

To classify all smooth minimal monomial Togliatti systems of cubics \( l \subset R := k[x_0, x_1, \cdots, x_n], \ n \geq 2. \)
\begin{itemize}
\item $n = 2$. The only smooth minimal monomial Togliatti system of cubics is $I = (a^3, b^3, c^3, abc) \subset k[a, b, c]$.
\item $n = 3$.
\end{itemize}

**Theorem**

Let $I \subset k[a, b, c, d]$ be a smooth minimal monomial Togliatti system of cubics. Then, up to a permutation of the coordinates, $I^{-1}$ is:

1. (23) $(a^2 b, a^2 c, a^2 d, ab^2, ac^2, ad^2, b^2 c, b^2 d, bc^2, bd^2, c^2 d, cd^2)$, $X$ is of degree 23, in $\mathbb{P}^{11}$, it is isomorphic to $\mathbb{P}^3$ blown up in the 4 coordinate points; or

2. (18) $(abc, abd, a^2 c, a^2 d, ac^2, ad^2, b^2 c, b^2 d, bc^2, bd^2, c^2 d, cd^2)$, $X$ is of degree 18, in $\mathbb{P}^{11}$, it is isomorphic to $\mathbb{P}^3$ blown up in the line \{c = d = 0\} and in the two points $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$; or

3. (13) $(abc, abd, acd, bcd, a^2 c, ac^2, a^2 d, ad^2, b^2 c, bc^2, b^2 d, bd^2)$, $X$ is of degree 13, in $\mathbb{P}^{11}$, it is isomorphic to $\mathbb{P}^3$ blown up in the two lines $\{a = b = 0\}$ and $\{c = d = 0\}$. 

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PROPOSITION (Mezzetti - MR - Ottaviani):

Consider a partition of \( n + 1: n + 1 = a_1 + a_2 + \cdots + a_s \) with \( n - 1 \geq a_1 \geq a_2 \geq \cdots \geq a_s \geq 1 \) and the monomial ideal

\[
I = (x_0, \cdots, x_{a_1-1})^3 + \cdots + (x_{n+1-a_s}, \cdots, x_n)^3 + J \text{ where }
\]

\[
J := (x_i x_j x_k \mid 0 \leq i < j < k \leq n \text{ and } \forall 1 \leq \lambda \leq s \quad \#(\{i, j, k\} \cap \{\sum_{\alpha \leq \lambda-1} a_{\alpha}, \cdots, \sum_{\alpha \leq \lambda} a_{\alpha} - 1\}) \leq 1).
\]

\( I \) is a smooth minimal monomial Togliatti system of cubics.

REMARK: \( \mu(I) = (\frac{a_1+2}{3}) + \cdots + (\frac{a_s+2}{3}) + \sum_{1 \leq i < j < h \leq s} a_i a_j a_h. \)

In particular, if \( a_1 = a_2 = \cdots = a_{n+1} = 1 \) or \( a_1 = n - 1 \) and \( a_2 = a_3 = 1 \), they have dimension \( n(n + 1) - 1 \) and we have a family of counterexamples to Ilardi’s conjecture.
CONJECTURE (Mezzetti - MR - Ottaviani):

Up to permutation of the coordinates, the above ideals are the only smooth minimal monomial Togliatti system of cubics.
THEOREM (Michałek - MR):

Let $I$ (or its inverse system $I^{-1}$) be a minimal smooth monomial Togliatti system of cubics. Then, up to a permutation of the coordinates, the pair $(I, I^{-1})$ is one of the ideals described in Proposition A. Moreover, $|I| \leq \binom{n+1}{3} + n + 1$ and if $|I| = \binom{n+1}{3} + n + 1$ then it corresponds to one of the following partitions:

- $n + 1 = (n - 1) + 1 + 1$,
- $n + 1 = 1 + 1 + \cdots + 1$,
- $4 = 2 + 2$. 

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OPEN PROBLEM:

To classify all smooth minimal monomial Togliatti systems $I \subset R := k[x_0, x_1, \cdots, x_n], n \geq 2$, of forms of degree $d \geq 4$.

THEOREM (Mezzetti - MR):

Let $I \subset R := k[x_0, x_1, \cdots, x_n], n \geq 2$, be a smooth minimal monomial Togliatti system of forms of degree $d \geq 4$. It holds

$$2n + 1 \leq \mu(I) \leq \binom{n + d - 1}{n - 1}.$$

E. Mezzetti and R.M. Miró-Roig. The minimal number of generators of a Togliatti system. Preprint arXiv:

THANK YOU!